Extreme events on networks and spatially extended regions

Ravindra E. Amritkar

[1]Institute of Infrastructure, Technology, Research and Management (IITRAM), Ahmedabad
ravin012@gmail.com

March 06, 2017

Plan of presentation

- Introduction
- Extreme events on networks
- Extreme events in continuous systems
- Conclusion

Introduction – Extreme events

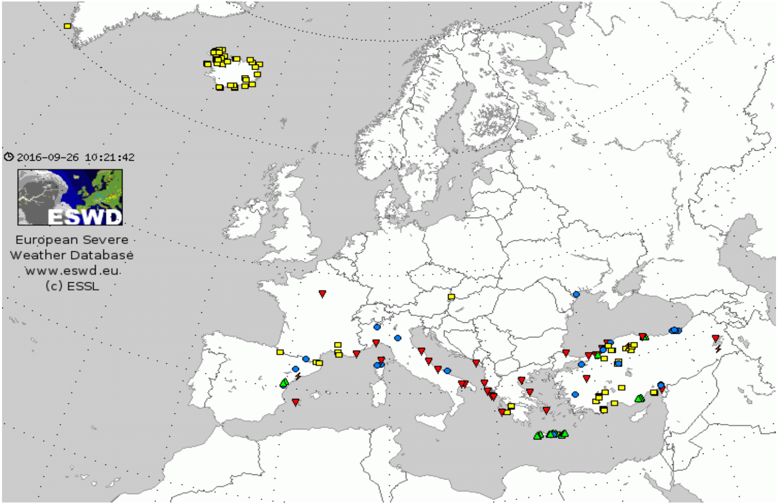
Flooding, Grimma, East-Germany, June 03, 2013

Train accident - Kanpur 20 Nov. 2016

Tornado, Hamburg, Germany, June 07, 2016

Earthquake Mag. 7.6, Sept. 21, 1999, Taiwan

Extreme events - spatial dependence

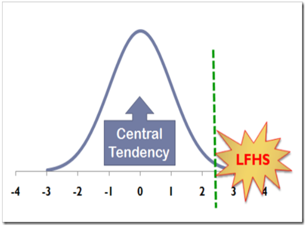


Extreme weather conditions in one weak over Europe, Sept. 19-26, 2016.
(red: tornado, yellow: extreme wind, green: hail, blue: heavy rain)

Extreme events

Basic features

- They are rare.
- They are recurrent.
- Which are inherent to the system under study.
- To which we can assign a variable (magnitude).



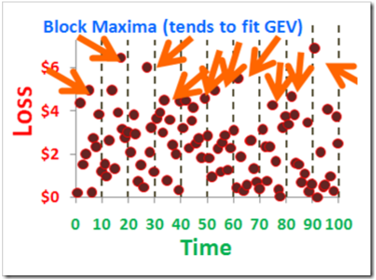
Events in the tail of the distribution

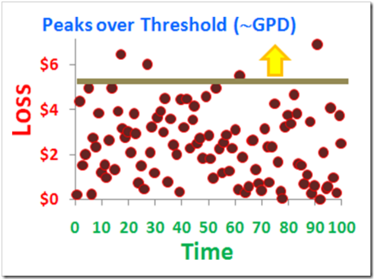
Definition

Given a probability distribution for the occurrence of events of given magnitude, an extreme event is an event which occurs in the tail of the distribution.

Extreme events

Two methods of analyzing extreme events



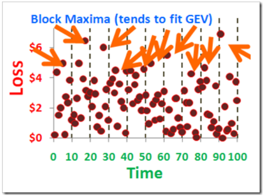


(a) Divide the time series in blocks of equal intervals
(b) An event of maximum size in each block is an extreme event. (Block size?)

An event which exceeds a threshold q is called as an extreme event.(Threshold q?)

Extreme events - statistical approach

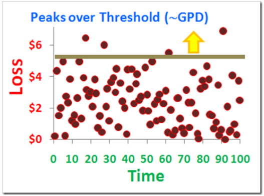
Limiting distributions for
independent identically distributed random variables



Generalized extreme event (GEV) distribution

$$f(x,\xi) = \frac{1}{\sigma}\left(1+\frac{\xi(x-\mu)}{\sigma}\right)^{-(\frac{1}{\xi}+1)} e^{-[1+\frac{\xi(x-\mu)}{\sigma}]^{-1/\xi}}$$

(ξ – shape, μ – location, σ – scale parameters)
Gumbel ($\xi = 0$), Fréchet ($\xi > 0$), Weibull ($\xi < 0$)



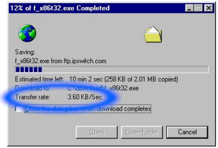
Generalized Pareto distribution

$$f(x,\xi,\mu,\sigma) = \frac{1}{\sigma}\left(1+\frac{\xi(x-\mu)}{\sigma}\right)^{-(\frac{1}{\xi}+1)}$$

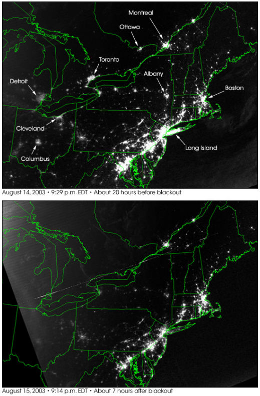
Extreme events on networks

Extreme events on networks

Traffic jam, China 12 days, 62 mile, aug 2010 (traffic network)



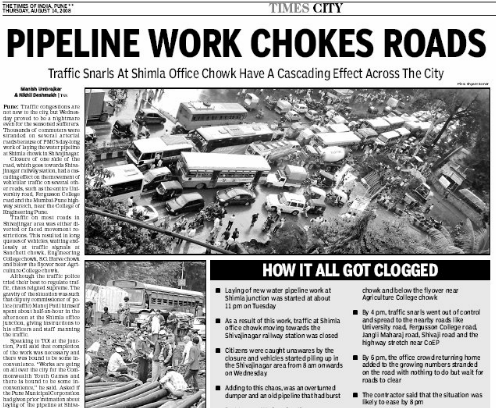
Internet slowdown (computer network)



USA blackout, Aug. 13,14, 2003 (Power grid)

These extreme events take place on some underlying network and hence, are our motivation behind studying Extreme Events on networks.

Extreme events - spatial dependence



THE TIMES OF INDIA, PUNE
THURSDAY, AUGUST 14, 2008

TIMES CITY

PIPELINE WORK CHOKES ROADS

Traffic Snarls At Shimla Office Chowk Have A Cascading Effect Across The City

Manish Umbrajkar & Nikhil Deshmukh | TNN

Pune: Traffic congestions are not new to the city but Wednesday proved to be a nightmare even for the seasoned sufferers. Thousands of commuters were stranded on several arterial roads because of PMC's day-long work of laying the water pipeline at Shimla chowk in Shivajinagar.

Closure of one side of the road, which goes towards Shivajinagar railway station, had a cascading effect on the movement of vehicular traffic on several other roads, such as the entire University road, Fergusson College road and the Mumbai-Pune highway stretch, near the College of Engineering Pune.

Traffic on most roads in Shivajingar area was either diverted or faced movement restrictions. This resulted in long queues of vehicles, waiting endlessly at traffic signals at Sancheti chowk, Engineering College chowk, S.G. Barve chowk and below the flyover near Agriculture College chowk.

Although the traffic police tried their best to regulate traffic, chaos reigned supreme. The gravity of the situation was such that deputy commissioner of police (traffic) Manoj Patil himself spent about half-an-hour in the afternoon at the Shimla office junction, giving instructions to his officers and staff manning the traffic.

Speaking to TOI at the junction, Patil said that completion of the work was necessary and there was bound to be some inconvenience. "Works are going on all over the city for the Commonwealth Youth Games and there is bound to be some inconvenience," he said. Asked if the Pune Municipal Corporation had given prior intimation about laying of the pipeline at Shiva-

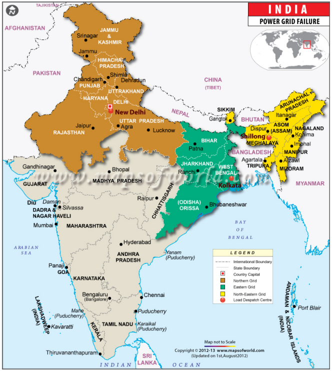
HOW IT ALL GOT CLOGGED

- Laying of new water pipeline work at Shimla junction was started at about 11 pm on Tuesday
- As a result of this work, traffic at Shimla office chowk moving towards the Shivajinagar railway station was closed
- Citizens were caught unawares by the closure and vehicles started piling up in the Shivajinagar area from 8 am onwards on Wednesday
- Adding to this chaos, was an overturned dumper and an old pipeline that had burst

chowk and below the flyover near Agriculture College chowk

- By 4 pm, traffic snarls went out of control and spread to the nearby roads like University road, Fergusson College road, Jangli Maharaj road, Shivaji road and the highway stretch near CoEP
- By 6 pm, the office crowd returning home added to the growing numbers stranded on the road with nothing to do but wait for roads to clear
- The contractor said that the situation was likely to ease by 8 pm

Cascading effect of traffic jam
(traffic network)



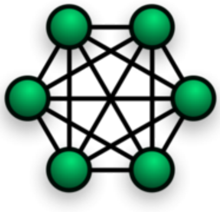
Hundreds of millions without power in India blackout
July 31, 2012
(power grid)

Types of networks

- Completely connected network
- Random networks
- Small world networks
- Scale free networks
- Percolation networks
- Bipartite networks
- Regular networks

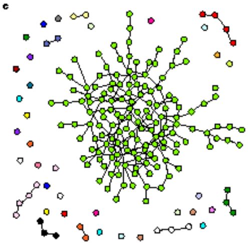
Networks

Completely connected



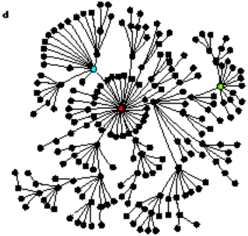
Each Node connected to all other nodes.

Random



Each pair of nodes connected with probability p.

Scale free



$$p(d) = d^{-\alpha}$$

Extreme events on network

Extreme events on networks: Statistical approach

The statistical approach can be extended by considering identically distributed random variables at different nodes.
Let $X = (x_1, x_2, \ldots, x_i, \ldots,)^T$ where x_i is a random variables at i-th node.
Different nodes can be coupled using network connections. E.g., In gaussian model, one can write the probability distribution function as

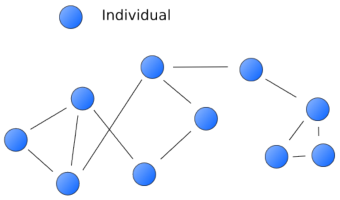
$$P(X) \propto exp\{-\alpha X^T A X\}$$

where A is the adjacency matrix. One can also introduce weighted connections.

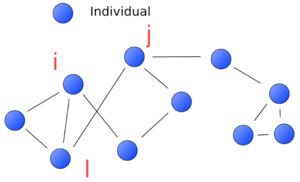
Extreme events on networks: Random walks

Model:

- A network of N nodes, E edges and W walkers.
- At every time step each walker takes a step to a neighboring node with some probability (unbiased or biased).
- Extreme event: The number of walkers on a node i exceeds a threshold q_i.



Walkers on a node



For an unbiased walk the asymptotic stationary probability for one walker is

$$p_j = \frac{k_j}{2E}, \quad k_j \text{ is } \textit{degree of node } j$$

Probability of w walkers on node i : Binomial distribution

$$f_i(w) = \begin{pmatrix} W \\ w \end{pmatrix} p_i^w \, (1 - p_i)^{W-w}$$

Mean no. of walkers on node i (flux) and its variance

$$\bar{w}_i = W \frac{k_i}{2E}, \qquad \sigma_i^2 = W \frac{k_i}{2E} \left(1 - \frac{k_i}{2E} \right)$$

Extreme events on networks

What is an extreme event on a node?
The threshold depends on the location / node.

A village square
(Malgudi days)

Delhi
Connaught place

Extreme events on networks

A village square
(Malgudi days)

Delhi
Connaught place

The threshold for extreme events depends on the node.

The threshold for node i

$$q_i = \bar{w}_i + m\sigma_i, \qquad m > 0$$

where $\bar{w}_i$ is the mean no. of walkers on node i (flux) and σ_i is the variance for each node

$$\bar{w}_i = W\frac{k_i}{2E}, \qquad \sigma_i^2 = W\frac{k_i}{2E}\left(1 - \frac{k_i}{2E}\right)$$

Extreme events on networks

Probability distribution of extreme events for a node of degree k and threshold q

$$\mathcal{F}(k) = \sum_{l=q}^{W} \binom{W}{l} p^l (1-p)^{W-l}, = I_p(\lfloor q \rfloor + 1, W - \lfloor q \rfloor).$$

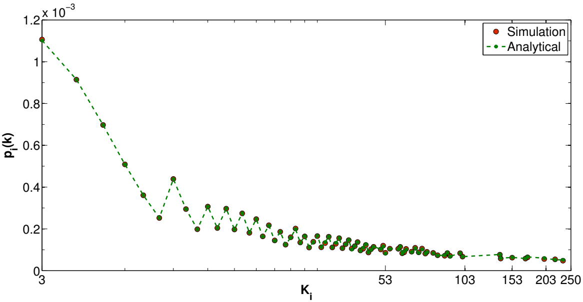
where the regularized incomplete beta function

$$I_p(\lfloor q \rfloor + 1, W - \lfloor q \rfloor) = \frac{1}{B(\lfloor q \rfloor + 1, W - \lfloor q \rfloor)} \int_0^p t^{\lfloor q \rfloor} (1-t)^{W - \lfloor q \rfloor - 1}$$

$B(a, b)$ is the beta function.

V. Kishore, M. S. Santhanam and REA, Phys. Rev. Lett. **106**, 188701 (2011).

Extreme events on networks



Distribution of extreme events as a function of the degree of a node

$N = 5000,\ E = 19815\ W = 2E$. Scale free network ($\gamma = 2.2$)

Small degree nodes

Small degree nodes have a higher probability of extreme events than the large degree nodes.

V. Kishore, M. S. Santhanam and REA, Phys. Rev. Lett. **106**, 188701 (2011).

Extreme events on networks

Fluctuating no. of walkers

If the total number of walkers changes with time, and is a uniform random variable in the interval $(W-\Delta, W+\Delta)$, then

Probability of w walkers on a node

$$f^{\Delta}(w) = \sum_{j=0}^{\widetilde{W}} \frac{1}{2\Delta+1} \begin{pmatrix} \widetilde{W}+j \\ w \end{pmatrix} p^{w} (1-p)^{\widetilde{W}+j-w}$$

Mean no. of walkers on a node (flux) and its variance

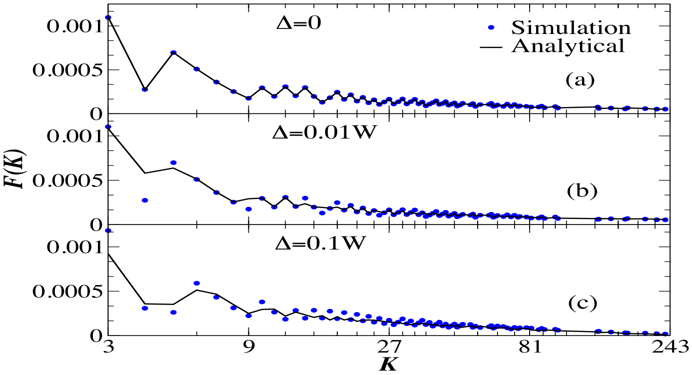
$$< f^{\Delta} > = < f >, \qquad \sigma_{\Delta}^{2} = < f > \left[1 + < f > \left(\frac{\Delta^2+\Delta}{3W^2} - \frac{1}{W} \right) \right]$$

Probability distribution of extreme events is

$$\mathcal{F}(k) \;=\; \sum_{j=0}^{2\Delta} \frac{1}{2\Delta+1} \sum_{l=q}^{\widetilde{W}+j} \binom{\widetilde{W}+j}{l} p^{l} \, (1-p)^{\widetilde{W}+j-l}$$

where $\widetilde{W} = W - \Delta$.

Extreme events on networks



Distribution of extreme events for fluctuating no. of walkers

Small degree nodes

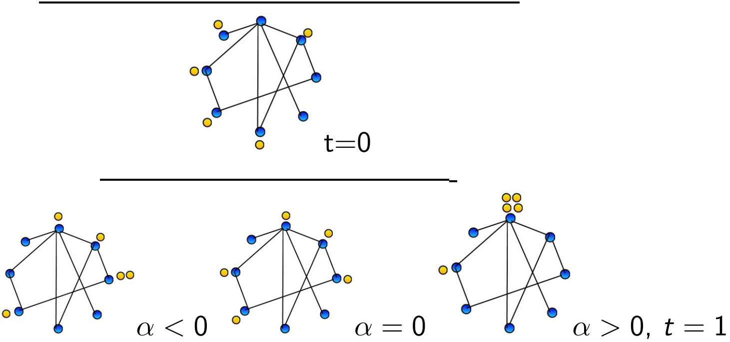
Small degree nodes have a higher probability of extreme events than the large degree nodes.

Extreme events on networks

Biased random walks

Probability of hopping from node i to j depends on the degree of j

$$b_{ij} \propto k_j^{\alpha}$$



Biased random walks

Stationary probability of finding walker at node j

$$p_j = \frac{k_j^{\alpha} \sum_{l=1}^{k_j} k_l^{\alpha}}{\sum_{m=1}^{N} \left(k_m^{\alpha} \sum_{l=1}^{k_m} k_l^{\alpha} \right)}.$$

We can define the generalized strength of j-th node to be

$$\phi_j = k_j^{\alpha} \sum_{i=1}^{k_j} k_i^{\alpha}.$$

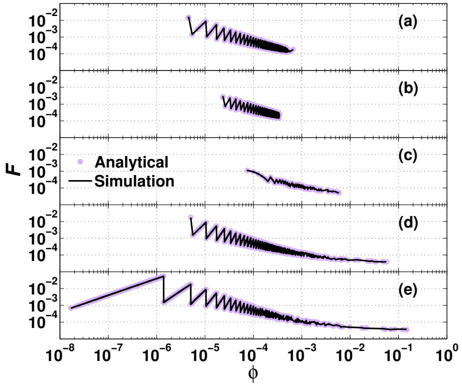
The stationary probability

$$p_j = \frac{\phi_j}{\sum_{l=1}^{N} \phi_l}.$$

Mean flux and variance

$$\langle f \rangle = W \frac{\phi}{\sum_{l=1}^{N} \phi_l}, \qquad \sigma^2 = W \frac{\phi}{\sum_{l=1}^{N} \phi_l} \left(1 - \frac{\phi}{\sum_{l=1}^{N} \phi_l} \right).$$

Extreme events on networks



$\alpha = -2$	Biased towards low degree nodes
$\alpha = -1$	
$\alpha = 0$	Standard Random walk
$\alpha = 1$	
$\alpha = 2$	Biased towards hubs

Extreme event probability as a function of strength for biased walks

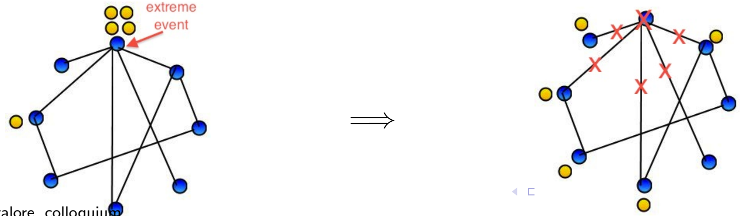
Small degree nodes

Nodes with small strength have a higher probability of extreme events than those with large strength.

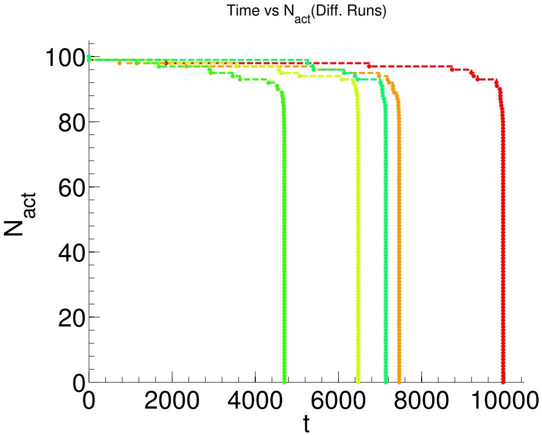
Network failure

Model of network failure

- Network with N nodes, E edges and W walkers
- Walkers perform unbiased or biased random walks.
- Extreme event – failure of a node
  - ⋆ A node i experiences an extreme event when the walkers on the node exceed the threshold $q_i =< f_i > +m\sigma_i$.
  - ⋆ The walkers on the said node walk to the nearest neighbors with the same walking rules.
  - ⋆ The said node and the edges connecting the node are removed.
- The walks continues.



Network failure - completely connected network



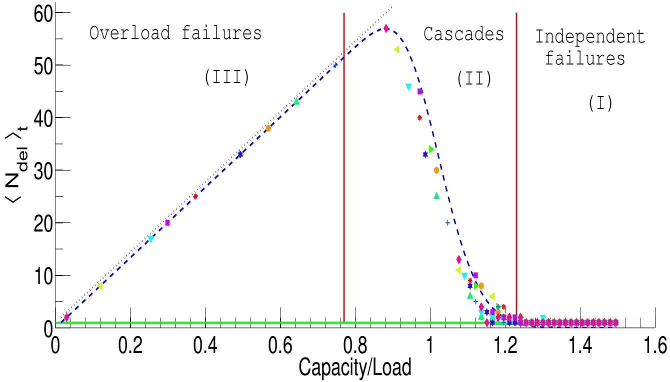
Network failure

Completely connected network

$N = 100,\ E = 4950,\ W = 9900,\ p = 1/100,\ q = 148\ (m = 5),\ \mathcal{F} = 1.19 \times 10^{-6}$

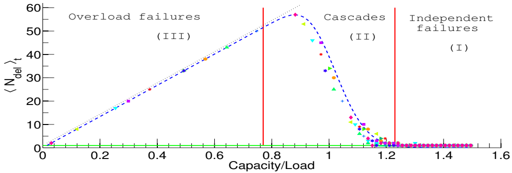
Network failure - completely connected network

Capacity of each node is the threshold	q_i.
Total load	W.
Total capacity of the network	$\mathcal{C} = \sum_{j=1}^{N} q_i$.
Capacity per unit load (Capacity/Load)	$= C/W$.



No. of deleted nodes at a time vs capacity/load

Network failure - completely connected network



Three regions

1. Independent failures: Individual uncorrelated failures at a time.

$$< N_{\mathrm{d}el} >< 1$$

2. Cascade failures: multiple node failure at successive times.
3. Overload failures: Single step failure of all the remaining nodes due to excessive load.

$$< N_{\mathrm{d}el} >\simeq N_{\mathrm{a}ct}$$

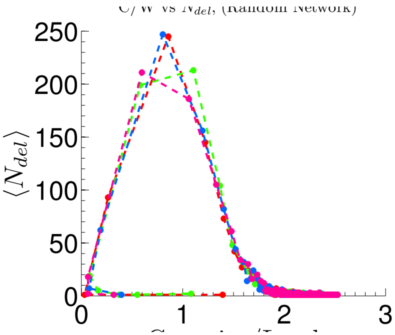
Network failure

Special features of the model of network failure

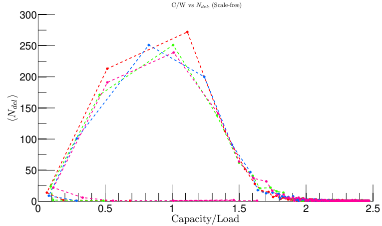
Our model	Other models
Driven by internal dynamics	External perturbation or input
Capacities are determined by the internal dynamics	Capacities are mostly randomly assigned
Extreme events occur due to inherent fluctuations of the internal dynamics	Extreme events occur due to external effects or load being large
Total failure of the network	Normally lead to partial failure of the network

Our model brings out the importance of internal fluctuations of the dynamics.

Network failure - random and SF networks



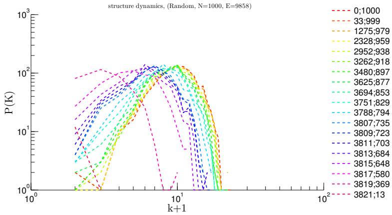
Random Network



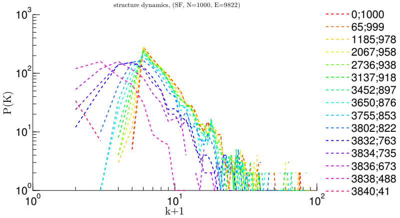
Scale-free network

Network failure - how the network changes

Effect on the degree distribution

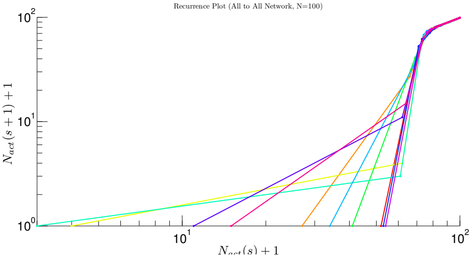


Random network

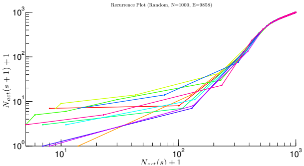


Scale-free network

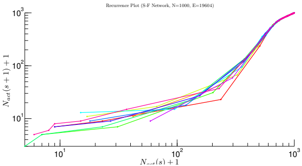
Network failure - return map



Completely connected network



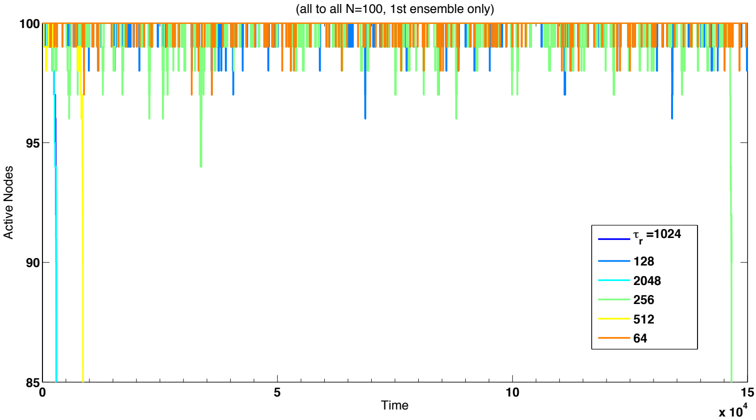
Random network



Scale-free network

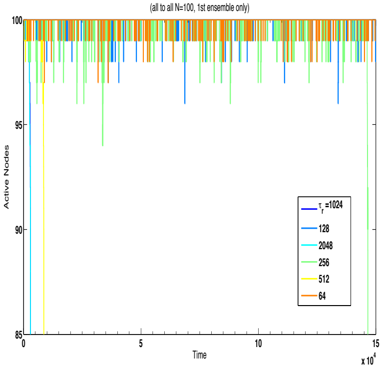
Network failure - repairing

Can one repair the nodes to prevent network failure?
Let τ_r be the repair time.

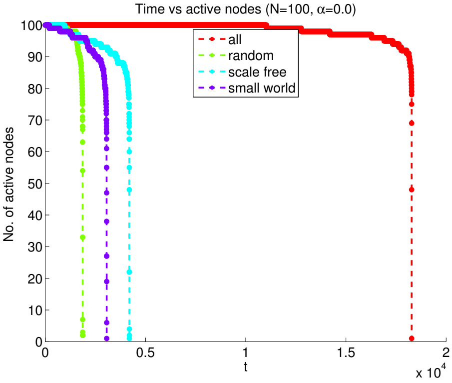


Network failure - repairing

Failure with and without repair



failure with repair



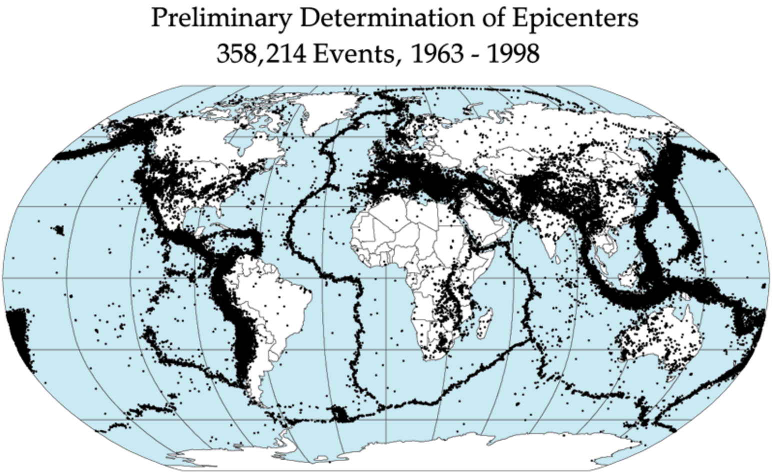
failure without repair

Network failure - repairing

- Except for very small repair time (< 10), the network fails.
- The failure times increase substantially.
- The nature of failure with and without repair is different.
- Without repair : failure through small degree nodes.
  With repair: failure through large degree nodes (hubs).

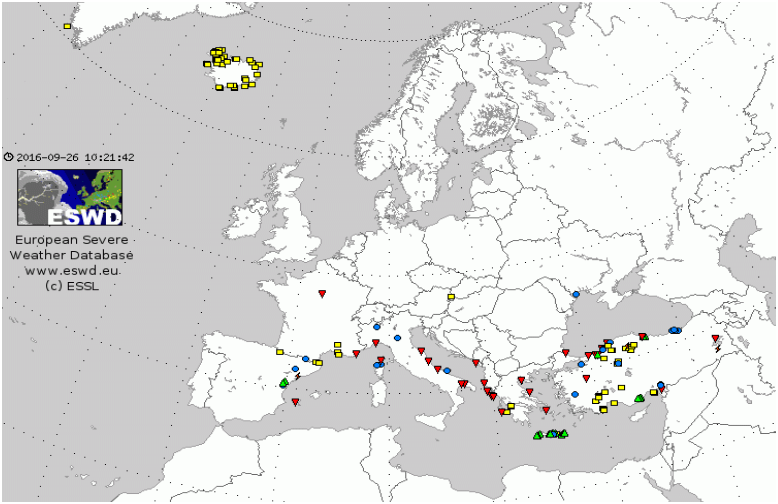
Extreme events on spatially extended regions

Extreme events - spatial dependence



Epicenters of earthquakes (mag > 5), 1963 to 1998.

Extreme events - spatial dependence



Extreme weather conditions in one weak over Europe, Sept. 19-26, 2016.
(red: tornado, yellow: extreme wind, green: hail, blue: heavy rain)

Model

Problem

(1) Extend the model to continuous space variables.
(2) Can some physical input be introduced?

Requirements of a model

1. A variable which depends on both space and time.
   - The magnitude of the variable represents the strength of the events.
   - The model should give probabilities of the variable as a function of space and time.
2. A function which gives the spatial properties of the terrain.

Our Model

Motion of Brownian particles in a potential.

Brownian particles in a potential

The motion of a Brownian particle in a one dimensional potential can be studied using Langevin equation.

$$\dot{x}(t) = h(x,t) + g(x,t)\Gamma(t)$$

where

$$< \Gamma(t) >= 0; \quad < \Gamma(t)\Gamma(t') >= \delta(t-t')$$

The probability distribution obeys Fokker-Planck equation.

$$\frac{\partial Q(x,t)}{\partial t} = -\frac{\partial}{\partial x} S(x,t)$$

$$S(x,t) = \left[\frac{\partial}{\partial x} D^{(1)} + \frac{\partial^2}{\partial x^2} D^{(2)}\right] Q(x,t)$$

where $S(x,t)$ is the probability current and

$$D^{(1)} = h(x,t) + \frac{\partial g(x,t)}{\partial x} g(x,t)$$

$$D^{(2)} = g^2(x,t)$$

Brownian particles in a potential

A Brownian particle of mass m in a potential $V(x)$ and high friction gives Smoluchowski equation,

$$\begin{aligned} D^{(1)} &= \frac{1}{m\gamma}F(x) = -\frac{1}{m\gamma}V'(x) \\ D^{(2)} &= D = \frac{kT}{m\gamma} \end{aligned}$$

where $\gamma = 1/\tau$ and τ is the relaxation time. The stationary solution when the probability current is zero is

$$Q_{st}(x) = Ae^{-\Phi(x)}$$

where

$$\Phi(x) = V(x)/D$$

("The Fokker-Planck Equation", H. Risken, Spriger)

Brownian particles in a potential

Consider a small interval $R = (x_1, x_2)$. The probability that a particle is in this interval is

$$p = \int_{x_1}^{x_2} Q_{st}(x)dx$$

Now consider W such independent Brownian particles.
Probability of w walkers in the interval R follows Binomial distribution

$$f_R(w) = \begin{pmatrix} W \\ w \end{pmatrix} p^w \, (1-p)^{W-w}$$

Mean no. of walkers in the interval R and its variance

$$\bar{w} = Wp, \qquad \sigma^2 = Wp(1-p)$$

Brownian particles - probability of extreme events

Define the threshold for extreme events as

$$q = \bar{w} + m\sigma$$

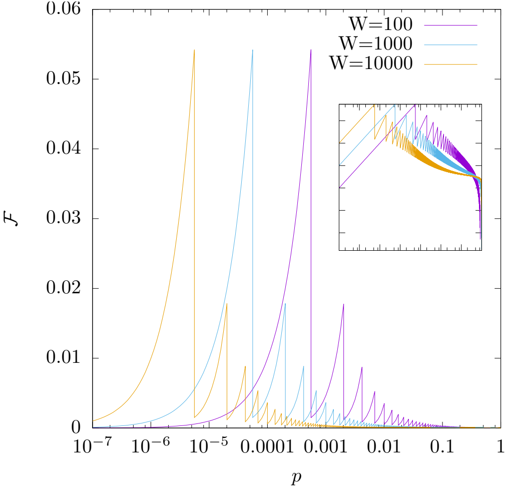
Then probability of observing an extreme event in the given interval is

$$\begin{aligned}\mathcal{F}(R) &= \sum_{l=\lfloor q \rfloor + 1}^{W} \binom{W}{l} p^l \, (1-p)^{W-l}, \\ &= I_p(\lfloor q \rfloor + 1, W - \lfloor q \rfloor)\end{aligned}$$

where the regularized incomplete beta function

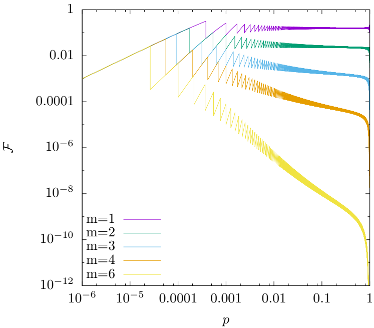
$$I_p(\lfloor q \rfloor + 1, W - \lfloor q \rfloor) = \frac{1}{B(\lfloor q \rfloor + 1, W - \lfloor q \rfloor)} \int_0^p t^{\lfloor q \rfloor} (1-t)^{W - \lfloor q \rfloor - 1}$$

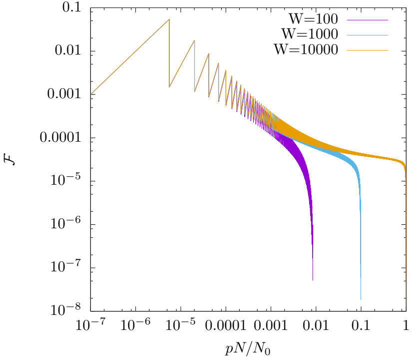
Brownian particles - probability of extreme events



Probability of extreme events $\mathcal{F}$
vs probability of Brownian particle in an interval p

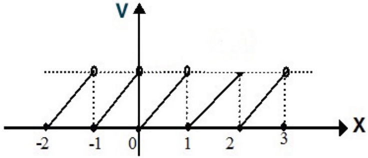
Brownian particles - probability of extreme events





Probability of extreme events $\mathcal{F}$
vs probability of Brownian particle in an interval p

Linear potential



We consider only one period

$$V(x) = \alpha x, \quad 0 \leq x < 1$$

Probability of a brownian particle being in $R = (x - dx/2, x + dx/2)$

$$p = \int_{x-dx/2}^{x+dx/2} Ae^{-cx} dx$$

where $c = \alpha \frac{m\gamma}{kT}$.

Linear potential

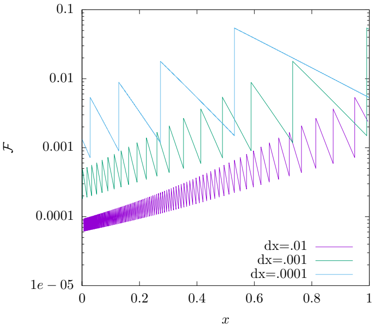
The threshold for extreme events is

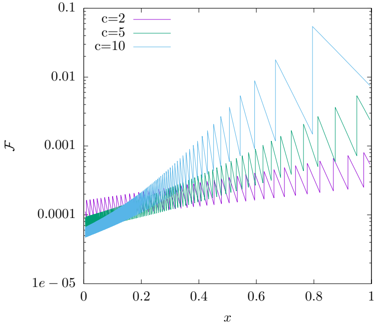
$$q = \bar{w} + m\sigma$$

where $\bar{w} = Wp,\ \sigma^2 = Wp(1-p)$.
The probability of observing an extreme event in the interval R is

$$\mathcal{F}(R) \quad = \quad I_p(\lfloor q \rfloor + 1, W - \lfloor q \rfloor)$$

Linear potential





Probability of extreme events $\mathcal{F}$ vs the location x

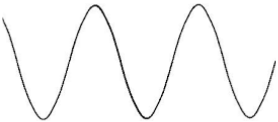
On the average, the probability of extreme events increases with increasing potential.

Extreme events vs location

Larger potential $\rightarrow$ Larger probability of extreme events

Smaller potential $\rightarrow$ Smaller probability of extreme events

Sinusoidal potential



We consider only one period

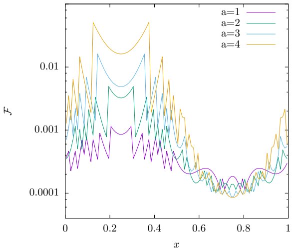
$$V(x) = \beta \sin(2\pi x), \quad 0 \leq x < 1$$

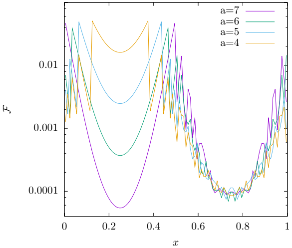
Probability of a brownian particle being in $R = (x - dx/2, x + dx/2)$

$$p = \int_{x-dx/2}^{x+dx/2} Ae^{-a\sin(x)} dx$$

where $a = \beta \frac{m\gamma}{kT}$.

Sinusoidal potential





Probability of extreme events $\mathcal{F}$ vs the location x

Extreme events vs location

Larger potential $\rightarrow$ Larger probability of extreme events
Smaller potential $\rightarrow$ Smaller probability of extreme events

Exception

For very large potentials (and/or very small interval) an opposite effect can be observed.

Conclusion

Networks

Smaller degree nodes have larger probability of extreme events
Large degree nodes (hubs) have smaller probability of extreme events

Continuous systems with potential

Larger potential has larger probability of extreme events
Lower potential has smaller probability of extreme events

Exception:

For very large potentials or very small regions the behavior changes.

Collaborators

1. Vimal Kishore, Stuttgart
2. M. S. santhanam, Pune
3. Wen-Jong Ma, Taipei
4. Chin-Kun Hu, Taipei

- Extreme events on complex networks
  V. Kishore, M. S. Santhanam and R. E. Amritkar,
  Phys. Rev. Lett. **106**, 188701 (2011).
- Extreme events and event size fluctuations in biased random walks on networks
  V. Kishore, M. S. Santhanam and R. E. Amritkar,
  Phys. Rev. E **85**, 056120 (2012).
- REA, W.-J. Ma, C.-K. Hu, Unpublished.


THANK YOU