Memory in Kardar-Parisi-Zhang growth: exact results via the replica Bethe ansatz, and experiments

P. Le Doussal (LPTENS)

- growth in plane, local stoch. rules => 1D KPZ class (integrability)

- discrete models in “KPZ class” => large time universality related to random matrix theory: Tracy Widom distributions of largest eigenvalue of GUE,GOE..
  Airy process, determinantal structure at level of one-time quantities

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Replica Bethe Ansatz method: integrable systems (Bethe Ansatz) +disordered systems(replica)

in math: discrete models => rigorous replica

with : Pasquale Calabrese (SISSA) Alberto Rosso (LPTMS Orsay)

Thomas Gueudre (Torino) Andrea de Luca (Oxford)

Thimothee Thiery (Leuwen)

large deviations KPZ/fermions

G. Schehr, S. Majumdar, D. Dean Alexandre Krajenbrink (LPTENS)

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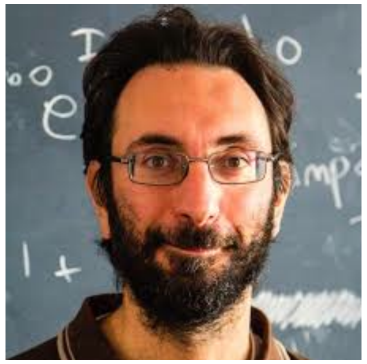
- two-time observables for models in 1d KPZ class ?

Jacopo de Nardis, PLD, arXiv:1612.08695
J. Stat. Mech. (2017) 053212
Tail of the two-time height distribution for KPZ growth in one dimension

Jacopo de Nardis, PLD, K. Takeuchi,
Phys. Rev. Lett. 118, 125701 (2017)
Memory and universality in interface growth

PLD arXiv:1709.06264 (2017)
Maximum of an Airy process plus Brownian motion and memory in KPZ growth



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Marton Kormos
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David Dean
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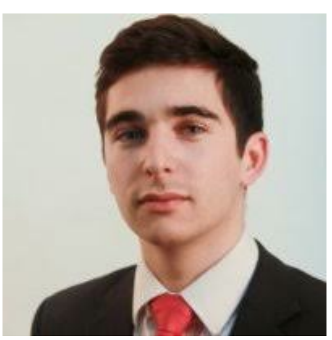


Alexandre Krajenbrink
(LPTENS)

Satya Majumdar + Gregory Schehr
(LPTMS Orsay)

Kazumaza Takeuchi (Tokyo U.)

Why is it interesting?

J. Baik, Z. Liu talk
multi-time
TASEP ring (2017)

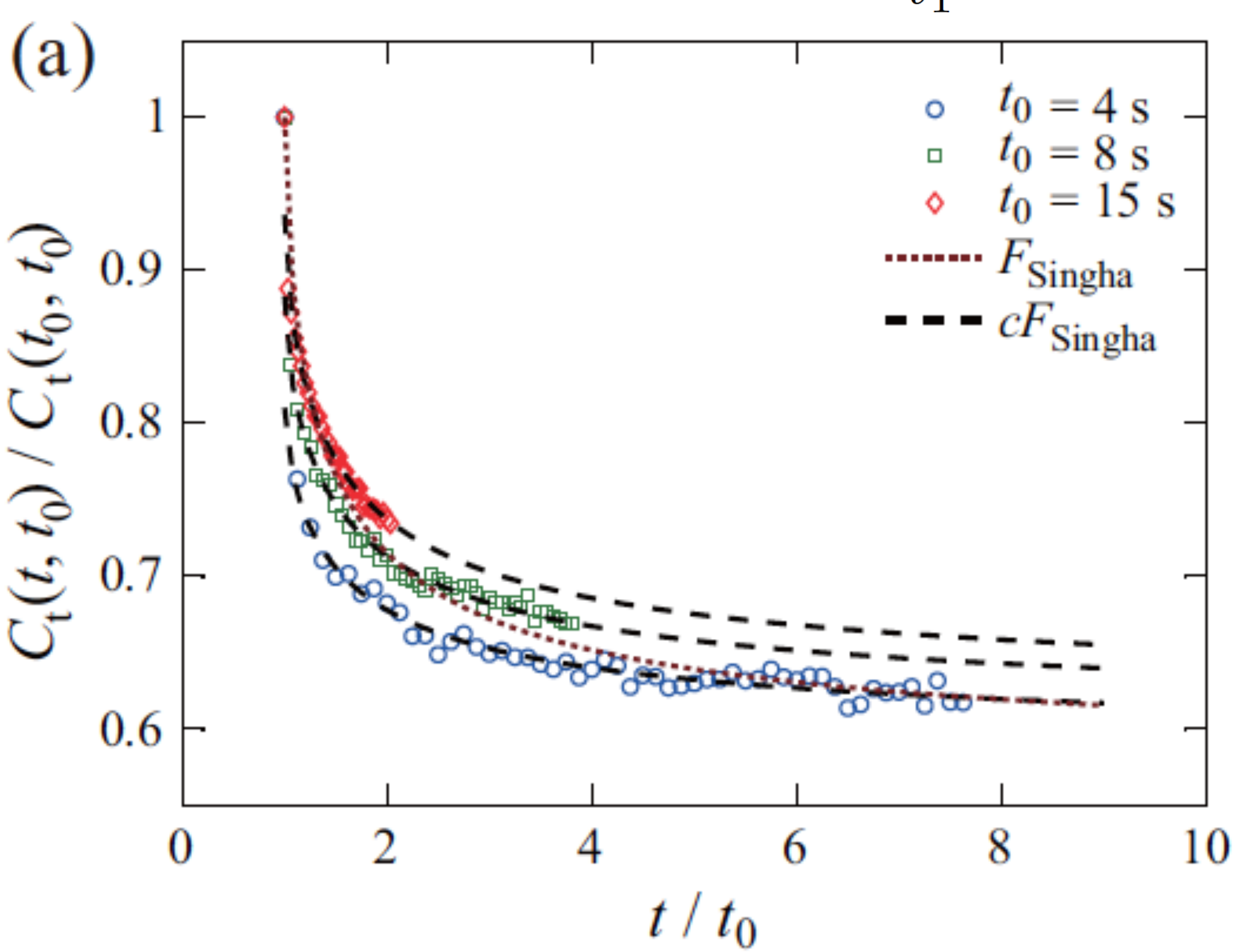
- what is large multi-time structure: determinantal?
- memory effect in time evolution: ergodicity breaking

two-time covariance ratio

droplet initial condition

$$\frac{\overline{h(0,t_1)h(0,t_2)}^c}{\overline{h(0,t_1)^2}^c} \xrightarrow[\substack{t_1,t_2 \to +\infty \\ \frac{t_2-t_1}{t_1}=\Delta}]{} C_\Delta \xrightarrow[\Delta \to \infty]{} C_{+\infty} \quad \begin{matrix} =0\,? \\ >0\,? \end{matrix}$$

persistent correlation



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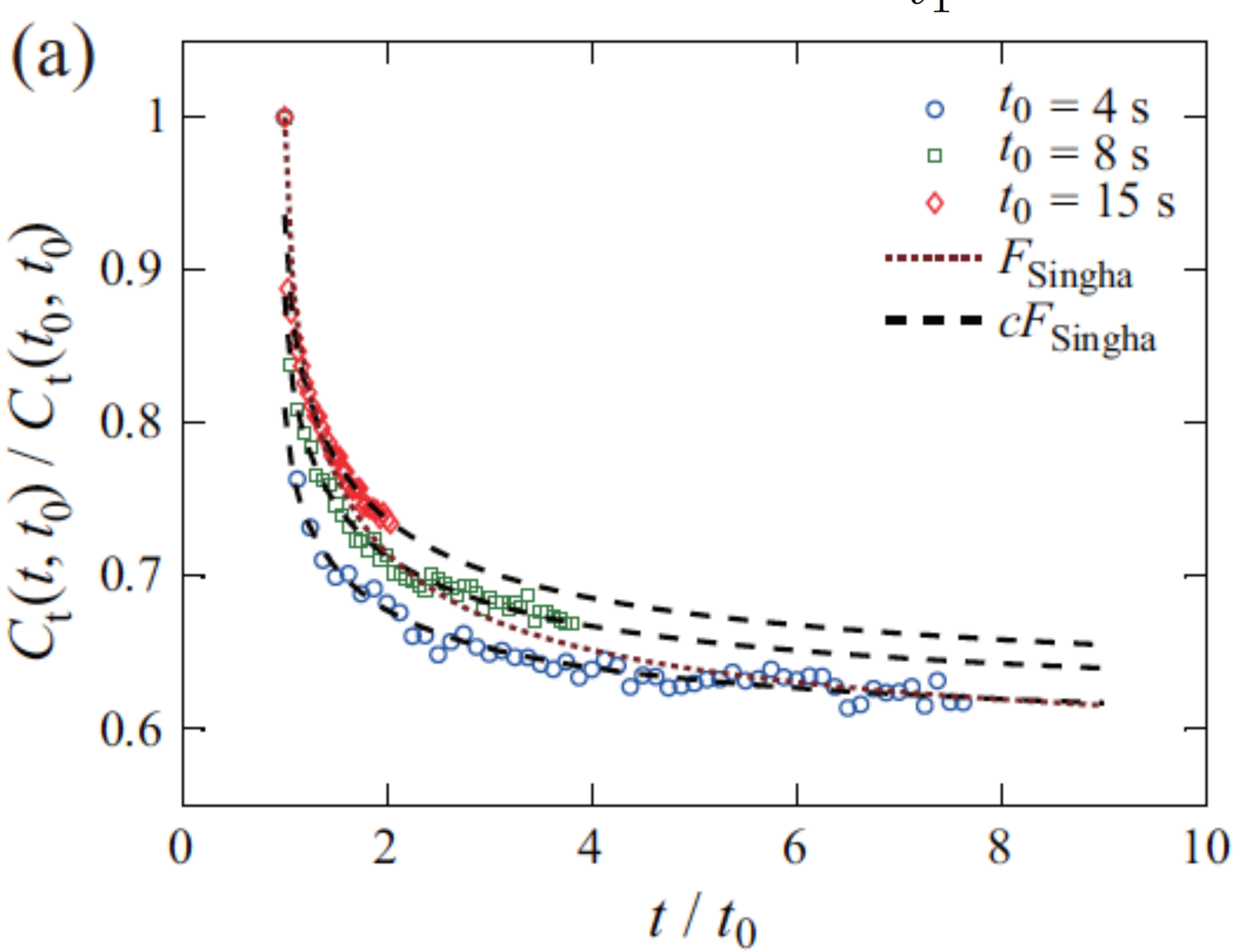
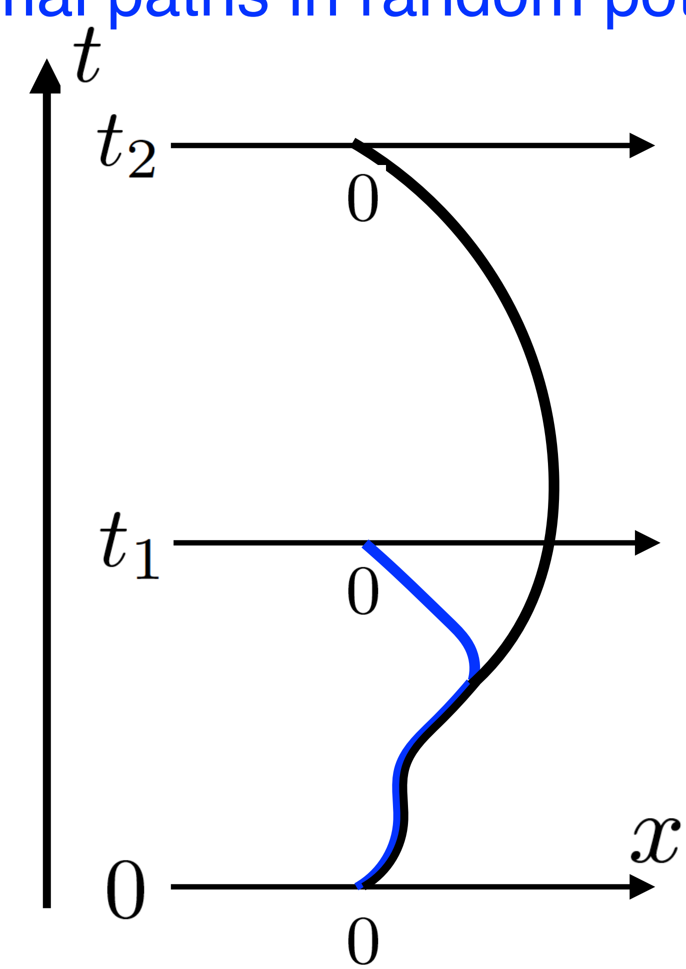


directed polymer: overlap/coalescence
optimal paths in random potential



Part I : one-time KPZ/DP: Replica Bethe Ansatz (RBA)

- KPZ equation, KPZ class, random matrices,Tracy Widom distributions.
- solving KPZ at any time by mapping to directed paths
  then using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => large time TW distrib. for KPZ height

- droplet initial condition => GUE
- flat initial condition => GOE
- half space initial condition => GSE
- stationary (Brownian) initial condition => Baik-Rains

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Part II: two-time KPZ via RBA

joint PDF $h(0,t_1)\ h(0,t_2)$

II a) direct approach any $\frac{t_2 - t_1}{t_1} = \Delta$

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exact tail of JPDF
for "large" positive $h(0,t_1)$

=> height difference crossover from stationary to GUE
=> agrees with experiments/numerics in broad range of $h(0,t_1)$

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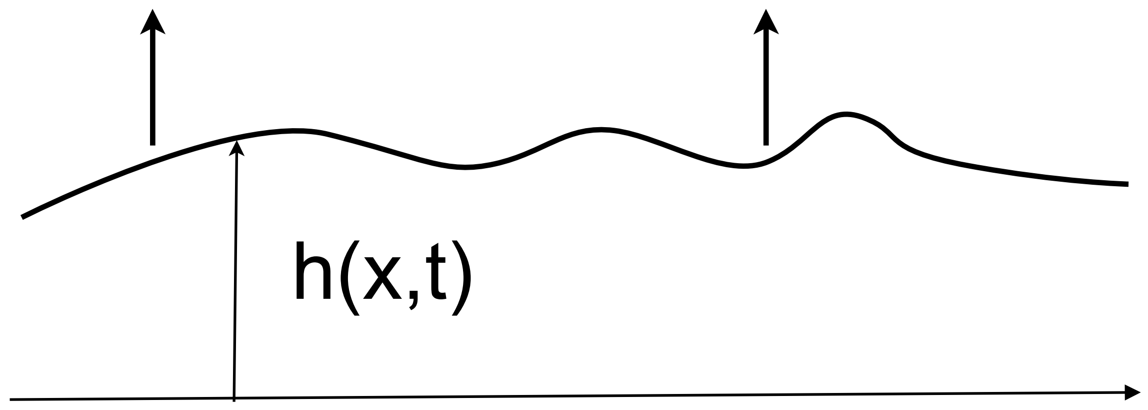
II b) limit $\Delta \to \infty$ exact JPDF of max of Airy process

⟶ persistent correlations in KPZ

a) b) mutually agree in matching region !

both RBA+approximations (magic recipe)

how to model a growing interface ?



neglect overhangs

large scale effective model

Edwards-Wilkinson

$$\partial_t h = \nu \partial_x^2 h + \eta(x,t) + v$$

surface tension noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

$$h_{\omega,q} = \frac{\eta_{\omega,q}}{\nu q^2 + i\omega}$$

$$\overline{hh}(q,\omega) = \frac{D}{\nu^2 q^4 + \omega^2}$$

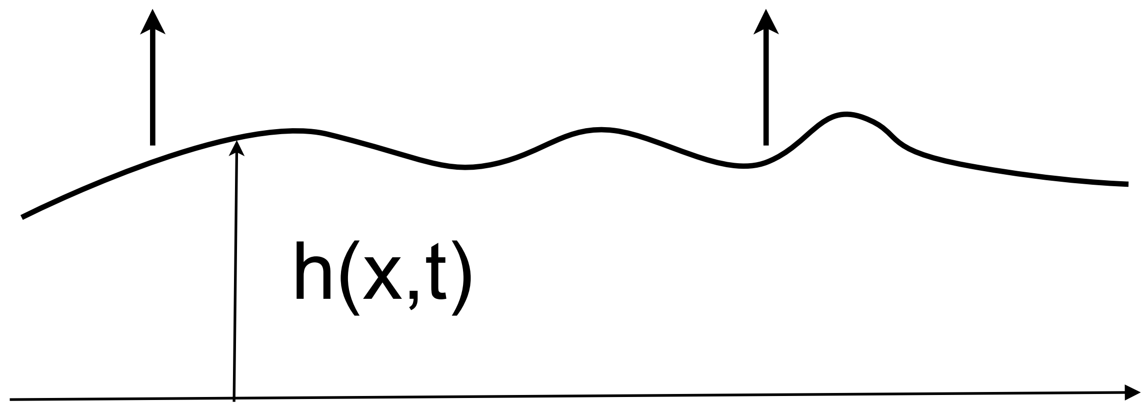
$$h \sim t^{1/4} \sim x^{1/2}$$

$$x \sim t^{1/2}$$

P(h) is gaussian, simple diffusive dynamics how to model a growing interface ?

neglect overhangs

large scale effective model



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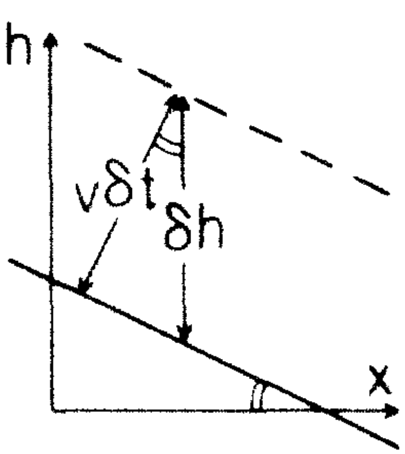
Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)



$$\frac{\partial h}{\partial t} = \frac{v}{\cos\theta}$$

$$= v\sqrt{1 + (\partial_x h)^2}$$

$$\approx v + \frac{v}{2}(\partial_x h)^2$$

non-linearity

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

noise

random deposition

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2} \quad x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

even at large time PDF depends on some broad features of initial condition

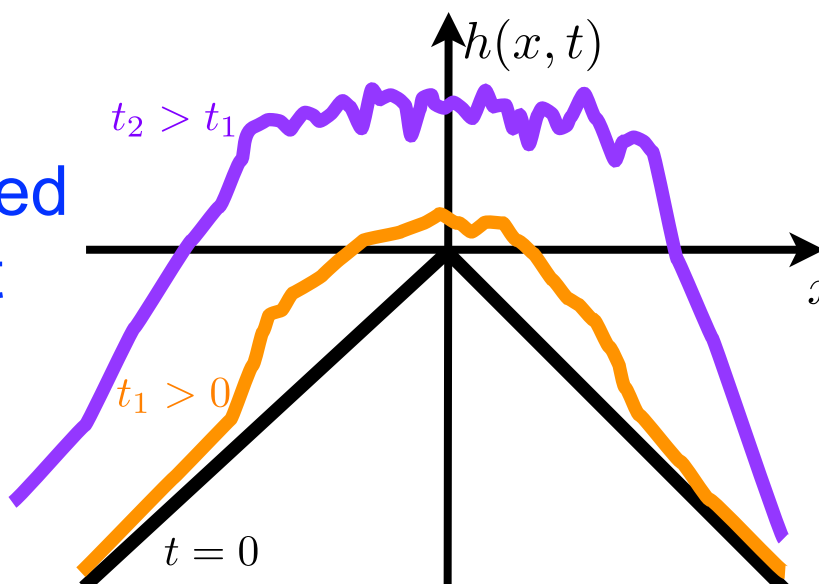
flat h(x,0) =0

wedge h(x,0) = - w |x| (droplet)

stationary h(x,0)=B(x)

related to different RMT ensembles !

wedge=curved =droplet

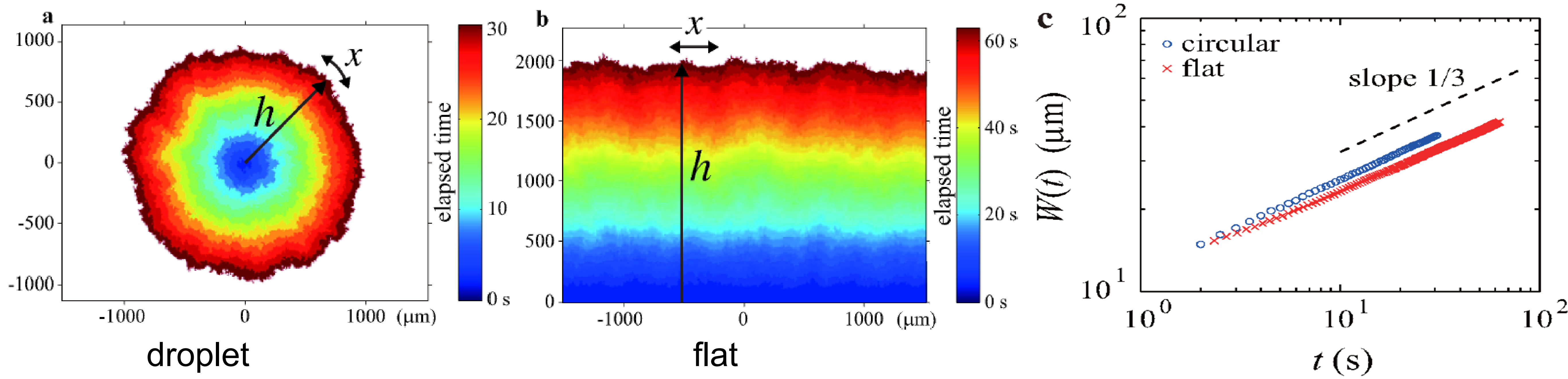


- short-time regime $\lambda_0 = 0$

Edwards Wilkinson P(h) gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t\to+\infty} v_\infty t + \chi\, t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

also reported in:

- slow combustion of paper — J. Maunuksela et al. PRL 79 1515 (1997)
- bacterial colony growth — Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)
- fronts of chemical reactions — S. Atis (2012)
- formation of coffee rings via evaporation — Yunker et al. PRL (2012)

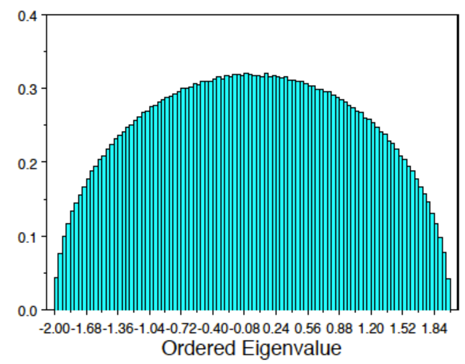
Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $i = 1,..N$

$$\beta = \begin{array}{ll} 1\ (\mathrm{GOE}) & \text{real symmetric} \\ 2\ (\mathrm{GUE}) & \text{hermitian} \\ 4\ (\mathrm{GSE}) & \text{symplectic} \end{array}$$

Universality large N :

- DOS: semi-circle law



- distribution of the largest eigenvalue Tracy Widom (1994)

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

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- 1 (GOE) real symmetric
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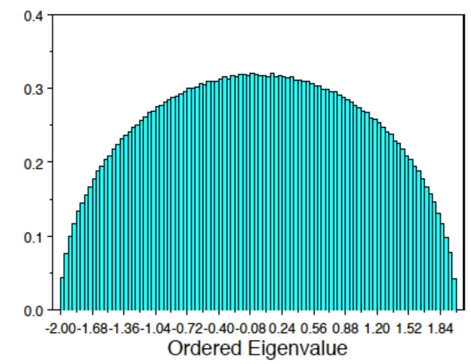

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CDF given by Fredholm determinants

GUE $F_2(s) = \mathrm{Det}(I - K)$ $\quad K(a,b) = K_{Ai}(a,b)\theta(a-s)\theta(b-s)$

$$K_{Ai}(a,b) = \int_0^{+\infty} dv Ai(a+v)Ai(b+v)$$

Airy kernel

GOE $K_1(x,y) = \theta(x)Ai(x+y+s)\theta(y)$

What is a Fredholm determinant?

$$K(x,y) = \theta(x)K(x,y)\theta(y)$$

$$Det[I-K] = e^{TrLn(I-K)} = e^{-\sum_{p=1}^{\infty}\frac{1}{p}TrK^p}$$

$$= 1 - TrK + \frac{1}{2}(TrK^2 - (TrK)^2) + ..$$

$$Det[I-K] = \sum_p \frac{(-1)^p}{p!}\int_{v_1>0,..v_p>0} det[K(v_i,v_j)]_{p\times p}$$

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$

Calculation of F1(s)

$$\sum_{j=1}^{m} w_j f(x_j) \approx \int_a^b f(x)\,dx$$

Gauss Legendre quadrature rule

$$d(z) = \det(I - zK\restriction_{L^2(a,b)})$$

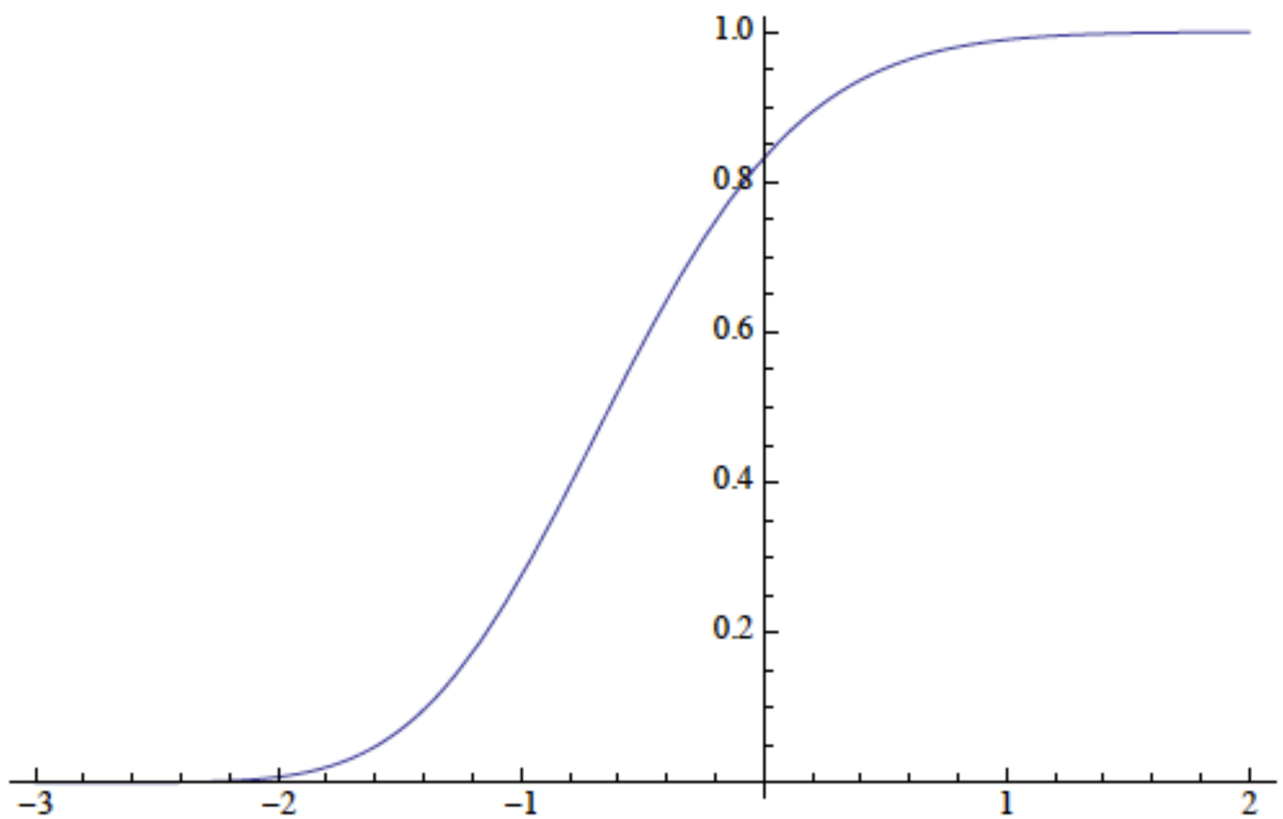
$$d_m(z) = \det\left(\delta_{ij} - z\, w_i^{1/2} K(x_i, x_j) w_j^{1/2}\right)_{i,j=1}^{m}$$

Bornemann (2009)

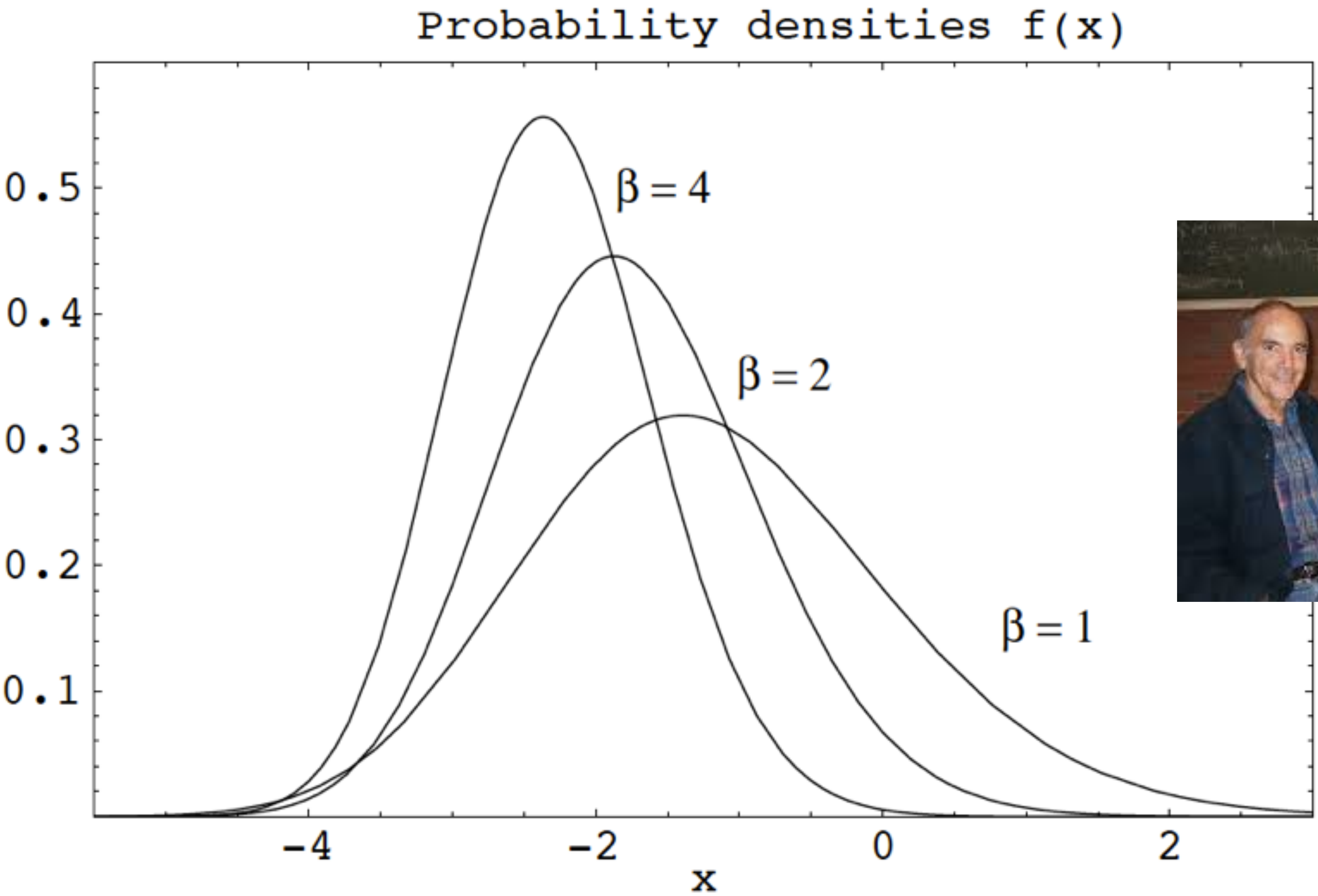
```
In[1]:= Needs["NumericalCalculus`"]
        GaussLegendre[a_, b_, m_] :=
         Module[{beta, T, V, c, d, e}, beta = Table[i / Sqrt[(2 i - 1) (2 i + 1)], {i, 1, m - 1}];
          T = DiagonalMatrix[beta, -1] + DiagonalMatrix[beta, 1];
          V = Eigensystem[N[T, 10]]; e = V[[2]]; d = Table[e[[i, 1]], {i, 1, m}];
          c = (V[[1]] + 1) / 2; {d^2 (b - a), (1 - c) a + b c}]
        FredholmDet[K_, z_, a_, b_, m_] := Module[{w, x}, {w, x} = GaussLegendre[a, b, m];
          w = Sqrt[w]; Det[IdentityMatrix[m] + (Transpose[{w}].{w}) Outer[K, x, x]]]

In[24]:= K[x_, y_] = -AiryAi[x + y];
         g[x0_, b_, m_] := N[FredholmDet[K, x0, x0, b, m]]
         g2 = Interpolation[Table[{x, g[x, 8, 20]}, {x, -10, 5, 0.2}]];
         Plot[g2[s], {s, -3, 2}]
```

Out[27]=



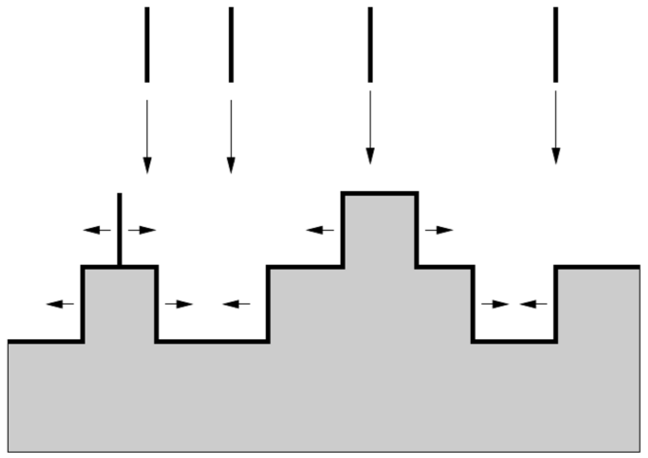
Tracy Widom distributions Tracy Widom (1994)



Exact results for height distributions for some discrete models in KPZ class

- PNG model

$$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$$



Baik, Deift, Johansson (1999)

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

droplet IC	GUE
flat IC	GOE

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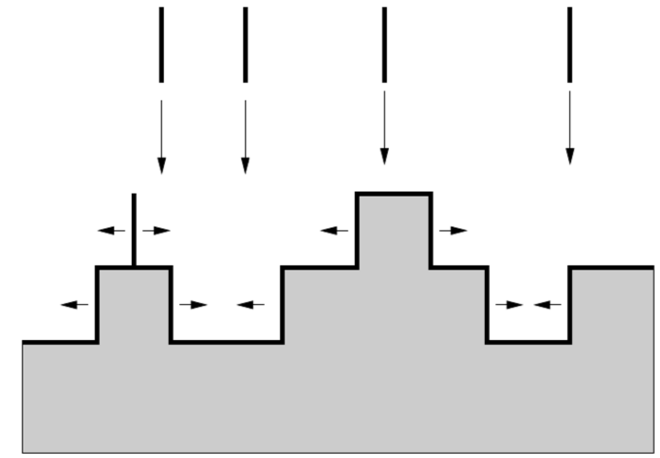
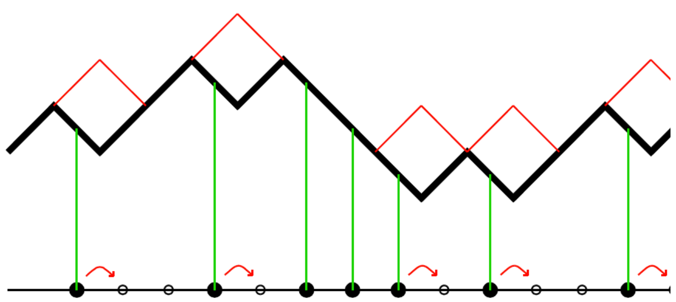
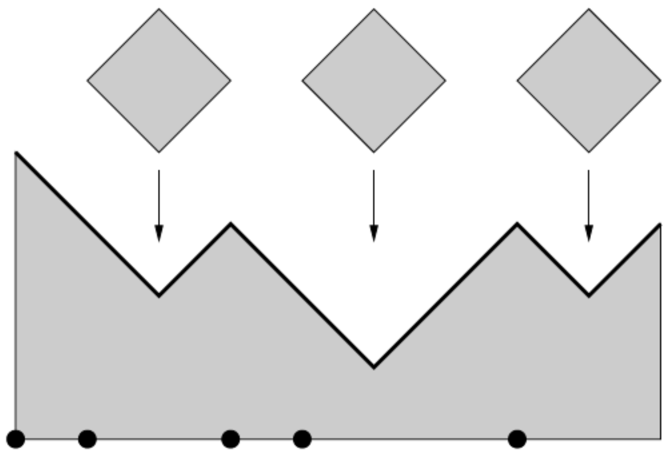
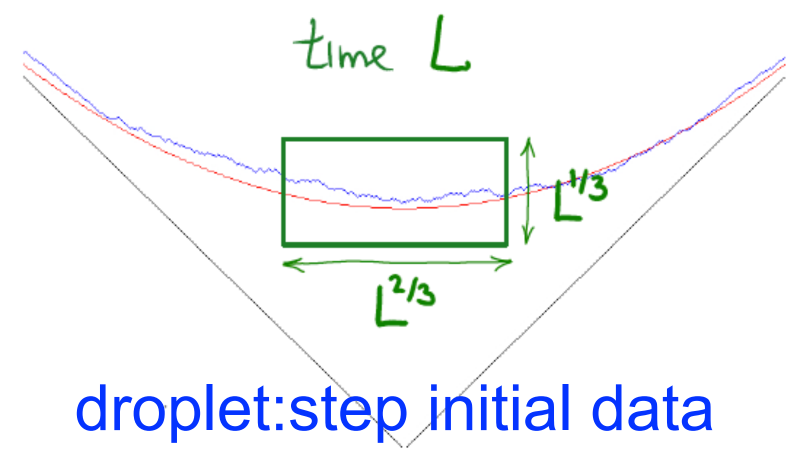


- similar results for totally asymmetric exclusion process (TASEP)

Johansson (1999), ...







droplet:step initial data

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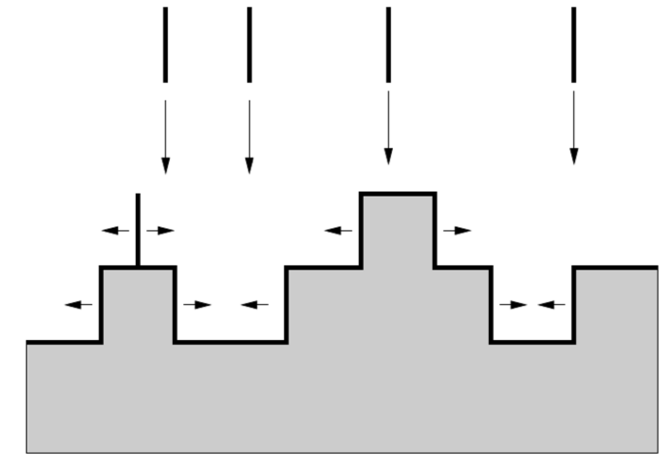

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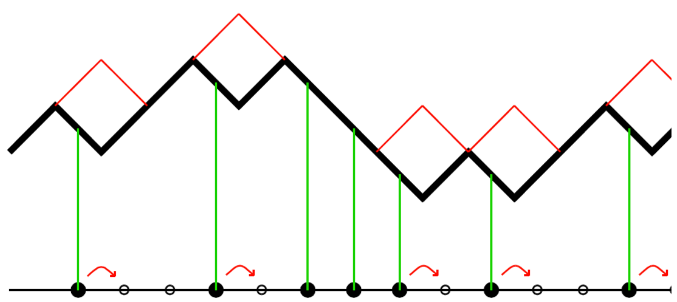
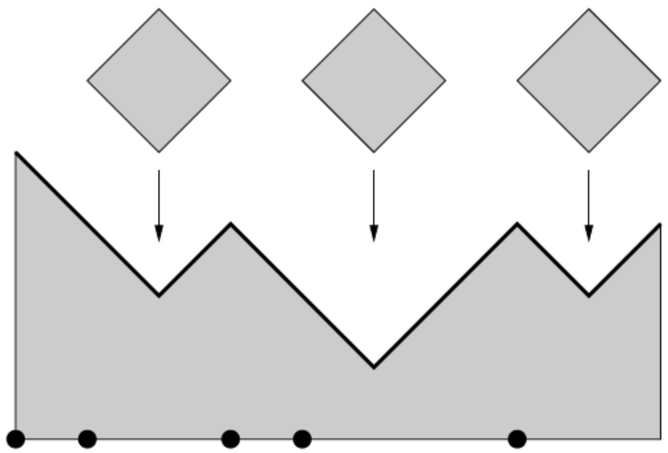
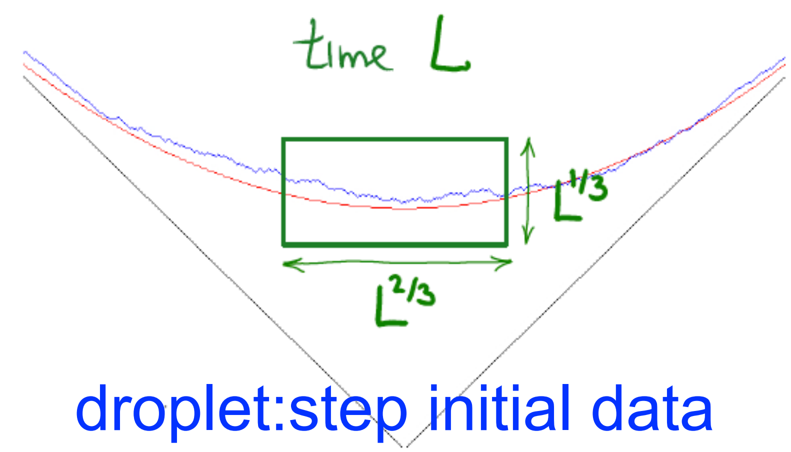






droplet:step initial data

- multi-space point correlations: Airy processes $A_2(y)$ $A_1(y)$ $\mathcal{A}_{\rm stat}(\hat{x})$

GUE GOE BR

$$(\Gamma t)^{-\frac{1}{3}}(h_{\rm drop}(x,t) - v_\infty t) \simeq \mathcal{A}_2(\hat{x}) - \hat{x}^2 \qquad \hat{x} = A\frac{x}{2t^{\frac{2}{3}}}$$

Airy2 process: stationary, reflection symmetric

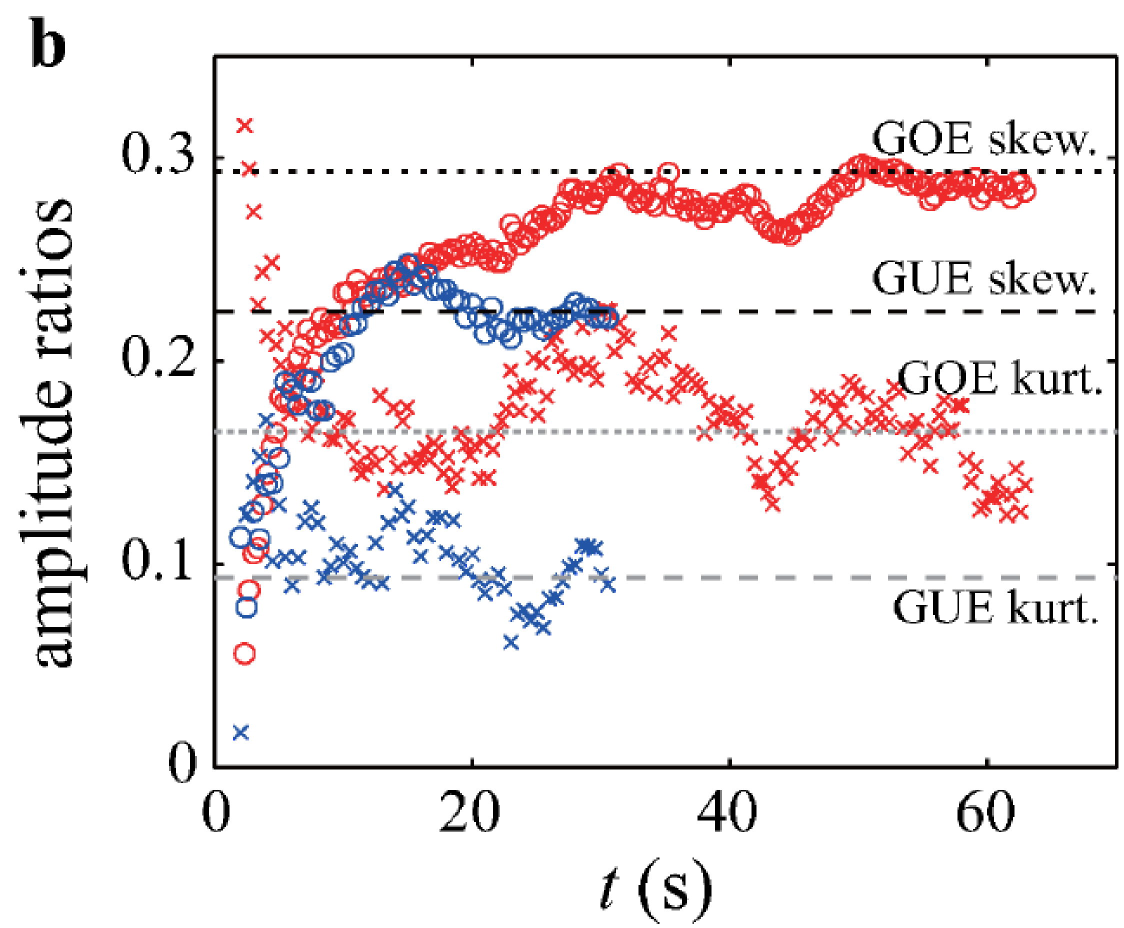
(scaled centered) trajectory of largest eigenvalue in DysonBM

m-spacepoint joint CDF is m*m matrix Fredholm determinant (extended Airy kernel)

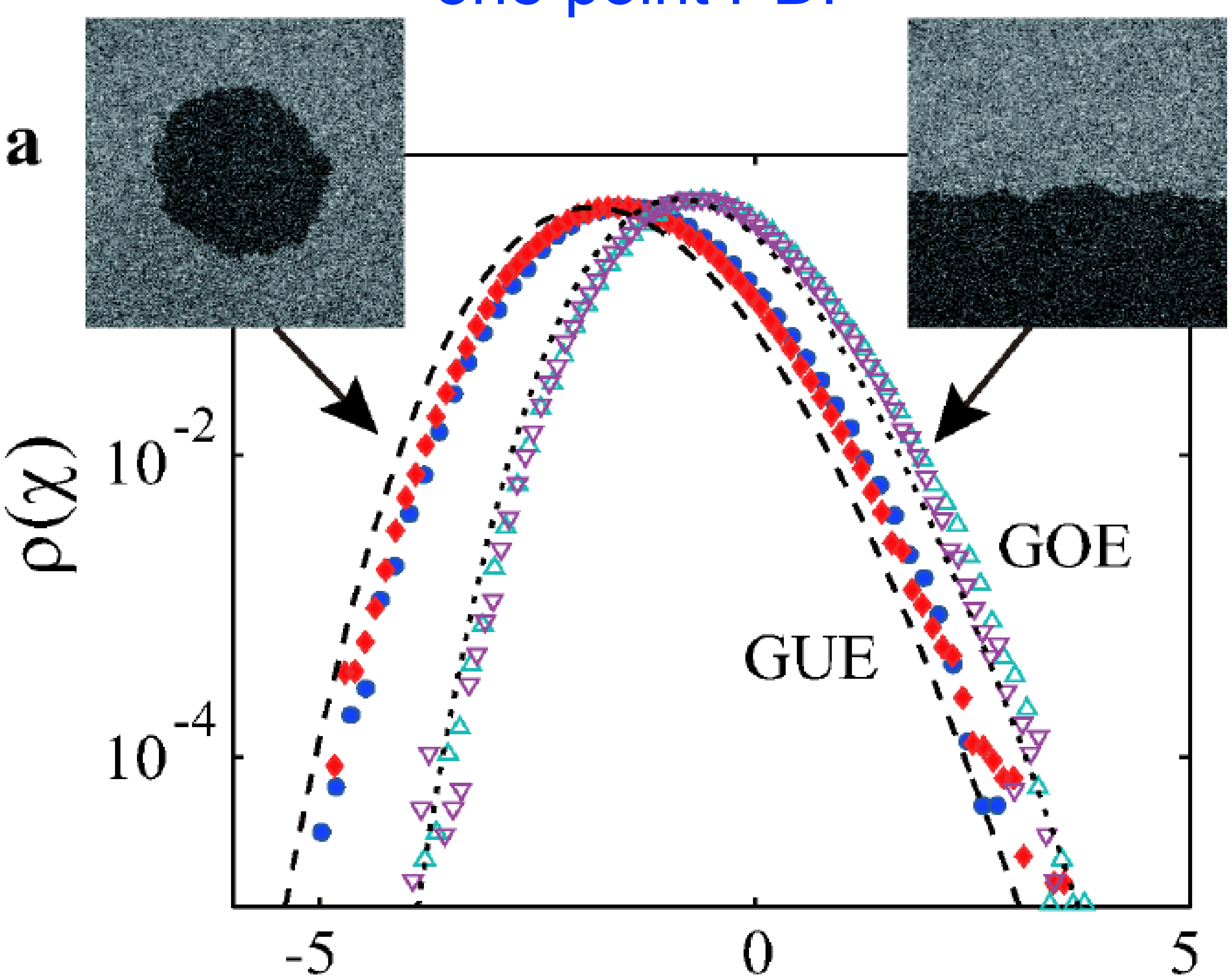
Takeuchi, Sano

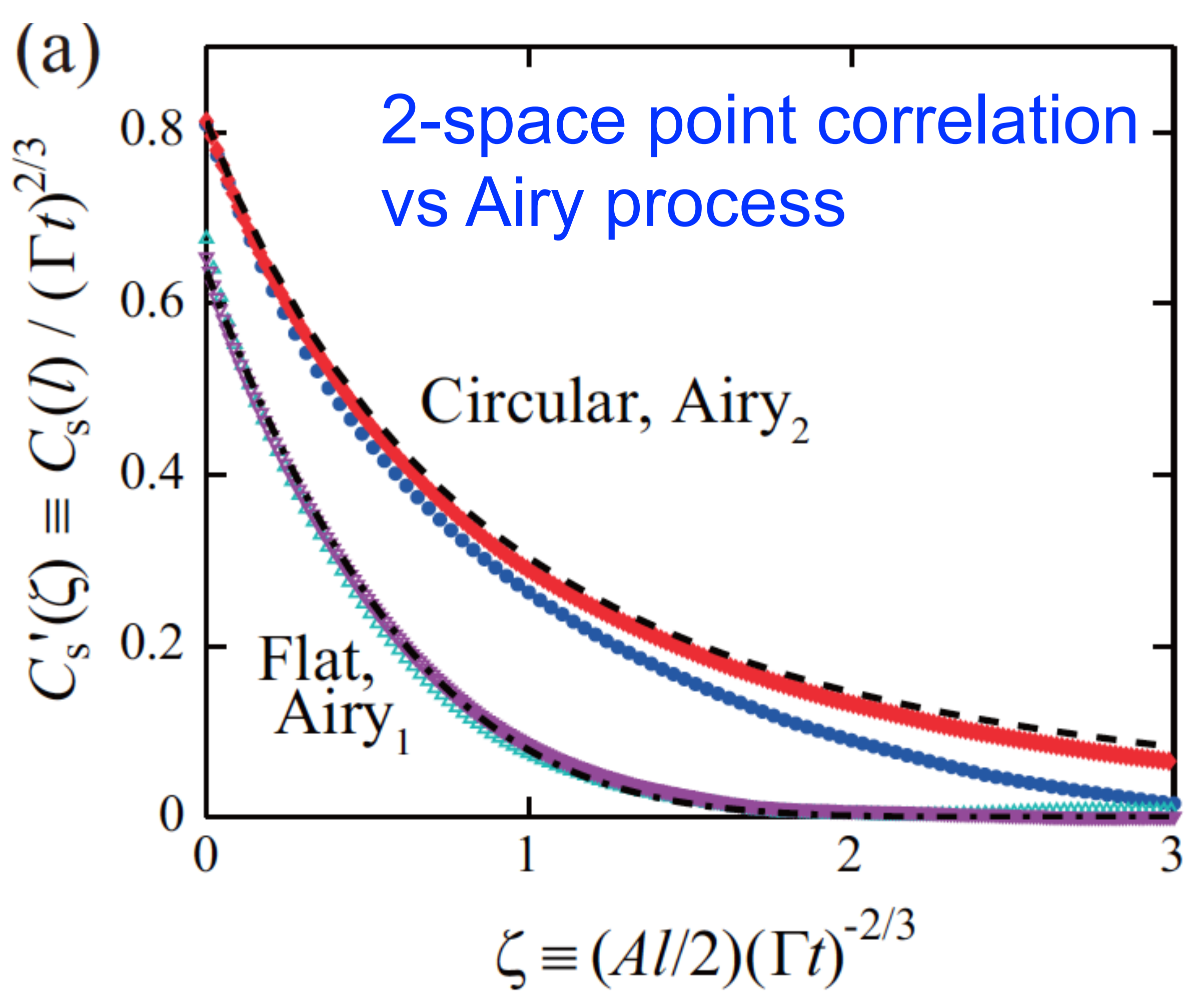
skewness = $$\frac{< (h- < h >)^3 >}{< (h- < h >)^2 >^{3/2}}$$

$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi,$$



one point PDF





part I

Cole Hopf mapping

KPZ equation ⟷ Continuum directed paths (polymers) in a random potential

Replica ↕

Quantum mechanics of bosons (imaginary time)

delta Bose gas solved

solved Bethe Ansatz

one-time KPZ height distributions ←

Kardar 87

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)

- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010) Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).

- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

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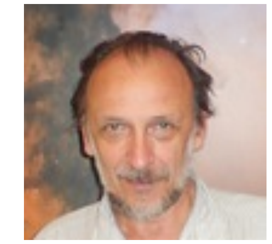

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- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)


- **Flat** KPZ/Continuum DP one free endpoint

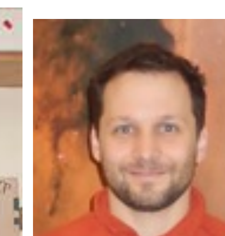
RBA: P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Ortmann, J. Quastel and D. Remenik, Comm. Pure App. Math. 70, 3 (2016), Annals of App. Prob. 16, 507 (2016)

- **Stationary** KPZ

RBA: T. Imamura, T. Sasamoto, Phys. Rev. Lett. 108, 190603 (2012) J. Stat. Phys. 150, 908-939 (2013).

Macdonald process: A. Borodin, I. Corwin, P. Ferrari and B. Veto, Math. Phys. Anal. Geom. 18(Art. 20), 95 (2015)

Cole Hopf mapping

KPZ equation: $$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

define: $$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

it satisfies SHE: $$\partial_t Z = \frac{T}{2}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$T = 2\nu$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)

Feynman Kac

$$Z(x,t|y,0) =$$

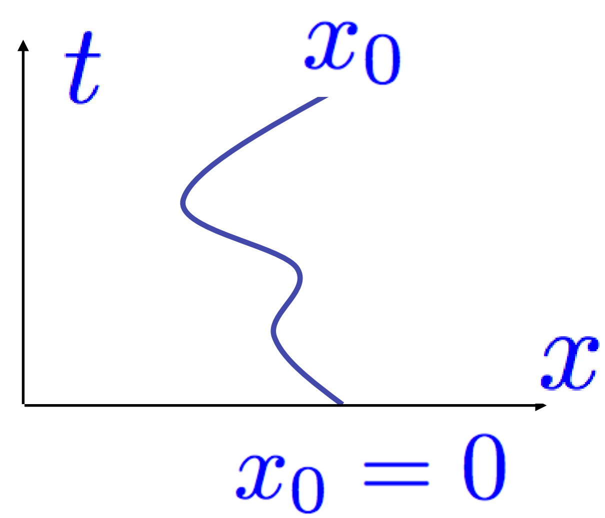
$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau \frac{\kappa}{2}(\frac{dx(\tau)}{d\tau})^2 + V(x(\tau),\tau)}$$

$$Z(x,y,t=0) = \delta(x-y)$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

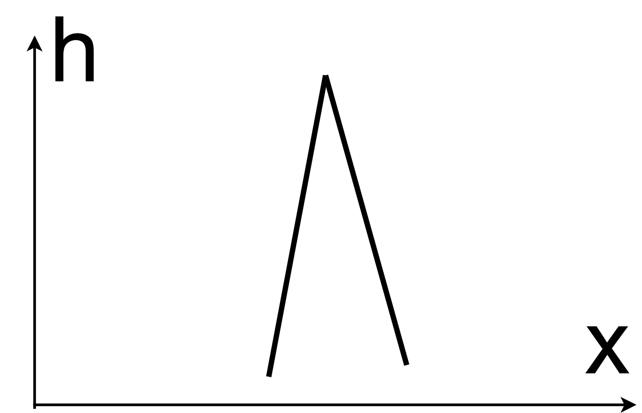
1) DP both fixed endpoints $Z(x_0,t|x_0,0)$



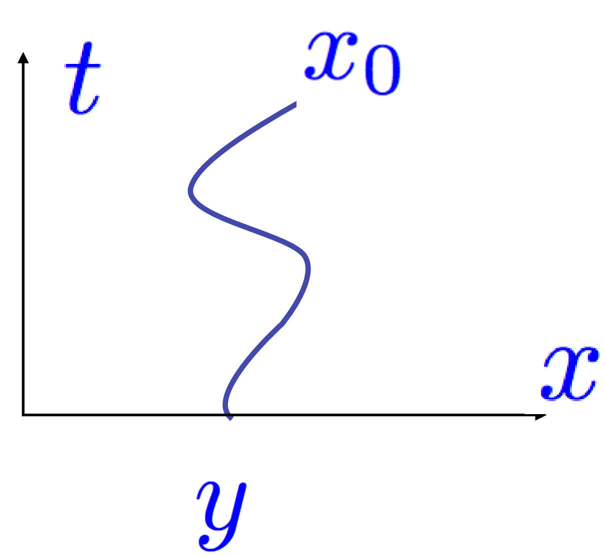
KPZ: narrow wedge <=> droplet initial condition

$$h(x,t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0,t|y,0)$



KPZ: flat initial condition

$$h(x,t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

calculate $$\overline{Z^n} = \int dZ\, Z^n P(Z) \qquad n \in \mathbb{N}$$

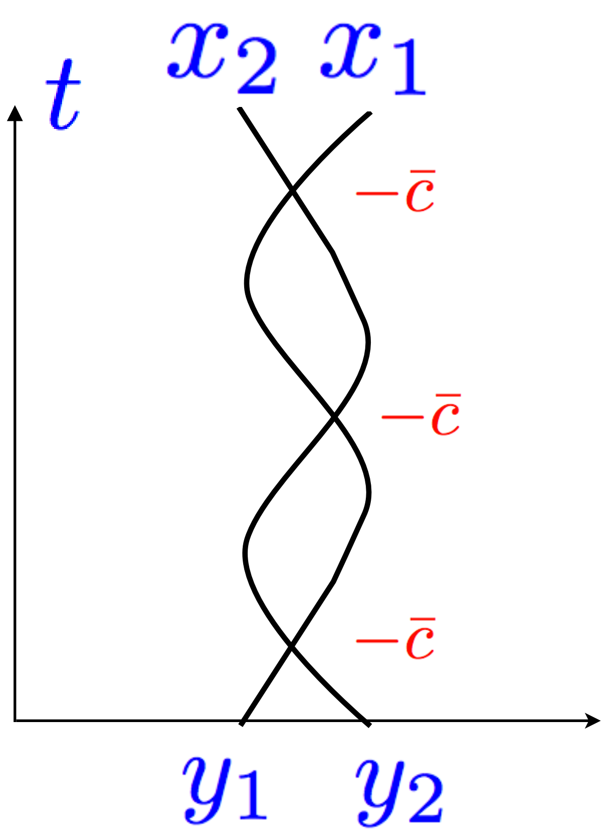
"guess" the probability distribution from its integer moments:

$$P(Z) \rightarrow P(\ln Z) \rightarrow P(h)$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1,t|y_1,0)..Z(x_n,t|y_n 0)} = \langle x_1,..x_n|e^{-tH_n}|y_1,..y_n\rangle$$

$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$



$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

drop the tilde..

$$H_n = -\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - 2\bar{c}\sum_{1\le i<j\le n}\delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963)

What do we need to solve KPZ with droplet initial condition ?

μ eigenstates

E_μ eigen-energies

fixed endpoint DP partition sum

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0 ... x_0|e^{-tH_n}|x_0, ..x_0 >$$

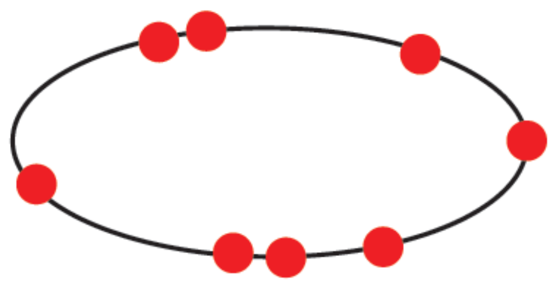
symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

we need:

- eigenfunctions at x0,..x0
- norms of eigenstates
- energies of eigenstates
- perform summation over eigenstates

LL model: n bosons on a ring with local delta attraction



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\leq i<j\leq n} \delta(x_i - x_j)$$

Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

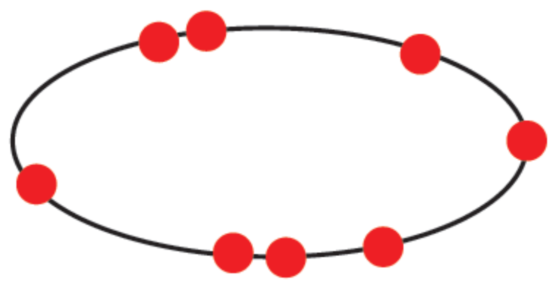
$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2$$

$$A_P = \prod_{n\geq\ell>k\geq1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

LL model: n bosons on a ring with local delta attraction



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They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell\neq j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1,..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|)$$

$$E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n\ln Z}} \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

n bosons+attraction => bound states

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$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

can it be continued in n ? NO !

information about the LARGE DEVIATION tail
of the distribution of “free energy”

f = - ln Z = - h

$$P(f) \sim_{f\to-\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

PLD, S. Majumdar, G. Schehr, arXiv1606.08509
Large deviations for the height in 1D Kardar-Parisi-Zhang growth at late times
EPL 113, 60004 (2016)

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

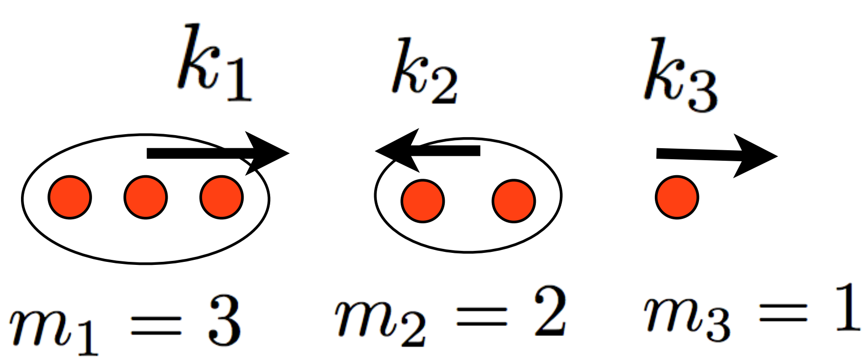
Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

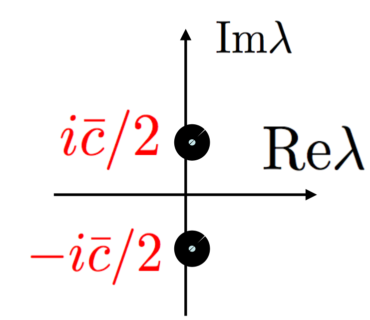
$$E_0(n) = -\frac{\bar{c}^2}{12} n(n^2-1)$$

need to sum over all eigenstates !

- all eigenstates are: All possible partitions of n into ns “strings” each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$\lambda_{j,a_j} = k_j + \frac{i\bar{c}}{2}(m_j + 1 - 2a_j) \qquad a_j = 1, ..m_j$$

$$j = 1, ..n_s$$

=> $$E_\mu = \sum_{j=1}^{n_s} (m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_\mu \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

$$\Psi_\mu(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\dots m_{n_s})_n} \qquad n = \sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2 c^2/4}{(k_i-k_j)^2 + (m_i+m_j)^2 c^2/4}$$

how to get P(ln Z) i.e. P(h) ?

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\ln Z = -\lambda f$$

f = - ln Z = - h random variable expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f - x)} = Prob(f > x)$$

how to get P(ln Z) i.e. P(h) ?

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$$\ln Z = -\lambda f$$

f = - ln Z = h   random variable expected O(1)

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$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

what we aim to calculate= Laplace transform of P(Z)

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1,\ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\le i<j\le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2\lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1,\ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2 + (m_i-m_j)^2\lambda^3}{(k_i-k_j)^2 + (m_i+m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

double Cauchy formula

$$det[\frac{1}{i(k_i-k_j)\lambda^{-3/2} + (m_i+m_j)}]$$

$$= \prod_{i<j} \frac{(k_i-k_j)^2 + (m_i-m_j)^2\lambda^3}{(k_i-k_j)^2 + (m_i+m_j)^2\lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s,x)=\prod_{j=1}^{n_s}\int_{v_j>0} dv_j \; det[K(v_j,v_\ell)] \qquad \lambda=(\frac{\bar{c}^2}{4}t)^{1/3}$$

$$K(v_1,v_2)= \; -\int\frac{dk}{2\pi}dyAi(y+k^2-x+v_1+v_2)e^{-ik(v_1-v_2)}\;\frac{e^{\lambda y}}{1+e^{\lambda y}}$$

$$g(x)=1+\sum_{n_s=1}^{\infty}\frac{1}{n_s!}Z(n_s,x) \;\; = Det[I+K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1,v_2)\equiv\theta(v_1)K(v_1,v_2)\theta(v_2)$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s, x) = \prod_{j=1}^{n_s} \int_{v_j>0} dv_j \, det[K(v_j, v_\ell)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1, v_2) = -\int \frac{dk}{2\pi} dy Ai(y + k^2 - x + v_1 + v_2) e^{-ik(v_1 - v_2)} \frac{e^{\lambda y}}{1 + e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x) = Det[I + K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1, v_2) \equiv \theta(v_1) K(v_1, v_2) \theta(v_2)$$

2) large time limit $\lambda = +\infty$ $\frac{e^{\lambda y}}{1 + e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v - v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

g(x)= $Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

GUE-Tracy-Widom distribution

Summary: one-time observables for models in KPZ class

and a tale of tails

droplet class initial condition, large time t

$$h(x=0,t) = -\frac{t}{12} + t^{1/3}\chi_2 + o(t^{1/3})$$

$$\mathrm{Prob}(\chi_2 < \sigma) = F_2(\sigma)$$

GUE-TW distribution

$$F_2(\sigma) = \mathrm{Det}[I - P_\sigma K_{\mathrm{Ai}} P_\sigma]$$

$$K_{\mathrm{Ai}}(v,v') = \int_0^{+\infty} dy \mathrm{Ai}(y+v)\mathrm{Ai}(y+v')$$

Tail ?

GUE-Tracy Widom distribution

$$F_2(\sigma) = \mathrm{Det}[I - P_\sigma K_{\mathrm{Ai}} P_\sigma] \qquad F_2'(\sigma_1) \sim \frac{e^{-\frac{4}{3}\sigma_1^{3/2}}}{8\pi\sigma_1} \quad \sigma_1 \gg 1$$

GUE-Tracy Widom distribution

$$F_2(\sigma) = \mathrm{Det}[I - P_\sigma K_{\mathrm{Ai}} P_\sigma] \qquad F_2'(\sigma_1) \sim \frac{e^{-\frac{4}{3}\sigma_1^{3/2}}}{8\pi\sigma_1} \quad \sigma_1 \gg 1$$

Tail approximant:

$$F_2(\sigma) = F_2^{(1)}(\sigma) + O(e^{-\frac{8}{3}\sigma^{3/2}}) \qquad \sigma \to +\infty$$

neglects terms of order $O(K_{\mathrm{Ai}}^2)$

$$F_2^{(1)}(\sigma) \equiv 1 - \mathrm{Tr}[P_\sigma K_{\mathrm{Ai}}] = 1 - \int_\sigma^{+\infty} dv K_{\mathrm{Ai}}(v,v)$$

$$F_2^{(1)}(\sigma) - 1 = O(e^{-\frac{4}{3}\sigma^{3/2}})$$

GUE-Tracy Widom distribution

$$F_2(\sigma) = \mathrm{Det}[I - P_\sigma K_{\mathrm{Ai}} P_\sigma] \qquad F_2'(\sigma_1) \sim \frac{e^{-\frac{4}{3}\sigma_1^{3/2}}}{8\pi\sigma_1} \quad \sigma_1 \gg 1$$

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$$F_2^{(1)}(\sigma) - 1 = O(e^{-\frac{4}{3}\sigma^{3/2}})$$

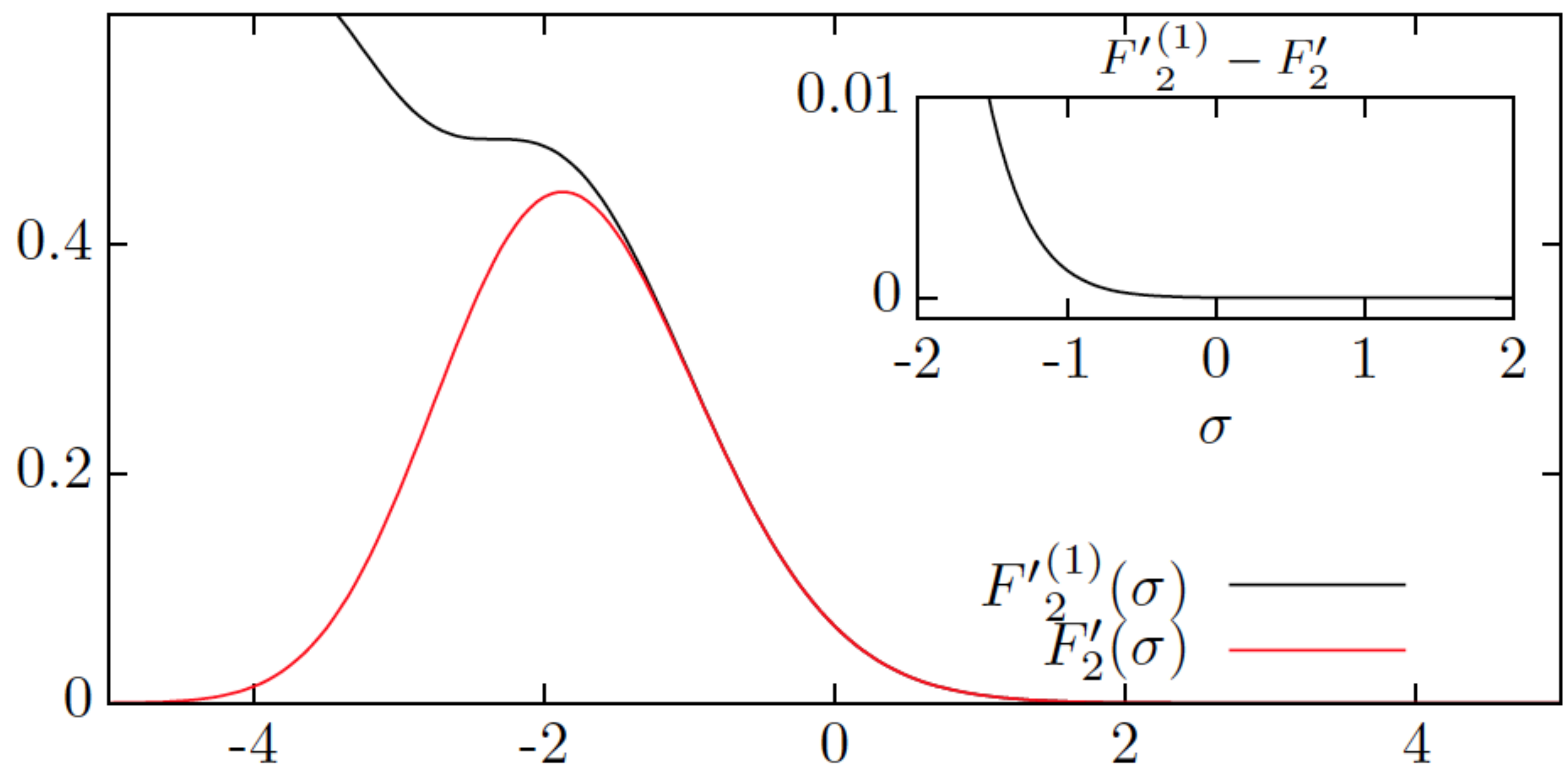

Figure 1. Plot of the Tracy-Widom distribution $F_2'(\sigma)$ (red line) compared with its tail for positive σ given by ${F'}_2^{(1)}(\sigma)$ (black line). Inset: difference between the Tracy-Widom distribution $F_2'(\sigma)$ and its tail ${F'}_2^{(1)}(\sigma)$, given in (12).

why is this tail approximant interesting ?

$$F_2^{(1)}(\sigma) \equiv 1 - \mathrm{Tr}[P_\sigma K_{\mathrm{Ai}}] = 1 - \int_\sigma^{+\infty} dv K_{\mathrm{Ai}}(v,v)$$

it corresponds to keeping only contributions of
one n-string when calculating generating function ns=1

<=> n particles all in a single bound state = the ground state of the Lieb Liniger model

neglects terms of order $O(K_{\mathrm{Ai}}^2)$ contributions of two mj-strings, ..

=> assume this property holds
for more complicated observables

Part II: two-time KPZ via RBA

II a) direct approach

with Jacopo de Nardis (ENS)

any ratio of times BUT only partial tail

Jacopo de Nardis, PLD, arXiv:1612.08695 J. Stat. Mech. (2017) 053212

Tail of the two-time height distribution for KPZ growth in one dimension

two-time problem: - KPZ equation w. droplet initial conditions

- 2 directed polymers with fixed endpoints

in same random potential

$$H_1 \equiv h(0,t_1) = \ln Z_\eta(0,t_1|0,0)$$

$$X = 0$$

$$H_2 \equiv h(X,t_2) = \ln Z_\eta(X,t_2|0,0)$$

two-time problem: - KPZ equation w. droplet initial conditions

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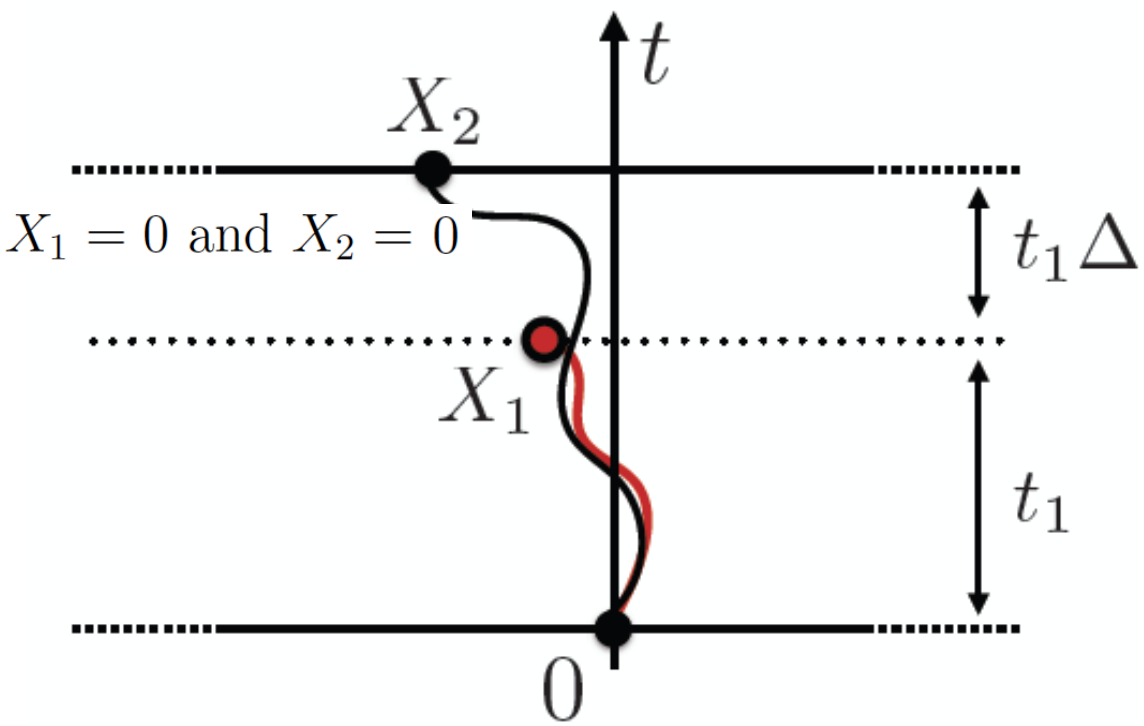

limit: $t_1 \to +\infty$

$t_2 \to +\infty$

$$\Delta = \frac{t_2 - t_1}{t_1}$$ fixed

two-time problem: - KPZ equation w. droplet initial conditions

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$$H_1 \equiv h(0,t_1) = \ln Z_\eta(0,t_1|0,0) \quad X=0$$

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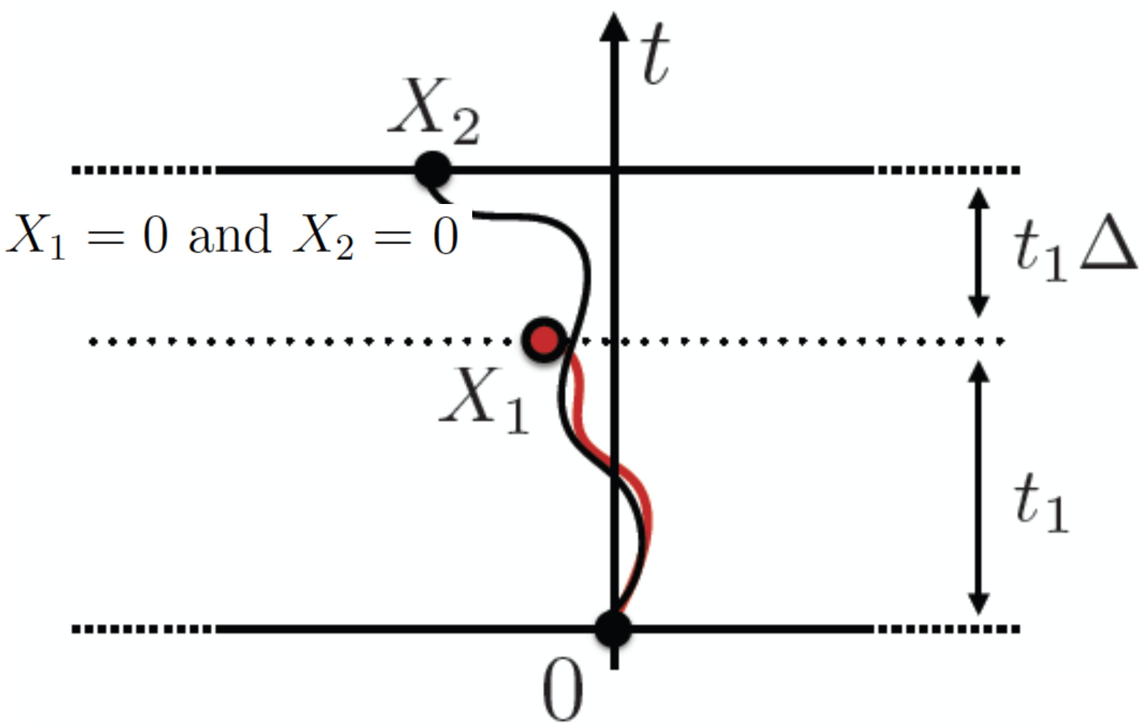
limit: $t_1 \to +\infty$

$t_2 \to +\infty$



$$\Delta = \frac{t_2 - t_1}{t_1} \quad \text{fixed}$$

define rescaled heights h1,h

$$H_1 = t_1^{1/3} h_1$$

$$H_2 - H_1 = (t_2 - t_1)^{1/3} h = \Delta^{1/3} t_1^{1/3} h$$

JPDF of rescaled heights

$$P_\Delta(\sigma_1, \sigma) = \lim_{t_1 \to +\infty} \overline{\delta(h_1 - \sigma_1)\delta(h - \sigma)}$$

Other works on two times

- Victor Dotsenko

V. Dotsenko, arXiv:1304.0626, J. Stat. Mech. (2013) P06017.
V. Dotsenko, arXiv:1507.06135
V. Dotsenko, arXiv:1603.08945, J.Phys. A: Math. Theor. 49 (2016) 27 LT01.

claim for the JPDF from continuum KPZ/DP model (droplet IC)

we believe: incorrect, many terms missing

- Kurt Johansson, arXiv:1502.00941

TWO TIME DISTRIBUTION IN BROWNIAN DIRECTED PERCOLATION

ABSTRACT. In the zero temperature Brownian semi-discrete directed polymer we study the joint distribution of two last-passage times at positions ordered in the time-like direction. This is the situation when we have the slow de-correlation phenomenon. We compute the limiting joint distribution function in a scaling limit. This limiting distribution is given by an expansion in determinants which is not a Fredholm expansion. A somewhat similar looking formula was derived non-rigorously in a related model by Dotsenko.

exact formula for the JPDF from semi-discrete DP model (droplet IC)

but VERY complicated formula!

- Ivan Corwin and Alan Hammond (in progress) asymptotics

- Patrick Ferrari and Herbert Spohn, arXiv:1602.00486

2-time second cumulant: asymptotics for droplet and flat, exact for stationary

- J. Baik, Z. Liu, multi-time TASEP ring (talk)

Main result: tail approximant of the joint PDF

$t_1 \to +\infty$
$t_2 \to +\infty$

$$P_\Delta(\sigma_1, \sigma) = \lim_{t_1 \to +\infty} \overline{\delta(h_1 - \sigma_1)\delta(h - \sigma)}$$

$$H_1 = t_1^{1/3} h_1$$

$$H_2 - H_1 = (t_2 - t_1)^{1/3} h$$

$$P_\Delta(\sigma_1, \sigma) = P_\Delta^{(1)}(\sigma_1, \sigma) + O(e^{-\frac{8}{3}\sigma_1^{3/2}})$$

$$\Delta = \frac{t_2 - t_1}{t_1}$$

↓

$O(e^{-\frac{4}{3}\sigma_1^{3/2}})$ uniformly in σ

Main result: tail approximant of the joint PDF

$t_1 \to +\infty$

$t_2 \to +\infty$

$$P_\Delta(\sigma_1, \sigma) = \lim_{t_1 \to +\infty} \overline{\delta(h_1 - \sigma_1)\delta(h - \sigma)} \qquad H_1 = t_1^{1/3} h_1 \qquad H_2 - H_1 = (t_2 - t_1)^{1/3} h$$

$$P_\Delta(\sigma_1, \sigma) = P_\Delta^{(1)}(\sigma_1, \sigma) + O(e^{-\frac{8}{3}\sigma_1^{3/2}}) \qquad \Delta = \frac{t_2 - t_1}{t_1}$$

$\downarrow$

$O(e^{-\frac{4}{3}\sigma_1^{3/2}})$ uniformly in σ

$$\boxed{P_\Delta^{(1)}(\sigma_1, \sigma) = \left(\partial_{\sigma_1}\partial_\sigma - \Delta^{-1/3}\partial_\sigma^2\right) f_\Delta(\sigma_1, \sigma)}$$

$$\boxed{f_\Delta(\sigma_1, \sigma) = \Delta^{1/3} F_2(\sigma) \mathrm{Tr}\left[P_\sigma K_{\sigma_1}^{(4)} P_\sigma (I - P_\sigma K_{\mathrm{Ai}} P_\sigma)^{-1}\right] - F_2(\sigma)\mathrm{Tr}[P_{\sigma_1} K_{\mathrm{Ai}}]}$$

$$\boxed{K_{\sigma_1}^{(4)}(u, v) = \int_0^\infty dy_1 dy_2 \mathrm{Ai}\left(-y_1 + u\right) K_{\mathrm{Ai}}(y_1 \Delta^{1/3} + \sigma_1, y_2 \Delta^{1/3} + \sigma_1) \mathrm{Ai}\left(-y_2 + v\right)}$$

Limits of the JPDF

$$\Delta = \frac{t_2 - t_1}{t_1} > 0$$

From the formula we show

very separated times

$t_2/t_1 \to +\infty$

$$\lim_{\Delta\to\infty} P^{(1)}_\Delta(\sigma_1,\sigma) = F_2^{(1)\prime}(\sigma_1)\ F_2'(\sigma)$$

H1, H2 are two independent GUE-TW

close times

$$\lim_{\Delta\to 0} P^{(1)}_\Delta(\sigma_1,\sigma) = F_2^{(1)\prime}(\sigma_1)\ F_0'(\sigma)$$

two independent variables:
H1 is GUE-TW, H2-H1 is Baik-Rains

stationary KPZ

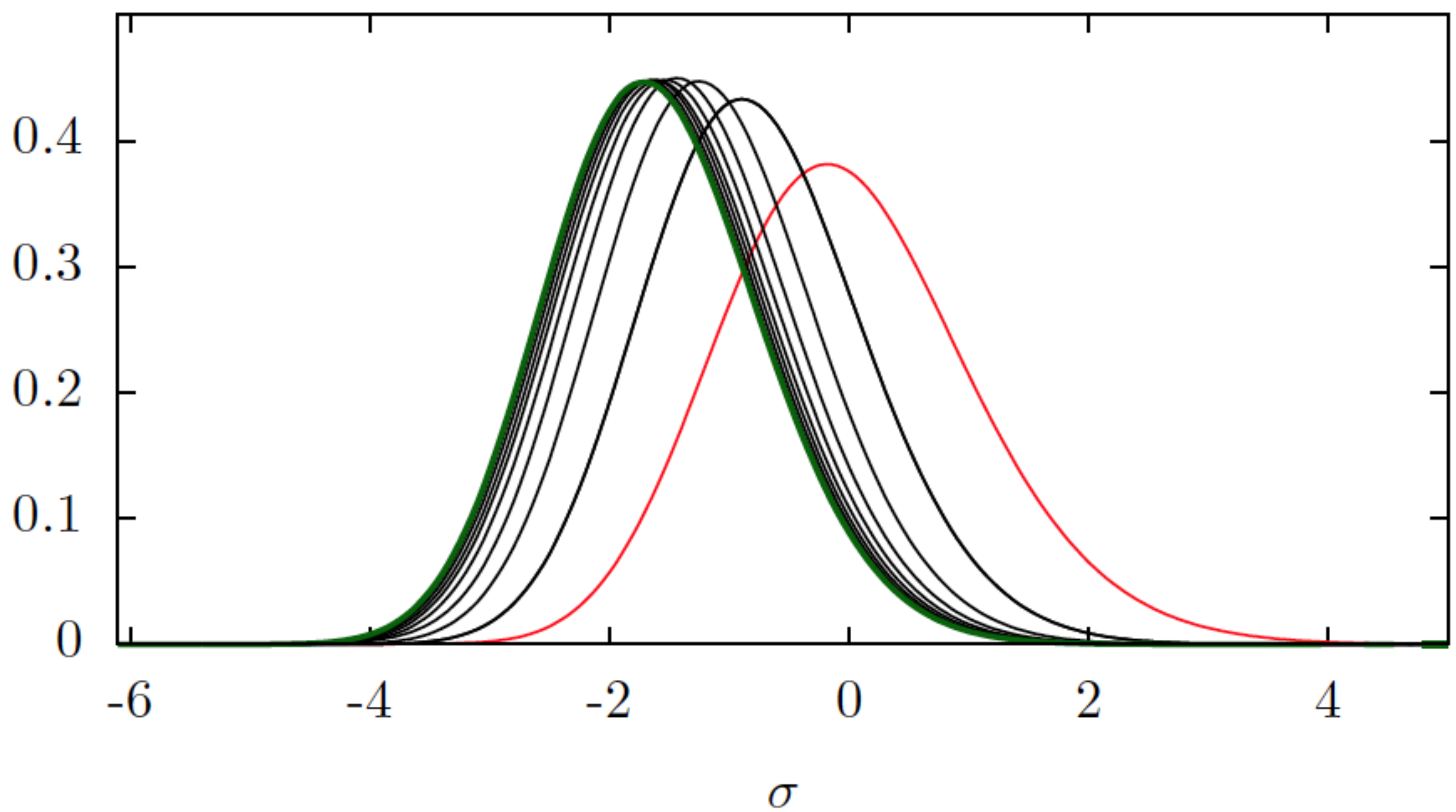
we obtain corrections

we argue exact (Airy processes)

$$\lim_{\Delta\to\infty} P_{\Delta,\infty}(\sigma_1,\sigma) = F_2'(\sigma_1)\ F_2'(\sigma)$$

$$\lim_{\Delta\to 0} P_{\Delta,\infty}(\sigma_1,\sigma) = F_2'(\sigma_1)\ F_0'(\sigma)$$

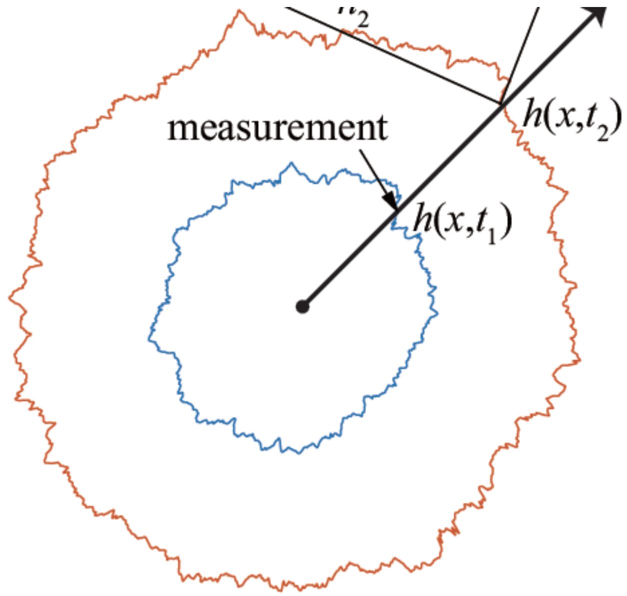
Crossover from Baik-Rains to GUE-Tracy Widom



$$P^{(1)}(\sigma|\sigma_1) \equiv \frac{P_{\Delta}^{(1)}(\sigma_1, \sigma)}{F_2^{(1)\prime}(\sigma_1)}$$

$$\int_{-\infty}^{+\infty} d\sigma P_{\Delta}(\sigma_1, \sigma) = F_2'(\sigma_1)$$

Figure 3. Plot of the conditional probability distribution $P^{(1)}(\sigma|\sigma_1)$, defined in (38), of the scaled height difference $h \equiv (H_2 - H_1)/t_1^{1/3} = \sigma$ for a fixed value of the height at the earlier time $h_1 \equiv H_1/t_1^{1/3} = \sigma_1 = 0$, as a function of σ. The various curves correspond to increasing values of $\Delta^{1/3} = (0.7k)$ with $k = 0, \ldots, 10$ (from right to left). The functions interpolate between the $\Delta = 0$ point (red line) which coincides with the Baik-Rains probability distribution $F_0'(\sigma)$, and the $\Delta \to \infty$ (green line) which corresponds to the GUE Tracy-Widom probability distribution $F_2'(\sigma)$.



Memory and universality in interface growth

Jacopo De Nardis,[1,*] Pierre Le Doussal,[2,†] and Kazumasa A. Takeuchi[3,‡]

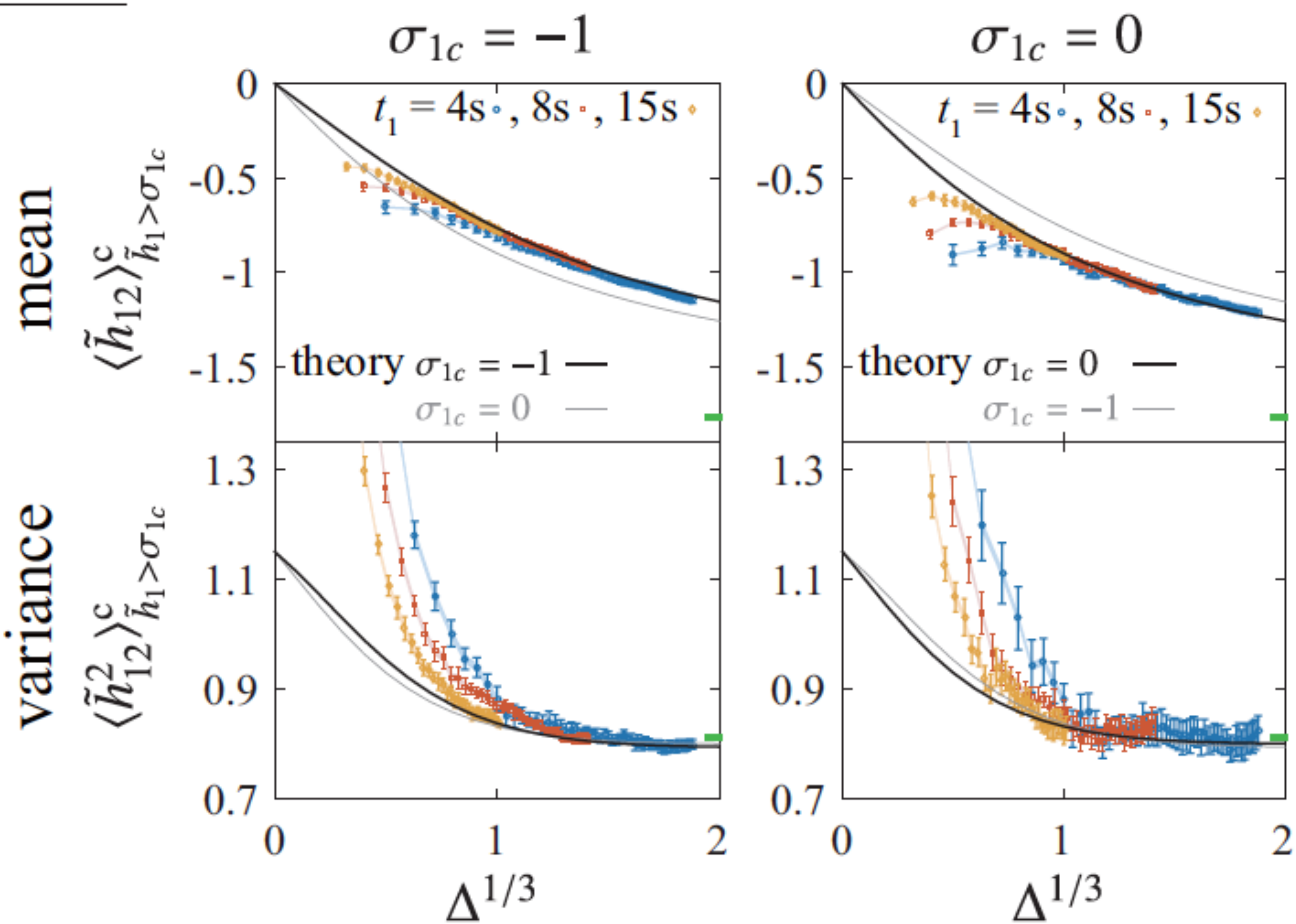
arXiv1611.04756
Phys. Rev. Lett. 118, 125701 (2017)

$$\tilde{h}_{12} := \frac{h(0,t_2) - h(0,t_1) - v_\infty t_1 \Delta}{(\Gamma t_1 \Delta)^{1/3}}$$

scaled height difference

moments of height difference conditioned to value of height at the earlier time

liquid-crystal experiment



want to calculate joint moments

$$Z_{n_1,n_2}(t_1,t_2) = \overline{Z_1^{n_1} Z_2^{n_2}} = \overline{Z_\eta(X_1,t_1|0,0)^{n_1} Z_\eta(X_2,t_2|0,0)^{n_2}}$$

want to calculate joint moments

$$Z_{n_1,n_2}(t_1,t_2) = \overline{Z_1^{n_1} Z_2^{n_2}} = \overline{Z_\eta(X_1,t_1|0,0)^{n_1} Z_\eta(X_2,t_2|0,0)^{n_2}}$$

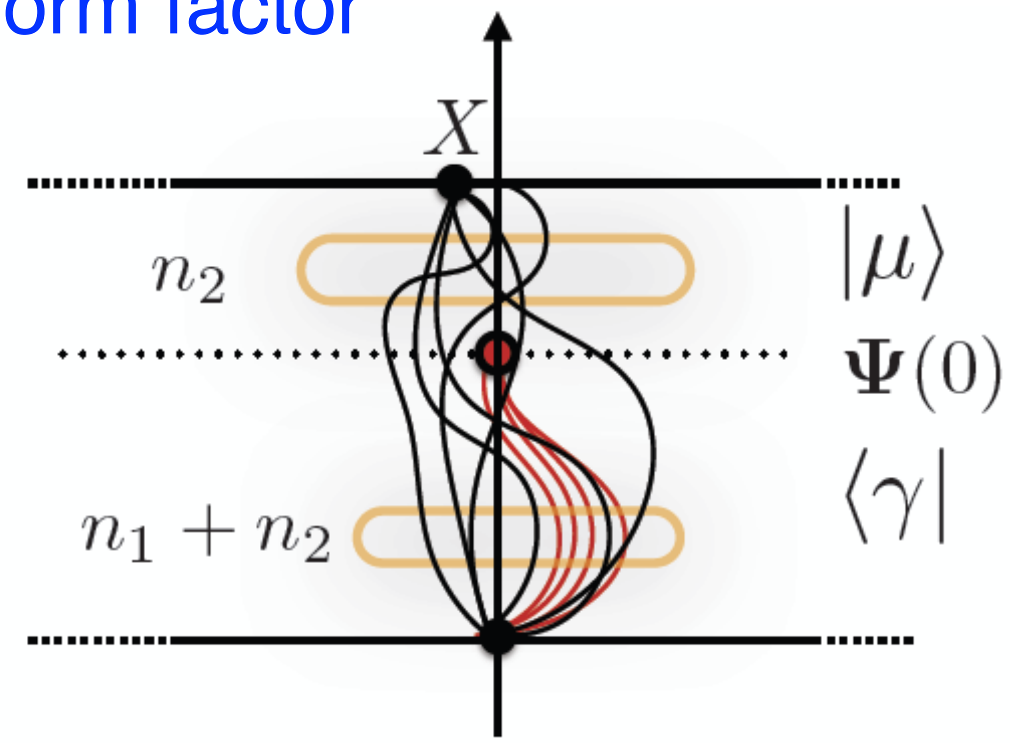
quantum mechanics (exact)

$$e^{-tH_n} = \sum_{\mu\in\Lambda_n} |\mu\rangle e^{-tE_\mu}\langle\mu|$$

$$Z_{n_1,n_2}(t_1,t_2) = \sum_{\mu\in\Lambda_{n_2},\gamma\in\Lambda_{n_1+n_2}} \frac{\psi^*_\gamma(0,..,0)\psi_\mu(X_2,..X_2)}{||\gamma||^2||\mu||^2} e^{-\Delta t_1 E_\mu - t_1 E_\gamma} F^{n_2;n_1+n_2}_{\mu;\gamma}$$

$$F^{n_2;n_1+n_2}_{\mu;\gamma} \equiv \prod_{\alpha=1}^{n_2} \int dy_\alpha e^{-w|y_j-X_1|} \psi^*_\mu(y_1,..,y_{n_2})\psi_\gamma(\underbrace{X_1,\dots,X_1}_{n_1},y_1,..,y_{n_2})$$

form factor



$n_1, n_2 \geq 0$

want to calculate joint moments

$$Z_{n_1,n_2}(t_1,t_2) = \overline{Z_1^{n_1} Z_2^{n_2}} = \overline{Z_\eta(X_1,t_1|0,0)^{n_1} Z_\eta(X_2,t_2|0,0)^{n_2}}$$

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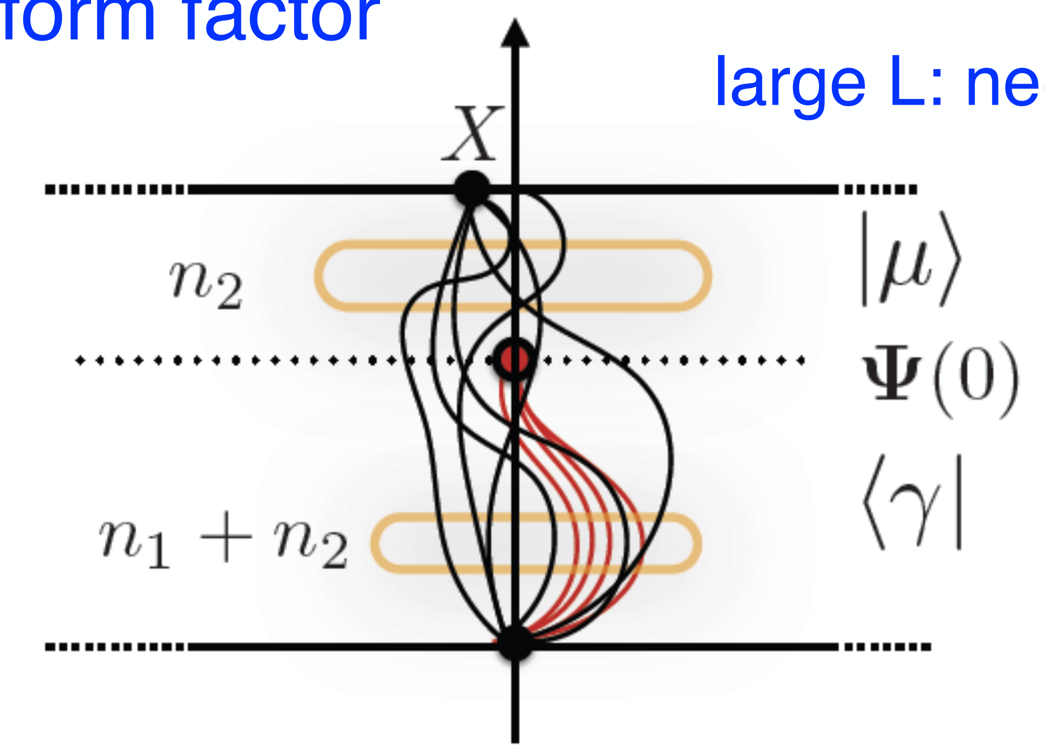
$$Z_{n_1,n_2}(t_1,t_2) = \sum_{\mu\in\Lambda_{n_2},\gamma\in\Lambda_{n_1+n_2}} \frac{\psi_\gamma^*(0,..,0)\psi_\mu(X_2,..X_2)}{||\gamma||^2||\mu||^2} e^{-\Delta t_1 E_\mu - t_1 E_\gamma} F_{\mu;\gamma}^{n_2;n_1+n_2}$$

$$F_{\mu;\gamma}^{n_2;n_1+n_2} \equiv \prod_{\alpha=1}^{n_2} \int dy_\alpha e^{-w|y_j-X_1|} \psi_\mu^*(y_1,..,y_{n_2}) \psi_\gamma(\underbrace{X_1,\ldots,X_1}_{n_1}, y_1,..,y_{n_2})$$

form factor



large L: need string form factors $F_{\mu;\gamma}^{n_2;n_1+n_2} = F_{\mathbf{p},\mathbf{m}^\mu;\mathbf{q},\mathbf{m}^\gamma}^{n_2;n_1+n_2}$

difficult!

$n_1, n_2 \geq 0$

want to calculate joint moments

$$Z_{n_1,n_2}(t_1,t_2) = \overline{Z_1^{n_1} Z_2^{n_2}} = \overline{Z_\eta(X_1,t_1|0,0)^{n_1} Z_\eta(X_2,t_2|0,0)^{n_2}}$$

quantum mechanics (exact)

$$e^{-tH_n} = \sum_{\mu\in\Lambda_n} |\mu\rangle e^{-tE_\mu}\langle\mu|$$

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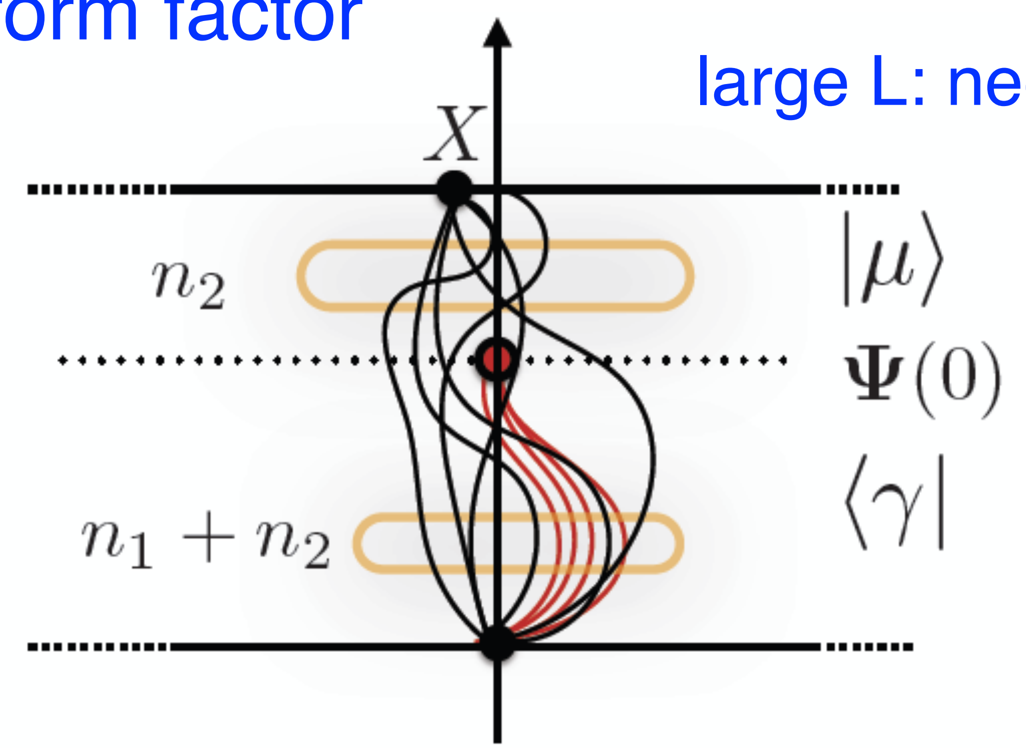
$$F_{\mu;\gamma}^{n_2;n_1+n_2} \equiv \prod_{\alpha=1}^{n_2} \int dy_\alpha e^{-w|y_j-X_1|} \psi_\mu^*(y_1,..,y_{n_2})\psi_\gamma(\underbrace{X_1,\dots,X_1}_{n_1},y_1,..,y_{n_2})$$

form factor



large L: need string form factors $F_{\mu;\gamma}^{n_2;n_1+n_2} = F_{\mathbf{p},\mathbf{m}^\mu;\mathbf{q},\mathbf{m}^\gamma}^{n_2;n_1+n_2}$

difficult!

in the sum keep only:

$|\mu\rangle = |\mathbf{p},\mathbf{m}\rangle$ arbitrary string state with any number of strings

$|\gamma\rangle = |q, n_1+n_2\rangle$ ONE n1+n2-string state

$n_1, n_2 \geq 0$ => 1) form factor simplifies, possible to carry the sum exactly !

=> 2) gives the TAIL of JPDF for large positive h1, arbitrary h

Part II: two-time KPZ via RBA

II b) using Airy process

no restriction on $h(0,t_1)\ h(0,t_2)$ BUT only at infinite time separation

$$\Delta \to +\infty$$

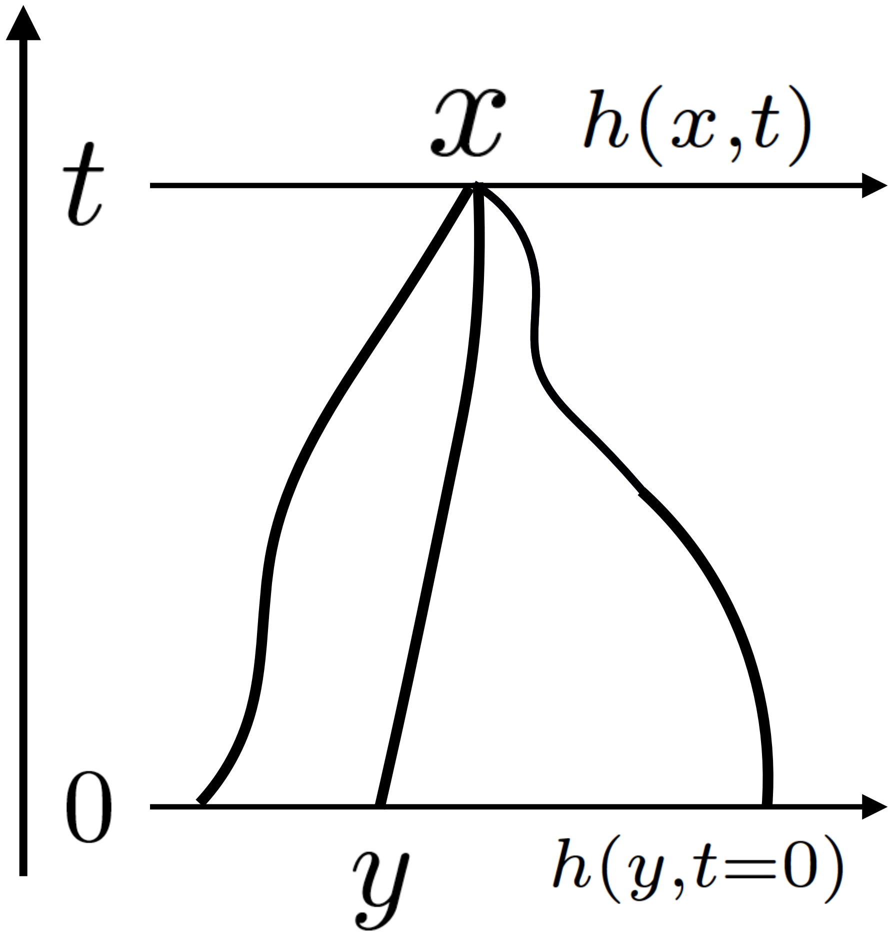
PLD arXiv:1709.06264 (2017)

Maximum of an Airy process plus Brownian motion and memory in KPZ growth

Large time limit of KPZ height for general initial condition

$\updownarrow$

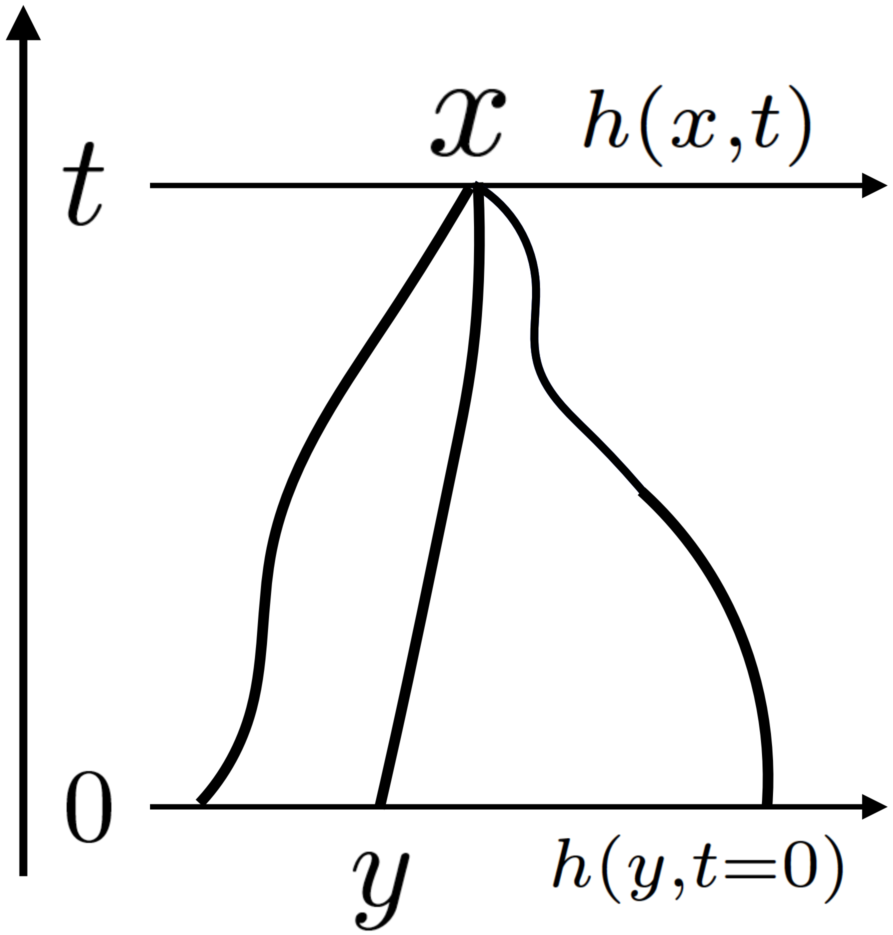
Variational problem using Airy process



general initial condition

$$e^{h(x,t)} = \quad \text{sum over paths}$$

$$\int dy Z_\eta(x,t|y,0)\, e^{h(y,t=0)}$$



general initial condition

$$e^{h(x,t)} = \text{ sum over paths}$$

$$\int dy Z_\eta(x,t|y,0)\, e^{h(y,t=0)}$$

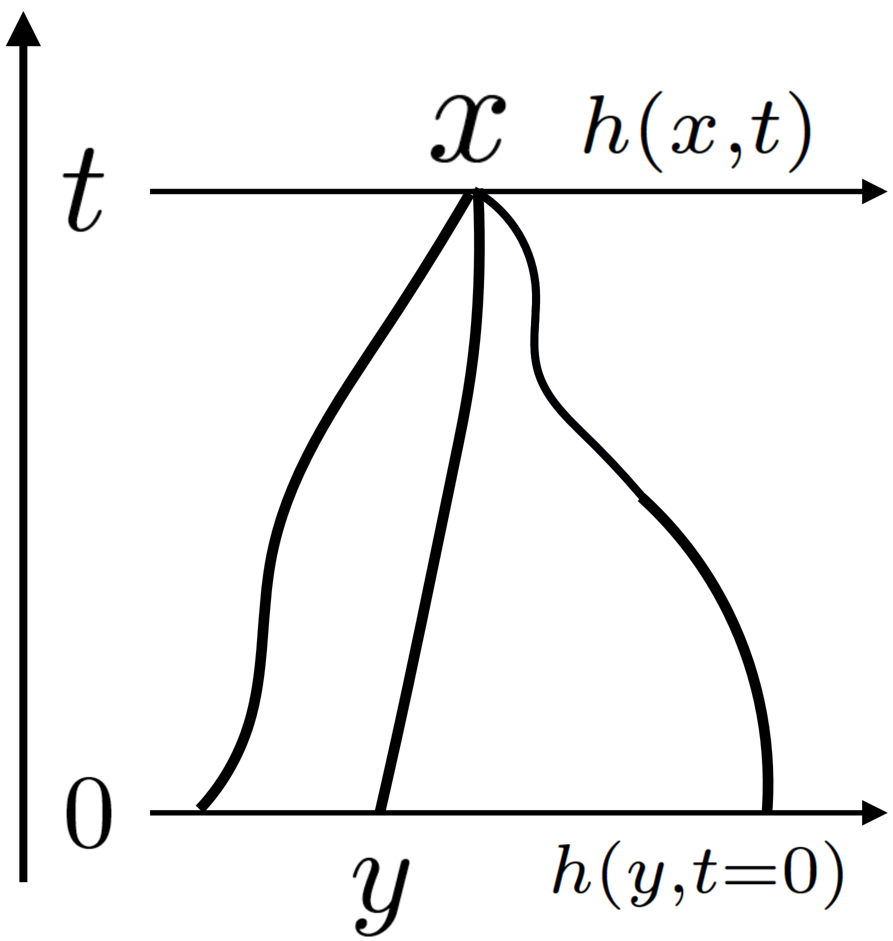
$\downarrow$ $\qquad$ $\downarrow$ choose

$$e^{t^{1/3}(\mathcal{A}_2(\hat{x}-\hat{y})-(\hat{x}-\hat{y})^2)} \qquad t^{1/3}\mathsf{h}_0(\frac{y}{2t^{1/3}})$$

$$t^{-1/3}h(0,t) \simeq \max_{\hat{y}} \left(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \mathsf{h}_0(\hat{y})\right)$$

$$\mathsf{h}_0(\hat{y}) \simeq t^{-1/3}h(2t^{2/3}\hat{y},0)$$

$$\hat{y} = \frac{y}{2t^{2/3}}$$



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$$\mathsf{h}_0(\hat{y}) \simeq t^{-1/3}h(2t^{2/3}\hat{y}, 0)$$

droplet $\mathsf{h}_0(0) = 0 \quad \mathsf{h}_0(\hat{y}) = -\infty \quad \hat{y} \neq 0$

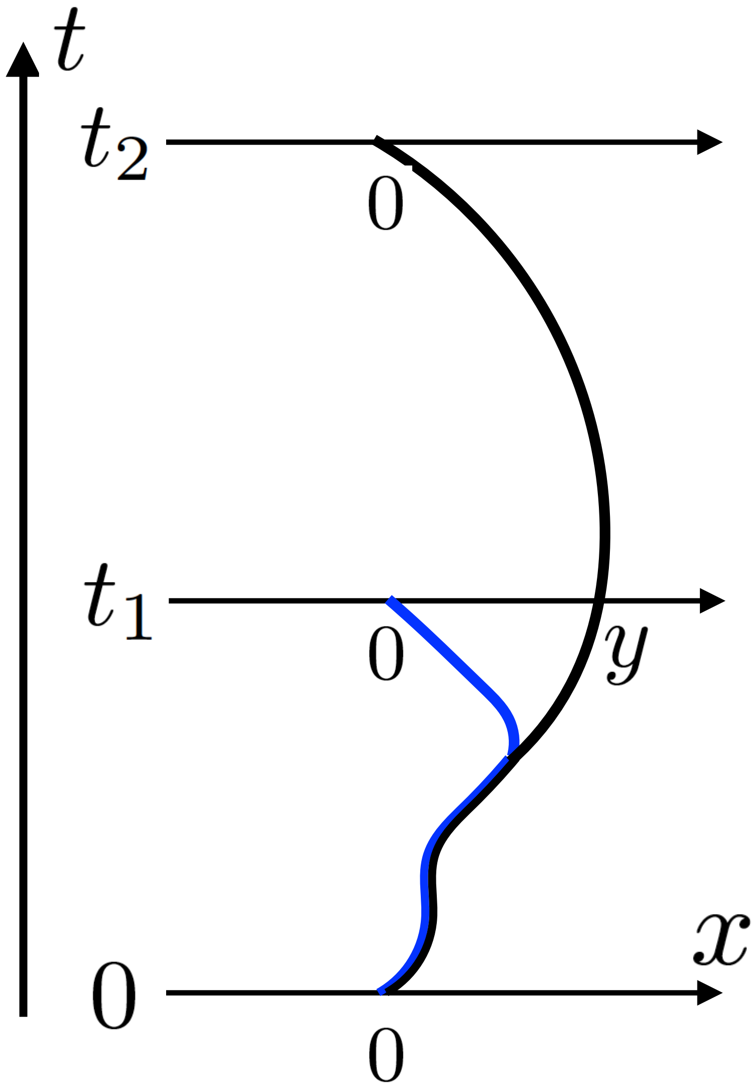
$$\hat{y} = \frac{y}{2t^{2/3}}$$

flat $\mathsf{h}_0(\hat{y}) = 0$

stationary $h(x,t=0) = B_0(x) \longrightarrow \mathsf{h}_0(\hat{y}) = \sqrt{2}B(\hat{y})$

$$\langle dB(x)^2 \rangle = dx$$

$$B(0) = 0$$



Two time KPZ via Airy processes

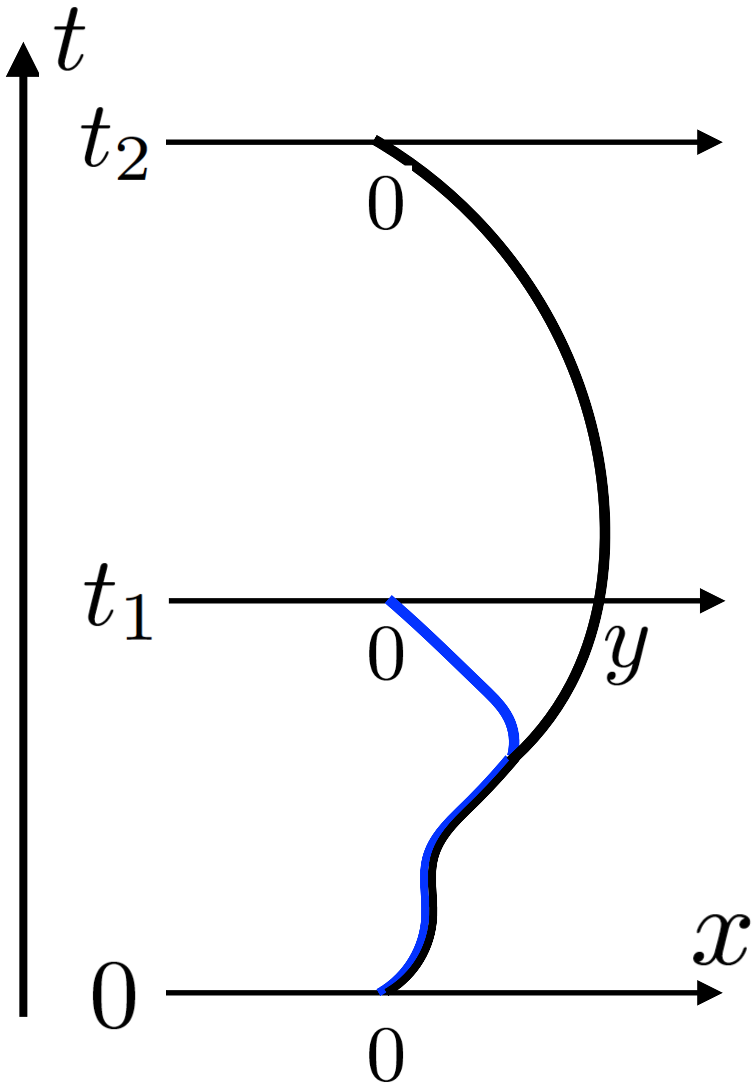
$t_1, t_2 \to +\infty \qquad \Delta = \frac{t_2 - t_1}{t_1} \quad$ fixed

$$t_1^{-1/3} h(0, t_1) \simeq \mathcal{A}_2(0)$$

$$\hat{y} = \frac{y}{2 t_1^{2/3}}$$

$$t_1^{-1/3} h(0, t_2) \simeq$$

$$\max_{\hat{y} \in \mathbb{R}} \left(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \Delta^{\frac{1}{3}} (\tilde{\mathcal{A}}_2(\frac{\hat{y}}{\Delta^{\frac{2}{3}}}) - \frac{\hat{y}^2}{\Delta^{\frac{4}{3}}}) \right)$$



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$$\max_{\hat{y}\in\mathbb{R}} \left(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \Delta^{\frac{1}{3}} (\tilde{\mathcal{A}}_2(\frac{\hat{y}}{\Delta^{\frac{2}{3}}}) - \frac{\hat{y}^2}{\Delta^{\frac{4}{3}}})\right)$$

$$\underset{\Delta\to+\infty}{\simeq} \Delta^{\frac{1}{3}} \tilde{\mathcal{A}}_2(0) + \max_{\hat{y}\in\mathbb{R}} \left(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2} B(\hat{y})\right) + O(\frac{1}{\Delta^{\frac{1}{3}}})$$

in large time separation limit $\Delta \to +\infty$

correlations between h(0,t1) and h(0,t2) are contained in the joint distribution

$$\underset{\text{JCDF}}{G(\sigma_1,\sigma_2)} = \mathrm{Prob}\left(\mathcal{A}_2(0) < \sigma_1, \ \max_{u\in\mathbb{R}} (\mathcal{A}_2(u) - u^2 + \sqrt{2} B(u)) < \sigma_2\right)$$

Explicit formula for JCDF

$$G(\sigma_1,\sigma_2) \;=\; \mathrm{Prob}(\mathcal{A}_2(0)<\sigma_1,\; \max_{u\in\mathbb{R}}(\mathcal{A}_2(u)-u^2+\sqrt{2}B(u))<\sigma_2)$$

$$= \; F_2(\sigma_1)Y_0(\sigma_1)\mathrm{Tr}[(I-P_{\sigma_1}K_{\mathrm{Ai}})^{-1}P_{\sigma_1}\mathrm{Ai}_{\sigma_2-\sigma_1}\mathrm{Ai}^T_{\sigma_2-\sigma_1}]$$

$\sigma_1 \le \sigma_2$

$$+F_2(\sigma_1)(\mathrm{Tr}[(I-P_{\sigma_1}K_{\mathrm{Ai}})^{-1}P_{\sigma_1}\mathrm{Ai}_{\sigma_2-\sigma_1}\mathcal{B}_0^T]-1)^2$$

$$\mathrm{Ai}_\sigma(u) \;=\; \mathrm{Ai}(u+\sigma)$$

marginals are

$$Y_{\hat{x}}(\sigma) := 1+\mathcal{L}_{\hat{x}}(\sigma)-\mathrm{Tr}[P_\sigma K_{\mathrm{Ai}}(I-P_\sigma K_{\mathrm{Ai}})^{-1}P_\sigma \mathcal{B}_{-\hat{x}}\mathcal{B}^T_{\hat{x}}]$$

$$\mathcal{L}_{\hat{x}}(\sigma)=\sigma-1-\hat{x}^2+\int_\sigma^{+\infty}dv(1-\mathcal{B}_{\hat{x}}(v)\mathcal{B}_{-\hat{x}}(v))$$

σ_1 GUE Tracy-Widom $F_2(\sigma_1)$

σ_2 Baik-Rains $F_0(\sigma_2)$

$$\mathcal{B}_w(v)=e^{\frac{1}{3}w^3-vw}-\int_0^{+\infty}dy\mathrm{Ai}(v+y)e^{wy}$$

$$G(\sigma_1,\sigma_2) \;=\; F_0(\sigma_2) \quad \sigma_1 \ge \sigma_2$$

$$F_0(\sigma) \;=\; \partial_\sigma(F_2(\sigma)Y_0(\sigma))$$

Consequence for 2-time KPZ: persistent correlations

$$C(t_1, t_2) = \frac{\overline{h(0,t_1)h(0,t_2)}^c}{\overline{h(0,t_1)^2}^c} \xrightarrow[t_1, t_2 \to +\infty]{} C_\Delta$$

$$C_{+\infty} = \frac{\langle \sigma_2 \sigma_1 \rangle^c}{\langle \sigma_1^2 \rangle}$$

$$\approx 0.6225 \pm 0.0015$$

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$$\approx 0.6225 \pm 0.0015$$

$$0.626 \pm 0.003$$

T. Halpin-Healy
num.sim.DP (2017)

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$$\approx 0.6225 \pm 0.0015$$

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T. Halpin-Healy
num.sim.DP (2017)

conditional mean

$$h := (h(0,t_2) - h(0,t_1))/(t_2-t_1)^{1/3}$$

$$\overline{h}_{h_1=\sigma_1} \simeq \kappa_1^{\rm GUE} + \frac{1}{\Delta^{1/3}} \langle \sigma_2 - \sigma_1 \rangle_{\sigma_1} + o(\frac{1}{\Delta^{1/3}})$$

check partial tail $\sigma_1 \gg 1$

$$p(\sigma_1,\sigma_2) \simeq -2\partial_{\sigma_2} K_{Ai}(\sigma_1,\sigma_2) - {\rm Ai}(\sigma_2)^2$$

$$\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1} \simeq \frac{[\int_{\sigma_1}^{+\infty} dy {\rm Ai}(y)]^2 - \int_{\sigma_1}^{+\infty} dy K_{\rm Ai}(y,y)}{K_{\rm Ai}(\sigma_1,\sigma_1)}$$

exactly the result of the part II a) paper with de Nardis

$$h := (h(0,t_2) - h(0,t_1))/(t_2 - t_1)^{1/3}$$

conditional mean

$$\overline{h}_{h_1>\sigma_{1c}} = \kappa_1^{\rm GUE} + \Delta^{-1/3}\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 > \sigma_{1c}}$$

$$\kappa_1^{\rm GUE} = -1.771$$

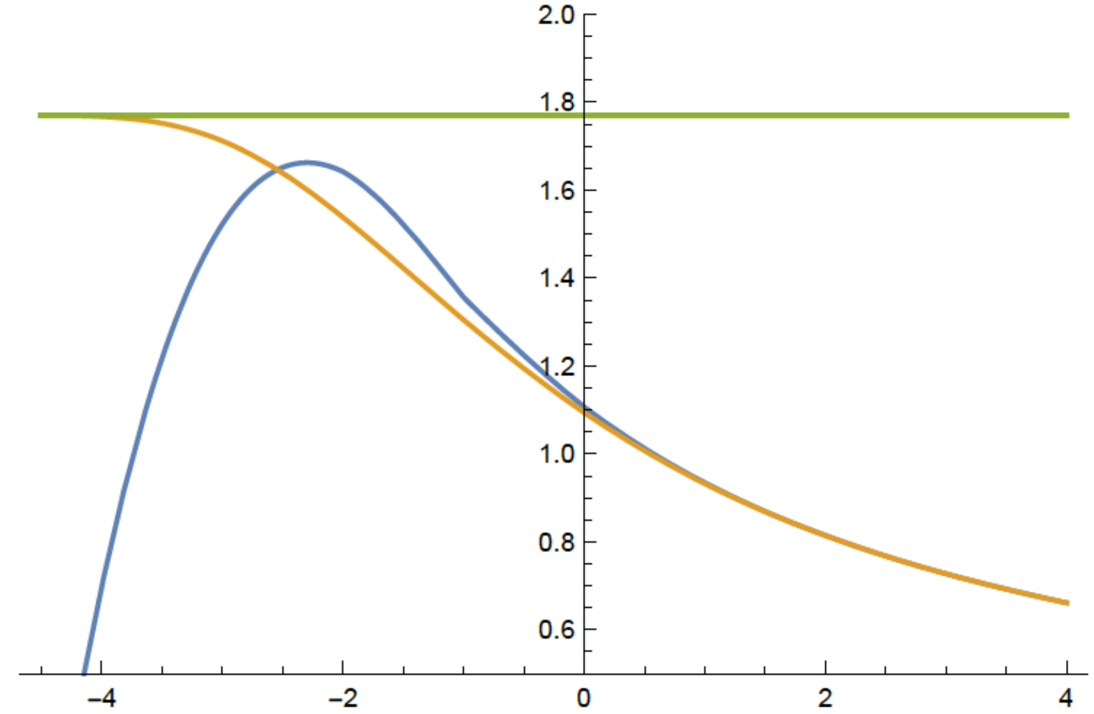
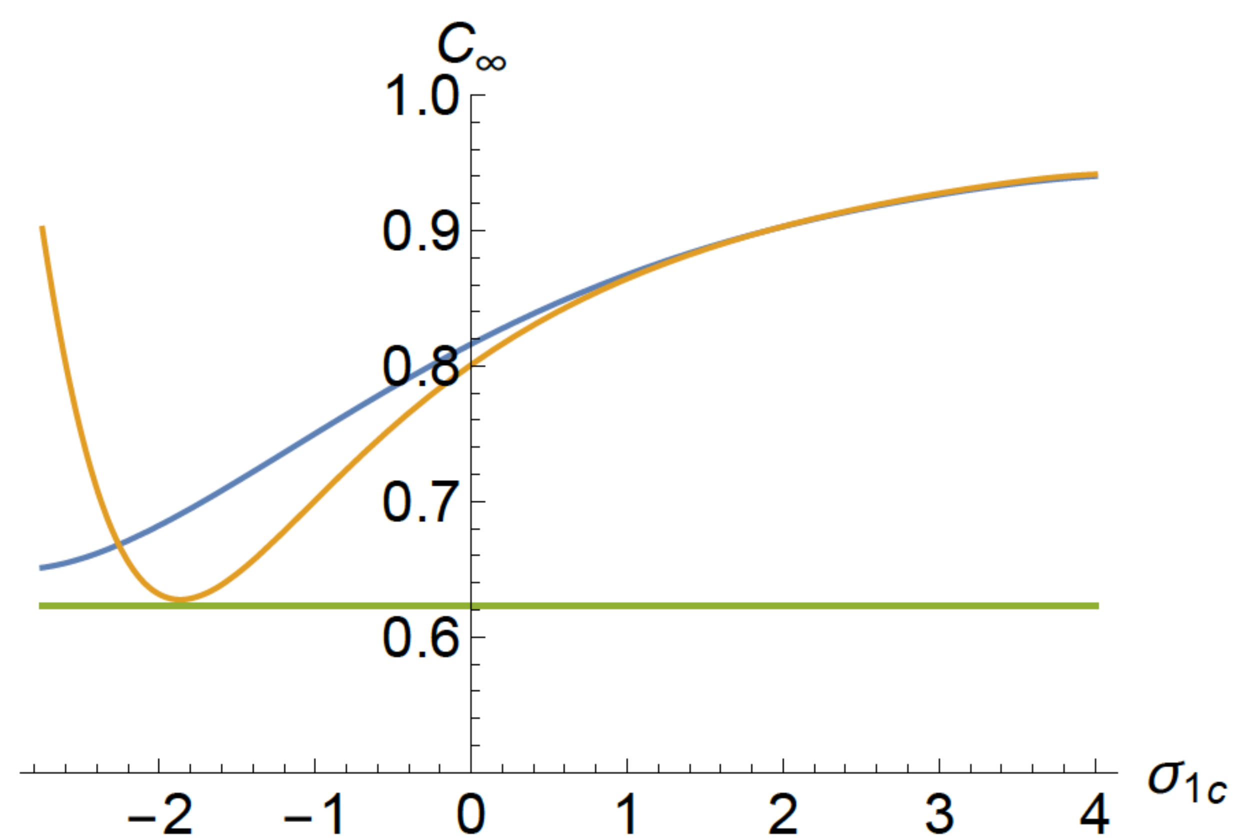


FIG. 2. Conditional average $\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 > \sigma_{1c}}$ (y axis) as a function of σ_{1c} (x axis), which described the averaged scaled KPZ

more detailed prediction
conditional covariance ratio

$$C_\infty(\sigma_{1c}) = \lim_{\Delta\to\infty} \frac{\overline{h_1 h_2}^{c}_{h_1>\sigma_{1c}}}{\overline{h_1^2}^{c}_{h_1>\sigma_{1c}}}$$



$$C_\infty \approx 0.6225 \pm 0.0015$$

$$C_\Delta(\sigma_{1c}) := \frac{\overline{h_1 h_2}^c_{h_1>\sigma_{1c}}}{\overline{h_1^2}^c_{h_1>\sigma_{1c}}}$$

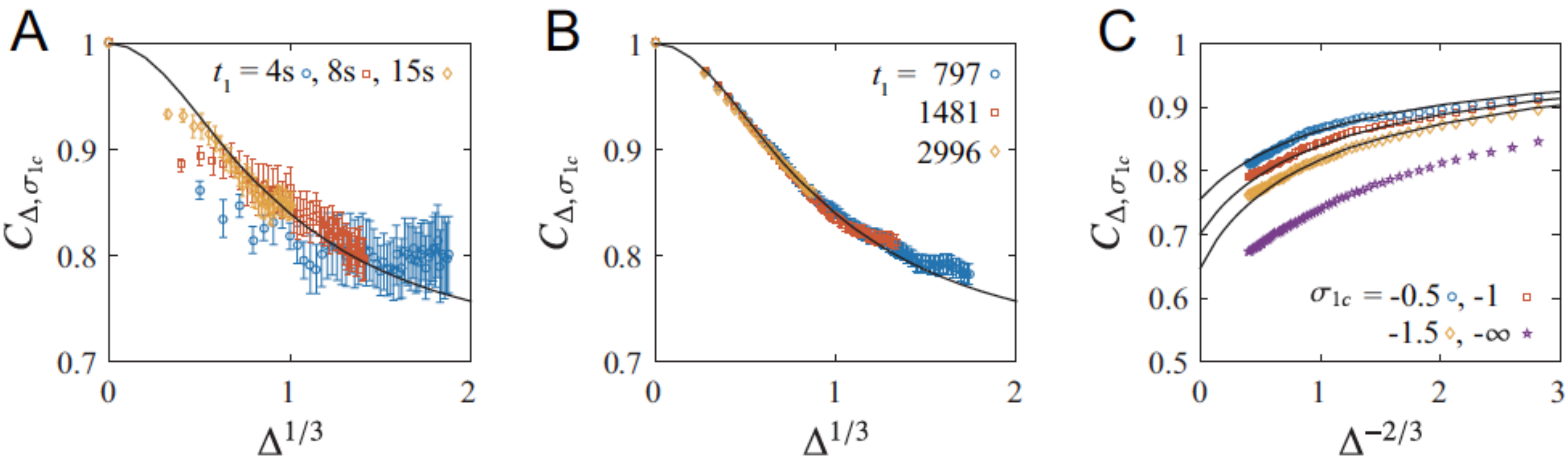


FIG. 4: The conditional covariance $C_{\Delta,\sigma_{1c}} = C(t_1, t_2)_{\tilde{h}_1>\sigma_{1c}} / C(t_1, t_1)_{\tilde{h}_1>\sigma_{1c}}$ ((3)). **a,b**, Experimental (A) and numerical (B) results for $\sigma_{1c} = -1$ and with varying t_1 (symbols), compared with the theoretical prediction (black line). The error bars indicate the standard errors. To reduce the effect of finite-time corrections, here we used such realizations that satisfy $\tilde{h}_1 > \tilde{h}_{1c}$ with $\mathrm{Prob}[\tilde{h}_1 \geq \tilde{h}_{1c}] = 1 - F_2(\sigma_{1c})$. **c**, Numerical data for $t_1 = 1008$ and for $\sigma_{1c} = -0.5, -1, -1.5$ and $-\infty$ (unconditioned). Error bars are omitted here for the sake of visibility. The black lines indicate the theoretical predictions for finite σ_{1c}. At large Δ and for any σ_{1c} they converge to their asymptotic values as $C_{\Delta\to\infty,\sigma_{1c}} + A_{\sigma_{1c}}\Delta^{-2/3} + B_{\sigma_{1c}}\Delta^{-1} + \ldots$. For $\sigma_{1c} = -\infty$ (the unconditioned case), the theory suggests a strictly positive asymptotic value, specifically $C_{\infty,-\infty} \approx 0.6$, which is consistent with the trend of the unconditioned data set in the panel **c** (purple stars).

J. de Nardis, PLD, K. Takeuchi
Phys. Rev. Lett. 118, 125701 (2017)

complicated exact expressions for some joint moments

$$\overline{Z_1^{n_1} Z_L^{n_L} Z_R^{n_R}}$$ ⟶ generating function

↓

large time + “decoupling assumption”

magic trick

$$\text{Prob}(\mathcal{A}_2(-\hat{x}) < \sigma_1, \hat{h}_L(\hat{x}) + \hat{x}^2 < \sigma_L, \hat{h}_R(\hat{x}) + \hat{x}^2 < \sigma_R)$$

$$\hat{h}_{L,R}(\hat{x}) = \max_{\hat{y}<0, \hat{y}>0} (\mathcal{A}_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}\hat{B}(\hat{y}))$$

Distribution of argmax of Airy minus parabola plus Brownian

$$H(\hat{x}) = \text{Prob}(\hat{y}_m > \hat{x}) \qquad \hat{y}_m = \text{argmax}_{\hat{y}\in\mathbb{R}} \left(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})\right)$$

$$\begin{aligned} H(-\hat{x}) = \int d\sigma F_2(\sigma) \Big(& Y_{\hat{x}}(\sigma) \\ & \times \text{Tr}[(I - P_\sigma K_{\text{Ai}})^{-1} P_\sigma (\text{Ai}' + \hat{x}\text{Ai})\text{Ai}^T] \\ & + (\text{Tr}[(I - P_\sigma K_{\text{Ai}})^{-1} P_\sigma \text{Ai}\mathcal{B}_{\hat{x}}^T] - 1) \\ & \times \text{Tr}[(I - P_\sigma K_{\text{Ai}})^{-1} P_\sigma (\text{Ai}' + \hat{x}\text{Ai})\mathcal{B}_{-\hat{x}}^T] \Big) \end{aligned}$$

application:midpoint probability
of very long DP

directed polymer from $(0,0)$ to $(0,t_2)$

position $x(t_1) = y$ at intermediate time t_1

$$\overline{P_{t_1,t_2}(y)}dy = \frac{\overline{Z(0,t_2|y,t_1)Z(y,t_1|0,0)}}{Z(0,t_2|0,0)}dy \to P_\Delta(\hat{y})d\hat{y}$$

$$P_{+\infty}(\hat{y})d\hat{y} = \text{Prob}(\hat{y}_m \in [\hat{y}, \hat{y} + dy[)$$

Distribution of argmax of Airy minus parabola plus Brownian

$$H(\hat{x}) = \text{Prob}(\hat{y}_m > \hat{x}) \qquad \hat{y}_m = \text{argmax}_{\hat{y}\in\mathbb{R}} \left(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})\right)$$

application:midpoint probability of very long DP

$$H(-\hat{x}) = \int d\sigma F_2(\sigma)\Big(Y_{\hat{x}}(\sigma) \times \text{Tr}[(I - P_\sigma K_{\text{Ai}})^{-1} P_\sigma(\text{Ai}' + \hat{x}\text{Ai})\text{Ai}^T] + (\text{Tr}[(I - P_\sigma K_{\text{Ai}})^{-1} P_\sigma \text{Ai}\mathcal{B}^T_{\hat{x}}] - 1) \times \text{Tr}[(I - P_\sigma K_{\text{Ai}})^{-1} P_\sigma(\text{Ai}' + \hat{x}\text{Ai})\mathcal{B}^T_{-\hat{x}}]\Big)$$

directed polymer from $(0,0)$ to $(0,t_2)$

position $x(t_1) = y$ at intermediate time t_1

$$\overline{P_{t_1,t_2}(y)}dy = \frac{\overline{Z(0,t_2|y,t_1)Z(y,t_1|0,0)}}{Z(0,t_2|0,0)}dy \to P_\Delta(\hat{y})d\hat{y}$$

$$P_{+\infty}(\hat{y})d\hat{y} = \text{Prob}(\hat{y}_m \in [\hat{y},\hat{y}+dy[)$$

C. Maes and T. Thiery, arXiv:1704.06909

Fluctuation Dissipation relations stationary Burgers
<=> variance of the height in stationary KPZ

$$\mathcal{P}(\hat{y}) = f_{KPZ}(\hat{y})$$

$$g(\hat{x}) = \langle\sigma^2\rangle_{F_0,\hat{x}} - \langle\sigma\rangle^2_{F_0,\hat{x}}$$

$$f_{KPZ}(\hat{y}) = \frac{1}{4}g''(\hat{y})$$

I checked that $$H(\hat{x}) = \frac{1}{2} - \frac{1}{4}g'(\hat{x})$$ $$g'(\pm\infty) = \pm 2$$

Conclusions

- used replica Bethe ansatz (RBA), well-tested method to calculate 1-time observables for the KPZ equation (large time convergence to TW)

- partial solution of the 2-times JPDF of heights (h1, h2) for the KPZ equation at large times

  I) obtained P(h1,h2-h1) for large h1>0, any t2/t1

  => predicted conditional cumulants of h2-h1 and conditional covariance ratio for large h1

  universal functions across KPZ class

  found to agree with experiments in broad range of h1

- calculated with RBA various joint distributions of max of Airy plus Brownian

  II) obtained 2 time covariance ratio (any h1) for large t2/t1

  agrees with I: RBA magic tricks work!

=> Memory effect for growth in expanding geometry

Next?

- still no answer if simple multi-time structure will emerge
- compare with other results: semi-discrete DP (Johansson) TASEP ring (Baik,Liu) , e.g. tails?
- push these RBA methods to get other multi-time observables

Perspectives/other works

- replica BA method

Airy process

stationary KPZ Sasamoto Inamura $t \to \infty$ $A_2(y)$

2 space points $Prob(h(x_1,t), h(x_2,t))$ Prohlac-Spohn (2011), Dotsenko (2013)

2 times $Prob(h(0,t), h(0,t'))$ Dotsenko (2013)

endpoint distribution of DP Dotsenko (2012) Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Connell, ..

q-TASEP $q \to 1$ avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

WASEP Bose gas moments as nested contour integrals

- sine-Gordon FT

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- Integrable lattice directed polymers

geometric T=0, log-gamma,beta T>0

Johansson (2000) Seppalainen (2012)
COSZ(2011) BCR(2013), Thiery, PLD(2014)
Barraquand,Corwin(2014) T. Thiery, PLD(2015)

connect to RWtimedepRE

KPZ at finite time and fermions at finite temperature in a trap

- KPZ equation with "droplet" initial condition

units: $t^* = 2(2\nu)^5/D^2\lambda_0^4$ scaled height: $\tilde{h}(0,t) = \frac{h(0,t) + \frac{t}{12}}{t^{1/3}}$

$h^* = 2\nu/\lambda_0$

Calabrese, Le Doussal, Rosso '10/Dotsenko '10/
Sasamoto, Spohn '10/Amir, Corwin, Quastel '11

$$g_t(s) := \langle \exp(-e^{t^{1/3}(\tilde{h}(0,t)-s)}) \rangle = \mathrm{Det}(1 - P_s K_{KPZ} P_s)$$

$t \to +\infty$ TW distrib

$\mathrm{Proba}(\tilde{h}(0,t) < s) = F_2(s)$

$$K_{KPZ}(a,b) = \int_{-\infty}^{+\infty} dv \frac{Ai(a+v)Ai(b+v)}{e^{-t^{1/3}v} + 1}$$

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- N non-interacting fermions in harmonic trap near edge

$T \sim N^{1/3} \to +\infty$ with fixed $b = \frac{\hbar\omega N^{1/3}}{T}$

position of rightmost fermion <=> KPZ height

Dean, PLD Majumdar, Schehr PRL (2015)

$$\frac{x_{max}(T) - r_{\text{edge}}}{w_N} =_{\text{inlaw}} \frac{h(0,t) + \frac{1}{12}t + G}{t^{1/3}}$$

$P(G) = e^{-G-e^{-G}}$

$w_N = \frac{N^{-1/6}}{\sqrt{2}\,\alpha}$

$b = t^{1/3}$

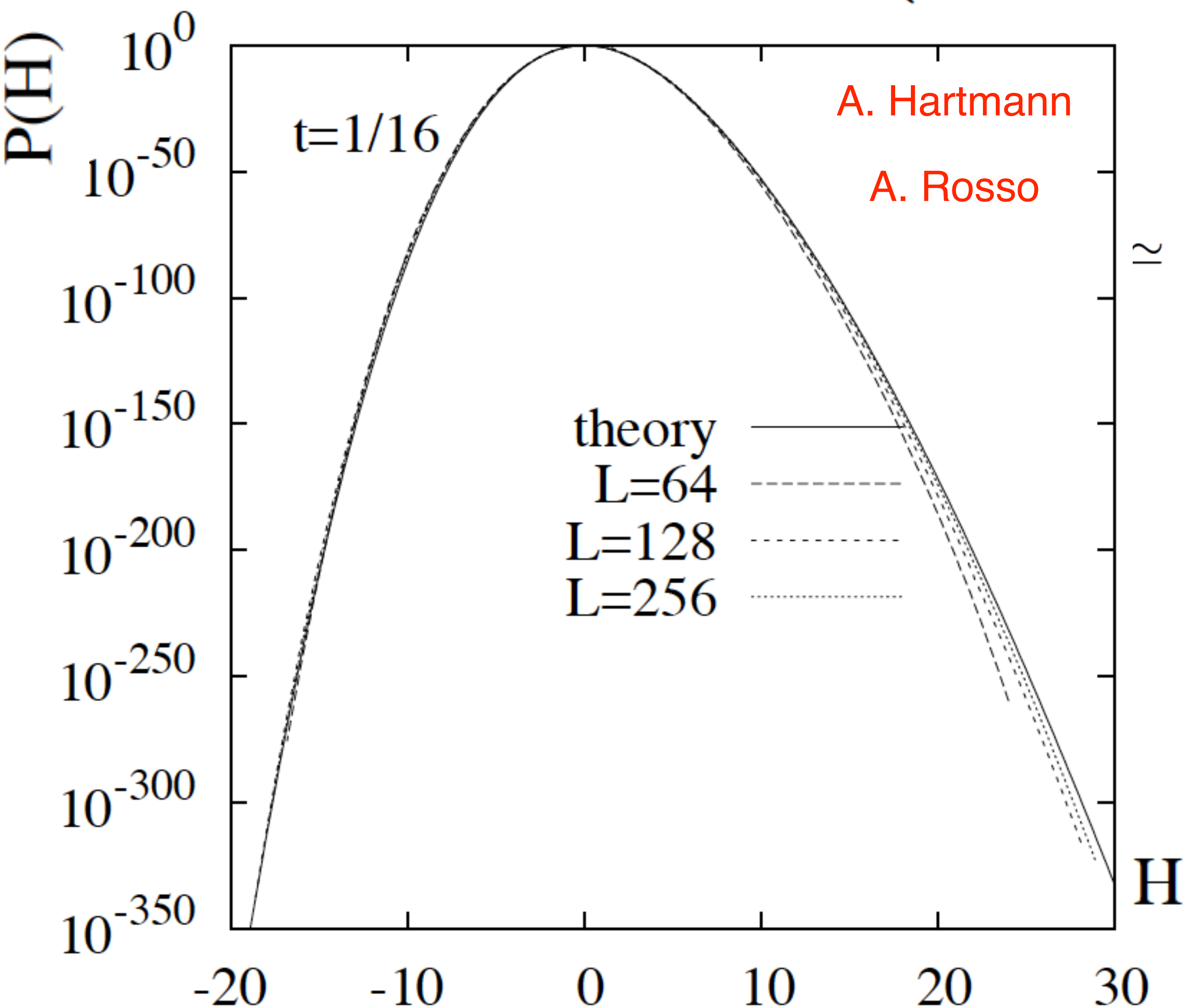
G is random variable with Gumbel distribution independent of h(0,t)

Two works I will NOT talk about !

Large deviations for KPZ (droplet IC) at short time

$$P(H,t) \sim \exp\left(-\frac{\Phi_{\rm drop}(H)}{\sqrt{t}}\right)$$

$$\Phi_{\rm drop}(H) = \begin{cases} \frac{-1}{\sqrt{4\pi}} \min\limits_{z\in[-1,+\infty[} [z e^H + Li_{\frac{5}{2}}(-z)], \; H \le H_c \\ \frac{-1}{\sqrt{4\pi}} \min\limits_{z\in[-1,0[} [z e^H + Li_{\frac{5}{2}}(-z) \\ \qquad - \frac{8\sqrt{\pi}}{3}(-\ln(-z))^{\frac{3}{2}}], \; H \ge H_c \end{cases} \quad (22)$$



$$\simeq \begin{cases} \frac{4}{15\pi}|H|^{5/2} , & H \to -\infty \\ \frac{H^2}{\sqrt{2\pi}} , & |H| \ll 1 \\ \frac{4}{3} H^{3/2} , & H \to +\infty . \end{cases}$$

EW (typical)

PLD, Majumdar, Rosso, Schehr, PRL 117, 070403 (2016)

N mutually avoiding paths in random potential

$\hat{\mathcal{Z}}_1(t)$ continuum partition sum of one directed polymer w. fixed endpoints at 0

$$\ln \hat{\mathcal{Z}}_1(t) \simeq -t/12 + \hat{\gamma}_1 t^{\;1/3}$$

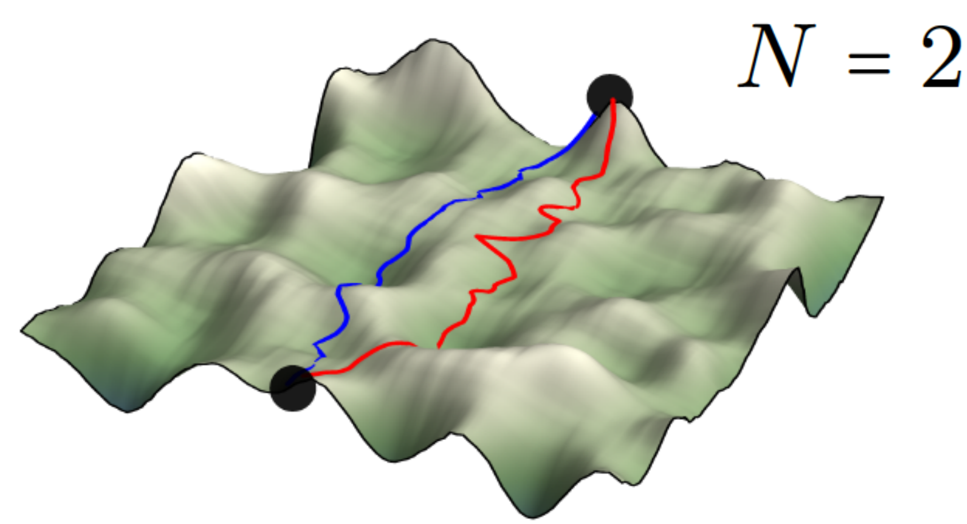
largest eigenvalues of a GUE random matrice

$\hat{\mathcal{Z}}_N(t)$ continuum partition sum of N non-crossing DP w. fixed endpoints at 0 in same random potential

CONJECTURE

$$\ln \hat{\mathcal{Z}}_N(t) \simeq -Nt/12 + t^{1/3}\hat{\zeta}^{(N)}$$

$$\hat{\zeta}^{(N)} \overset{\text{in law}}{\equiv} \sum_{i=1}^{N} \hat{\gamma}_i =: \hat{\gamma}$$



$\hat{\gamma}_1, \ldots, \hat{\gamma}_N$ N largest eigenvalues of a GUE random matrice

T=0 semidiscrete DP model — Yor, O' Connell, Doumerc (2002)

T>0 continuum (KPZeq) — Andrea de Luca, PLD, arXiv1606.08509, Phys. Rev. E 93, 032118 (2016) and 92, 040102 (2015)

Generalized Bethe Ansatz

Universal distribution of conductance in 2D localized phase

Somoza, Ortuno, Prior (2007)

$$H = \sum_i \epsilon_i c_i^+ c_i - t \sum_{<ij>} c_i^+ c_j + c_j^+ c_i$$

$$\ln g = -\frac{2L}{\xi} + \alpha\left(\frac{L}{\xi}\right)^{1/3} \chi_2$$

ξ localization length

L system size

χ random variable with Tracy Widom distribution

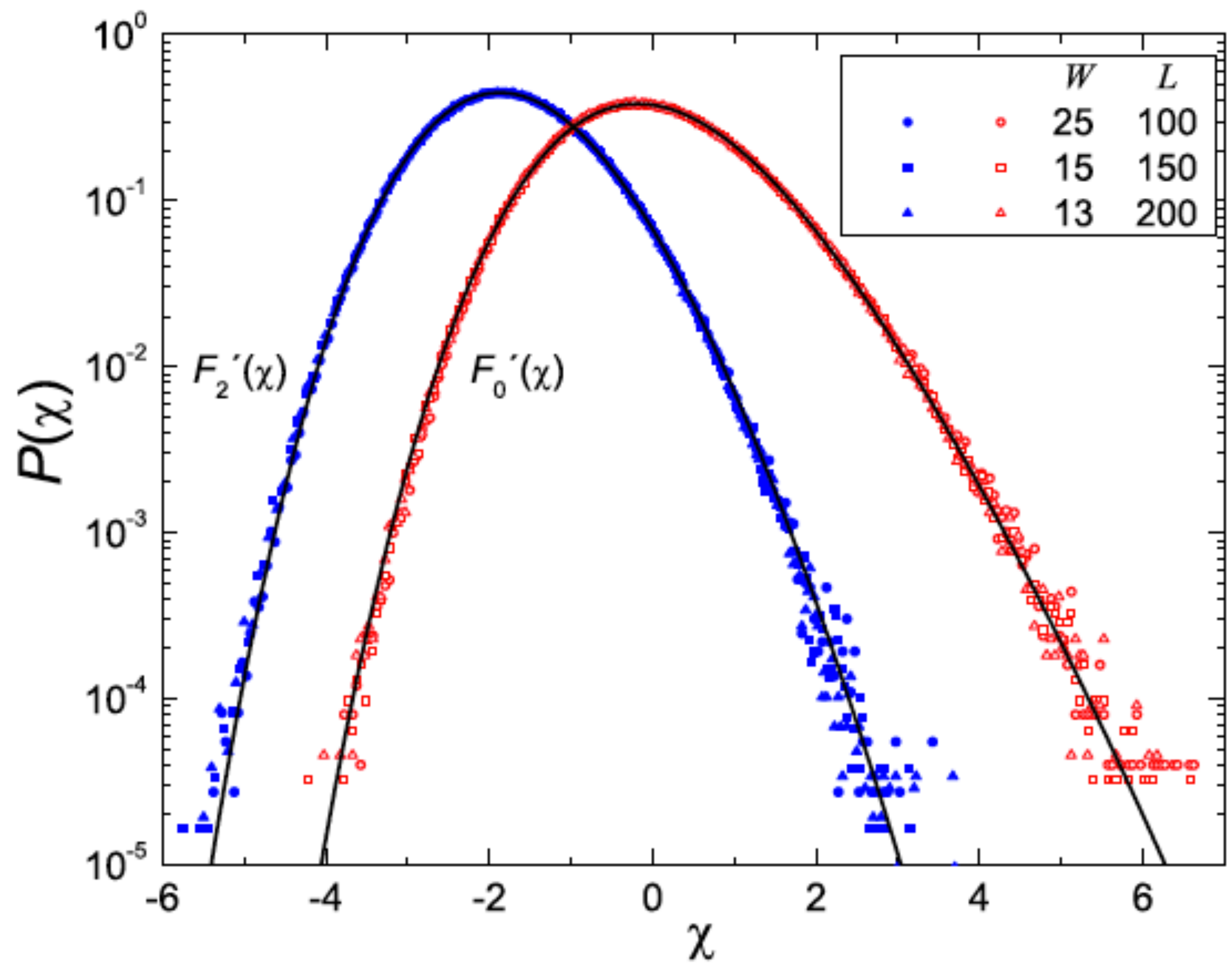


FIG. 1 (color online). Histograms of $\ln g$ versus the scaled variable χ for several sizes and disorders of the Anderson model with narrow (solid symbols) and wide (empty symbols) leads. The continuous lines correspond to $F_2'(\chi)$ and $F_0'(\chi)$.

i

Summary:

for droplet initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + (\frac{t}{t^*})^{1/3}\chi$$

similar (more involved)

χ at large time has the same distribution as the largest eigenvalue of the GOE

$$t^* = \frac{8(2\nu)^5}{D^2\lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

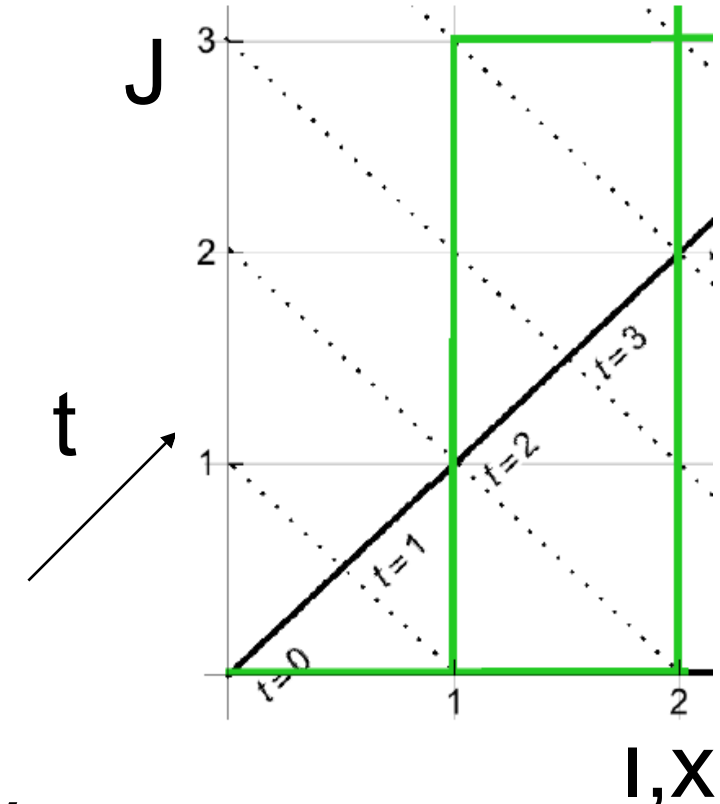
decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

GSE ? KPZ in half-space

Integrable directed polymer (DP) on square lattice

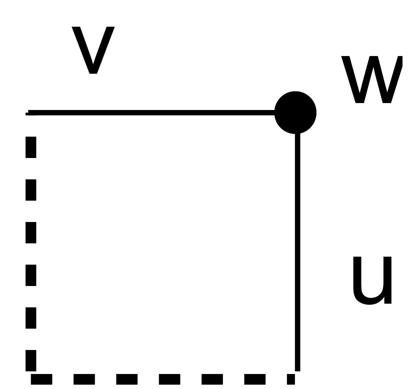
$$Z_t(x) = \sum_{\pi:(0,0)\to(x,t)} \prod_{(x',t')\in\pi} w_{x',t'}$$



- log-Gamma DP on-site weights $w \in [0,+\infty[$

inverse Gamma distribution $P(w) = \frac{1}{\Gamma(\gamma)} w^{-1-\gamma} e^{-1/w}$

Seppalainen (2012) Brunet
COSZ(2011) BCR(2013), Thiery, PLD(2014)



- Strict-Weak DP $u \in [0,+\infty[\quad v = 1$

Gamma distribution $P(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u}$

Corwin,Seppalainen,Shen(2014)
O'Connell,Ortmann(2014)

- Beta DP $u, v \in [0,1] \quad v = 1-u$

Beta distribution $p_{\alpha,\beta}(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \qquad \alpha > 0 \text{ and } \beta > 0$

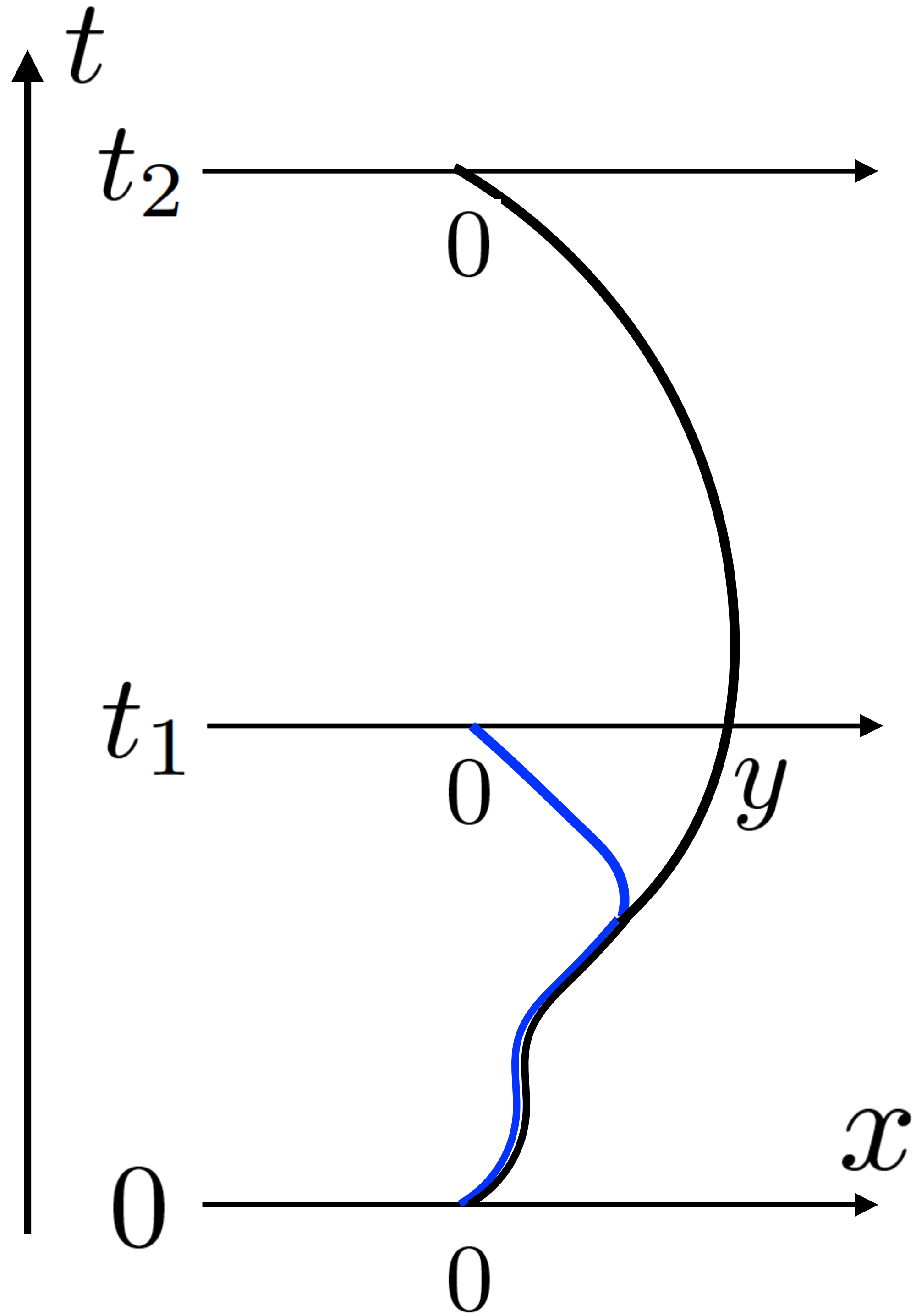
Barraquand,Corwin(2014)

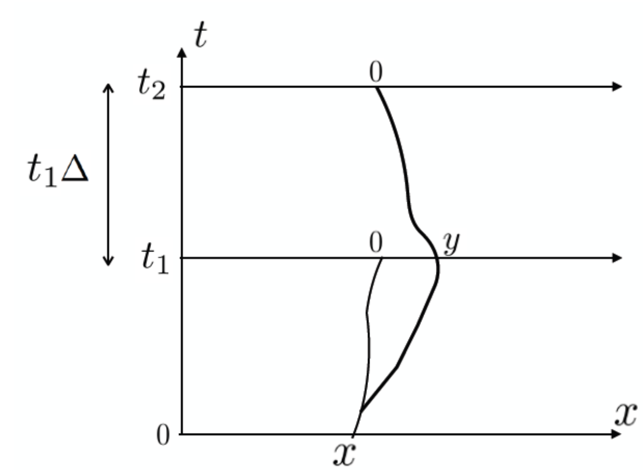
- Inverse-Beta DP $u \in [1,+\infty[\quad v \in [0,+\infty[\quad v = u-1$

inverse-Beta distribution

$$\tilde{p}_{\gamma,\beta}(u) = \frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma)\Gamma(\beta)} \frac{1}{u^{1+\gamma}} \left(1 - \frac{1}{u}\right)^{\beta-1} \qquad \gamma := 1-(\alpha+\beta)$$

T. Thiery, PLD(2015)


t
t2
0
t1
0
y
x
0
0


t
t2
0
t1Δ
t1
0
y
x
0
x