Noncommutative Geometric Invariant Theory

Lecture 3
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Introduction to lecture 3

In the proof of Laudal's structure theorem, we actually lift the restriction of the pro-versal family explicitly by defining its A-module structure. That is, with the notation above, let $H(\mathcal{M}) = H = (H_{ij})$. Then we have defined the pro-versal family $M_H \in \mathrm{Def}_M(H)$ by the morphism

$$\eta : A \to (H_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) = \mathrm{End}_H(M_H).$$

Notice that $End_H(M_H)$ is a k-algebra, and that $M_1, \ldots, M_r$ are exactly the the simple $\mathrm{End}_H(M_H)$-modules. We actually have the following:

Introduction to lecture 3

Theorem

(A generalized Burnside's theorem) Let A be a finite dimensional k-algebra, k algebraically closed. Let $\mathcal{M} = \{M_1, \dots, M_r\}$ be the family of simple (right) A-modules. Then the morphism of the versal family

$$\eta : A \to \mathcal{O}^A(\mathcal{M}) = \mathrm{End}_H(M_H)$$

is an isomorphism.

Proof.

The proof can be found in the book [3]. For short, we state that the injectivity follows by the theory of iterated extensions which computes the kernel, and the surjectivity then follows by the Wedderburn-Malcev structure theorem. □

Introduction to lecture 3

One of the main consequences of Theorem 1 is that the $\mathcal{O}$-construction is closed. This is the content of the next result which will be essential in the construction of noncommutative affine schemes.

Corollary

Let $\mathcal{M} = \{M_1, \dots, M_r\}$ be a set of r finite dimensional, right A-modules. Then $\mathcal{M}$ is the set of simple $\mathcal{O}^A(\mathcal{M})$-modules, and

$$\mathcal{O}^{\mathcal{O}^A(\mathcal{M})}(\mathcal{M}) \simeq \mathcal{O}^A(\mathcal{M}),$$

i.e. the $\mathcal{O}$-construction is closed.

Introduction to lecture 3

Proof.

First, notice that

$$\mathcal{O}^A(\mathcal{M}) = \operatorname{End}_H(M_H) = (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)) \rightarrow \oplus_{i=1}^r \operatorname{Hom}_k(M_i, M_i)$$

so that the M_i's are right $\mathcal{O}^A(\mathcal{M})$-modules. Burnside's theorem states that when k is algebraically closed, M is simple if and only if the structure morphism is onto, proving that in this case, $\mathcal{M}$ is exactly the set of simple $\mathcal{O}^A(\mathcal{M})$-modules.

We have that $\mathcal{O}^A(\mathcal{M})/\operatorname{I}^n$ is finite dimensional for $n \geq 0$. It follows from Theorem 1 that

$$\mathcal{O}^A(\mathcal{M})/\operatorname{I}^n \xrightarrow{\sim} O^{O^A(\mathcal{M})}(\mathcal{M})/\operatorname{I}^n$$

is an isomorphism for each n. By the completeness of $\mathcal{O}^A(\mathcal{M})$ we have

$$\mathcal{O}^A(\mathcal{M}) = \varprojlim_n \mathcal{O}^A(\mathcal{M})/\operatorname{I}^n \simeq \varprojlim_n \mathcal{O}^{O^A(\mathcal{M})}(\mathcal{M})/\operatorname{I}^n \simeq \mathcal{O}^{\mathcal{O}^A(\mathcal{M})}(\mathcal{M})$$

Commutative Affine Schemes

We are looking for a definition that can be generalized. The main obstacle is that noncommutative k-algebras lack the utility of localization. Using deformation theory, one of our main results is that we can define the localization of A in an element $f \in A$. In this subsection we recall the functorial construction of sheaves and schemes in the ordinary noncommutative situation.

Commutative Affine Schemes

We define sheaves on a topological space by limits: Let X be a topological space. Let $\mathbf{Top}(X)$ be the category with objects the open subsets of X and morphisms the inclusions. A presheaf in a category $\mathbf{C}$ on X is a contravariant functor

$$\mathcal{F} : \mathbf{Top}(X) \to \mathbf{C}.$$

Such a presheaf is called a *sheaf* if in addition

$$\mathcal{F}(U) = \varprojlim_{V \subseteq U} \mathcal{F}(V)$$

for each open $U \subseteq X$.

Notice that the universal properties of the limit gives the properties of existence and uniqueness given on elements in Hartshorne [5].

For a commutative k-algebra A which we assume to be a finitely presented domain, the affine scheme structure $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ can be defined as follows. First of all the Zariski topology $X = \operatorname{Spec} A$ is generated by the open sets $D(f),\ f \in A$.

Commutative Affine Schemes

Consider the canonical morphism

$$\eta_f : A \to \prod_{\mathfrak{p} \in D(f)} \hat{A}_{\mathfrak{p}} = \lim_{\substack{\longleftarrow \\ \mathfrak{p} \in D(f)}} \hat{A}_{\mathfrak{p}}.$$

Lemma

Let $O(D(f))$ be the subring of $\lim\limits_{\substack{\longleftarrow \\ \mathfrak{p} \in D(f)}} \hat{A}_{\mathfrak{p}}$ generated by $\eta_f(A)$ and $\eta(f)^{-1}$.

Then $O(D(f)) \simeq A_f$.

Proof.

Because $\eta_f(f)$ is a unit in $\lim\limits_{\substack{\longleftarrow \\ \mathfrak{p} \in D(f)}} \hat{A}_{\mathfrak{p}}$ there is a homomorphism

$\phi : A_f \to O(D(f))$ given by $\phi(a/f^n) = \eta_f(a)\eta(f)^{-n}$. This homomorphism is both surjective and injective, so an isomorphism. □

Commutative Affine Schemes

The following definition is equivalent to the one given in Hartshorne [5].

Definition

Let A be a commutative k-algebra. Let $\mathrm{Spec}(A)$ be the set of prime ideals $\mathfrak{p} \subset A$ with the topology generated by the open sets $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \notin \mathfrak{p}\}$, $f \in A$. We define a sheaf of rings on $\mathrm{Spec}(A)$ by

$$\mathcal{O}_{\mathrm{Spec}(A)}(U) = \varprojlim_{D(f) \subseteq U} O(D(f)).$$

Then $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a locally ringed space coinciding with the ordinary affine scheme associated to A.

Nonommutative Affine Varieties

For varieties, i.e. when A is a finitely generated integral domain, finitely presented over an algebraically closed field k of characteristic 0, the ringed space $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}\,A}) \simeq (\max(A), \mathcal{O}_{\max(A)})$ can be generalized directly to the the noncommutative situation. The general embedding of $\mathbf{Sch}_k$ in $\mathbf{ncSch}_k$ will be considered later in these lectures.

Definition

We let $\mathrm{Simp}(A)$ denote the set of all simple (right) A-modules. For $f \in A$ we define the basic open subset $D(f) \subseteq \mathrm{Simp}(A)$ by

$$D(f) = \{M \in \mathrm{Simp}(A) \mid \eta_M(f) : M \to M \text{ is invertible } \}$$

where for each $M \in \mathrm{Simp}(A)$, $\eta_M(f)(m) = m \cdot f$ is multiplication by f.

Nonommutative Affine Varieties

The proof of the following fact is straight forward.

Lemma

The family of sets $D(f)$, $f \in A$, is a basis for a topology on $\mathrm{Simp}(A)$.

For an associative k-algebra A, for a set $\mathcal{M} = \{M_1, \dots, M_r\}$ of simple A-modules, Corollary 2 says that the k-algebra

$$O^A(\mathcal{M}) = \mathrm{End}_{H(\mathcal{M})}(\mathcal{M})$$

has exactly the simple right A-modules $\mathcal{M} = \{M_1, \dots, M_r\}$.

Nonommutative Affine Varieties

There exists a k-algebra homomorphism

$$A \overset{\eta_{\mathcal{M}}}{\to} O^A(\mathcal{M})$$

defining the semi-universal family.

Definition

We call the multi-pointed algebra $A_{\mathcal{M}} = O^A(\mathcal{M})$ the multi-localization of A in $\mathcal{M}$.

Nonommutative Affine Varieties

Lemma

i) *If* $\mathcal{M} \subseteq D(f)$ *then* $\eta_{\mathcal{M}}(f)$ *is a unit in* $A_{\mathcal{M}}$.

ii) *Let* $\eta_f : A \to \varprojlim_{\mathcal{M} \subset D(f)} A_{\mathcal{M}}$ *be the limit of the morphisms* $\eta_{\mathcal{M}} : A \to A_{\mathcal{M}}$.

Then $\eta_f(f)$ *is a unit.*

Proof.

Because η_M is given deformation theoretically as $\eta_{\mathcal{M}}(a) = a + \xi(a)$ with $\xi(a)$ in the Jacobson radical, it follows by definition of $D(f)$ that $\eta_M(f)$ is invertible, i.e. a unit in $A_{\mathcal{M}}$. Now ii) follows from the sheaf condition: An element that is a unit on all stalks can be lifted to a unit globally. □

Nonommutative Affine Varieties

Definition

Let A be an associative k-algebra, $f \in A$. Then we define A_f as the subring of $\varprojlim_{\mathcal{M} \subset D(f)} A_{\mathcal{M}}$ generated by $\eta_f(A)$ together with $\eta_f^{-1}(f)$.

Definition

A ringed space $(X, \mathcal{O}_X)$ is called a *multi-locally ringed space* if for each finite set of points P the stalk $\mathcal{O}_{X,P}$ is a multi-pointed ring. A morphism $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is *multi-local* if the limit morphism $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a morphism of multi-pointed rings.

Nonommutative Affine Varieties

Definition

(Noncommutative affine scheme) Let A be an associative k-algebra. Let $\text{Simp}(A)$ be the set of simple, right A-modules with the topology generated by the base $\{D(f),\ f \in A\}$. Let $\mathcal{O}_{\text{Simp}(A)}$ be the sheaf of k-algebras defined by $\mathcal{O}_{\text{Simp}(A)}(U) = \varprojlim_{D(f)\subseteq U} A_f$. Then

$$(\text{Simp}(A), \mathcal{O}_{\text{Simp}(A)})$$

is a multi-locally ringed space called the affine noncommutative scheme associated to A.

Notice that it follows that the stalk of the sheaf $\mathcal{O}_{\text{Simp}(A)}$ in the finite point-set $\mathcal{M}$ is $\mathcal{O}_{\text{Simp}(A),\mathcal{M}} = \varinjlim_{\mathcal{M}\subset P(U)} \mathcal{O}_{\text{Simp}(A)}(U) = A_{\mathcal{M}}$ so that the ringed space above is indeed multi-local.

Example

If A is commutative and finitely presented, then it is of the form $A = k[x_1, \ldots, x_n]/I$ for an ideal I, and there is a bijective correspondence between $X = \mathrm{Simp}(A)$ and the closed points of $\mathrm{Spec}(A)$, given by

$$M \mapsto \mathrm{Ann}(M).$$

In fact, $M = A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m} \subseteq A$, and $A/\mathfrak{m} \simeq A/\mathfrak{m}'$ as right A-modules if and only if the maximal ideal $\mathfrak{m} = \mathfrak{m}'$ coincide. For any $f \in A$, we have that $M \cdot f = M$ if $M \cdot f \neq 0$, since $Mf \subseteq M$ is an A-submodule in the commutative case. Therefore, a simple module $M = A/\mathfrak{m} \in D(f)$ if and only if $f \notin \mathfrak{m}$. It follows that when $X = \mathrm{Simp}(A)$ has the Jacobson topology and $\mathrm{Spec}(A)$ has the Zariski topology, the bijective correspondence between $X = \mathrm{Simp}(A)$ and the closed points in $\mathrm{Spec}(A)$ is a homeomorphism.

Example

Let $X = \text{Simp}(A)$ for the noncommutative algebra A given by

$$A = \begin{pmatrix} k[x_{11}] & \langle x_{12} \rangle \\ 0 & k[x_{22}] \end{pmatrix}.$$

In other words, A is a tensor algebra of the k^2-bimodule V, where $\dim_k V_{ij} = 1$ for $(i,j) = (1,1), (1,2), (2,2)$ and $V_{21} = 0$. Then we have that

$$X = \{M_\alpha : \alpha \in k\} \cup \{N_\beta : \beta \in k\} = \mathbb{A}^1 \coprod \mathbb{A}^1$$

where $M_\alpha = k[x_{11}]/(x_{11} - \alpha)$ and $M_\beta = k[x_{22}]/(x_{22} - \beta)$. As a topological space, X is a disjoint union of the two affine lines, since we have that $D(e_1) = \{M_\alpha : \alpha \in k\}$ and $D(e_2) = \{N_\beta : \beta \in k\}$ are the connected components of X.

Example

The multi-localization in the finite set $\mathcal{M} = \{M_\alpha, N_\beta\}$ is given as

$$A_{\mathcal{M}} = \begin{pmatrix} H_{11}(\mathcal{M}) \otimes_k \operatorname{End}_k(M_\alpha, M_\alpha) & H_{12}(\mathcal{M}) \otimes_k \operatorname{Hom}_k(M_\alpha, N_\beta) \\ H_{21}(\mathcal{M}) \otimes_k \operatorname{Hom}_k(N_\beta, M_\alpha) & H_{22}(\mathcal{M}) \otimes_k \operatorname{End}_k(N_\beta, N_\beta) \end{pmatrix}.$$

This is easily computable as there are no non-trivial relations in the definition of A, and as the two simple modules under consideration is both of k-dimension 1. We get

$$A_{\mathcal{M}} = \begin{pmatrix} H_{11}(\mathcal{M}) & H_{12}(\mathcal{M}) \\ H_{21}(\mathcal{M}) & H_{22}(\mathcal{M}) \end{pmatrix} = \begin{pmatrix} k \ll x_{11} \gg & \ll x_{12} \gg \\ 0 & k \ll x_{22} \gg \end{pmatrix}.$$

Example

Also notice that this extends to finite sets with more than two elements, e.g. for $\mathcal{M}' = \{M_\alpha, M_{\alpha'}, N_\beta\}$ we get

$$A_{\mathcal{M}} = \begin{pmatrix} k \ll x_{11} \gg & 0 & \ll x_{13} \gg \\ 0 & k \ll x_{22} \gg & \ll x_{23} \gg \\ 0 & 0 & k \ll x_{33} \gg \end{pmatrix},$$

for $\mathcal{M}'' = \{M_\alpha, N_\beta, N_{\beta'}\}$ we get

$$A_{\mathcal{M}} = \begin{pmatrix} k \ll x_{11} \gg & \ll x_{12} \gg & \ll x_{13} \gg \\ 0 & k \ll x_{22} \gg & 0 \\ 0 & 0 & k \ll x_{33} \gg \end{pmatrix},$$

and there are (natural) canonical restriction morphisms

$$r_{\mathcal{M} \subseteq \mathcal{M}'} : A_{\mathcal{M}'} \to A_{\mathcal{M}}, \; r_{\mathcal{M} \subseteq \mathcal{M}''} : A_{\mathcal{M}'} \to A_{\mathcal{M}}.$$

Example - Geometric Algebras

We consider the class of finitely presented tensor-algebras with commutative k-algebras on the diagonal. These are essential for solving problems in (commutative) algebraic geometry by noncommutative algebraic geometry, and we call them *geometric algebras*.
These algebras are most conveniently introduced by examples, where the generalization is clear.
Consider a line and a parabola in 2-space intersecting in the origin.

$$y = x^2 \text{ and } y = cx.$$

We identify the points in the intersection of these curves.

Example - Geometric Algebras

$$R = \frac{\begin{pmatrix} k[t_{11}(1), t_{11}(2)] & k\langle t_{12}(1)\rangle \\ k\langle t_{21}(1)\rangle & k[t_{22}(1), t_{22}(2)] \end{pmatrix}}{\langle t_{12}(1)(t_{22}(1) - t_{22}^2(2)),\ (t_{22}(1) - ct_{22}(2))t_{21}(1)\rangle} =: \frac{F\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}{(f_{12}, f_{21})} \tag{1}$$

We will study its affine nc scheme $(\mathrm{Simp}(R), \mathcal{O}_{\mathrm{Simp}(R)})$, and eventually, prove that $\mathcal{O}(\mathrm{Simp}(R)) \simeq R$.
In general, the simple modules are the disjoint union of the simple modules on the diagonal. In our case this gives

$$\mathrm{Simp}(R) = V(k[t_{11}(1), t_{11}(2)]) \coprod V(k[t_{22}(1), t_{22}(2)]) = V_1 \coprod V_2$$

where we for a k-algebra A let $V(A)$ be its variety.

Example - Geometric Algebras

There is a possibility for having relations on the diagonal,and the generalization is clear. Put

$$M_i(a,b) = k[t_{ii}(1), t_{ii}(2)]/(t_{ii}(1) - a, t_{ii}(2) - b)$$

for $i = 1, 2$ and $a, b \in k$. As the topology on each of the affine varieties V_i is well known, we find that the topology on $\mathrm{Simp}(R)$ is the product topology of V_1 and V_2, just saying that $\mathrm{Simp}(R) = V_1 \coprod V_2$ as a topological space.

Example - Geometric Algebras

From the Tangent Lemma in Lecture 1 it follows that for $R = F/\mathfrak{f}$ where $\mathfrak{f} = (f_{ij}(l_{ij}))$ is a finitely generated two-sided ideal we have that

$$\mathrm{Ext}^1_R(M_i(P), M_j(Q)) = \mathrm{Ext}^1_F(M_i(P), M_j(Q))/(\frac{\partial \mathfrak{f}}{\partial \underline{t}_{ij}}(P, Q))$$

where the ideal in the quotient means the derivations of all polynomials in $\mathfrak{f}$ in all the variables $t_{ij}(l_{ij})$ and evaluated in the point P on the left, Q on the right. This again says that the relations in $\mathfrak{f}$ must satisfy $\partial(\mathfrak{f}) = 0$.

Example - Geometric Algebras

It is well known that for A a finitely presented commutative k-algebra, k algebraically closed, $\mathfrak{m}_1, \mathfrak{m}_2$ two different maximal ideals,

$$\mathrm{Ext}^1_A(A/\mathfrak{m}_1, A/\mathfrak{m}_2) = 0.$$

From this it follows that if we have two different points P, Q in the same entry on the diagonal, that is the set of simple modules $\{M_i(P), M_i(Q)\},\ P \neq Q$, then

$$\mathrm{Ext}^1_R(M_i(P), M_i(Q)) = \mathrm{Ext}^1_{k[t_{ii}(1), t_{ii}(2)]}(M_i(P), M_i(Q)) = 0.$$

Example - Corollary

Let $R = F/\mathfrak{f}$ be geometric. Then there are the following possibilities for pairs of simple (one-dimensional, right) k-modules and their tangent spaces:

A duplicate of a point in one diagonal entry. Then

$$\mathrm{Ext}^1_R(M_i(P), M_i(P)) = \mathrm{Ext}^1_F(M_i(P), M_i(P))/(\frac{\partial \mathfrak{f}_{ii}}{\partial \underline{t}_{ii}}(P)).$$

Two different points $P \neq Q$ in the same entry. Then

$$\mathrm{Ext}^1_R(M_i(P), M(Q)) = 0.$$

Two points P and Q in different entries of the diagonal. Then

$$\mathrm{Ext}^1_R(M_i(P), M_j(Q)) = \mathrm{Ext}^1_F(M_i(P), M_j(Q))/(\frac{\partial \mathfrak{f}}{\partial \underline{t}_{ij}}(P, Q)).$$

Example - Geometric Algebras

We will start by applying the general concepts to our present example, with the given geometric algebra R from (1). We start by considering two points
$\{M_1(a,b), M_2(c,d)\}$ for a particular choice of coordinates on the diagonal. We make a linear coordinate change, so that it suffices, without loss of generality, to consider the set of points

$$\mathcal{M}(x,y) = \{M_1(0,0) = M_1, M_2(x,y)\},\ (x,y) \in k^2. \qquad (2)$$

The computations given by this choice of a finite set of simple modules will give us the ability to state the results for the other necessary computations (other selections of finite point-sets):

Example - Geometric Algebras

(1.1)

$$\mathrm{Ext}^1_R(M_1, M_1) = \mathrm{Ext}^1_F(M_1, M_1) = k\frac{\partial}{\partial t_{11}(1)} \oplus k\frac{\partial}{\partial t_{11}(2)}$$

(1.2)

$$\mathrm{Ext}^1_R(M_1, M_2(x,y)) = (k\frac{\partial}{t_{12}(1)})/(\frac{\partial f_{12}}{\partial t_{12}(1)})$$

(2.1)

$$\mathrm{Ext}^1_R(, M_2(x,y), M_1) = (k\frac{\partial}{t_{21}(1)})/(\frac{\partial f_{21}}{\partial t_{21}(1)})$$

(2.2)

$$\begin{aligned}\mathrm{Ext}^1_R(M_2(x,y), M_2(x,y)) &= \mathrm{Ext}^1_F(M_2(x,y), M_2(x,y)) \\ &= k\frac{\partial}{\partial t_{22}(1)} \oplus k\frac{\partial}{\partial t_{22}(2)}\end{aligned}$$

Example - Geometric Algebras

From this we read that the tangent space dimensions are $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ if the point is $(x, y) = (0, 0)$, otherwise the tangent space dimension is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. It follows that it is sufficient to consider the set of two simple modules given by

$$\{M_1(0,0), M_2(0,0)\} = \{M_1, M_2\}.$$

We now use the algorithm given in [3] to compute the hull of the deformation functor in this finite point-set, the result is

Example - Geometric Algebras

$$\hat{R}_{\mathcal{M}} = \varprojlim S_n = \frac{\begin{pmatrix} k[\![t_{11}(1), t_{11}(2)]\!] & k \ll t_{12}(1) \gg \\ k \ll t_{21}(1) \gg & k[\![t_{22}(1), t_{22}(2)]\!] \end{pmatrix}}{\langle t_{12}(1)(t_{22}(1) - t_{22}^2(2)),\ (t_{22}(1) - ct_{22}(2))t_{21}(1) \rangle}$$

$$=: \frac{F\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)}{(f_{12}, f_{21})}.$$

We have an injection

$$\iota : R \hookrightarrow \hat{R}_{\mathcal{M}},$$

and the image is the (algebraic) multi-pointed ring

$$R_{\mathcal{M}} = \operatorname{im}(\iota) \subseteq \hat{R}_{\mathcal{M}}.$$

Geometric Algebras

This example can be generalized to any geometric k-algebra R, and we have the localization morphisms

$$R \to R_{\mathcal{M}}$$

for every finite set of simple modules $\mathcal{M}$, and because the construction of the pro-representing hull is natural, this is universal, making R unique.

Corollary

For a geometric k^r-algebra R,

$$\mathcal{O}_{\operatorname{Simp} R}(\operatorname{Simp} R) = R.$$

Proof.

Given any other geometric QAR R' with multi-localizations $R'_{\mathcal{M}}$, this has to contain all relations and so the localization morphisms factors through R. □

Geometric Algebras

Proposition

Let S be a singularity. Then, adding tangents as in the guiding example tells us that that there exists a geometric algebra $(\mathrm{Simp}(R), \mathcal{O}_{\mathrm{Simp}\, R}) \rightarrow (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$ that is a rational morphism to the singularity, and eventually a noncommutative resolution.

Noncommutative Varieties

Definition

A noncommutative (nc) variety is a multi-locally ringed space $(X, \mathcal{O}_X)$ such that X has a covering of open affine subsets, i.e. $X = \cup_i U_i$ such that

$$(U_i, \mathcal{O}_X|_{U_i}) \simeq (\mathrm{Simp}(A_i), \mathcal{O}_{\mathrm{Simp}(A_i)})$$

for some k-algebras A_i, separated and of finite type over k, running through an index set. A morphism of nc schemes, is a morphism in the category of multi-locally ringed spaces.

The noncommutative deformation theory tells us that we can study the geometry of a k-algebra A by its representations. Thus we already have the very short and precise definition of a noncommutative variety.

Noncommutative Varieties

Definition

Let A be an integral, separated k-algebra of finite type over k. A family of A-modules $\mathcal{M}$ is called an affine variety for A if the pro-versal family

$$\eta_{\mathcal{M}} : A \to \mathcal{O}^A(\mathcal{M})$$

is an isomorphism of k-algebras.

The background for this definition is the generalized Burnside's theorem, Theorem 1, which implies that for a finite dimensional k-algebra A we have that $\mathrm{Simp}(A)$ is a scheme for A.

Noncommutative Varieties

Proposition

When A is a geometric k-algebra, $\mathrm{Simp}(A)$ *is an affine varity for A.*

Proof.

We have that $\iota : A \to \mathcal{O}_{\mathsf{Simp}}(D(1)) = O_{\mathsf{Simp}}(\mathrm{Simp}(A))$ is injective because A is geometric, and surjective onto its image. □

Noncommutative Varieties

When the ordinary commutative schemes X are algebraic varieties over k, i.e. integral, separated schemes of finite type over k, the embedding of schemes works directly because of the reconstruction theorem $A \simeq \mathcal{O}_{\mathrm{Simp}\, A}(\mathrm{Simp}\, A)$ which is Proposition 2.6 in [5] stating that the maximal ideals are sufficient for reconstructing the k-algebra. This says that the two definitions are equivalent.

Corollary

Let X be a commutative variety. Then the set of closed points $\mathrm{Simp}(X) \subset X$ is a variety for X.

Proof.

This means that $\mathrm{Simp}(X)$ is locally a scheme for A for an open covering by $\mathrm{Spec}(A)$. This follows from Proposition 2. □

Remark

We would like to give a remark on noncommutative schemes that does not come from commutative ones. This means that there is no algebraic (= finitely generated) k-algebra A such that $\mathcal{O}(\mathrm{Simp}(A)) \simeq A$. This is even worse: There is no reductive group action on an affine commutative variety resulting in these noncommutative schemes, so they are not even *derived from commutative schemes*.
The most trivial example of this is the noncommutative k-algebra in two variables,

$$A = k\langle x, y \rangle.$$

There exist simple modules of any dimension, see Eriksen [2], and so the ring of observables is necessarily a complete noncommutative k^r-algebra which is not the completion of a k^r-algebra with one-dimensional simple modules only on the diagonal.

Deformations due to diagrams

To give the stated embedding in the general situation, we need to refine the families of modules.

Definition

A diagram of A-modules is a set of right A-modules $\mathcal{M} = \{M_i\}_{i \in I}$, together with a set of A-module homomorphisms $\Gamma_{ij} \subseteq \text{Hom}_A(M_i, M_j)$ for each pair of modules. The idempotents $e_i \in \text{End}_A(M_i)$ is supposed to be included in the diagram. We will write $\underline{c} = (\mathcal{M}, \Gamma)$, and $|\underline{c}| = \mathcal{M}$.

The details can be found in the proceedings, we just indicate that this gives the generalization of scheme-theory in general.

Deformations due to diagrams

Let $V = \mathbf{t}(\mathrm{Def}_{\mathcal{M},k[\Gamma]})$ be the tangent space of $\mathrm{Def}_{\mathcal{M},k[\Gamma]}$ and let $W = (W_{ij})$ with $W_{ij} = k^{d_{ij}}$. There are natural k-linear maps $\kappa_{ij} : V_{ij}^* \to W_{ij}$ given by

$$\kappa_{ij}(\psi_{ij}^*) = (\psi_{ij}^*(\phi_{ij}(l)))$$

where $\{\phi_{ij}(l) : 1 \leq l \leq d_{ij}\}$ are the morphisms from M_i to M_j in the diagram $\underline{c}$. Since $\mathrm{Def}_{\mathcal{M},k[\Gamma]}$ is unobstructed, there is an induced morphism $\kappa : H(\mathcal{M}, k[\Gamma]) \to \mathbf{T}(W)$, where $\mathbf{T}(W)$ is the tensor algebra of W over k^r (the matrix algebra F generated by the k^r-bimodule W), and $\ker(\kappa) \subseteq H(\mathcal{M}, k[\Gamma])$ is an ideal.

Deformations due to diagrams

Let us also denote by $\ker(\kappa)$ the ideal in $H(\mathcal{M}, A[\Gamma])$ generated by the image of $\ker(\kappa) \subseteq H(\mathcal{M}, k[\Gamma])$ under the ring homomorphism $H(\mathcal{M}, k[\Gamma]) \to H(\mathcal{M}, A[\Gamma])$.

Definition

We define $H(\underline{c}) = H(\mathcal{M}, A[\Gamma])/\ker(\kappa)$, and the *ring of observables* of the diagram $\underline{c}$ to be

$$\mathcal{O}(\underline{c}) = \operatorname{im}(\eta_{\underline{c}}) \subseteq (H(\underline{c})_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)).$$

Deformations due to diagrams

For a commutative k-algebra A, let

$$\mathrm{Spec}^*(A) = \{A \overset{\psi_{\mathfrak{p}}}{\to} A/\mathfrak{p} : \mathfrak{p} \in \mathrm{Spec}(A)\},$$

where the morphisms $\psi_{\mathfrak{p}} \in \mathrm{Hom}_A(A, A/\mathfrak{p})$ is the set of all A-linear homomorphisms.

Theorem

For k algebraically closed of characteristic 0, for A a commutative, finitely generated k-algebra A,

$$\mathcal{O}(\mathrm{Spec}^*(A)) \simeq A.$$

Summing up

Definition

Let A be a k-algebra. A diagram $\underline{c}$ of A-modules is called an affine scheme for A if the pro-versal family

$$\eta_{\underline{c}} : A \to \mathcal{O}(\underline{c})/R$$

is an isomorphism of k-algebras, where R is the ideal of relations in the diagram.

We give the general definition of a noncommutative affine scheme.

Definition

We define $\mathrm{Simp}^*(A)$ as the diagram $\mathrm{Simp}^*(A) = \{A\} \cup \mathrm{Simp}(A)$ with quiver $\Gamma^* = \cup_{M_i, M_j \in \mathrm{Simp}(A)} \mathrm{Ext}^1_A(M_i, M_j) \cup \{A \to M | M \in \mathrm{Simp}(A)\}$.

Summing up

Lemma

If $\eta_{\mathrm{Simp}^*(A)}$ is injective, $\mathrm{Simp}^*(A)$ *is a scheme for* A.

Proof.

This follows by definition. It is also proved by O.A. Laudal in [6]. □

Definition

We give $\mathrm{Simp}^*(A)$ the topology induced by the Jacobson topology. On this topological space we consider a multi-locally ringed space $\mathcal{O}_{\mathrm{Simp}^*}$ defined as in definition 11, and we call this a noncommutative affine scheme. A noncommutative scheme is a multi-locally ringed space, locally isomorphic as such to a noncommutative affine scheme.

Summing up

Proposition

The category of k-schemes is naturally, full, and faithfully embedded in the category of noncommutative k-schemes, i.e.

$$\mathbf{sch}_k \subset \mathbf{ncSch}_k$$

Proof.

Consider a k-scheme X with an open affine cover by $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. Consider the diagram $\operatorname{Simp}^*(A)$ defined as above. The open affine embed fully as $(\operatorname{Simp}^*(A), \mathcal{O}_{\operatorname{Simp}^*(A)})$. The reconstruction holds, and then the reconstruction lemma proves it is faithful. Notice that this procedure formally adds enough generic points. □

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