Noncommutative algebraic invariant theory
Notes for the school on Cluster Algebras
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0.1 Foreword

Some time ago, algebraic representation theory and algebraic geometry was two quite separate fields in mathematics. The representation theory is traditionally (some may think) an approximation of category theory, while algebraic geometry is (some may feel) an approximation of differential geometry. The classification of representations gives valuable applications to physics by the representations of time-spaces, while the classification of algebraic spaces (curves, surfaces, etc.) gives information on the relations between the different representations. Thus algebraic geometry also have some applications to physics, more than as a differential geometry.

One can say that many results from representation theory is about the equivalence (or derived equivalence) of categories of representations, while algebraic geometry is about proving the existence of moduli schemes of special families of representations (geometric points). Because this is essentially equivalent problems, algebraic geometry has obvious relations to physics, more than as an approximation to differential geometry.

In classical deformation theory, one approximates the local structure of moduli spaces, and in some very special cases, we can construct moduli spaces by gluing local spaces.

Inspired by the representation theory and its physical interpretation, O.A. Laudal looked for a generalization of the deformation theory to deformations over rings that are not necessarily commutative. This lead to the genius (though intuitive) task of deforming a set of r modules simultaneously. Then at once, one got the multilocalization of points, and a noncommutative algebraic geometry could be formed, based on the theory of representation. Thus the two theories are working more closely together than ever.

This text proves that techniques from algebraic geometry and representation theory can be woven together to a theoretical framework for physics.

Chapter 1

Introduction

In this text we apply noncommutative deformation theory to general moduli problems.

It is well known that ordinary deformation theory of modules applies to the theory of moduli, and that it solves problems in very special algebraic situations.

In most algebraic situations, e.g. geometric invariant theory, the ordinary deformation theory is not sufficient. Olav Arnfinn Laudal generalized the deformation functor

$$\mathsf{Def}_M : \ell \to \mathbf{Sets},$$

which goes from the category of local artinian (pointed) k-algebras to the category of sets and where M is an A-module, to a *noncommutative deformation functor*

$$\mathsf{Def}_{\mathcal{M}} : \mathbf{a}_r \to \mathbf{Sets},$$

which goes from the category of r-pointed, not necessarily commutative, artinian k-algebras to the category of sets, and where $\mathcal{M} = \{M_1, \ldots, M_r\}$ is a family of r (right) A-modules. The study of this generalization is interesting in its own rights, and it turns out that it more or less solves the problems in geometric invariant theory (e.g. when an action of a group is not free).

The Chapters 2 to 6 considers the noncommutative geometric invariant theory. The theory and all its belongings was invented by O. A. Laudal, and all the examples are developed under his supervision.

In Chapter 7 we observe that we can define a dynamical structure by letting a Lie-algebra act on the tangent space of the (noncommutative) moduli scheme, thereby introducing dynamical invariants in algebraic geometry. As everything else in this text, this idea is completely due to Laudal. He was also the one that accepted that this algebraic dynamical structure was not sufficient in the analytic case, i.e. for algebraic spaces. Thus the chapter 7 is called *Pre-dynamic GIT*.

In Chapter 8, we give a solution to the above algebraic challenges, by inventing a motivic algebraic theory based on deformation theory. We give a simplicial noncommutative theory, defining dynamical invariants going beyond the ones in Chapter 7.

We start by giving the necessary technicalities and results from noncommutative deformation theory for modules. Then we give its main application, the noncommutative schemes as moduli of its simple modules. The main objective of this text is to apply the deformation theory to orbit spaces. At first, we construct the moduli of orbits under an action of a linear group which is the noncommutative geometric invariant theoretic quotient X/G of a scheme X under the action of a linear group G. This quotient exists as an orbit space for all reductive groups G and proves that noncommutative algebraic geometry solves the problem with non-stable points.

As pin-pointed by Gunnar Fløystad at theUniversity of Bergen, Norway, under the Abel lectures 2018 in Oslo, a usual difference between pure mathematics and applied mathematics, is the introduction of time. O. Arnfinn Laudal has taken the consequence of this, and introduces dynamics into noncommutative algebraic geometry. This is the second main theme of this work. We consider the action of a Lie algebra on the tangent sheaf on a noncommutative scheme, and the orbits under this Lie-algebra action is the moduli of integral curves, thereby introducing time as a metric on the moduli space.

Chapter 2

Preliminaries

Let k be a field, algebraically closed of sufficiently high characteristic. We recall the necessary background for noncommutative geometry, which can be found in all details in the book [3]. However, we work in the category of modules rather than in a general additive, abelian category.

2.1 Basics

2.1.1 Justification

Already at the beginning of the school for which these notes are written, it became clear that the participants in the best cases had graduate knowledge of commutative algebra. Thus we include a justification for dealing with deformation theory and some necessary basics from category theory. We apply the category theory to define algebraic schemes in an effective way.

Let $\mathcal{C}$ be a class of mathematical objects that we are going to study. This can even be objects from our physical universe that are observed (represented) as mathematical objects. Also, as our ability to observe is discrete, we claim that it is sufficient to study algebraic objects. So, to study any observed object mathematically, we assign some parameters to the object, saying that we have to study the set of functions defined on an object X. Let $\mathcal{O}_k(X)$ be the ring of functions with values in the ground field k, assumed to be algebraically closed of characteristic 0, e.g. $k = \mathbb{C}$. Of course, we admit that this is not the complete story, neither the correct story, but it gives a justification, or interpretation, of the importance of studying one-dimensional representations in the commutative situation. If the parameters of the object are noncommutative, the functions are defined on noncommuative domains, and so we have to generalize to finite-dimensional representations of associative algebras $A = A(X)$. An interpretation of A is that it is a ring of functions with values in $n \times n$-matrices (upto similarity) over the ground field k, that is a k-algebra homomorphism

$$A \longrightarrow \mathrm{End}_k(V), \ \dim_k V = n.$$

Thus V is an A-module, and in general we should gain a lot from the study of the class $\mathcal{C}$ of A-modules V, which some prefer to call the class of n-dimensional representations if A is noncommutative and $n > 1$.

2.1.2 Categories and functors

Very briefly, a category $\mathcal{C}$ consists of, i) a class of objects $\mathrm{ob}(\mathcal{C})$, (ii) for each pair (A, B) of objects in $\mathcal{C}$, a set $\mathrm{Mor}(A, B)$ such that for an additional object $C \in \mathrm{ob}(\mathcal{C})$ there are associative compositions $\mathrm{Mor}(A, B) \circ \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$, (iii) for each $A \in \mathrm{ob}(\mathcal{C})$ there is an element $\mathrm{id}_A \in \mathrm{Mor}(A, A) = \mathrm{End}(A)$ such that $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$.

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is a map $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ together with a map for each pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$ $F : \mathrm{Mor}(A, B) \to \mathrm{Mor}(F(A), F(B))$ that respect compositions.

A morphism of functors $F, G : \mathcal{C} \to \mathcal{D}$, or a natural transformation of functors, $\phi : F \to G$ consist of, for each object $A \in \mathrm{ob}(\mathcal{C})$ a morphism $\phi(A) \in \mathrm{Mor}(F(A), G(A))$ such that for a morphism $\psi : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(\psi)"'] & G(A) \arrow[d, "G(\psi)"] \\
F(B) \arrow[r, "\phi(B)"'] & G(B)
\end{tikzcd}$$

Notice that reversing all arrows gives the definition of a contravariant functor.

2.1.3 Limits in categories

A projective (inverse) system in a category $\mathcal{C}$ is a subfamily

$$\{A_i\}_{i \in I} \subseteq \mathrm{ob}(\mathcal{C})$$

indexed over a poset I together with morphisms $f_{ij} : A_j \to A_i,\ i \leq j$ such that, i) $f_{ii} = \mathrm{id}$, ii) $f_{ij} \circ f_{jk} = f_{ik}$. Notice that the morphisms f_{ij} goes the inverse way, and that reversing those arrows gives an inductive (direct) system. We give the definition of projective system because this is the most used.

Example 1. *Let $\mathfrak{m} \subset A$ be a maximal ideal in a commutative ring A. Then $\{A/\mathfrak{m}^i\}_{i \in \mathbb{N}}$ is a projective system, drawn as*

$$\cdots \to A/\mathfrak{m}^{i+1} \to A/\mathfrak{m}^i \to A/\mathfrak{m}^{i-1} \to \cdots \to A/\mathfrak{m}$$

Definition 1. *Given a projective system $\{A_i\}_{i \in I} \subseteq \mathrm{ob}(\mathcal{C})$. Then the projective limit is the object $\varprojlim_{i \in I} A_i$ defined as the unique smallest object mapping naturally to all A_i, i.e.*

$$\begin{tikzcd}
 & Y \arrow[dl, bend right, "\psi_j"'] \arrow[d, "\exists! u"] \arrow[dr, bend left, "\psi_i"] & \\
 & \varprojlim_{i\in I} A_i \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
A_j \arrow[rr, "f_{ij}"'] & & A_i
\end{tikzcd}$$

Considering an inductive system, reversing all arrows, and finding the biggest object, we get the inductive limit.

Example 2. *The completion of a ring A in an ideal $\mathfrak{a}$ is the projective limit*

$$\hat{A}_{\mathfrak{a}} = \varprojlim_{i\in\mathbb{N}} A/\mathfrak{a}^i.$$

And yes, this is the topological definition considering the Zariski topology.

Finally, we recall that a category $\mathcal{C}$ is called *small* when $\mathrm{ob}(\mathcal{C})$ is a set, it is called *abelian* if it has a kernels and coimages and if $\mathrm{Mor}(A, B)$ is a group for all $A, B \in \mathrm{ob}(\mathcal{C})$. It is called a *Grothendieck Category* if it acts like the category of A-modules over a k-algebra A, that is, it is abelian, and all limits exists. Also recall that a categorical (co)product is a (co)limit over certain (co)systems.

2.1.4 Schemes

Let A be commutative ring with unit. The spectrum of all prime ideals in A is defined as

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subseteq A | \mathfrak{p} \text{ prime } \}.$$

The sets

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \notin \mathfrak{p}\},\ f \in A,$$

is a base for the Zariski topology on $\mathrm{Spec}(A)$:

$$D(f) \cap D(g) = D(fg),\ D(f) \cup D(g) = D(f+g).$$

Also notice that $D(f) = \mathrm{Spec}(A_f)$ where $A_f = S^{-1}A$ with the multiplicative system $S = \{f^n | n \in \mathbb{N}\}$. As the open subsets $\mathbf{Top}(\mathrm{Spec}(A))$ of $\mathrm{Spec}(A)$ is a projective system, we can define a functor

$$\mathcal{O}_{\mathrm{Spec}(A)} : \mathbf{Top}(\mathrm{Spec}(A)) \to \mathbf{Rings},\ \mathcal{O}_{\mathrm{Spec}(A)}(U) = \varprojlim_{D(f)\subseteq U} A_f.$$

As a direct consequence of the universal properties of the projective limit, we obtain.

Lemma 1. *i)* $\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} := \varprojlim_{D(f)\ni\mathfrak{p}} A_f = A_{\mathfrak{p}}$, *the localization of* A *in the prime ideal* $\mathfrak{p}$. *ii)* $\mathcal{O}_{\mathrm{Spec}\,A}(\mathrm{Spec}\,A) \cong A$.

Definition 2. *A scheme* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X : \mathbf{Top}(X) \to \mathbf{Rings}$ *is a functor, is called a* Scheme *if* X *can be covered by open subsets* U *such that*

$$(U, \mathcal{O}_X|_U) \cong (\mathrm{Spec}\,A_U, \mathcal{O}_{\mathrm{Spec}\,A_U})$$

for a family of rings A_U.

2.2 Quiver algebras with relations

In ordinary algebraic geometry, the basic objects are quotients of the finitely generated polynomial rings $A = k[x_1, \ldots, x_d] \simeq \mathrm{sym}_k(V),\ V = \sum_{l=1}^{d} kx_l$. It is proved in [3] and [4] that the basic objects in noncommutative algebraic geometry are the quotients of quiver algebras, which are enhancements of the category of finitely generated k-algebras.

Definition 3. *Consider the left and right* k^r*-module* $V = (V_{ij})_{1\le i,j\le r}$ *with* $V_{ij} = \sum_{l_{ij}=1}^{d_{ij}} \alpha_{ij}(l_{ij})t_{ij}(l_{ij})$. *We let the quiver algebra in the variables* $t_{ij}(l_{ij}),\ 1 \le i,j \le r,\ 1 \le l_{ij} \le d_{ij}$ *be the* k^r*-algebra*

$$F\langle (d_{ij})\rangle = \mathrm{T}_{k^r}(V) = \begin{pmatrix} k\langle \underline{t}_{11}\rangle & \cdots & \langle \underline{t}_{1r}\rangle \\ \vdots & \ddots & \vdots \\ \langle \underline{t}_{r1}\rangle & \cdots & k\langle \underline{t}_{rr}\rangle \end{pmatrix}.$$

Here $T_{k^r}(V)$ *denotes the tensor algebra of* V *over* k^r*, and we use the notation* $\underline{t}_{ij} = t_{ij}(1), t_{ij}(2), \ldots, t_{ij}(d_{ij})$. *A quotient* $P = F\langle (d_{ij})\rangle/\mathfrak{a}$ *of* $F\langle (d_{ij})\rangle$ *by a two-sided ideal* $\mathfrak{a}$ *is called a* quiver algebra with relations*, or a QAR for short.*

We see that F is the algebra generated by the idempotents e_i, the $r \times r$-matrix with 1 in the (i,i)-entry and zero elsewhere, and the variables $t_{ij}(l_{ij})$, under the (only) relations $t_{ij}(l_{ij}) \cdot t_{kl}(l_{kl}) = 0,\ j \ne k$.

In invariant theory, and in the theory of moduli in general, the local tangent spaces are of importance. In the commutative situation, for $X = \mathrm{Spec}\,A$ the tangent space in the (closed) point $\mathfrak{m}$ is determined by $(\mathfrak{m}/\mathfrak{m}^2)^* = \hom_k(A, k[x]/(x^2))$, because after localization (or when A is a local ring), $A = k \oplus \mathfrak{m}$ as k-vector space. The usual notation is

$$k[x]/(x^2) = k[\varepsilon],$$

and it is called the *ring of dual numbers.*

This is also true in the noncommutative setting althoug localization doesn't make sense.

Definition 4. *We define the noncommutative k^r-algebra of dual numbers as*

$$E = T_{k^r}(V)/(t_{ij})^2$$

where $V = (V_{ij})_{1\leq i,j\leq r}$, $V_{ij} = k$.

Notice that with the notation above, this says

$$E = \begin{pmatrix} k\langle t_{11}\rangle & \cdots & \langle t_{1r}\rangle \\ \vdots & \ddots & \vdots \\ \langle t_{r1}\rangle & \cdots & k\langle t_{rr}\rangle \end{pmatrix} /(t_{ij})^2.$$

2.3 The category of r-pointed Artinian k-algebras

In the commutative situation, defining schemes (studying moduli spaces) by deformation theory builds on the category ℓ of local Artinian k-algebras. This definition is generalized.

Definition 5. *We define the category $\mathbf{a}_r$ where the objects are are diagrams*

$$\begin{tikzcd} k^r \arrow[r, "\iota"] \arrow[dr, "\mathrm{id}"'] & S \arrow[d, "\rho"] \\ & k^r \end{tikzcd}$$

where S is a finitely generated k-algebra, and where $\mathrm{I}^n(S) = \ker^n(\rho) = 0$ for some $n \in \mathbb{N}$, and $\rho \circ \iota = \mathrm{id}$. The morphisms are k-algebra homomorphisms ϕ commuting in the diagram

$$\begin{tikzcd} k^r \arrow[r] \arrow[rr, bend left] \arrow[dr] & S \arrow[r, "\phi"] \arrow[d] & S' \arrow[dl] \\ & k^r & \end{tikzcd}$$

Every $S \in \mathrm{ob}(\mathbf{a}_r)$ is Artinian with exactly r simple (right) S-modules. We call $\mathbf{a}_r$ the category of r-pointed Artinian k-algebras.

Example 3. *Let $V = (V_{ij})_{1\leq i,j\leq r}$ be a finite dimensional vector space, and let $F = T_{k^r}(V)$ be the tensor algebra. Let $\rho : S \to k^r$ be the morphism sending each basis element $t_{ij}(l_{ij})$ to 0. Consider any ideal $\mathfrak{a} \subseteq \mathrm{I}(F) = \ker(\rho)$, and let $S = F/(\mathfrak{a} + \mathrm{I}(F)^n)$*

Then we have homomorphisms

$$\begin{tikzcd} k^r \arrow[r, "\iota"] \arrow[dr, "\mathrm{id}"'] & S \arrow[d, "\rho"] \\ & k^r \end{tikzcd}$$

and S is an object of $\mathbf{a}_r$.

Lemma 2. *Let $S \in \mathrm{ob}(\mathbf{a}_r)$, and put*

$$S_{ij} = e_i S e_j$$

where $e_i = \iota(e_i)$ in the inclusion morphism $k^r \hookrightarrow S$. Then it follows that $S \simeq (S_{ij})$ as matrix algebra.

Proof. The isomorphism is given by sending $s \mapsto (e_i s e_j)$ with inverse

$$(s_{ij}) \mapsto \sum_{1 \le i,j \le r} s_{ij} = s.$$

□

2.4 Noncommutative Deformation Theory

In the commutative situation, the existence of a deformation theory is given in M. Schlessingers classical article, [8]. We generalize this from $\ell = \mathbf{a}_1$ to $\mathbf{a}_r$ for any natural number $r \in \mathbb{N}$.

In the following, A is a finitely generated, associative k-algebra with unit, k algebraically closed of characteristic 0, $\mathcal{M} = \{M_1, \dots, M_r\}$ is a set of r right A-modules, and

$$M = M_1 \oplus \cdots \oplus M_r.$$

Definition 6. *The noncommutative deformation functor*

$$\mathsf{Def}_M : \mathbf{a}_r \to \mathbf{Sets}$$

is given by

$$\mathsf{Def}_M(S) = \{S \otimes_k A\text{-modules } M_S | M_S \text{ is } S\text{-flat, } k^r \otimes_S M_S \simeq M\} / \sim$$

where the equivalence is given by the existence of an isomorphism commuting in the diagram

$$\begin{tikzcd} M_S \arrow[rr, "\sim"] \arrow[dr] & & M'_S \arrow[dl] \\ & M & \end{tikzcd}$$

Notice that the flatness replaces continuity in the analytic setting.

Lemma 3. *Let $S \in \mathrm{ob}(\mathbf{a}_r)$. Then an $S \otimes_k A$-module M_S is S-flat if and only if*

$$M_S \simeq S \otimes_{k^r} M$$

as S-module. With the notation $S = (S_{ij})$ this says $M_S \simeq (S_{ij} \otimes M_j)$ as $S = (S_{ij})$-module.

Proof. We refer to the book [3] for the proof based on local freeness writing up the long exact sequences. □

The following lemma is not very hard to prove, but is essential.

Lemma 4. *Let $\mathcal{M} = \{M_1, \dots, M_r\}$ be a family of A-modules. Then for the noncommutative k^r-algebra of dual numbers, we have*

$$\mathsf{Def}_M(E) \simeq (\mathrm{Ext}^1_A(M_i, M_j))_{1 \le i,j \le r}$$

Proof. To give a deformation M_E of $M = M_1 \oplus \cdots \oplus M_r$ to

$$E = \begin{pmatrix} k\langle t_{11}\rangle & \cdots & \langle t_{1r}\rangle \\ \vdots & \ddots & \vdots \\ \langle t_{r1}\rangle & \cdots & k\langle t_{rr}\rangle \end{pmatrix} / (t_{ij})^2,$$

that is to give an element $M_E \in \mathsf{Def}_M(E)$, is equivalent to give a k-algebra homomorphism $\eta_M : A \to \mathrm{End}_E(E \otimes_{k^r} M)$ induced by its restriction

$$\eta_M : A \to \mathrm{Hom}_k(M, (E_{ij} \otimes_k M_j)) \tag{2.1}$$

lifting the structure morphism of M. For each $t_{ij} \in E$, let $\xi_{ij} : A \to \mathrm{Hom}_k(M_i, M_j)$ be given by $\xi_{ij}(a) = \eta_M(a)|_{t_{ij}}$, saying that

$$\eta_A(a) = \mathrm{id} \cdot a + \sum_{1 \le i,j \le 1} t_{ij} \otimes \xi_{ij}(a).$$

The associativity of η_M then proves that for each $1 \le i, j \le r$,

$$\xi_{ij} \in \mathrm{Der}_k(A, \mathrm{Hom}_k(M_i, M_j))/\,\mathrm{Inner} = \mathrm{Ext}^1_A(M_i, M_j).$$

Conversely, any matrix of derivations ξ_{ij} as above gives a k-linear homomorphism η_M as in (2.1), and is extended by E-linearity to give a deformation $M_E \in \mathsf{Def}_M(E)$. □

In deformation theory, the main concern is to construct a space containing all deformations, that is to find the smallest ideal $J \subseteq F$ such that there is a deformation $M_U \in \mathsf{Def}_M(U)$, $U = F/J$, where F is a quiver algebra. The method is to construct the ideal step by step by decomposing the morphism $\rho : S \to k^r$ into small(er) morphisms.

Definition 7. *A surjection $\phi : R \twoheadrightarrow S$ in $\mathbf{a}_r$ is called* small *if* $(\ker \phi)\,\mathrm{I}(R) = 0$.

Any surjection in $\mathbf{a}_r$ can be decomposed in small morphisms: Given

$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, "\rho_R"'] & & S \arrow[dl, "\rho_S"] \\ & k^r & \end{tikzcd}$$

and assume that $I(R)^n = I(S)^m = 0$ with $n > m$. (Otherwise, it is small enough). Then we can decompose ϕ in small morphisms due to the following diagram:

$$\begin{tikzcd}
R = R/\,\mathrm{I}(R)^n \arrow[r, "\phi"] \arrow[d] & S/\,\mathrm{I}(S)^m \\
R/\,\mathrm{I}(R)^{n-1} \arrow[ur] \arrow[d] & \\
\vdots \arrow[d] & \\
R/\,\mathrm{I}(R)^{m-1} \arrow[uuur] &
\end{tikzcd}$$

We need to use the procategory of $\mathbf{a}_r$: This is the category $\hat{\mathbf{a}_r}$ where the objects are the projective limits of objects in $\mathbf{a}_r$, i.e. $\hat{S} \in \mathrm{ob}(\hat{\mathbf{a}_r}) \Leftrightarrow \hat{S} = \varprojlim S_n,\ S_n \in \mathrm{ob}(\mathbf{a}_r)$.

We find no harm in recalling the following very well known and essential fact.

Lemma 5. *(Yoneda) Let $F : \mathcal{C} \to$ **Sets** be a covariant functor. Then there is a bijection for every $C \in \mathrm{ob}(\mathcal{C})$,*

$$F(C) \xrightarrow{\sim} \mathrm{Mor}(\mathrm{Mor}(C, -), F)$$

where the right-hand set is the set of natural transformations of functors.

Proof. The map sending $\xi \in F(C)$ to $\phi_\xi : \mathrm{Mor}(C, -) \to F$ given by

$$\phi_\xi(C \xrightarrow{\eta} D) = F(\eta)(\xi) \in F(D)$$

is a natural bijection. □

In our situation, this gives a bijection for each $H \in \hat{\mathbf{a}_r}$,

$$\mathsf{Def}_M(H) \xrightarrow{\sim} \mathrm{Mor}(\mathrm{Mor}(H, -), \mathsf{Def}_M).$$

Definition 8. *We will say that a morphism $\phi : \mathrm{Mor}(H, -) \to \mathsf{Def}_M$ is an isomorphism at tangent level if $\phi(E)$ is an isomorphism when E is the ring of dual numbers. We will call it smooth (essentially surjective) if for all small morphisms $S \twoheadrightarrow R$, if $\phi_R \in \mathrm{Mor}(H, R)$ maps to $M_R \in \mathsf{Def}_M(R)$, if M_R lifts to $M_S \in \mathsf{Def}_M(S)$, then there exists a $\phi_S \in \mathrm{Mor}(H, S)$ mapping to ϕ_R and M_S simultaneously:*

$$\begin{tikzcd}
\exists \phi_S \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & \in \mathrm{Mor}(H, S) \arrow[r] \arrow[d] & \mathsf{Def}_M(S) \ni \arrow[d] & M_S \arrow[d, mapsto] \\
\phi_R \arrow[rrr, mapsto, bend right] & \in \mathrm{Mor}(H, R) \arrow[r] & \mathsf{Def}_M(R) \ni & M_R
\end{tikzcd}$$

In this case we say that ϕ_R (and M_R) can be lifted to *S.*

For representations of functors, the best we can do is the following.

Definition 9. *Consider a* couple *$(H, \tilde{M})$, with $H \in \mathrm{ob}(\hat{\mathbf{a}_r})$, $\tilde{M} \in \mathsf{Def}_M(H)$, and let $\phi_{\tilde{M}} : \mathrm{Mor}(H, -) \to \mathsf{Def}_M$ be the induced morphism. Then $(H, \tilde{M})$ is called:*

(i) Prorepresenting *if $\phi_{\tilde{M}}$ is an isomorphism of functors on $\hat{\mathbf{a}_r}$.*

(ii) A prorepresenting hull *if $\phi_{\tilde{M}}$ is smooth and an isomorphism on the tangent level.*

The following lemma follows without proof.

Lemma 6. *Prorepresenting objects are unique up to unique isomorphism when they exist. Hulls are unique upto non-unique isomorphism when they exist.*

Prorepresenting hulls are the model of the local rings on a moduli scheme. This is because of the following: Let $X = \operatorname{Spec} A$ be an affine scheme over k, and let $\mathfrak{m}_x \subset A$ be a geometric point. Then by definition, $\hat{\mathcal{O}}_{X,\mathfrak{m}_x}$ is a prorepresenting hull for $\mathsf{Def}_{A/\mathfrak{m}_x} : \hat{\mathbf{a}_1} \to \mathbf{Sets}$. Because of this we also call a prorepresenting hull *a local formal moduli* of the objects of interest.

From the definition, it follows exactly as in M. Schlessingers classical article [8] that a hull H is the smallest object such that we can lift a point in the tangent space to H. Given an obstruction theory as below, we can construct the k-algebras representing the local formal moduli explicitly.

2.5 Obstruction Theory

We include this section into the preliminaries. Chapter 3 will give alternative formulations of the results in this sections, and the proofs will be more detailed. We just give a first version with sketch of proofs here.

Lemma 7. *Let*

$$0 \to (I_{ij}) \to S = (S_{ij}) \xrightarrow{\phi} (R_{ij}) = R \to 0$$

be a small morphism in $\mathbf{a}_r$, and let $M_R \in \mathsf{Def}_M(R)$. Then there exists an obstruction

$$o(\phi, M_R) \in (\mathrm{Ext}^2_A(M_i, I_{ij} \otimes_k M_j)) = (I_{ij} \otimes_k \mathrm{Ext}^2_A(M_i, M_j))$$

such that M_R can be lifted to S if and only if $o(\phi, M_R) = 0$.

Proof. A detailed and strict proof can be found in [3]. Here we will just give the core: A lifting of $M_R = (R_{ij} \otimes_k M_j)$ to S is an $S \otimes_k A$-module structure on

$M_S = (S_{ij} \otimes_k M_j)$. This is equivalent to defining a structure morphism, a k-algebra homomorphism η_S lifting the structure morphism η_R as in the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & I_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j) \arrow[d] \\
A \arrow[r, "\eta_S"] \arrow[dr, "\rho_R"'] & (S_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) \arrow[d, "\phi"] \\
 & (R_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) \arrow[d] \\
 & 0
\end{tikzcd}$$

Choose any k-linear lifting $\tilde{\eta}_S$ of η_R, which is possible because ϕ is onto. Then for each pair $a, b \in A$,

$$\tilde{\eta}_S(ab) - \tilde{\eta}_S(a)\tilde{\eta}_S(b) \in I_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j),$$

representing

$$o(\phi, M_R) \in I_{ij} \otimes_k \mathrm{HH}^2(A, \mathrm{Hom}_k(M_i, M_j).$$

If $o(\phi, M_R) = 0$, there is a $\xi : A \to \mathrm{Hom}_k(M_i, M_j)$ mapping to $o(\phi, M_R)$. Put $\eta_S = \tilde{\eta}_S + \xi$, and

$$\begin{aligned}
\eta_S(ab) - \eta_S(a)\eta_S(b) &= \tilde{\eta}(ab) + \xi(ab) - (\tilde{\eta}_S(a) + \xi(a))(\tilde{\eta}_S(b) + \xi(b)) \\
&= o(\phi, M_R) - d\xi = 0.
\end{aligned}$$

□

Theorem 1. *(Laudal Structure Theorem) Let* $T^l = T_{k^r}((\mathrm{Ext}^l_A(M_i, M_j)^*)$, $l = 1, 2$, *where* T *denotes the tensor algebra and* $(-)^*$ *denotes the dual. Then the obstruction theory defines an obstruction morphism*

$$o : T^2 \to T^1$$

such that

$$H = T^1 \otimes_{T^2} k^r$$

is a prorepresenting hull for Def_M.

Before giving the rather technical proof, with abuse of notation, this says that $T^1 \simeq k^r(t_{ij}(l_{ij}))$, $T^2 \simeq k^r(y_{ij}(h_{ij}))$, and finally

$$H \simeq k^r(t_{ij}(l_{ij}))/(f_{ij}(h_{ij}))$$

where $f_{ij}(h_{ij}) = o(y_{ij}(h_{ij}))$.

The reason for giving the proof is that it also proves that there exists an algorithm for computing the hull.

Proof. Put $S_2 = k^r\langle t_{ij}(l_{ij})\rangle/\,\mathrm{I}^2$. Then we have an isomorphism $\mathsf{Def}_M(S_2) = (\mathrm{Ext}^1_A(M_i, M_j)^*)$. This says

$$\mathrm{Mor}(S_2, S_2/\,\mathrm{I}^2) \simeq \mathsf{Def}_M(S_2).$$

A sequence of elements $\underline{\alpha} = (\alpha_{ij}(l_{ij})) \in (\mathrm{Ext}^1_A(M_i, M_j))$ defines a deformation $M_2(\underline{\alpha}) \in \mathsf{Def}_M(S_2)$. Let B_2' be the set of all monomials of degree 2 in the $t_{ij}(l_{ij})$ and consider

$$\pi_2' : R_3 = k^r\langle t_{ij}(l_{ij})\rangle/\,\mathrm{I}^3 \to k^r\langle t_{ij}(l_{ij})\rangle/\,\mathrm{I}^2 = S_2.$$

Then we can write

$$o(M_2(\underline{\alpha}), \pi_2') = \sum_{\underline{t}\in B_2'} \underline{t} \otimes \langle \underline{\alpha}; \underline{t}\rangle \in (I_{ij} \otimes_k \mathrm{Ext}^2_A(M_i, M_j)).$$

We call $M_2(\underline{\alpha})$ a defining system for the second order generalized Massey products $\langle \underline{\alpha}; \underline{t}\rangle$, $\underline{t} \in B_2'$.

Choose bases $\{y_{ij}(m_{ij})\}$ for the dual spaces $\mathrm{Ext}^2_A(M_i, M_j)^*$. Write

$$o(M_2(\underline{\alpha}), \pi_2') = \sum_{\underline{t}\in B_2'} \underline{t} \otimes \langle \underline{\alpha}; \underline{t}\rangle = \sum_{t\in B_2'} y_{ij}(m_{ij})(\langle \underline{\alpha}; \underline{t}\rangle)\underline{t} \otimes_k y^*_{ij}(m_{ij}).$$

We put

$$f^2_{ij}(m_{ij}) = \sum_{t\in B_2'} y_{ij}(m_{ij})(\langle \underline{\alpha}; \underline{t}\rangle)\underline{t},\ S_3 = R_3/(f^2_{ij}(m_{ij})),$$

and we let $\pi_2 : S_3 \to S_2$ be the induced morphism. Then we have the following diagram:

$$\begin{tikzcd}
\mathrm{Mor}(S_3, S_3/\,\mathrm{I}^3) \arrow[r] \arrow[d] & \mathsf{Def}_M(S_3) \arrow[d] \\
\mathrm{Mor}(S_3, S_3/\,\mathrm{I}^2) \arrow[r, "\sim"] & \mathsf{Def}_M(S_3/\,\mathrm{I}^2)
\end{tikzcd}$$

proving the partial (stepwise) smoothness.

Now we start the lifting procedure. Here we will construct new polynomials inductively, killing the obstructions and proving smoothness. The point is that the polynomials in each degree will be the old ones with some higher degrees added. Choose a monomial basis $B_2 \subseteq B_2'$ for $\ker \pi_2$, and put $\overline{B}_2 = \overline{B}_1 \cup B_2$ where $\overline{B}_1$ is the set of all monomials of degree 0 and 1. Then $o(M_2(\underline{\alpha}), S_3) = 0$. Assume that S_{N_1} has been constructed such that $M_2(\underline{\alpha})$ can be lifted to $M_{N-1}(\underline{\alpha}) \in \mathsf{Def}_M(S_{N-1})$. Also assume that monomial bases B_{N-2} and $\overline{B}_{N-2}$ have been constructed to satisfy the below conditions. Put

$$R_N = k^r\langle \underline{t}\rangle/\,\mathrm{I}^N + \mathrm{I}(f^{N-1}_{ij}(m_{ij})) \overset{\pi_N'}{\to} S_{N-1}.$$

We can write
$$\ker \pi'_N = (f_{ij}^{N-1}(m_{ij}))/\,\mathrm{I}(f_{ij}^{N-1}(m_{ij})) \oplus I_N$$
with $I_N = \mathrm{I}^{N-1}/(\mathrm{I}^N + \mathrm{I}^{N-1} \cap (f_{ij}^{N-1}(m_{ij}))$. Pick a monomial basis B'_{N-1} for I_N, such that for $\underline{t} \in B'_{N-1}$ we have that $\underline{t} = \underline{u} \cdot \underline{s}$ or $\underline{t} = \underline{s} \cdot \underline{u}$ for some $\underline{u} \in B_{N-2}$. Put $\overline{B}'_{N-1} = \overline{B}_{N-2} \cup B'_{N-1}$. For every monomial $\underline{u}$ with degree less than N we have a unique relation in R_N:
$$\underline{u} = \sum_{t \in \overline{B}'_{N-1}} \beta'_{\underline{t},\underline{u}} \underline{t} + \sum_{i,j,m_{ij}} \beta'_{\underline{u}} f_{ij}^{N-1}(m_{ij}),$$
and the next obstruction for lifting is
$$\begin{aligned} o(M_{N-1}(\underline{\alpha}), \pi'_N) = & \sum_{i,j,m_{ij}} f_{ij}^{N-1}(m_{ij}) \otimes y_{ij}(m_{ij})^* \\ & + \Big(\sum_{\underline{t} \in B'_{N-1}} c_{i,j,m_{ij},\underline{t}}\ \underline{t}\Big) \otimes \sum_{i,j,m_{ij}} y_{ij}(m_{ij})^*. \end{aligned}$$
We call $M_{N-1}(\alpha)$ a defining system for the generalized Massey products
$$\langle \underline{\alpha}; \underline{t} \rangle = \sum_{i,j,m_{ij}} c_{i,j,m_{ij},\underline{t}} y_{ij}(m_{ij})^* \in \mathrm{Ext}^2_A(M_i, M_j),\ \underline{t} \in B'_{N-1}.$$

To continue, put
$$f_{ij}^N(m_{ij}) = f_{ij}^{N-1}(m_{ij}) + \sum_{\underline{t} \in B'_{N-1}} y_{ij}(m_{ij})(\langle \underline{\alpha}; \underline{y} \rangle),$$
let $S_N = R_N/(f_{ij}^N(m_{ij}))$, and let $\pi_N : S_N \to S_{N-1}$ be the natural homomorphism. Choose a monomial basis $B_{N-1} \subseteq B'_{N-1}$ for $\ker \pi_N$, put $\overline{B}_{N-1} = B_{N-1} \cup \overline{B}_{N-2}$, and continue by induction.

It now follows that if we let $\underline{x}^* = \underline{\alpha}$ be a basis for $(\mathrm{Ext}^1_A(M_i, M_j))$, we have that
$$H(\mathcal{M}) = F/(f_{ij}(m_{ij})),$$
with
$$f_{ij}(m_{ij}) = \sum_{l=0}^{\infty} \sum_{\underline{t} \in B_{N+l}} y_{ij}(\langle \underline{x}^*; \underline{t} \rangle \underline{t},$$
is a prorepresenting hull, or local formal moduli, of Def_M. $\square$

Chapter 3

Computation of pro-representing hulls

Recall that k denotes an algebraically closed field of sufficiently high characteristic.

Given a Grothendieck category $\mathcal{C}$ (that is one in which all limits exists), the procategory is the category $\hat{\mathcal{C}}$ where the objects are projective limits of objects in $\mathcal{C}$.

Let $\mathfrak{m}_x$ be a closed point in a commutative affine space $\operatorname{Spec} A$. Then recall that the tangent space in that point is

$$\operatorname{Der}_k(A, A/\mathfrak{m}_x) \simeq \operatorname{Hom}_k(A, k[\varepsilon])) = k[\varepsilon]/(\varepsilon^2).$$

We generalize this to the noncommutative case by letting E be the quiver algebra with r nodes and one arrow $t_{ij} : n_i \to n_j$ for all $1 \leq i, j \leq r$, divided by the square of its radical:

$$E = \left\langle \begin{pmatrix} k[t_{11} & \langle t_{12}\rangle & \cdots & \langle t_{1r}\rangle \\ \langle t_{21} & k[t_{22}] & \cdots & \langle t_{2r}\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle t_{r1}\rangle & \langle t_{r2}\rangle & \cdots & k[t_{rr}] \end{pmatrix} \right\rangle /(t_{ij})^2 \tag{3.1}$$

Definition 10. *The tangent space of the deformation functor is defined as*

$$T_{\mathsf{Def}_M} = \mathsf{Def}_M(E)$$

We recall the definition of the objects that are of our main interest:

Definition 11. *An object $\hat{H} \in \mathsf{ob}(\hat{\mathbf{a}_r})$ is called a* pro-representing hull*, or a* local formal moduli*, if the induced natural transformation*

$$h_{\hat{H}} = \operatorname{Mor}(H, -) \to \mathsf{Def}_M$$

is smooth (definition 8), and an isomorphism at the tangent level (definition 10).

Theorem 2. *There exists a prorepresenting hull $\hat{H}$ for the noncommutative deformation functor*

$$\mathsf{Def}_M : \hat{\mathbf{a}_r} \to \mathbf{Sets}$$

The main objective of this chapter, is to give a proof of Theorem 2, together with a guiding example illustrating the output of the proof, which is an algorithm for explicitly computing the prorepresenting hull. The basis of the proof is what is called an *obstruction theory.* The main ingredient of the proof of the theorem is then to kill the obstructions for lifting smoothly.

Lemma 8. *Let $\pi : R \twoheadrightarrow S$ be a small morphism in $\mathbf{a}_r$. Let $M_S \in \mathsf{Def}_M(S)$. Then there exists an* obstruction*, that is, an element*

$$o(\pi, M_S) \in (\mathrm{Ext}^2_A(M_i, M_j))$$

such that $o(\pi, M_S) = 0$ if and only if M_S can be lifted to $M_{S'}$.

We will prove Lemma 8 in two equivalent ways. In section 3.1 we give the proof of Theorem 2, and we give a guiding example of the induced algorithm for computing the prorepresenting hull using the first proof of the obstruction lemma. Then in section 3.2 we do the algorithm over again, using the second version of the proof. Notice that it is the comparison of the two techniques that gives the main result.

Proof. Here follows the proof where we use Hochschild cohomology for computing Ext's. This is the more direct proof, not using projective resolutions. So let $M_S \in \mathsf{Def}_M(S)$ be given by its structure morphism

$$\rho_S : S \otimes_k A \to \mathrm{End}_S(S \otimes_{k^r} M),$$

where we notice that ρ_S is S-linear and M_S is S-flat. Because of this, the $S \otimes_k A$-module structure is given by its structure as A-module

$$\rho_S : A \to \mathrm{End}_S(S \otimes_{k^r} M),$$

and ρ_S is a k^r-algebra homomorphism. Choose any lifting of ρ_S to $\pi : R \twoheadrightarrow S$. That is, choose any R-linear lifting ρ'_R commuting in the diagram

$$\begin{tikzcd} A \arrow[r, "\rho'_R"] \arrow[dr, "\rho_S"'] & \mathrm{End}_R(R \otimes_{k^r} M) \arrow[d] \\ & \mathrm{End}_S(S \otimes_{k^r} M) \end{tikzcd} .$$

Notice that ρ'_R can always be found as $\rho_S(a)$ is determined by its restriction to M, $\rho_S(a)|_M : M \to S \otimes_{k^r} M$ and π is surjective. The problem is really to find a lifting ρ'_R that is a k^r-algebra homomorphism.

For each pair $a \otimes b \in A^{\otimes 2}$, by R-linearity, we can put

$$\psi(a \otimes b) = \rho'_R(ab) - \rho'_R(a)\rho'_R(b) : M \to R \otimes_{k^r} M.$$

This gives an element $\psi \in \operatorname{Hom}_k(A^{\otimes 2}, \operatorname{Hom}_k(M_i, I_{ij} \otimes_{k^r} M))$. Notice that every $S \in \mathsf{ob}(\mathbf{a}_r)$ is isomorphic to the matrix algebra $S \simeq (S_{ij})$ with $S_{ij} = e_i S e_j$. Then $I = (I_{ij}) = \ker(\pi)$ in the short exact sequence

$$0 \to (I_{ij}) \to (R_{ij}) \xrightarrow{\pi} (S_{ij}) \to 0.$$

It is straight forward computation to check that ψ is a Hochschild 2-cocycle, and its class is defined to be the *obstruction* for lifting M_S to R by π :

$$\begin{aligned} o(\pi, M_S) &= \overline{\psi} \in \mathrm{HH}^2(A, \operatorname{Hom}_k(M_i, I_{ij} \otimes_k M_j)) \\ &= (\mathrm{HH}^2(A, \operatorname{Hom}_k(M_i, M_j)) \otimes_k I_{ij}) \simeq (\operatorname{Ext}^2_A(M_i, M_j) \otimes_k (I_{ij})). \end{aligned}$$

If $o(\pi, M_S) = 0$ we can choose a 1-cocycle

$$\xi \in \operatorname{Hom}_k(A, \operatorname{Hom}_k(M_i, I_{ij} \otimes_k M_j))$$

such that $d(\xi) = \psi$. Put

$$\rho_R = \rho'_R + \xi,$$

and ρ_R is a k^r-algebra homomorphism defining a lifting of M_S to R. □

Proof. Here follows the proof where we use the Yoneda complex for computing Ext's: Given $M_S \in \mathsf{Def}_M(S)$, choose an $S \otimes_k A$-free resolution

$$0 \longleftarrow M_S \xleftarrow{\mu} (S \otimes A)^{n_0} \xleftarrow{d_1^S} (S \otimes A)^{n_1} \xleftarrow{d_2^S} (S \otimes A)^{n_2} \xleftarrow{d_3^S} \cdots \tag{3.2}$$

By tensoring over S by k, we get for each $M_i \in \mathcal{M} = \{M_1, \dots, M_r\}$ a free resolution of A-modules:

$$0 \longleftarrow M_i \xleftarrow{\mu_i} A^{n_{i0}} \xleftarrow{d_{i1}} A^{n_{i1}} \xleftarrow{d_{i2}} A^{n_{i2}} \xleftarrow{d_{i3}} \cdots$$

This follows from the free resolution of $M = \oplus_{i=1}^r M_i$,

$$0 \longleftarrow M \xleftarrow{\mu} A^{n_0} \xleftarrow{d_1} A^{n_1} \xleftarrow{d_2} A^{n_2} \xleftarrow{d_3} \cdots$$

with $n_l = \sum_{i=1}^r n_{il}$ and $d_l = \oplus_{i=1}^r d_i$. This can be written as

$$0 \leftarrow \begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & M_r \end{pmatrix} \leftarrow \begin{pmatrix} A^{n10} & 0 & \dots & 0 \\ 0 & A^{n20} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & A^{nr0} \end{pmatrix} \xleftarrow{d_1} \cdots$$

with

$$d_i = \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{21} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & d_{r1} \end{pmatrix}.$$

By the small surjectivity of $\pi : R \twoheadrightarrow S$, we can always lift the sequence (3.2) to a sequence of $(R \otimes_k A)$-modules. This says that we can choose a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & I \otimes_k M \arrow[l] \arrow[d] & I \otimes A^{n_0} \arrow[l, "\mathrm{id} \otimes \mu"'] \arrow[d] & I \otimes A^{n_1} \arrow[l, "\mathrm{id} \otimes d_1"'] \arrow[d] & I \otimes A^{n_2} \arrow[l, "\mathrm{id} \otimes d_2"'] \arrow[d] & \cdots \arrow[l, "\mathrm{id} \otimes d_3"'] \\
0 & M_R \arrow[l] \arrow[d] & (R \otimes_k A)^{n_0} \arrow[l, "\mu^R"'] \arrow[d] & (R \otimes_k A)^{n_1} \arrow[l, "\tilde{d}_1^R"'] \arrow[d] & (R \otimes_k A)^{n_2} \arrow[l, "\tilde{d}_2^R"'] \arrow[d] & \cdots \arrow[l, "\tilde{d}_3^R"'] \\
0 & M_S \arrow[l] \arrow[d] & (S \otimes_k A)^{n_0} \arrow[l, "\mu^S"'] \arrow[d] & (S \otimes_k A)^{n_1} \arrow[l, "d_1^S"'] \arrow[d] & (S \otimes_k A)^{n_2} \arrow[l, "d_2^S"'] \arrow[d] & \cdots \arrow[l, "d_3^S"'] \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
\tag{3.3}
$$

The R-linear composition $\tilde{d}_i^R \circ \tilde{d}_{i-1}^R$ is determined by its action on A, that is, by the homomorphism

$$\tilde{d}_i^R \circ \tilde{d}_{i-1}^R : A^{n_i} \to (I \otimes A)^{n_{i-2}} \xrightarrow{\simeq} I \otimes A^{i_{n-2}}$$

for each $i \geq 2$. The composition above gives the element

$$\psi = \{\tilde{d}_i^R \circ \tilde{d}_{i-1}^R\}_{i \geq 2} \in \oplus_{i \geq 2} \operatorname{Hom}_A(A^{n_i}, I \otimes A^{n_{i-2}}) \simeq I \otimes (\oplus_{i \geq 2} \operatorname{Hom}_A(A^{n_i}, A^{n_{i-2}})).$$

This is a Hochshild 2-cocycle, and the obstruction for lifting M_S by π is

$$o(M_S, \pi) = \overline{\psi} = \mathrm{cl}(\{\tilde{d}_i^R \circ \tilde{d}_{i-1}^R\}_{i \geq 2}) \in (I_{ij} \otimes_k \mathrm{HH}^2(A, \operatorname{Hom}_k(M_i, M_j))).$$

As the cohomology is independent of the choice of resolution, so is the defined obstruction.

If M_S can be lifted to R, then we choose a lifting M_R with an $R \otimes_k A$-free resolution which by tensoring over R by S gives a free $S \otimes_k A$-resolution of M_S. Then, as the original complex is a resolution, it follows that the obstruction $\overline{\psi} = 0$. This proves the only if part.

For the other direction, assume that $\overline{\psi} = 0$. Then there exists a

$$\xi \in \operatorname{Hom}_k(A, I_{ij} \otimes \operatorname{Hom}_k(M_i, M_j))$$

such that $d(\xi) = \psi$. Put

$$d_i^R = \tilde{d}_i^R + b \otimes \xi_i, \; b \in I.$$

Then, because π is small, $I^2 = 0$ and $d_i^R \circ d_{i-1}^R = 0$, and so the middle horizontal sequence of the diagram (3.3) with $\tilde{d}^R$ replaced by d^R is a complex. Taking the long exact sequence of the short exact sequence of complexes (3.3), we find that the cohomology of the middle sequence vanish, so that it is a resolution. Also, as $M_R = H^0((R \otimes_k A)^{n\bullet})$ fits in a short exact sequence with R-flat modules in both ends, it is an R-flat lifting of M_S, i.e. an element $M_R \in \mathsf{Def}_M(R)$ mapping to $M_S \in \mathsf{Def}_M(S)$. $\square$

We end the preliminaries fixing some notation: Let $L = (l_{ij})$ be an $r \times r$-matrix with integer coefficients. Then we denote the quiver algebra with r nodes and l_{ij} arrows from node i to node j, including the idempotents, as

$$k\langle L\rangle = \begin{pmatrix} k\langle t_{11}(1),\dots,t_{11}(l_{11})\rangle & \langle t_{12}(1),\dots,t_{12}(l_{12})\rangle & \cdots & \langle t_{1r}(1),\dots,t_{1r}(l_{1r})\rangle \\ \langle t_{21}(1),\dots,t_{21}(l_{21})\rangle & k\langle t_{22}(1),\dots,t_{22}(l_{22})\rangle & \cdots & \langle t_{2r}(1),\dots,t_{2r}(l_{2r})\rangle \\ \vdots & \vdots & & \vdots \\ \langle t_{r1}(1),\dots,t_{r1}(l_{21})\rangle & \langle t_{r2}(1),\dots,t_{r2}(l_{r2})\rangle & \cdots & k\langle t_{rr}(1),\dots,t_{rr}(l_{rr})\rangle \end{pmatrix}.$$

When we use the notation $k[L]$, all the (free) polynomial algebras on the diagonal are commutative, so that the notation $\langle\cdot\rangle$ is replaced by square brackets $[\cdot]$.

The following lemma is just a computation.

Lemma 9. *Let $M_1,\dots,M_r$ be 1-dimensional points along the diagonal in $k\langle L\rangle$, $L \in \mathsf{M}(r,\mathbb{N})$, i.e.*

$$M_i = k\langle t_{ii}(1),\dots,t_{ii}(l_{ii})\rangle/(t_{ii}(1)-\alpha_{1i},\dots,t_{ii}(l_{ii})-\alpha_{l_{ii}}),$$

$\alpha_l \in k,\ 1 \le l \le l_{ii}$. Then

$$\operatorname{Ext}^1_{k\langle L\rangle}(M_i,M_j) = \operatorname{HH}^1(k\langle L\rangle, \operatorname{Hom}_k(M_i,M_j)) = \oplus_{l=1}^{l_{ij}} k\partial_{t_{ij}(l)},$$

where $\partial_{t_{ij}(l)}$ is differentiation with respect to $t_{ij}(l)$, $1 \le i,j \le r$.

Proof. This is an explicit computation, recalling the action of $k\langle L\rangle$ on $\operatorname{Hom}_k(M_i,M_j)$. $\square$

3.1 Proof of Theorem 2, Hochschild version

We will give the proof in parallel with an example, and so when going from the proof to the example, we will label it as such. When the reader then reads Section 3.2, he can read this section again, and call the examples from that section.

Let A be an associative k-algebra, let $\mathcal{M} = \{M_1,\dots,M_r\}$ be a set of r right A-modules.

Example 4.

$$A = \begin{pmatrix} k[t_{11}(1),t_{11}(2)] & \langle t_{12}\rangle \\ 0 & k[t_{22}] \end{pmatrix} /(t_{11}(1)t_{12} - t_{12}t_{22}, t_{11}^3(2) - t_{12}t_{22}^2),$$

$$M_1 = A/(t_{11}(1), t_{11}(2), \langle t_{12}\rangle, k[t_{22}]),\ M_2 = A/(k[t_{11}(1)], k[t_{11}(2)], \langle t_{12}\rangle, t_{22}).$$

Notice that M_1 and M_2 are the origins on the diagonal. There is no harm in thinking of the open sets on the diagonal as a Grothendieck site, and the space of points as a stack with a cleavage given by the entries off the diagonal. ■

Now, choose a k-basis for $\mathrm{Ext}^1_A(M_i, M_j)) = \langle \xi_{ij}(1), \ldots, \xi_{ij}(l_{ij})\rangle$ and put

$$S_1 = k\langle l_{ij}\rangle / \mathrm{I}^2(k\langle l_{ij}\rangle).$$

Then the restriction of the mini-versal family $\tilde{M}$ to S_1 is the deformation of $M = \oplus_{i=1}^r M_i$ in $\mathsf{Def}_M(S_1)$ given by

$$\rho_1 : A \longrightarrow \mathrm{End}_{S_1}(S_1 \otimes_{k^r} M),$$

where $\rho_1(a)$ is induced by

$$\rho_1(a) : M \to S_1 \otimes_{k^r} M = (S_{ij} \otimes_k M_j),$$
$$\rho_1(a)(m) = (1 \otimes m \cdot a + \sum_{l=1}^{l_{ij}} t_{ij}(l) \otimes \xi_{ij}(l)(m)).$$

This gives an isomorphism

$$\mathrm{Mor}(H, S_1) \xrightarrow{\sim} \mathsf{Def}_M(S_1)$$

and is the basis for constructing H.

Example 5. *In the example we have $A = k\begin{pmatrix} 2 & 1 \\ 0 & 1\end{pmatrix}$ and we use Lemma 9 to choose a basis for $(\mathrm{Ext}^1_A(M_i, M_j))$. That is, we choose the basis*

$$\{\partial_{t_{ij}(l)}, \ldots, \partial_{t_{ij}(l_{ij})}\}$$

As $\{M_1, M_2\}$ are the origins on the diagonal, we have that

$$\begin{aligned}\partial_{t_{12}}(t_{11}(1)t_{12} - t_{12}t_{22}) &= t_{11}(1)\partial_{t_{12}}(t_{12}) + \partial_{t_{12}}(t_{11}(1))t_{12} - t_{12}\partial_{t_{12}}(t_{22}) \\ -\partial_{t_{12}}(t_{12})t_{22} &= t_{11}(1) - t_{22} = 0,\end{aligned} \tag{3.4}$$

because $t_{11}(1) = t_{22} = 0$ in this example. Also

$$\partial_{t_{12}}(t_{11}^3(2)t_{12} - t_{12}t_{22}^2) = t_{11}^3(2) - t_{22}^2 = 0, \tag{3.5}$$

which says that the Ext^1*-dimension drops outside the parametrization.*■

Back to the general situation: Let $\overline{B}_1$ be the obvious monomial basis for S_1, explicitly that is $\{1, \underline{t}_{ij}\}$ where $\underline{t}_{ij}$ is short for $t_{ij}(1), \ldots, t_{ij}(l_{ij})$. Put

$$R_2 = k\langle l_{ij}\rangle / \mathrm{I}^3 \xrightarrow{\pi_2'} S_1$$

and choose a monomial basis B_2' for $\ker \pi_2'$;

$$B_2' = \{\underline{t} | \deg \underline{t} = 2\}.$$

Let M_{S_1} be the restriction of the mini-versal family $\widetilde{M}$ to S_1 previously defined. Then, using Lemma 8 we have that

$$o(M_{S_1}, \pi_2') = \sum_{\underline{t} \in B_2'} \langle \underline{\xi}; \underline{t} \rangle \otimes \underline{t} \in (\mathrm{Ext}^2_A(M_i, M_j) \otimes_k I_{ij}). \tag{3.6}$$

We call M_{S_1} a defining system for the second order *Massey products*

$$\langle \underline{\xi}; \underline{t} \rangle, \ \underline{t} \in B_2'.$$

Example 6. *Going into the example again, we write up a basic sequence of maps just to keep track on the order of composition:*

$$A \to t_{ij} \otimes \xi_{ij} \to t_{ij}(t_{jk} \otimes \xi_{jk}(\xi_{ij})) = t_{ij}t_{jk} \otimes \xi_{ij}\xi_{jk}.$$

In this example, the 2. *order Massey products is nothing but the cup-products which we can write as follows, when we choose a* k*-basis for* $A/\mathrm{I}^3(A)$ *corresponding to* B_2'.

$$\partial_{t_{11}(1)} \cup \partial_{t_{11}(1)} = (t_{11}(1) \otimes t_{11}(1))^\vee = -d((t_{11}^2(1))^\vee) = 0$$

This is because

$$d(t_{11}^2(1)^\vee)(a \otimes b) = a(t_{11}(1)^\vee(b) - t_{11}^2(1)^\vee(ab) + t_{11}^2(1)^\vee(a)b.$$

In the same way, we get for the others,

$$\begin{aligned}
\partial_{t_{11}(1)} \cup \partial_{t_{11}(2)} &= (t_{11}(1) \otimes t_{11}(2))^\vee \neq 0 \\
\partial_{t_{11}(2)} \cup \partial_{t_{11}(1)} &= (t_{11}(2) \otimes t_{11}(1))^\vee \neq 0 \\
\partial_{t_{11}(1)} \cup \partial_{t_{12}} &= (t_{11}(1) \otimes t_{12})^\vee \neq 0 \\
\partial_{t_{11}(2)} \cup \partial_{t_{12}} &= (t_{11}(2) \otimes t_{12})^\vee = -d((t_{11}(2)t_{12})^\vee) = 0 \\
\partial_{t_{12}} \cup \partial_{t_{22}} &= (t_{12} \otimes t_{22})^\vee \neq 0
\end{aligned}$$

■

In general, choose dual bases $\{y_{ij}(m)\}_{m=1}^{e_{ij}^2}$ for the dual spaces $\mathrm{Ext}^2_A(M_i, M_j)^\vee$. Then we can write

$$o(M_{S_1}, \pi_2') = \sum_{\underline{t} \in B_2'} \langle \underline{\xi}; \underline{t} \rangle \otimes \underline{t} = (\sum_m y_{ij}^\vee(m) \otimes \sum_{\underline{t} \in B_2'} y_{ij}(m)(\langle \underline{\xi}; \underline{t} \rangle)\underline{t}). \tag{3.7}$$

Put

$$f_{ij}^2(m) = \sum_{\underline{t} \in B_2'} y_{ij}(m)(\langle \underline{\xi}; \underline{t} \rangle)\underline{t},$$

and put $S_2 = R_2/(\underline{f}_{ij}^2) \xrightarrow{\pi_2} S_1$. Choose a monomial basis $B_2 \subseteq B_2'$ for $\ker \pi_2$ and put $\overline{B}_2 = \overline{B}_1 \cup B_2$. Also notice that now $o(M_1, \pi_2) = 0$.

Example 7. *In our quiding example,*

$$d((t_{11}(1)t_{11}(2))^\vee) = (t_{11}(1) \otimes t_{11}(2))^\vee + (t_{11}(2) \otimes t_{11}(2))^\vee,$$
$$d((t_{11}(1)t_{12})^\vee) = (t_{11}(1) \otimes t_{12})^\vee + (t_{12} \otimes t_{22})^\vee.$$

We choose one of each of these as dual basis elements, i.e.

$$y_{11} = (t_{11}(1) \otimes t_{11}(2))^\vee,\ y_{12} = (t_{11}(1) \otimes t_{12})^\vee.$$

Then we have

$$f_{11}^2 = t_{11}(1)t_{11}(2) - t_{11}(2)t_{11}(1),\ f_{12}^2 = t_{11}(1)t_{12} - t_{12}t_{22}.$$

We let

$$B_2 = B_2' \setminus \{t_{11}(2)t_{11}(1), t_{12}t_{22}\},$$

and then $o(M_1, \pi_2) = 0$. ■

Because we actually choose bases, we have a unique relation in S_2. For any monomial $\underline{u} \in S_2$,

$$\underline{u} = \sum_{\underline{m} \in \overline{B}_2} \beta_{\underline{u},\underline{m}}\, \underline{m}.$$

For any $\underline{u} \in B_2$ we have that

$$\sum_{\underline{n} \in B_2'} \beta_{\underline{n},\underline{u}} \langle \underline{\xi}; \underline{n} \rangle = \overline{\beta}_{\overline{u}} = 0$$

because $o(M_{S_1}, \pi_2) = 0$. For each $\underline{u} \in B_2$, choose an $\alpha_{\underline{u}} \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(M_i.M_j))$ such that $d(\alpha_{\underline{u}}) = -\beta_{\underline{u}}$. This defines a lifting of M_{S_1} to S_2 by putting

$$\rho_2 : A \to \mathrm{End}_{S_2}(S_2 \otimes_k A),\ \rho_2(a) = \sum_{\underline{n} \in \overline{B}_2} \underline{n} \otimes \alpha_{\underline{n}},$$

where $\alpha_1 = \mathrm{id},\ \alpha_{t_{ij}(l)} = \xi_{ij}(l)$.

Example 8. *In the example, for the cup-products that are identically zero, there are no relations and we choose the corresponding* $\alpha_{\underline{t}} = -(\underline{t})^\vee$. *The non-zero cup-products are represented in the relations*

$$\beta_{t_{11}(1)t_{11}(2)} = (t_{11}(1) \otimes t_{11}(2))^\vee + (t_{11}(2) \otimes t_{11}(1))^\vee = d((t_{11}(1)t_{11}(2))^\vee)$$
$$\beta_{t_{11}(1)t_{12}} = (t_{11}(1) \otimes t_{12})^\vee + (t_{12} \otimes t_{22})^\vee = d((t_{11}(1)t_{12})^\vee).$$

We let the explicit definition of ρ_2 *be given by intuition.* ■

Now follows the general induction step, which is the tricky part: Assume that S_{N-1} has been constructed such that M_{S_1} can be lifted to $M_{S_{N-1}}$. Also assume that monomial bases B_{N-1} and $\overline{B}_{N-1}$ has been chosen due to the claims in the following induction step. Put

$$R_N = k\langle l_{ij}\rangle/(\mathrm{I}^{N+1} + \mathrm{I}(f_{ij}^{N-1}) + (f_{ij}^{N-1})\,\mathrm{I}) \xrightarrow{\pi'_N} S_{N-1}.$$

Write

$$\ker \pi'_N = (f_{ij}^{N-1})/(\mathrm{I}(f_{ij}^{N-1}) + (f_{ij}^{N-1})\,\mathrm{I}) \oplus J_N$$

where

$$J_N = \mathrm{I}^N \,/\, \mathrm{I}^{N+1} + \mathrm{I}^N \cap (f_{ij}^{N-1}))$$

and chose a monomial basis B'_N for J_N such that for each $\underline{t} \in B'_N$ we have $\underline{t} = \underline{u} \cdot \underline{s}$ or $\underline{t} = \underline{s} \cdot \underline{u}$ for some $\underline{u} \in B_{N-1}$. Put

$$\overline{B}'_N = \overline{B}_{N-1} \cup B'_N.$$

Then for any monomial $\underline{u}$ with $\deg u < N+1$, we have a unique relation in R_N :

$$\underline{u} = \sum_{\underline{t} \in B_N} \beta'_{\underline{t},\underline{u}} \underline{t} + \sum_{i,j,m} \beta'_{\underline{u}} f_{ij}^{N-1}(m)$$

and we have that

$$\begin{aligned} &o(M_{S_{N-1}}, \pi'_N) \\ &= \sum_{\underline{t} \in B_N} \beta'_{\underline{t},\underline{u}} \underline{t} + \sum_{i,j,m} y_{ij}(m)^\vee \otimes f_{ij}^{N-1}(m) + \sum_{i,j,m} y_{ij}(m)^\vee \otimes (\sum_{\underline{t} \in B'_{N-1}} c_{i,j,m,\underline{t}} \otimes \underline{t}). \end{aligned}$$

Then $M_{S_{N-1}}$ is called a defining system for the Massey products

$$\langle \underline{\xi}; \underline{t}\rangle = \sum_{i,j,m} c_{i,j,m,\underline{t}} y_{ij}(m)^\vee \in \mathrm{Ext}_A^2(M_i, M_j),\ \underline{t} \in B'_{N-1}.$$

For the induction step, we put

$$f_{ij}^N(m) = f_{ij}^{N-1}(m) + \sum_{\underline{t} \in B'_{N-1}} y_{ij}(m)(\langle \underline{\xi}; \underline{t}\rangle)\underline{t},$$

we put

$$S_N = R_N/(f_{ij}^N(m)),\ \pi_N : S_N \to S_{N-1}.$$

We choose a monomial basis $B_N \subseteq B'_N$ for $\ker \pi_N$ and put $\overline{B}_N = B_n \cup \overline{B}_{N-1}$. We can lift $M_{S_{N-1}}$ to M_{S_N} because the obstruction vanish, and we continue by induction. We procrastinate the conclusion until we have finished the example.

Example 9. *We start by commenting on the computational aspect of liftings because this most naturally belongs to the example. The identity* $o(M_{S_{N-1}}, \pi_N) = 0$ *translates to: For each* $\underline{m} \in B_N$,

$$b_{\underline{m}} = \sum_{l=0}^{N-2} \sum_{\underline{n} \in B'_{2+l}} \beta_{\underline{n},\underline{m}} \langle \underline{\xi}; \underline{n} \rangle = 0.$$

To define a lifting is then equivalent to, for each $\underline{m} \in B_N$, *choose* $\alpha_{\underline{m}} \in (\mathrm{Hom}_k(A, \mathrm{Hom}_k(M_i, M_j)))$ *such that* $d(\alpha_{\underline{m}}) = -b_{\underline{m}}$.

With this comment, we are ready to compute the third order Massey products: We start by choosing a monomial basis B'_3, *that is, we take all third degree monomials except* $t_{ij}\{t_{11}(2)t_{11}(1), t_{12}t_{22}\}t_{ij}$. *Then the prime condition is satisfied, and the following products are computed:*

$$\begin{aligned}
\langle \underline{\xi}; t_{11}^3(1) \rangle &= \alpha_{t_{11}(1)} \alpha_{t_{11}^2(1)} + \alpha_{t_{11}^2(1)} \alpha_{t_{11}(1)} = d((t_{11}^3(1))^\vee) = 0 \\
\langle \underline{\xi}; t_{11}^2(1) t_{11}(2) \rangle &= \alpha_{t_{11}^2(1)} \alpha_{t_{11}(2)} + \alpha_{t_{11}(1)} \alpha_{t_{11}(1)t_{11}(2)} = d((t_{11}^2(1) t_{11}(2))^\vee) = 0 \\
&\vdots \\
\langle \underline{\xi}; t_{12} t_{22}^2 \rangle &= \alpha_{t_{12}} \alpha_{t_{22}^2} + \alpha_{t_{12}t_{22}} \alpha_{t_{22}} \neq 0 \text{ (because in } A,\ t_{12}t_{22}^2 = t_{11}^3(2) t_{12}) \\
&\vdots
\end{aligned}$$

Notice in particular that we have to choose defining systems that are non-zero all the way. We get

$$f_{12}^3 = f_{12}^2 = t_{11}(1)t_{11}(2) - t_{11}(2)t_{11}(1),\ f_{12}^3(1) = t_{11}(1)t_{12} - t_{12}t_{22},\ f_{12}^3(2) = t_{12}t_{22}^2.$$

We continue the algorithm by by choosing bases, choosing liftings, and start computing 4^{th} *order Massey products. Everything works out as trivial as before, exept*

$$\langle \underline{\xi}; t_{11}^3(2) t_{12} \rangle = -y_{12}(2)^\vee,$$

so that

$$f_{12}^3(2) = t_{12}t_{22}^2 - t_{11}^3(2) t_{12}.$$

All other fourth order relations stay unaltered, and these are the final relations. ■

We have proved the following.

Theorem 3. *The noncommutative deformation functor*

$$\mathsf{Def}_M : \mathbf{a}_r \to \mathbf{Sets}$$

is prorepresented by

$$\hat{H} = \varinjlim_n k\langle l_{ij} \rangle / (f_{ij}^{2+n}(l_{ij}))$$

with proversal family given by $\tilde{M} = \varinjlim_n M_{S_{1+n}}$.

Example 10. *In the example, this follows by the computation above:*

$$\hat{H}_M = \frac{k\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}{(t_{11}(1)t_{11}(2) - t_{11}(2)t_{11}(1), t_{11}(1)t_{12} - t_{12}t_{22}, t_{11}^3(2)t_{12} - t_{12}t_{22}^2)}$$

3.2 Proof of Theorem 2, Yoneda version

Now, in the example of Section 3.1 included in the proof of the theorem, we replace all the computations in the Hochschild complex to the Yoneda complex. Recall the example,

$$A = \begin{pmatrix} k[t_{11}(1), t_{11}(2)] & \langle t_{12} \rangle \\ 0 & k[t_{22}] \end{pmatrix} / (t_{11}(1)t_{12} - t_{12}t_{22}, t_{11}^3(2) - t_{12}t_{22}^2),$$

$$M_1 = A/(t_{11}(1), t_{11}(2), \langle t_{12} \rangle, k[t_{22}]),\ M_2 = A/(k[t_{11}(1)], k[t_{11}(2)], \langle t_{12} \rangle, t_{22}).$$

We have that M_1 and M_2 are the origins on the diagonal.

We need free resolutions, and include a k-basis for $\operatorname{Ext}_A^1(M_i, M_j)$ in the diagram:

$$\begin{tikzcd}
0 & M_1 \arrow[l] & A \arrow[l] & A^4 \arrow[l, "d_1^1"'] \arrow[dl, shift right, "\xi_{11}^1(1)"'] \arrow[dl, shift left, "\xi_{11}^1(2)"] & A^3 \arrow[l, "d_2^1"'] \arrow[dl, shift right, "\xi_{11}^2(1)"'] \arrow[dl, shift left, "\xi_{11}^2(2)"] & \dots \arrow[l] \\
0 & M_1 \arrow[l] & A \arrow[l] & A^4 \arrow[l, "d_1^1"'] \arrow[dl, "\xi_{12}^1"'] & A^3 \arrow[l, "d_2^1"'] \arrow[dl, "\xi_{12}^2"'] & \dots \arrow[l] \\
0 & M_2 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "d_1^2"'] \arrow[dl, "\xi_{22}^1"'] & 0 \arrow[l] \arrow[dl] & \dots \arrow[l] \\
0 & M_2 \arrow[l] & A \arrow[l] & A^2 \arrow[l, "d_1^2"'] & 0 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

$$
\begin{aligned}
& d_1^1 = (t_{11}(1), t_{11}(2), t_{12}, e_2),\ d_2^1 = \begin{pmatrix} t_{11}(2) & t_{12} & 0 \\ -t_{11}(1) & 0 & t_{11}^2(2)t_{12} \\ 0 & -t_{22} & -t_{22}^2 \\ 0 & 0 & 0 \end{pmatrix} \\
& d_1^2 = (e_1, t_{22}) \\
& \xi_{11}^1(1) = (1,0,0,0),\ \xi_{11}^2(1) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
& \xi_{11}^1(2) = (0,1,0,0),\ \xi_{11}^2(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -t_{11}(2)t_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
& \xi_{12}^1 = (0,0,1,0),\ \xi_{12}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & t_{22} \end{pmatrix} \\
& \xi_{22}^1 = (0,1)
\end{aligned}
$$

Notice that we have only included the non-trivial relations, e.g. $e_2 \cdot t_{12} = 0$ is not included above, as those above are the only one that influence the computation.

3.2.1 Second order Massey products (cup-products)

$$
\begin{aligned}
& \langle t_{11}(1)t_{11}(2) \rangle = (-1,0,0) \\
& \langle t_{11}(2)t_{11}(1) \rangle = (1,0,0) \\
& \langle t_{11}^2(2) \rangle = (0,0,-t_{11}(2)t_{12}) = d\left(\left\{\begin{matrix} \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}\right.\right) \\
& \langle t_{11}(1)t_{12} \rangle = (0,-1,0) \\
& \langle t_{12}t_{22} \rangle = (0,1,t_{22}) = (0,1,0) + d\left(\left\{\begin{matrix} \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}\right.\right)
\end{aligned}
$$

This gives the relations

$$f_{11} = t_{11}(1)t_{11}(2) - t_{11}(2)t_{11}(1),\ f_{12}(1) = t_{11}(1)t_{12} - t_{12}t_{22}.$$

We choose monomial bases the following way: $B_2 = B_2' \backslash \{t_{11}(2)t_{11}(1), t_{11}(1)t_{12}\}$.

We choose a defining system:

$$\alpha_{t_{11}^2(2)} = \begin{cases} \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}, \ \alpha_{t_{12}t_{22}} = \begin{cases} \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{cases}.$$

We choose all other degree two to be 0. This is then a defining system for the Massey products $\langle \underline{m} \rangle$, $\underline{m} \in B_3'$,

$$B_3' = \{\underline{t} : |\underline{t}| = 3\} \setminus t_{ij}(l) \cdot \{t_{11}(2)t_{11}(1), t_{11}(1)t_{12}\} \cdot t_{ij}(l),$$

meaning multiplication on one of the right or left side.

3.2.2 Third order Massey products

This must be $\deg 1 + \deg 2 = \deg 3$, so because of the choice of defining system, only the following needs computation:

$$\langle t_{11}^2(2)t_{11}(2) \rangle = (1,0,0,0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0,0,t_{12}) = d \left(\begin{cases} (0) \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{cases} \right)$$

$$\langle t_{12}t_{22} \cdot t_{22} \rangle = (0,1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (0,0,-1).$$

This gives the following relations.

$$f_{11} = t_{11}(1)t_{11}(2) - t_{11}(2)t_{11}(1),\ f_{12}(1) = t_{11}(1)t_{12} - t_{12}t_{22},\ f_{12}(2) = t_{12}t_{22}^2.$$

So we choose $B_3 = B_3' \setminus \{t_{12}t_{22}^2\}$, and choose the defining system

$$\alpha_{t_{11}^3(2)} = \begin{cases} \begin{pmatrix} 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}$$

3.2.3 Fourth order Massey products

All deg 2 times deg 2 vanish, so we are left with the following possibilities:

$$\langle t_{11}^3(2)t_{11}(1)\rangle = 0$$
$$\langle t_{11}^3(2)t_{11}(2)\rangle = (0,0,1,0)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix} = (0,0,-1,0).$$

This gives the relations:

$$f_{11} = t_{11}(1)t_{11}(2) - t_{11}(2)t_{11}(1),\ f_{12}(1) = t_{11}(1)t_{12} - t_{12}t_{22},$$
$$f_{12}(2) = t_{12}t_{22}^2 + t_{11}^3(2)t_{12}.$$

These are seen to be the final relations: We can write up the corresponding complex and see directly that $d^2 = 0$. This follows because all 4^{th} order defining systems can be chosen to be zero. Any degree five or higher must be combinations of the defining systems with one of degree at least four, and this proves the condition. Notice the magical $+$-sign in the last relation. This could eventually be fixed by sending $t_{11}(2)$ to $-t_{11}(2)$.

3.3 Local representability of the deformation functor

Computing the local formal moduli of the family of simple A-modules when A is a QAR by the two different methods, we observe that we can put a criterion on the basis-elements generating the tangent space, implying that the lifting procedure eventually stops. It then follows that the power series giving the relations in $\hat{H}(M_1,\ldots,M_r)$ are polynomials.

The criterion we put on the basis-elements of

$$\langle\overline{\xi}_1,\ldots,\overline{\xi}_l\rangle = H^1(\mathrm{YC}^\bullet(M,M)) \cong H^1(\mathrm{HC}^\bullet(M,M)) \cong \mathrm{Ext}^1_A(M,M)$$

says that it is possible to choose representatives $\{\xi_1,\ldots,\xi_l\}$ for the cohomology classes such that the resulting representatives of the cup-products never differ by a non-zero coboundary. This says that we exclude the possibility that

$$\xi_i\xi_j - \xi_{i'}\xi_{j'} = d(\alpha) \neq 0.$$

We observe that then either of the Yoneda and Hochschild complexes are L_∞-algebras, so we choose that name for our criterion.

Definition 12. *Let A be quiver algebra with relations, and let $P_1,\ldots,P_r$ be a set of r simple A-modules. Then we call $\hat{H}^A(P)$ the completion of A in $P = \oplus_{i=1}^r P_i$.*

Remark 1. *In the paper [4], we prove that this completion is complete, that is, completing once more in the same set of points results in an isomorphic algebra. This justifies that we can call this the completion of A.*

Definition 13. *A functor $F : \mathbf{a}_r \to \mathbf{Sets}$ is called locally representable if there exists a (finitely generated) quiver algebra A with relations, such that $\hat{A}_{P_1,\dots,P_r}$ pro-represents F. We call A an algebraization of $\hat{A}$.*

Theorem 4. *Let A be an associative, finitely generated k-algebra. Assume that $\mathcal{M} = \{M_1, \dots, M_r\}$ is a set of r right, finitely generated A-modules. Let $M = M_1 \oplus \cdots \oplus M_r$. The noncommutative deformation functor*

$$\mathsf{Def}_M : \mathbf{a}_r \to \mathbf{Sets}$$

is locally representable if and only if the tangent space of the noncommutative deformation functor gives a true ∞-structure.

Proof. Assume that the tangent space of the noncommutative deformation functor gives a true ∞-structure. We have the following commutative diagram of free resolutions and 1-cycles $\xi_i,\ 1 \leq i \leq \dim_k$ representing a basis for $\mathrm{Ext}^1_A(M, M)$:

$$\begin{tikzcd}
0 & M \arrow[l] & A^{n_0} \arrow[l] & A^{n_1} \arrow[l] \arrow[dl, "\xi_1^1"'] \arrow[dl, shift left=2, "\xi_1^l"] & A^{n_2} \arrow[l] \arrow[dl, "\xi_2^1"'] \arrow[dl, shift left=2, "\xi_2^l"] & A^{n_3} \arrow[l] \arrow[dl, "\xi_3^1"'] \arrow[dl, shift left=2, "\xi_3^l"] & \cdots \arrow[l] \arrow[dl] \arrow[dl, shift left=2] \\
0 & M \arrow[l] & A^{n_0} \arrow[l] & A^{n_1} \arrow[l] & A^{n_2} \arrow[l] & A^{n_3} \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{3.8}$$

It is always enough to consider the two first mappings $\{\xi_1^i, \xi_2^i\}$ in the complex, and the criterion gives that we might assume that the cup-products never differ by a non-zero coboundary. This says that we can choose a second order defining system with first coordinate equal to 0, i.e. $\alpha_{\underline{t}} = \{0, \alpha_{\underline{t},2}\}$. By induction, the criterion holds for each lifting, i.e. all higher order generalized matric Massey products, and all higher order defining systems can be choosen on the same form. In addition, the degree of of the entries in each defining system drops, and they eventually vanish, proving that the process stops.

Conversely, assume that H represents the noncommutative deformation functor locally by localizing in $P = P_1 \oplus \cdots \oplus P_r$. By Lemma 9, we can choose a resolution of P on the form (3.8) with A replaced by H. By Morita equivalence, that is tensoring over H by A, we get a resolution satisfying the criterion.

□

Corollary 1. *If the tangent space of of the noncommutative deformation functor gives a true ∞-structure, the prorepresenting hull is determined by the generalized Massey products in the DGLA governing the deformation theory.*

Chapter 4

Noncommutative schemes

In the proof of Laudal's structure theorem, Theorem 1, we actually lift the restriction of the pro-versal family explicitly by defining its A-module structure. That is, with the notation above, let $H(\mathcal{M}) = H = (H_{ij})$. Then we have defined the pro-versal family $M_H \in \mathsf{Def}_M(H)$ by the morphism

$$\eta : A \to (H_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)) = \operatorname{End}_H(M_H).$$

Notice that $End_H(M_H)$ is a k-algebra, and that $M_1, \dots, M_r$ are exactly the the simple $\operatorname{End}_H(M_H)$-modules. We actually have the following:

Theorem 5. *(A generalized Burnside's theorem) Let A be a finite dimensional k-algebra, k algebraically closed. Let $\mathcal{M} = \{M_1, \dots, M_r\}$ be the family of simple (right) A-modules. Then the morphism of the versal family*

$$\eta : A \to \mathcal{O}^A(\mathcal{M}) = \operatorname{End}_H(M_H)$$

is an isomorphism.

Proof. The proof can be found in the book [3]. For short, we state that the injectivity follows by the theory of iterated extensions which computes the kernel, and the surjectivity then follows by the Wedderburn-Malcev structure theorem. □

One of the main consequences of Theorem 5 is that the $\mathcal{O}$-construction is closed. This is the content of the next result which will be essential in the construction of noncommutative affine schemes.

Corollary 2. *Let $\mathcal{M} = \{M_1, \dots, M_r\}$ be a set of r finite dimensional, right A-modules. Then $\mathcal{M}$ is the set of simple $\mathcal{O}^A(\mathcal{M})$-modules, and*

$$\mathcal{O}^{\mathcal{O}^A(\mathcal{M})}(\mathcal{M}) \simeq \mathcal{O}^A(\mathcal{M}),$$

i.e. the $\mathcal{O}$-construction is closed.

Proof. First, notice that

$$\mathcal{O}^A(\mathcal{M}) = \mathrm{End}_H(M_H) = (H_{ij} \otimes_k \mathrm{Hom}_k(M_i, M_j)) \to \oplus_{i=1}^r \mathrm{Hom}_k(M_i, M_i)$$

so that the M_i's are right $\mathcal{O}^A(\mathcal{M})$-modules. Burnside's theorem states that when k is algebraically closed, M is simple if and only if the structure morphism is onto, proving that in this case, $\mathcal{M}$ is exactly the set of simple $\mathcal{O}^A(\mathcal{M})$-modules.

We have that $\mathcal{O}^A(\mathcal{M})/\,\mathrm{I}^n$ is finite dimensional for $n \geq 0$. It follows from Theorem 5 that

$$\mathcal{O}^A(\mathcal{M})/\,\mathrm{I}^n \xrightarrow{\sim} O^{O^A(\mathcal{M})}(\mathcal{M})/\,\mathrm{I}^n$$

is an isomorphism for each n. By the completeness of $\mathcal{O}^A(\mathcal{M})$ we have

$$\mathcal{O}^A(\mathcal{M}) = \varprojlim_n \mathcal{O}^A(\mathcal{M})/\,\mathrm{I}^n \simeq \varprojlim_n \mathcal{O}^{O^A(\mathcal{M})}(\mathcal{M})/\,\mathrm{I}^n \simeq \mathcal{O}^{\mathcal{O}^A(\mathcal{M})}(\mathcal{M})$$

□

4.1 Commutative affine schemes

We are looking for a definition that can be generalized. The main obstacle is that noncommutative k-algebras lack the utility of localization. Using deformation theory, one of our main results is that we can define the localization of A in an element $f \in A$. In this subsection we recall the functorial construction of sheaves and schemes in the ordinary noncommutative situation.

We define sheaves on a topological space by limits: Let X be a topological space. Let $\mathbf{Top}(X)$ be the category with objects the open subsets of X and morphisms the inclusions. A presheaf in a category $\mathbf{C}$ on X is a contravariant functor

$$\mathcal{F} : \mathbf{Top}(X) \to \mathbf{C}.$$

Such a presheaf is called a *sheaf* if in addition

$$\mathcal{F}(U) = \varprojlim_{V \subseteq U} \mathcal{F}(V)$$

for each open $U \subseteq X$.

Notice that the universal properties of the limit gives the properties of existence and uniqueness given on elements in Hartshorne [5].

For a commutative k-algebra A which we assume to be a finitely presented domain, the affine scheme structure $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$ can be defined as follows. First of all the Zariski topology $X = \mathrm{Spec}\, A$ is generated by the open sets $D(f),\ f \in A$. Consider the canonical morphism

$$\eta_f : A \to \prod_{\mathfrak{p} \in D(f)} \hat{A}_{\mathfrak{p}} = \varprojlim_{\mathfrak{p} \in D(f)} \hat{A}_{\mathfrak{p}}.$$

Lemma 10. *Let $O(D(f))$ be the subring of $\varprojlim_{\mathfrak{p}\in D(f)} \hat{A}_{\mathfrak{p}}$ generated by $\eta_f(A)$ and $\eta(f)^{-1}$. Then $O(D(f)) \simeq A_f$.*

Proof. Because $\eta_f(f)$ is a unit in $\varprojlim_{\mathfrak{p}\in D(f)} \hat{A}_{\mathfrak{p}}$ there is a homomorphism $\phi : A_f \to O(D(f))$ given by $\phi(a/f^n) = \eta_f(a)\eta(f)^{-n}$. This homomorphism is both surjective and injective, so an isomorphism. □

The following definition is equivalent to the one given in Hartshorne [5].

Definition 14. *Let A be a commutative k-algebra. Let $\operatorname{Spec} A$ be the set of prime ideals $\mathfrak{p} \subset A$ with the topology generated by the open sets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A | f \notin \mathfrak{p}\}$, $f \in A$. We define a sheaf of rings on $\operatorname{Spec} A$ by*

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \varprojlim_{D(f)\subseteq U} O(D(f)).$$

Then $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a locally ringed space coinciding with the ordinary affine scheme associated to A.

4.2 Noncommutative affine varieties

For varieties, i.e. when A is a finitely generated integral domain, finitely presented over an algebraically closed field k of characteristic 0, the ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \simeq (\max(A), \mathcal{O}_{\max(A)})$ can be generalized directly to the the noncommutative situation. The general embedding of $\mathbf{Sch}_k$ in $\mathbf{ncSch}_k$ is considered in section 4.7.

Definition 15. *We let $\operatorname{Simp} A$ denote the set of all simple (right) A-modules. For $f \in A$ we define the basic open subset $D(f) \subseteq \operatorname{Simp} A$ by*

$$D(f) = \{M \in \operatorname{Simp} A \mid \eta_M(f) : M \to M \text{ is invertible } \}$$

where for each $M \in \operatorname{Simp} A$, $\eta_M(f)(m) = m \cdot f$ is multiplication by f.

The proof of the following fact is straight forward.

Lemma 11. *The family of sets $D(f)$, $f \in A$, is a basis for a topology on $\operatorname{Simp} A$.*

For an associative k-algebra A, for a set $\mathcal{M} = \{M_1, \dots, M_r\}$ of simple A-modules, Corollary 2 says that the k-algebra

$$O^A(\mathcal{M}) = \operatorname{End}_{H(\mathcal{M})}(\mathcal{M})$$

has exactly the simple right A-modules $\mathcal{M} = \{M_1, \dots, M_r\}$. There exists a k-algebra homomorphism

$$A \overset{\eta_{\mathcal{M}}}{\to} O^A(\mathcal{M})$$

defining the semi-universal family.

Definition 16. *We call the multi-pointed algebra $A_{\mathcal{M}} = O^A(\mathcal{M})$ the multi-localization of A in $\mathcal{M}$.*

Lemma 12. *i) If $\mathcal{M} \subseteq D(f)$ then $\eta_{\mathcal{M}}(f)$ is a unit in $A_{\mathcal{M}}$.*

ii) Let $\eta_f : A \to \varprojlim_{\mathcal{M} \subset D(f)} A_{\mathcal{M}}$ be the limit of the morphisms $\eta_{\mathcal{M}} : A \to A_{\mathcal{M}}$. Then $\eta_f(f)$ is a unit.

Proof. Because η_M is given deformation theoretically as $\eta_{\mathcal{M}}(a) = a + \xi(a)$ with $\xi(a)$ in the Jacobson radical, it follows by definition of $D(f)$ that $\eta_M(f)$ is invertible, i.e. a unit in $A_{\mathcal{M}}$. Now ii) follows from the sheaf condition: An element that is a unit on all stalks can be lifted to a unit globally. $\square$

Definition 17. *Let A be an associative k-algebra, $f \in A$. Then we define A_f as the subring of $\varprojlim_{\mathcal{M} \subset D(f)} A_{\mathcal{M}}$ generated by $\eta_f(A)$ together with $\eta_f^{-1}(f)$.*

Definition 18. *A ringed space $(X, \mathcal{O}_X)$ is called a* multi-locally ringed space *if for each finite set of points P the stalk $\mathcal{O}_{X,P}$ is a multi-pointed ring. A morphism $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is* multi-local *if the limit morphism $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a morphism of multi-pointed rings.*

Definition 19. *(Noncommutative affine scheme) Let A be an associative k-algebra. Let $\operatorname{Simp} A$ be the set of simple, right A-modules with the topology generated by the base $\{D(f),\ f \in A\}$. Let $\mathcal{O}_{\operatorname{Simp} A}$ be the sheaf of k-algebras defined by $\mathcal{O}_{\operatorname{Simp}(A)}(U) = \varprojlim_{D(f) \subseteq U} A_f$. Then*

$$(\operatorname{Simp} A, \mathcal{O}_{\operatorname{Simp} A})$$

is a multi-locally ringed space called the affine noncommutative scheme associated to A.

Notice that it follows that the stalk of the sheaf $\mathcal{O}_{\operatorname{Simp} A}$ in the finite point-set $\mathcal{M}$ is $\mathcal{O}_{\operatorname{Simp} A, \mathcal{M}} = \varinjlim_{\mathcal{M} \subset P(U)} \mathcal{O}_{\operatorname{Simp}(A)}(U) = A_{\mathcal{M}}$ so that the ringed space above is indeed multi-local.

Example 11. *If A is commutative and finitely presented, then it is of the form $A = k[x_1, \ldots, x_n]/I$ for an ideal I, and there is a bijective correspondence between $X = \operatorname{Simp} A$ and the closed points of $\operatorname{Spec} A$, given by*

$$M \mapsto \operatorname{Ann}(M)$$

for any simple module $M \in X = \operatorname{Simp} A$. In fact, $M = A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m} \subseteq A$, and $A/\mathfrak{m} \simeq A/\mathfrak{m}'$ as right A-modules if and only if the maximal ideal $\mathfrak{m} = \mathfrak{m}'$ coincide. For any $f \in A$, we have that $M \cdot f = M$ if $M \cdot f \neq 0$, since $Mf \subseteq M$ is an A-submodule in the commutative case. Therefore, a simple module $M = A/\mathfrak{m} \in D(f)$ if and only if $f \notin \mathfrak{m}$. It follows that when

$X = \operatorname{Simp} A$ has the Jacobson topology and $\operatorname{Spec} A$ has the Zariski topology, the bijective correspondence between $X = \operatorname{Simp} A$ and the closed points in $\operatorname{Spec} A$ is a homeomorphism.

Example 12. *Let $X = \operatorname{Simp} A$ for the noncommutative algebra A given by*

$$A = \begin{pmatrix} k[x_{11}] & \langle x_{12} \rangle \\ 0 & k[x_{22}] \end{pmatrix}.$$

In other words, A is a tensor algebra of the k^2-bimodule V, where $\dim_k V_{ij} = 1$ for $(i,j) = (1,1), (1,2), (2,2)$ and $V_{21} = 0$. Then we have that

$$X = \{M_\alpha : \alpha \in k\} \cup \{N_\beta : \beta \in k\} = \mathbb{A}^1 \coprod \mathbb{A}^1$$

where $M_\alpha = k[x_{11}]/(x_{11} - \alpha)$ and $M_\beta = k[x_{22}]/(x_{22} - \beta)$. As a topological space, X is a disjoint union of the two affine lines, since we have that $D(e_1) = \{M_\alpha : \alpha \in k\}$ and $D(e_2) = \{N_\beta : \beta \in k\}$ are the connected components of X.

The multi-localization in the finite set $\mathcal{M} = \{M_\alpha, N_\beta\}$ is given as

$$A_{\mathcal{M}} = \begin{pmatrix} H_{11}(\mathcal{M}) \otimes_k \operatorname{End}_k(M_\alpha, M_\alpha) & H_{12}(\mathcal{M}) \otimes_k \operatorname{Hom}_k(M_\alpha, N_\beta) \\ H_{21}(\mathcal{M}) \otimes_k \operatorname{Hom}_k(N_\beta, M_\alpha) & H_{22}(\mathcal{M}) \otimes_k \operatorname{End}_k(N_\beta, N_\beta) \end{pmatrix}.$$

This is easily computable as there are no non-trivial relations in the definition of A, and as the two simple modules under consideration is both of k-dimension 1. We get

$$A_{\mathcal{M}} = \begin{pmatrix} H_{11}(\mathcal{M}) & H_{12}(\mathcal{M}) \\ H_{21}(\mathcal{M}) & H_{22}(\mathcal{M}) \end{pmatrix} = \begin{pmatrix} k \ll x_{11} \gg & \ll x_{12} \gg \\ 0 & k \ll x_{22} \gg \end{pmatrix}.$$

Also notice that this extends to finite sets with more than two elements, e.g. for $\mathcal{M}' = \{M_\alpha, M_{\alpha'}, N_\beta\}$ we get

$$A_{\mathcal{M}} = \begin{pmatrix} k \ll x_{11} \gg & 0 & \ll x_{13} \gg \\ 0 & k \ll x_{22} \gg & \ll x_{23} \gg \\ 0 & 0 & k \ll x_{33} \gg \end{pmatrix},$$

for $\mathcal{M}'' = \{M_\alpha, N_\beta, N_{\beta'}\}$ we get

$$A_{\mathcal{M}} = \begin{pmatrix} k \ll x_{11} \gg & \ll x_{12} \gg & \ll x_{13} \gg \\ 0 & k \ll x_{22} \gg & 0 \\ 0 & 0 & k \ll x_{33} \gg \end{pmatrix},$$

and there are (natural) canonical restriction morphisms

$$r_{\mathcal{M} \subseteq \mathcal{M}'} : A_{\mathcal{M}'} \to A_{\mathcal{M}},\ r_{\mathcal{M} \subseteq \mathcal{M}''} : A_{\mathcal{M}'} \to A_{\mathcal{M}}.$$

4.3 Noncommutative affine schemes of geometric algebras

The main objective for the present geometric definition of nc affine schemes is the interpretation to the class of finitely presented tensor-algebras with commutative k-algebras on the diagonal. These are essential for solving problems in (commutative) algebraic geometry by noncommutative algebraic geometry, and we call them *geometric algebras*.

These algebras are most conveniently introduced by examples, where the generalization is clear.

4.4 Guiding example

Consider a line and a parabola in 2-space intersecting in the origin.

$$y = x^2 \text{ and } y = cx.$$

We identify the points in the intersection of these curves. Consider the following k-algebra.

$$R = \frac{\begin{pmatrix} k[t_{11}(1), t_{11}(2)] & k\langle t_{12}(1)\rangle \\ k\langle t_{21}(1)\rangle & k[t_{22}(1), t_{22}(2)] \end{pmatrix}}{\langle t_{12}(1)(t_{22}(1) - t_{22}^2(2)),\ (t_{22}(1) - ct_{22}(2))t_{21}(1)\rangle} =: \frac{F\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}{(f_{12}, f_{21})} \tag{4.1}$$

We will study its affine nc scheme $(\operatorname{Simp} R, \mathcal{O}_{\operatorname{Simp} R})$, and eventually, prove that $\mathcal{O}(\operatorname{Simp} R) \simeq R$.

In general, the simple modules are the disjoint union of the simple modules on the diagonal. In our case this gives

$$\operatorname{Simp} R = V(k[t_{11}(1), t_{11}(2)]) \coprod V(k[t_{22}(1), t_{22}(2)]) = V_1 \coprod V_2$$

where we for a k-algebra A let $V(A)$ be its variety. There is a possibility for having relations on the diagonal, and the generalization is clear. Put

$$M_i(a, b) = k[t_{ii}(1), t_{ii}(2)]/(t_{ii}(1) - a, t_{ii}(2) - b)$$

for $i = 1, 2$ and $a, b \in k$. As the topology on each of the affine varieties V_i is well known, we find that the topology on $\operatorname{Simp} R$ is the product topology of V_1 and V_2, just saying that $\operatorname{Simp} R = V_1 \coprod V_2$ as a topological space.

Recall the following from [2].

Lemma 13. *Consider the geometric $r \times r$ algebra*

$$F = F(d_{ij}) = \begin{pmatrix} k[t_{11}(1), \ldots, t_{11}(d_{11})] & \cdots & \langle t_{1r}(1), \ldots, t_{1r}(d_{1r})\rangle \\ \vdots & \ddots & \vdots \\ \langle t_{r1}(1), \ldots, t_{r1}(d_{r1})\rangle & \cdots & k[t_{rr}(1), \ldots, t_{rr}(d_{rr})] \end{pmatrix}.$$

Let

$$M_i(P), M_j(Q)$$

be two simple F-modules, each represented by its corresponding point at its corresponding entry on the diagonal. Then

$$\operatorname{Ext}^1_F(M_i(P), M_j(Q)) = \sum k \frac{\partial}{\partial t_{ij}(l)}(P, Q).$$

From Lemma 13 it follows that for $R = F/\mathfrak{f}$ where $\mathfrak{f} = (f_{ij}(l_{ij}))$ is a finitely generated two-sided ideal we have that

$$\operatorname{Ext}^1_R(M_i(P), M_j(Q)) = \operatorname{Ext}^1_F(M_i(P), M_j(Q))/(\frac{\partial \mathfrak{f}}{\partial \underline{t}_{ij}}(P, Q))$$

where the ideal in the quotient means the derivations of all polynomials in $\mathfrak{f}$ in all the variables $t_{ij}(l_{ij})$ and evaluated in the point P on the left, Q on the right. This again says that the relations in $\mathfrak{f}$ must satisfy $\partial(\mathfrak{f}) = 0$.

It is well known that for A a finitely presented commutative k-algebra, k algebraically closed, $\mathfrak{m}_1, \mathfrak{m}_2$ two different maximal ideals,

$$\operatorname{Ext}^1_A(A/\mathfrak{m}_1, A/\mathfrak{m}_2) = 0.$$

From this it follows that if we have two different points P, Q in the same entry on the diagonal, that is the set of simple modules $\{M_i(P), M_i(Q)\}$, $P \neq Q$, then

$$\operatorname{Ext}^1_R(M_i(P), M_i(Q)) = \operatorname{Ext}^1_{k[t_{ii}(1), t_{ii}(2)]}(M_i(P), M_i(Q)) = 0.$$

We sum up the content of this section in the following.

Corollary 3. *Let $R = F/\mathfrak{f}$ be geometric. Then there are the following possibilities for pairs of simple (one-dimensional, right) k-modules and their tangent spaces:*

A duplicate of a point in one diagonal entry. Then

$$\operatorname{Ext}^1_R(M_i(P), M_i(P)) = \operatorname{Ext}^1_F(M_i(P), M_i(P))/(\frac{\partial \mathfrak{f}_{ii}}{\partial \underline{t}_{ii}}(P)).$$

Two different points $P \neq Q$ in the same entry. Then

$$\operatorname{Ext}^1_R(M_i(P), M(Q)) = 0.$$

Two points P and Q in different entries of the diagonal. Then

$$\operatorname{Ext}^1_R(M_i(P), M_j(Q)) = \operatorname{Ext}^1_F(M_i(P), M_j(Q))/(\frac{\partial \mathfrak{f}}{\partial \underline{t}_{ij}}(P, Q)).$$

We will start by applying the general concepts to our present example, with the given geometric algebra R from (4.1). We start by considering two points $\{M_1(a,b), M_2(c,d)\}$ for a particular choice of coordinates on the diagonal. We make a linear coordinate change, so that it suffices, without loss of generality, to consider the set of points

$$\mathcal{M}(x,y) = \{M_1(0,0) = M_1, M_2(x,y)\},\ (x,y) \in k^2. \tag{4.2}$$

The computations given by this choice of a finite set of simple modules will give us the ability to state the results for the other necessary computations (other selections of finite point-sets).

From Corollary 3 we find that the tangent space in our example is given by

(1.1)
$$\mathrm{Ext}^1_R(M_1, M_1) = \mathrm{Ext}^1_F(M_1, M_1) = k\frac{\partial}{\partial t_{11}(1)} \oplus k\frac{\partial}{\partial t_{11}(2)}$$

(1.2)
$$\mathrm{Ext}^1_R(M_1, M_2(x,y)) = (k\frac{\partial}{t_{12}(1)})/(\frac{\partial f_{12}}{\partial t_{12}(1)})$$

(2.1)
$$\mathrm{Ext}^1_R(, M_2(x,y), M_1) = (k\frac{\partial}{t_{21}(1)})/(\frac{\partial f_{21}}{\partial t_{21}(1)})$$

(2.2)
$$\mathrm{Ext}^1_R(M_2(x,y), M_2(x,y)) = \mathrm{Ext}^1_F(M_2(x,y), M_2(x,y)) = k\frac{\partial}{\partial t_{22}(1)} \oplus k\frac{\partial}{\partial t_{22}(2)}$$

From this we read that the tangent space dimensions are $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ if the point is $(x,y) = (0,0)$, otherwise the tangent space dimension is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

It follows that it is sufficient to consider the set of two simple modules given by

$$\{M_1(0,0), M_2(0,0)\} = \{M_1, M_2\}.$$

We now use the algorithm given in [3] to compute the hull of the deformation functor in this finite point-set, the result is

$$\hat{R}_{\mathcal{M}} = \underset{\longleftarrow}{\lim} S_n = \frac{\begin{pmatrix} k[[t_{11}(1), t_{11}(2)]] & k \ll t_{12}(1) \gg \\ k \ll t_{21}(1) \gg & k[[t_{22}(1), t_{22}(2)]] \end{pmatrix}}{\langle t_{12}(1)(t_{22}(1) - t_{22}^2(2)),\ (t_{22}(1) - ct_{22}(2))t_{21}(1)\rangle} =: \frac{F\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)}{(f_{12}, f_{21})}.$$

We have an injection

$$\iota : R \hookrightarrow \hat{R}_{\mathcal{M}},$$

and the image is the (algebraic) multi-pointed ring

$$R_{\mathcal{M}} = \mathrm{im}(\iota) \subseteq \hat{R}_{\mathcal{M}}.$$

This example can be generalized to any geometric k-algebra R, and we have the localization morphisms

$$R \to R_{\mathcal{M}}$$

for every finite set of simple modules $\mathcal{M}$, and because the construction of the pro-representing hull is natural, this is universal, making R unique.

Corollary 4. *For a geometric k^r-algebra R,*

$$\mathcal{O}_{\mathrm{Simp}\,R}(\mathrm{Simp}\,R) = R.$$

Proof. Given any other geometric QAR R' with multi-localizations $R'_{\mathcal{M}}$, this has to contain all relations and so the localization morphisms factors through R. □

Proposition 1. *Let S be a singularity. Then, adding tangents as in the guiding example 4.4 tells us that that there exists a geometric algebra $(\mathrm{Simp}\,R, \mathcal{O}_{\mathrm{Simp}\,R}) \to (\mathrm{Spec}\,S, \mathcal{O}_{\mathrm{Spec}\,S})$ that is a rational morphism to the singularity, and eventually a noncommutative resolution.*

4.5 Noncommutative varieties

Definition 20. *A noncommutative (nc) variety is a multi-locally ringed space $(X, \mathcal{O}_X)$ such that X has a covering of open affine subsets, i.e. $X = \cup_i U_i$ such that*

$$(U_i, \mathcal{O}_X|_{U_i}) \simeq (\mathrm{Simp}(A_i), \mathcal{O}_{\mathrm{Simp}(A_i)})$$

for some k-algebras A_i, separated and of finite type over k, running through an index set. A morphism of nc schemes, is a morphism in the category of multi-locally ringed spaces.

The noncommutative deformation theory tells us that we can study the geometry of a k-algebra A by its representations. Thus we already have the very short and precise definition of a noncommutative variety.

Definition 21. *Let A be an integral, separated k-algebra of finite type over k. A family of A-modules $\mathcal{M}$ is called an affine variety for A if the pro-versal family*

$$\eta_{\mathcal{M}} : A \to \mathcal{O}^A(\mathcal{M})$$

is an isomorphism of k-algebras.

The background for this definition is the generalized Burnside's theorem, Theorem 5, which implies that for a finite dimensional k-algebra A we have that $\mathrm{Simp}\,A$ is a scheme for A.

Proposition 2. *When A is a geometric k-algebra,* $\operatorname{Simp} A$ *is an affine variety for A.*

Proof. We have that $\iota : A \to \mathcal{O}_{\mathrm{Simp}}(D(1)) = O_{\mathrm{Simp}}(\operatorname{Simp} A)$ is injective because A is geometric, and surjective onto its image. □

When the ordinary commutative schemes X are algebraic varieties over k, i.e. integral, separated schemes of finite type over k, the embedding of schemes works directly because of the reconstruction theorem $A \simeq \mathcal{O}_{\operatorname{Simp} A}(\operatorname{Simp} A)$ which is Proposition 2.6 in [5] stating that the maximal ideals are sufficient for reconstructing the k-algebra. This says that the two definitions are equivalent.

Corollary 5. *Let X be a commutative variety. Then the set of closed points* $\operatorname{Simp}(X) \subset X$ *is a variety for X.*

Proof. This means that $\operatorname{Simp} X$ is locally a scheme for A for an open covering by $\operatorname{Spec} A$. This follows from Proposition 2. □

Remark 2. *We would like to give a remark on noncommutative schemes that does not come from commutative ones. This means that there is no algebraic (= finitely generated) k-algebra A such that $\mathcal{O}(\operatorname{Simp} A) \simeq A$. This is even worse: There is no reductive group action on an affine commutative variety resulting in these noncommutative schemes, so they are not even* derived from commutative schemes.

The most trivial example of this is the noncommutative k-algebra in two variables,

$$A = k\langle x, y\rangle.$$

There exist simple modules of any dimension, see Eriksen [2], and so the ring of observables is necessarily a complete noncommutative k^r-algebra which is not the completion of a k^r-algebra with one-dimensional simple modules only on the diagonal.

4.6 Deformations due to diagrams

To give the stated embedding in the general situation, we need to refine the families of modules.

Definition 22. *A diagram of A-modules is a set of right A-modules $\mathcal{M} = \{M_i\}_{i \in I}$, together with a set of A-module homomorphisms $\Gamma_{ij} \subseteq \operatorname{Hom}_A(M_i, M_j)$ for each pair of modules. The idempotents $e_i \in \operatorname{End}_A(M_i)$ is supposed to be included in the diagram. We will write $\underline{\mathsf{c}} = (\mathcal{M}, \Gamma)$, and $|\underline{\mathsf{c}}| = \mathcal{M}$.*

Consider the diagram $\underline{\mathsf{c}}$ where $\mathcal{M} = \{M_1, \ldots, M_r\}$ is a finite set of right A-modules together with a finite path of A-module homomorphisms Γ, consisting of the idempotents $\{e_1, \ldots, e_r\}$ together with $\gamma_{ij}(l_{ij}) \in \operatorname{Hom}_A(M_i, M_j)$, $1 \leq i, j \leq r$, $1 \leq l_{ij} \leq d_{ij}$. We let $k[\Gamma]$ denote the path algebra, and we let $A[\Gamma] = k[\Gamma] \otimes_k A$. Then $M = \oplus_{i=1}^r M_i$ is a right $A[\Gamma]$-module, with action given by $m_i \cdot e_j = \delta_{ij} m_i$ and $M \cdot I(A[\Gamma]) = 0$, where $I(A[\Gamma])$ denotes all paths of length at least one. This also makes the set $\mathcal{M} = \{M_1, \dots, M_r\}$ of A-modules to a set of right $A[\Gamma]$-modules, commuting with the actions of A and $k[\Gamma]$ by the embeddings $A \overset{\Delta}{\to} A[\Gamma]$ and $k[\Gamma] \hookrightarrow A[\Gamma]$ (notice that $\Delta(a) = \sum_{i=1}^{r} ae_i$). The k-algebra we are going to define depends on the definitions of the rings $\mathcal{O}(\mathcal{M}, A[\Gamma])$ and $\mathcal{O}(\mathcal{M}, k[\Gamma])$ (see Theorem 5) which are the *ring of immediate observables* of the family of $A[\Gamma]$-modules, respectively $k[\Gamma]$-modules $\mathcal{M}$. The second is a particular coincidence of the first with $A = k$, it will suffice to consider the first one. Because the tangent space determines the base space of $\mathcal{O}(\mathcal{M}, A[\Gamma])$, that will be our starting point. We start by computing $\mathrm{Ext}^1_{A[\Gamma]}(M_i, M_j)$ for all pairs of $A[\Gamma]$-modules, using the isomorphism

$$\begin{aligned}\mathrm{Ext}^1_{A[\Gamma]}(M_i, M_j) &\simeq \mathrm{HH}^1(A[\Gamma], \mathrm{Hom}_k(M_i, M_j)) \\ &\simeq \mathrm{Der}_k(A[\Gamma], \mathrm{Hom}_k(M_i, M_j))/\,\mathrm{Inner},\end{aligned}$$

where we let $\mathrm{Hom}_k(M_i, M_j)$ be the $A[\Gamma]$-module with left-right action $(a \cdot \phi)(v_i) = \phi(v_i \cdot a)$ and $(\phi \cdot a)(v_i) = \phi(v_i) \cdot a$.

$\mathrm{Ext}^1_{A[\Gamma]}(M_i, M_i)$

A derivation $\delta : A[\Gamma] \to \mathrm{Hom}_k(M_i, M_i)$ is given by its action on the generators as k-algebra, and its action on A. We find $\delta(e_i) = \delta(e_i^2) = 2\delta(e_i) = 0$, and for $j \neq i$, $\delta(e_j) = \delta(e_j^2) = e_j\delta(e_j) + \delta(e_j)e_j = 0$. For $(k, l) \neq (i, i)$, we have for $k \neq i$ that

$$\delta(\gamma_{kl}) = \delta(e_k\gamma_{kl}) = e_k\delta(\gamma_{kl}) + \delta(e_k)\gamma_{kl} = 0,$$

and for $l \neq i$ that

$$\delta(\gamma_{kl}) = \delta(\gamma_{kl}e_l) = \gamma_{kl}\delta(e_l) + \delta(\gamma_{kl})e_l = 0.$$

There are no restrictions on $\delta(\gamma_{ii}(l))$. Thus we have one $\alpha(l) \in \mathrm{End}_k(M_i)$ for each l, $1 \leq l \leq d_{ii}$. We find that $\mathrm{ad}_\alpha(\gamma_{ii}(l)) = [\gamma_{ii}(l), \alpha] = \alpha\gamma_{ii}(l) - \gamma_{ii}(l)\alpha = 0$ by the very definition of the action of $A[\Gamma]$ on M, and so the inner derivations only have influence on the derivation $\delta : A \to \mathrm{End}_k(M)$ determining a derivation. We conclude that

$$\mathrm{Ext}^1_{A[\Gamma]}(M_i, M_i) \cong \mathrm{Ext}^1_A(M_i, M_i) \oplus \mathrm{End}_k(M_i)^{d_{ii}}.$$

$\mathrm{Ext}^1_{A[\Gamma]}(M_i, M_j)$

In this case, if $\delta : A \to \mathrm{Hom}_k(M_i, M_j)$ is any derivation, we find $\delta(e_p) = 0$, $p \neq i, j$ and $\delta(e_i) + \delta(e_j) = 0$. For $p \neq i$, we find $\delta(\gamma_{pq}) = \delta(e_p\gamma_{pq}) = e_p\delta(\gamma_{pq}) + \delta(e_p)\gamma_{pq} = 0$, for $q \neq j$ we have $\delta(\gamma_{pq}) = \delta(\gamma_{pq}e_q) = \gamma_{pq}\delta(e_q) + \delta(\gamma_{pq})e_q = 0$. There are no restrictions on $\delta(\gamma_{ij}(l))$. As $[e_i, \delta(e_i)] = \delta(e_i)$, $[e_j, \delta(e_i)] = -\delta(e_i)$, $[\gamma_{ij}, \delta(e_i)] = 0$, it follows that the derivation given by $\delta(e_i) = \alpha$, $\delta(e_j) = -\alpha$, and $\delta(\gamma_{ij}(l)) = 0$ is an inner derivation. The relation $ae_k - e_ka = 0$ implies $a\delta(e_k) + \delta(a)e_k - e_k\delta(a) - \delta(e_k)a = 0$ and gives $\delta(a) = [a, \delta(e_i)]$ for $k = i$,

$\delta(a) = -[a, \delta(e_j)]$ for $k = j$, and nothing for $k \neq i, j$. This means that any derivation is an inner derivation with respect to A.

Finally, the relation $\gamma_{pq} a - a\gamma_{pq} = 0$ gives $\gamma_{pq}\delta(a) + \delta(\gamma_{pq})a - a\delta(\gamma_{pq}) - \delta(a)\gamma_{pq} = 0 \Leftrightarrow \delta(\gamma_{pq})a = a\delta(\gamma_{pq})$ which says that $\delta(\gamma_{pq})$ is A-linear. We conclude that

$$\mathrm{Ext}^1_{A[\Gamma]}(M_i, M_j) \cong \mathrm{Hom}_A(M_i, M_j)^{d_{ij}}.$$

We have proved:

Lemma 14. *For $1 \leq i, j \leq r$, the tangent space of $\mathsf{Def}_{\mathcal{M},A[\Gamma]}$ is given by*

$$\mathsf{t}(\mathsf{Def}_{\mathcal{M},A[\Gamma]})_{ij} \cong \begin{cases} \mathrm{Hom}_A(M_i, M_j)^{d_{ij}}, \ i \neq j \\ \mathrm{Ext}^1_A(M_i, M_i) \oplus \mathrm{End}_A(M_i)^{d_{ii}}, \ i = j. \end{cases}$$

Lemma 15. *The deformation functor $\mathsf{Def}_{\mathcal{M},k[\Gamma]}$ is unobstructed.*

Proof. Let $I \to R \overset{\pi_R}{\to} S \to 0$ be a small morphism in $\mathbf{a}_r$. Let $M_S \in \mathsf{Def}_{\mathcal{M},k[\Gamma]}$ be given by the element $\sigma_S : k[\Gamma] \to \mathrm{End}_S(S \otimes_k M)$, and let σ'_R be a lifting of σ_S. Then the obstruction for lifting M_S is given by the element

$$\overline{\psi}_R \in (I_{ij} \otimes_k \mathrm{HH}^2(k[\Gamma], \mathrm{Hom}_k(V_i, V_j)),$$

which is represented by $\psi_R : k[\Gamma] \otimes_k k[\Gamma] \to (I_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$ given by

$$\psi_R(a, b) = \sigma'_R(ab) - a\sigma'_R(b) - \sigma'_R(a)b.$$

Notice that this factors through the kernel I because σ_R restricts to σ_S, so that ψ_R restricts to 0 on S. The morphism ψ_R is k-bilinear. Assume that, relative to a basis for the kernel I of π_R,

$$\psi(a, b) = i_1 i_2 \otimes \sigma'_R(ab) - (i_1 \otimes_k \sigma'_R(a))(i_2 \otimes_k \sigma'_R(b)) = -i_1 i_2 \otimes_k \sigma'_R(a)\sigma'_R(b) \neq 0.$$

Here σ'_R is interpreted as a k-linear homomorphism $V_i \to S_{ij} \otimes_k V_j$ so that σ'_R does not take values in $I \otimes_k \mathrm{Hom}_k(V_i, V_j)$. That is, S is finite dimensional, and σ_S is defined on a basis for S, while R is an extended vector space, $R = I \otimes_k S$ as k-vector space. Now, define $\xi : k[\Gamma] \to (I_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$ by $\xi(ab) = -\sigma'_R(a)\sigma'_R(b)$, all other values 0. This is possible on basis elements because there are no relations between elements in in the path algebra $k[\Gamma]$. Then we have

$$d(\xi)(a \otimes b) = \xi(a) - \xi(ab) + \xi(b) = \sigma'_R(a)\sigma'_R(b),$$

proving that the obstruction $o(\pi_R, M_S) = 0$ (in cohomology), and we are done. $\square$

Example 13. $A = \begin{pmatrix} k & x_{12} \\ x_{21} & k \end{pmatrix}$, *$\mathcal{M} = \{k, k\}$ where the x_{ij} acts trivially on the $V_i = k$, $i = 1, 2$. The tangent space is given according to lemma 13, that is $d_{x_{12}}$, $d_{x_{21}} : A \to \mathrm{Hom}_k(V_i, V_j)$ and we find the cup-products to be*

$$d_{x_{12}} \cup d_{x_{21}}(x_{12} \otimes_k x_{21}) = d_{x_{12}}(x_{12}) d_{x_{21}}(x_{21}) = 1,$$

all other generators are zero. So we choose defining system $\xi_{x_{12}x_{21}} = (x_{12} \otimes x_{21})^\vee$ *and let* $d_2 = d + \xi_{x_{12}x_{21}}$ *and can continue the lifting procedure as there are no relations. That is* $H \simeq A$.

Example 14. *Same as above, but with the relation* $x_{ij}x_{ji} = 0$. *This means that we cannot define any dual of* $\xi_{x_{12}x_{21}} = (x_{12} \otimes x_{21})$, *because this element is zero in* A. *Then the obstruction is different from* 0, *and we have to divide out by* $x_{12}x_{21}$ *in the construction of the multi-local formal moduli, i.e.* $H \simeq A$.

Let $V = \mathsf{t}(\mathsf{Def}_{\mathcal{M},k[\Gamma]})$ be the tangent space of $\mathsf{Def}_{\mathcal{M},k[\Gamma]}$ and let $W = (W_{ij})$ with $W_{ij} = k^{d_{ij}}$. There are natural k-linear maps $\kappa_{ij} : V_{ij}^* \to W_{ij}$ given by

$$\kappa_{ij}(\psi_{ij}^*) = (\psi_{ij}^*(\phi_{ij}(l)))$$

where $\{\phi_{ij}(l) : 1 \leq l \leq d_{ij}\}$ are the morphisms from M_i to M_j in the diagram $\underline{\mathsf{c}}$. Since $\mathsf{Def}_{\mathcal{M},k[\Gamma]}$ is unobstructed, there is an induced morphism $\kappa : H(\mathcal{M}, k[\Gamma]) \to \mathbf{T}(W)$, where $\mathbf{T}(W)$ is the tensor algebra of W over k^r (the matrix algebra F generated by the k^r-bimodule W), and $\ker(\kappa) \subseteq H(\mathcal{M}, k[\Gamma])$ is an ideal.

Let us also denote by $\ker(\kappa)$ the ideal in $H(\mathcal{M}, A[\Gamma])$ generated by the image of $\ker(\kappa) \subseteq H(\mathcal{M}, k[\Gamma])$ under the ring homomorphism $H(\mathcal{M}, k[\Gamma]) \to H(\mathcal{M}, A[\Gamma])$.

Definition 23. *We define* $H(\underline{\mathsf{c}}) = H(\mathcal{M}, A[\Gamma])/\ker(\kappa)$, *and the* ring of observables *of the diagram* $\underline{\mathsf{c}}$ *to be*

$$\mathcal{O}(\underline{\mathsf{c}}) = \operatorname{im}(\eta_{\underline{\mathsf{c}}}) \subseteq (H(\underline{\mathsf{c}})_{ij} \otimes_k \operatorname{Hom}_k(M_i, M_j)).$$

Remark 3. *If the quiver* Γ *is a quiver with relations* R, *we also denote by* $\ker(\kappa)$ *the ideal in* $H(\mathcal{M}, A[\Gamma])/R$ *generated by the image of* $\ker(\kappa) \subseteq H(\mathcal{M}, k[\Gamma])$ *under the ring homomorphism* $H(\mathcal{M}, k[\Gamma]) \to H(\mathcal{M}, A[\Gamma])/R$.

For a commutative k-algebra A, let

$$\operatorname{Spec}^*(A) = \{A \xrightarrow{\psi_{\mathfrak{p}}} A/\mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} A\},$$

where the morphisms $\psi_{\mathfrak{p}} \in \operatorname{Hom}_A(A, A/\mathfrak{p})$ is the set of all A-linear homomorphisms.

Theorem 6. *For* k *algebraically closed of characteristic* 0, *for* A *a commutative, finitely generated* k*-algebra* A,

$$\mathcal{O}(\operatorname{Spec}^*(A)) \simeq A.$$

Proof. The proof of the existence of a prorepresenting hull is constructive, thus we will give a constructive proof of this result as well.

The prorepresenting hull $H(\operatorname{Spec}^*(A))$ of the diagram $\operatorname{Spec}^*(A)$ is the direct limit of $H(\underline{\mathsf{c}})$ for all *finite* subdiagrams $\underline{\mathsf{c}} \subseteq \operatorname{Spec}^*(A)$. In turn, this is defined as the quotient of $H(|\underline{c}|, A[\Gamma])$ by the image of the kernel of $\kappa : H(|\underline{c}|, k[\Gamma]) \to \mathbf{T}(W)$, i.e.

$$H(\underline{c}) = H(|\underline{c}|, A[\Gamma]) / \operatorname{im}(\ker(\kappa)).$$

Let $\underline{c} \subseteq \operatorname{Spec}^*(A)$ be a finite subdiagram, $\underline{c} = (|\underline{c}|, \Gamma)$. First, consider $H(\underline{c}, A[\Gamma])$. For diagrams $\underline{c}$ where A is not a node, after dividing out by $\operatorname{im}(\ker\kappa)$, the tangent space looks like

$$\begin{pmatrix} \operatorname{Ext}^1_A(A/\mathfrak{p}_1, A/\mathfrak{p}_1) & & 0 \\ & \ddots & \\ 0 & & \operatorname{Ext}^1_A(A/\mathfrak{p}_r, A/\mathfrak{p}_r) \end{pmatrix},$$

which is the product of the domains strictly contained in A. When A is one out of two nodes in a diagram we get

$$\begin{pmatrix} 0 & \psi_{\mathfrak{p}} \\ 0 & 0 \end{pmatrix},$$

where $\mathfrak{p}$ is a prime ideal, $\mathfrak{p} \neq 0$.

Constructing the multi-pointed formal moduli, all the time dividing out by $\operatorname{im}(\ker\kappa)$ at each level, we obtain the ring $H(\underline{c})$. Taking the direct limit, we get $H(\operatorname{Spec}^*(A))$.

The global sections, or the regular functions, is then represented by

$$\mathcal{O}(\operatorname{Spec}^*(A)) = \lim_{\substack{\longleftarrow \\ \underline{c} \subset \operatorname{Spec}^*(A)}} H(\underline{c}) \otimes_{k^{r(|\underline{c}|)}} (\operatorname{Hom}_k(A/\mathfrak{p}_i, A/\mathfrak{p}_j)).$$

The construction gives $\mathcal{O}(\operatorname{Spec}^*(A))$ the maximum of relations, i.e. the maximal ideal for $A \to \lim\limits_{\substack{\longleftarrow \\ \underline{c} \subset \operatorname{Spec}^*(A)}} \operatorname{Hom}_{H(\underline{c})}(H(\underline{c}) \otimes_{k^{r(|\underline{c}|)}} (\bigoplus\limits_{\mathfrak{p} \in \underline{c}} A/\mathfrak{p}))$ to be lifted, which proves that it is an isomorphism, that is

$$A \simeq \lim_{\substack{\longleftarrow \\ U \subseteq \operatorname{Spec}^* A}} \mathcal{O}(U)$$

where $\mathcal{O}(U) = \lim\limits_{\substack{\longleftarrow \\ \mathfrak{p} \in U}} A_{\mathfrak{p}} = \lim\limits_{\substack{\longleftarrow \\ \mathfrak{p} \in U}} (\operatorname{Hom}(A \otimes_k A/\mathfrak{p}))$.

notice that this is the homomorphism $A \to \mathcal{O}(\operatorname{Spec}^*(A))$ given in Hartshorne [5], because $\operatorname{Der}_k(A, A/\mathfrak{p}) \simeq (\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^* \subseteq (A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^*$.

□

4.7 The general definition of noncommutative schemes

Definition 24. *Let A be a k-algebra. A diagram $\underline{c}$ of A-modules is called an affine scheme for A if the pro-versal family*

$$\eta_{\underline{c}} : A \to \mathcal{O}(\underline{c})/R$$

is an isomorphism of k-algebras, where R is the ideal of relations in the diagram.

We give the general definition of a noncommutative affine scheme.

Definition 25. *We define* $\mathrm{Simp}^*(A)$ *as the diagram* $\mathrm{Simp}^*(A) = \{A\} \cup \mathrm{Simp}\, A$ *with quiver* $\Gamma^* = \cup_{M_i, M_j \in \mathrm{Simp}\, A} \mathrm{Ext}^1_A(M_i, M_j) \cup \{A \to M | M \in \mathrm{Simp}\, A\}$.

Lemma 16. *If* $\eta_{\mathrm{Simp}^*(A)}$ *is injective,* $\mathrm{Simp}^*(A)$ *is a scheme for* A.

Proof. This follows by definition. It is also proved by O.A. Laudal in [6]. □

Definition 26. *We give* $\mathrm{Simp}^*(A)$ *the topology induced by the Jacobson topology. On this topological space we consider a multi-locally ringed space* $\mathcal{O}_{\mathrm{Simp}^*}$ *defined as in definition 19, and we call this a noncommutative affine scheme. A noncommutative scheme is a multi-locally ringed space, locally isomorphic as such to a noncommutative affine scheme.*

Proposition 3. *The category of* k*-schemes is naturally, full, and faithfully embedded in the category of noncommutative* k*-schemes, i.e.*

$$\mathbf{sch}_k \subset \mathbf{ncSch}_k$$

Proof. Consider a k-scheme X with an open affine cover by $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$. Consider the diagram $\mathrm{Simp}^*(A)$ defined as above. The open affine embed fully as $(\mathrm{Simp}^*(A), \mathcal{O}_{\mathrm{Simp}^*(A)})$. The reconstruction holds, and then Lemma 16 proves it is faithful. Notice that this procedure formally adds enough generic points. □

4.8 Tangent Spaces of QARs

Consider the QAR

$$\begin{tikzcd}
A = \begin{pmatrix} k[t_{11}] & \langle t_{12} \rangle \\ \langle t_{21} \rangle & k[t_{22}] \end{pmatrix} \arrow[r] \arrow[d] & k[t_{11}]/(t_{11} - \alpha) = V_1(\alpha) \, . \\
k[t_{22}]/(t_{11} - \beta) = V_2(\beta) &
\end{tikzcd}$$

This is a very simple form of a QAR, and the simple modules is exactly the simple modules on the diagonal, represented by the quotients of their maximal ideals. The canonical morphisms are given by the usual evaluation of generators. We have a k-algebra A and two right A-modules $M_1 = M_1(\alpha)$, $M_2 = M_2(\beta)$, $\alpha, \beta \in k$.

We will compute the tangent space of this k-algebra in the (semi-) points (M_1, M_2), using the well known identity

$$\mathrm{Ext}^1_A(M_i, M_j) = \mathrm{HH}^1(A, \mathrm{Hom}_k(M_i, M_j)).$$

This implies in particular that

$$\mathrm{Ext}^1_A(M_i, M_j) = \mathrm{Der}_k(A, \mathrm{Hom}_k(M_i, M_j)) / \mathrm{Inner},$$

where $\mathrm{Inner} = \{\mathrm{ad}(\phi) | \mathrm{ad}(\phi)(a) = a\phi - \phi a\}$ is the subspace of inner derivations. Before starting the computation, we recall that the bi-module structure on $\mathrm{Hom}_k(M_i, M_j)$ is given by

$$(a \cdot \phi)(v) = \phi(va),\ (\phi \cdot a)(v) = \phi(v) \cdot a.$$

Any derivation is determined by its action on a generator set, we let e_i denote the idempotents.

$\mathrm{Ext}^1_A(M_1, M_1)$. For any derivation $\delta : A \to \mathrm{End}_k(V_1)$ we have that

$$\begin{aligned} a_{12} \in A_{12} = e_1 A e_2 \Rightarrow \delta(a_{12}) = \delta(a_{12}e_2) = \delta(a_{12})e_2 + a_{12}\delta(e_2) = 0, \\ a_{21} \in A_{21} \Rightarrow \delta(a_{21}) = \delta(e_2 a_{21}) = \delta(e_2)a_{21} + e_2\delta(a_{21}) = 0, \\ a_{22} \in A_{22} \Rightarrow \delta(a_{22}) = \delta(a_{22}e_2) = \delta(a_{22})e_2 + a_{22}\delta(e_2) = 0. \end{aligned}$$

Then, as $0 = \delta(1) = \delta(e_1 + e_2) = \delta(e_1)$, we have that

$$\mathrm{Ext}^1_A(M_1, M_2) = \langle d_{t_{11}} \rangle.$$

$\mathrm{Ext}^1_A(M_1, M_2)$. For any derivation $\delta : A \to \mathrm{Hom}_k(V_1, V_2)$ we have that

$$\begin{aligned} a_{11} \in A_{11} \Rightarrow \delta(a_{11}) = \delta(a_{11}e_1) = \delta(a_{11})e_1 + a_{11}\delta(e_1) = a_{11}\delta(e_1) = \mathrm{ad}_{\delta(e_1)}(a_{11}) \\ a_{22} \in A_{22} \Rightarrow \delta(a_{22}) = \delta(e_2 a_{22}) = \delta(e_2)a_{22} + e_2\delta(a_{22}) = \delta(e_2)a_{22} = -\mathrm{ad}_{\delta(e_2)}(a_{22}) \\ a_{21} \in A_{21} \Rightarrow \delta(a_{21}) = \delta(a_{21}e_1) = \delta(a_{21})e_1 + a_{21}\delta(e_1) = 0. \end{aligned}$$

It follows that

$$\mathrm{Ext}^1_A(M_1, M_2) = \langle d_{t_{12}} \rangle.$$

The two final computations of $\mathrm{Ext}^1_A(M_2, M_1)$ and $\mathrm{Ext}^1_A(M_2, M_2)$ is exactly similar, and we can generalize to the following lemma.

Lemma 17. *Let $F = T_{k^r}(V)$ be the QAR from definition 3. Let $\mathcal{M} = \{M_1, \ldots, M_r\}$ be r diagonal points (one-dimensional simple modules), i.e.,*

$$M_i = k\langle t_{ii}(1), \ldots, t_{ii}(d_{ii})\rangle / (t_{ii}(1) - \alpha_{ii}(1), \ldots, t_{ii}(d_{ii}) - \alpha_{ii}(d_{ii})).$$

Then the tangent space of F in the point M is

$$TF_M = \begin{pmatrix} \langle d_{\underline{t}_{11}(l_{11})} \rangle & \cdots & \langle d_{\underline{t}_{1r}(l_{1r})} \rangle \\ \vdots & \ddots & \vdots \\ \langle d_{\underline{t}_{r1}(l_{r1})} \rangle & \cdots & \langle d_{\underline{t}_{rr}(l_{rr})} \rangle \end{pmatrix}.$$

With this preliminaries, we are ready to give a relevant example.

4.9 Example

This example illustrates the generalized Burnside's theorem (Theorem 5) and the definition of the generalized Massey products. After the computation, we give a geometric interpretation of the resulting noncommutative affine scheme.

Let

$$A = \begin{pmatrix} k[t_{11}(1), t_{11}(2)] & \langle t_{12}(1), t_{12}(2)\rangle \\ 0 & k[t_{22}] \end{pmatrix} / (f_{12})$$

where $f_{12} \in A_{12} = e_1 A e_2$ is the polynomial

$$f_{12} = t_{11}(1)t_{12}(2) - t_{11}(2)t_{12}(1) - 2t_{12}(2)t_{22} + t_{12}(1)t_{22}^2.$$

We see that the two origins on the diagonal is a point on the noncommutative curve f_{12}, i.e., we consider the set of simple modules $\mathcal{M} = \{M_1, M_2\}$ where

$$M_1 = k[t_{11}(1), t_{11}(2)]/(t_{11}(1), t_{11}(2)) = k,\ M_2 = k[t_{22}]/(t_{22}).$$

The computation of the tangent space is completed by lemma17, so the computation starts by letting

$$S_2 = \begin{pmatrix} k[t_{11}(1), t_{11}(2)] & \langle t_{12}(1), t_{12}(2)\rangle \\ 0 & k[t_{22}] \end{pmatrix} / \, \mathrm{I}^2$$

and letting the restriction of the universal lifting to S_2 be given by the corresponding basis for $(\mathrm{Ext}^1_A(M_i, M_j))$.

Before the computation starts, be sure to recall that

$$d_{t_{ij}} : A \to \mathrm{Hom}_k(M_i, M_j)$$

so that the order of composition is given by $d_{t_{ij}}(a)d_{t_{jk}}(b)$.

$$\begin{aligned} \langle t_{11}^2\rangle : d_{t_{11}} \cup d_{t_{11}}(t_{11} \otimes t_{11}) &= d_{t_{11}}(t_{11}) \circ d_{t_{11}}(t_{11}) = 1. \\ d((t_{11}^2)^*)(a \otimes b) &= a(t_{11}^2)^*(b) - (t_{11}^2)^*(ab) + (t_{11}^2)^*(a)b \Rightarrow \\ \langle t_{11}^2\rangle &= (t_{11} \otimes t_{11})^* = d((t_{11}^2)^*) = 0. \end{aligned}$$

$$\begin{aligned} \langle t_{11}(1)t_{11}(2)\rangle : d_{t_{11}(1)} \cup d_{t_{11}(2)} &= (t_{11}(1) \otimes t_{11}(2))^* \\ & d((t_{11}(1)t_{11}(2))^*) + d_{t_{11}(2)} \cup d_{t_{11}(1)} \Rightarrow \\ & \langle t_{11}(1)t_{11}(2)\rangle = \langle t_{11}(2)t_{11}(1)\rangle + d((t_{11}(1)t_{11}(2))^*). \end{aligned}$$

Similarly,

$$\begin{aligned} \langle t_{11}(1)t_{12}(1)\rangle &= d((t_{11}(1)t_{12}(1))^*) = 0 \\ \langle t_{11}(1)t_{12}(2)\rangle &= -\langle t_{11}(2)t_{12}(1)\rangle + d((t_{11}(2)t_{12}(1))^*) \\ \langle t_{11}(2)t_{12}(2)\rangle &= d((t_{11}(2)t_{12}(2))^*) = 0 \\ \langle t_{12}(1)t_{22}\rangle &= d((t_{12}(1)t_{22})^*) = 0 \\ \langle t_{12}(2)t_{22}\rangle &= -\frac{1}{2}\langle t_{11}(1)t_{12}(2)\rangle + d((t_{12}(2)t_{22})^*) \end{aligned}$$

There is only one choice of monomial bases at this stage, and the computations above give us

$$S_3 = \frac{\begin{pmatrix} k[t_{11}(1), t_{11}(2)] & \langle t_{12}(1), t_{12}(2)\rangle \\ 0 & k[t_{22}] \end{pmatrix}}{(t_{11}(1)t_{12}(2) + t_{11}(2)t_{12}(1) - 2t_{12}(2)t_{22}) + \mathrm{I}^3}$$

Now, we choose bases for this algebra S_2 and R_3 due to the algorithm (conditions) given in the proof of Theorem 1. Then the defining system is given by the choices in the computations above, and we continue to the next step. Here everything is identically zero by the relations in A, except for

$$\langle t_{12}(1)t_{22}^2\rangle = \langle t_{11}(1)t_{12}(2)\rangle + d((t_{12}(1)t_{22}^2)^*).$$

Then this turns out to be rather easy (modulo all the indexes) as long as we choose equal monomial bases on both sides. We see that we are through, and that

$$\hat{H} = \hat{A}$$

with A as a natural algebraization of $\hat{H}$.

4.10 Generalized Matric Massey Products

To compute the local formal moduli of a family of finitely generated A-modules $\mathcal{M} = \{M_i\}_{i=1}^r$ in the general case, it turns out that the most convenient way is to work in the *Yoneda complex*. In this section we describe this technique.

Consider a family $\mathcal{M} = \{M_1, \dots, M_r\}$ of A-modules. For each $1 \leq i \leq r$ choose a projective (or free) resolution of M_i,

$$0 \leftarrow M_i \leftarrow L_0^i \overset{d_0^i}{\leftarrow} L_1^i \overset{d_1^i}{\leftarrow} \cdots$$

Definition 27. *The* Yoneda complex *is given by*

$$\mathrm{YC}^p(M_i, M_j) = \prod_{s \geq p} \mathrm{Hom}_A(L_s^i, L_{s-p}^j)$$

for $p \geq 0$, with differential $d^p : \mathrm{YC}^p(M_i, M_j) \to \mathrm{YC}^{p+1}(M_i, M_j)$ given by

$$d^p(\{\xi_s\}) = \{d_s^i \circ \xi_{s-1} - (-1)^p \xi_s \circ d_{s-p}^j\}.$$

The following is then an easily proven fact.

Lemma 18. $\mathrm{YH}^p(M_i, M_j) := \mathrm{h}^p(\mathrm{YC}^{\cdot}(M_i, M_j)) \simeq \mathrm{Ext}_A^p(M_i, M_j)$.

Proposition 4. *Let $S \in \mathbf{a}_r$. To give a deformation (lifting) of $M = \oplus_{i=1}^r$ to S is equivalent to give a lifting of complexes*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & I \otimes_{k^r} L_0 \arrow[d] & I \otimes_{k^r} L_1 \arrow[l] \arrow[d] & I \otimes_{k^r} L_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & & S \otimes_{k^r} L_0 \arrow[d] & S \otimes_{k^r} L_1 \arrow[l] \arrow[d] & S \otimes_{k^r} L_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & M \arrow[l] & L_0 \arrow[l] \arrow[d] & L_1 \arrow[l] \arrow[d] & L_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

where $L_p = \oplus_{i=1}^r L_p^i$ and $I = (I_{ij})$ is the radical of S.

Proof. The proof is given in the book [3] and is by induction on the n such that $\mathrm{I}^n(S) = 0$. Here we give a sketch of the induction step, factoring $\rho : S \to k^r$ in small morphisms. Thus we assume $\pi : R \to S$ is a small morphism in $\mathbf{a}_r$, and we assume that $M_S \in \mathsf{Def}_M(S)$ is given. Let $\ker(\pi) = I = (I_{ij})$. Then we can always choose an $S \otimes_k A$-free resolution of M_S, and by the surjectivity of π, we can lift the differential to R:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & C \arrow[l] \arrow[d] & I \otimes_{k^r} L_0 \arrow[l] \arrow[d] & I \otimes_{k^r} L_1 \arrow[l, "I\otimes d_0"'] \arrow[d] & I \otimes_{k^r} L_2 \arrow[l, "I\otimes d_1"'] \arrow[d] & \cdots \arrow[l] \\
0 & M_R \arrow[l] \arrow[d] & R \otimes_{k^r} L_0 \arrow[l] \arrow[d] & R \otimes_{k^r} L_1 \arrow[l, "d_0^R"'] \arrow[d] & R \otimes_{k^r} L_2 \arrow[l, "d_1^R"'] \arrow[d] & \cdots \arrow[l, "d_2^R"'] \\
0 & M_S \arrow[l] \arrow[d] & S \otimes_{k^r} L_0 \arrow[l] \arrow[d] & S \otimes_{k^r} L_1 \arrow[l, "d_0^S"'] \arrow[d] & S \otimes_{k^r} L_2 \arrow[l, "d_1^S"'] \arrow[d] & \cdots \arrow[l, "d_2^S"'] \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

If $(d^R)^2 = 0$, the fact that π is small, implies by the long exact sequence, that $M_R = H^0(d^R)$ is a flat lifting of M_S, and that d^R is a resolution of M_R. $\square$

The proof immediately proves the following.

Lemma 19. *With the notation above, the obstruction for lifting M_S to R is given by*

$$o(M_S, \pi) = (d_0^R)^2 \in (I_{ij} \otimes_k \operatorname{Ext}_A^2(M_i, M_j)).$$

The obstruction is independent of the choice of resolution $(L^S., d^S.)$ and proves that $\operatorname{Ext}_A^1(M_i, M_j)$ gives Def_M a structure of principal homogeneous space.

Chapter 5

Noncommutative GIT

5.1 Basic Definitions

At this stage of the developing theory, we will mostly consider the affine situation where an affine algebraic group $\underline{G} = \operatorname{Spec} \mathcal{O}_G$ acts on an affine (commutative) scheme $X = \operatorname{Spec} A$. Again we will assume that all schemes are over k, algebraically closed of characteristic 0, and we notice that most of the results can be generalized to an arbitrary field k with $\operatorname{char} k >> 0$.

An action of G on X is a morphism of schemes $\overline{\nabla} : \underline{G} \times_k X \to X$, and this induces for each $g \in G$ a morphism $\overline{\nabla}_g : X \to X$ which again gives $\nabla_g : A \to A$ which is usually called the *dual action.* As $\overline{\nabla}$ is supposed to be associative, and the identity when multiplying with the identity, this yields for the dual action as well:

$$\nabla_{g_1 g_2}(a) = \nabla_{g_1}(\nabla_{g_2}(a)),\ \nabla_{\mathrm{id}} = \mathrm{id}\,.$$

Example 15. *Let $X = \mathbb{A}_k^n = \operatorname{Spec}(k[x_1, \ldots, x_n]) = \operatorname{Spec} k[\underline{x}]$. For $g \in G \subseteq \mathsf{GL}(n)$, we have that $\phi : k[\underline{x}] \to k[\underline{x}]$ given by $\phi(\underline{x}) = g \cdot \underline{x}$ maps the point $(\underline{x} - \underline{a})$ to*

$$\phi^{-1}(\underline{x} - \underline{a}) = (g^{-1} \cdot x - a) = (\underline{x} - \overline{\nabla}_g(a)).$$

Thus $\nabla_g = \phi$, and we have an argument for computing in the intuitive way.

Definition 28 (Geometric Quotient)**.** *Let G be an algebraic group acting on a variety X. Then*

$$\phi : X \to Y = X/G$$

is a geometric quotient *of X by G if the geometric fibres of ϕ are precisely the orbits of the geometric points of X, that is, in the diagram*

$$\begin{tikzcd} G \times_k X \arrow[r, "\overline{\nabla}"] \arrow[d, "p_2"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"'] & Y \end{tikzcd}$$

the following conditions are fulfilled:

i) *Commutativity:* $\phi \circ \overline{\nabla} = \phi \circ p_2$

ii) *We have that* ϕ *is onto and submersive:* $U \subseteq Y$ *open* $\Leftrightarrow \phi^{-1}(U) \subseteq X$ *open .*

iii) $\mathcal{O}_Y = \mathcal{O}_X^G$.

Example 16. *With the usual notation, when the* k*-algebra* A *is noetherian, we have that*

$$A^G = k[s_1, \dots, s_r]/\mathfrak{a},$$

saying that the invariant ring of A *by* G *is generated by* $s_1, \dots, s_r$*. Then* Spec A *is a geometric quotient if the invariants* $s_1, \dots, s_r$ *separates orbits. This follows directly from iii) in definition 28 together with the inclusion* $A^G \hookrightarrow A$*, and is (most probably) the origin of the definition.*

5.2 Fine Moduli for Orbits

In general, for a family of objects parametrized by schemes

$$\mathcal{F} : \mathbf{Sch}_k \to \mathbf{Sets},$$

a fine moduli for the objects $\mathcal{F}(\operatorname{Spec} k)$ is a couple $\mathcal{U} \in \mathcal{F}(\mathcal{M})$ such that the morphism of functors

$$\phi_{\mathcal{U}} : \operatorname{Mor}(-, \mathcal{M}) \to \mathcal{F}$$

given by $\phi_{\mathcal{U}}(S)(\psi : S \to M) = \mathcal{F}(\psi)(\mathcal{U})$ is an isomorphism of functors. Translated to the affine situation, an affine fine moduli is an affine scheme Spec A such that the closed points are in one-to-one correspondence with the objects, i.e., it is generated by the invariants, and such that any other ring B factors through A. In particular, if the objects of interest are G-orbits, an affine fine moduli of schemes is a geometric quotient, but not vice versa, as we will see.

Definition 29. *Let* A *be a* k*-algebra with an action* $G \times A \to A$*. An* $A - G$ *module* M *is a (right or left or both)* A*-module together with an action*

$$\nabla : G \times M \to M$$

such that for all $g \in G$*,* $\nabla_g(am) = \nabla_g(a)\nabla_g(m)$

Let $A[G]$ be the free A-module over G, that is

$$A[G] = A^G = \underset{i \in G}{\oplus} a_g \cdot g.$$

We will give $A[G]$ a k-algebra structure such that the category of $A-G$-modules is equivalent to the category of $A[G]$-modules. This is satisfied if the associativity holds for all $A - G$ modules M as defined above:

$$\begin{aligned}(a_g g \cdot a_h h)m &= a_g g(a_h h \cdot m) = a_g \nabla_g(a_h \nabla_h(m)) = a_g \nabla_g(a_h)\nabla_g(\nabla_h(m)) \\ &= a_g \nabla_g(a_h)\nabla_{gh}(m)\end{aligned} \tag{5.1}$$

Thus this is fulfilled when we give the following definition.

Definition 30. *Let G be a group with an action on a k-algebra A. Then the* skew group algebra $A[G]$ *is the free A-module indexed by G, and with the product*

$$ag \cdot bh = a\nabla_g(b)gh.$$

From the explaining computation (5.1), we then have:

Lemma 20. *The categories of $A-G$ modules and $A[G]$-modules are equivalent,*

$$\text{mod}_{A-G} \cong \text{mod}_{A[G]}.$$

Consider the orbit $Gx \subset X = \operatorname{Spec} A$, and let $\overline{Gx}$ be its closure. Then the ideal $I(\overline{Gx})$ is a G-invariant ideal which is an $A-G$ module. This is simply because $\nabla : G \times X \to X$ is continuous, and so is the map $\nabla(-,x) : G \to X$ sending g to $g \cdot x$ for every x. Thus $\nabla^{-1}(\overline{Gx}) \subseteq G$ is closed, implying that $I(\overline{Gx})$ is G-invariant, and also that

$$G/\operatorname{Iso}(x) \simeq \overline{Gx},$$

where $\operatorname{Iso}(x) = \{g \in G | gx = x\}$ denotes the *isotropy group* of x. Thus we get the useful fact that the dimension of the orbit of x is the codimension of the isotropy subgroup of x in G.

Lemma 21. *The orbits in X under the action of G are in one-to-one correspondence with their closures, which again corresponds to a set of G-invariant ideals which is a certain family of* $\operatorname{rk} 1$ *$A[G]$-modules.*

Proof. The correspondence is given by $o(x) = Gx \mapsto A/I(Gx)$, the rest follows by Hilbert's Nullstellensatz. □

In this affine situation, we consider the ring of invariants A^G, and $\operatorname{Spec}(A^G)$ is a geometric quotient, see the book of Mumford, Fogarty and Kirwan [7]. However, it is certainly no fine moduli, as the examples below will show. To find a fine moduli, we construct the noncommuative affine moduli of $A[G]$-modules corresponding to the orbits. This is a geometric quotient, which is also a fine moduli.

Lemma 22. *Let X^G be a (noncommutative) affine scheme that is a fine moduli for the A/G-modules*

$$\{A/\mathfrak{a}_x | \mathfrak{a}_x = I(Gx),\ x \in X\}.$$

Then X^G is an affine, noncommutative, geometric quotient of X by G.

Proof. From the reconstruction corollary (2) it follows that

$$\Gamma(X^G, \mathcal{O}_{X^G}) \simeq A^G.$$

The rest follows from the definitions of affine noncommutative schemes and geometric quotients. □

5.3 Constructive Method for Noncommutative GIT

Before giving applications, we outline the essentials of the construction in this subsection. We begin with the affine scheme $\operatorname{Spec} A$, with its group action $G \times X \to X$. We start by computing the invariant ring A^G which is generated by a finite number of invariants, $a_1, \dots, a_d$ when A is noetherian, i.e. such that $g \cdot a_i = a_i$, $1 \leq i \leq d$. The radical $\mathrm{I}(\mathfrak{a})$ of the ideal $\mathfrak{a} = (a_1 - \alpha_1, \dots, a_d - \alpha_d)$ is the ideal of the the algebraic set containing the closures of all orbits of points with these invariants.

Construct a composition series of $A[G]$-submodules (invariant ideals)

$$0 \subseteq \mathrm{I}(\mathfrak{a}) = \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_{n-1} \subset \mathfrak{a}_n = A.$$

Each $\mathfrak{a}_i/\mathfrak{a}_{i-1},\ 1 \leq i \leq n$ is $A[G]$-simple, so each of the (radical) ideals in this composition series corresponds to the closure of an orbit. We get an $A[G]$ composition series

$$0 \subseteq A/\mathfrak{a}_1 \to A/\mathfrak{a}_2 \to \cdots \to A/\mathfrak{a}_n = A,$$

and the noncommutative fine moduli is

$$\mathcal{O}_H(\mathcal{M}),\ \mathcal{M} = \{A/\mathfrak{a}_i\}_{i=1}^n.$$

In the preliminaries, it is explained how we compute the obstructions by projective resolutions, working in the Yoneda complex rather than in the Hochschild complex. This is directly applied to the situation with $A[G]$-modules, with some simplifying facts.

To compute the rings $\mathcal{O}_H(\mathcal{M})$, we are in need of projective $A[G]$-resolutions of the modules $M_i = A/\mathfrak{a}_i$ above. Because M_i is finitely generated as A-module, we might choose an $A[G]$-free resolution. However, we have an easier way.

Lemma 23. *Assume that $P = A^n$ is an $A[G]$-module. Then P is projective as $A[G]$-module.*

Proof. To prove that P is projective, we will prove that $\operatorname{Hom}_{A[G]}(P, -)$ is exact. This means that in the below diagram, ϕ can be lifted to ξ,

$$\begin{tikzcd}
& P \arrow[d, "\phi"] \arrow[dl, dashed, "\xi"'] & \\
M \arrow[r, "\rho"'] & N \arrow[r] & 0
\end{tikzcd}$$

Because P is A-free, such a ξ is determined by $\xi(e_i) = m_i$, where $\rho(m_i) = \phi(e_i)$. Then ξ is determined as an $A[G]$-module homomorphism by extending it by linearity, i.e., $\xi(\nabla_g(e_i)) = \nabla_g(\xi(e_i))$ for each $g \in G$. Thus the lifting ξ exists, and P is $A[G]$ projective. $\square$

From this lemma, it follows that any A-free resolution of an $A[G]$-module M is also an $A[G]$-projective resolution. The technique is given in the proof of the following.

Corollary 6. *Let M be an $A[G]$-module. Then an A-free resolution can be lifted to an $A[G]$-projective resolution.*

Proof. For each $g \in G$, we can lift the complex due to the following:

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "\nabla_g"'] & A^{n_0} \arrow[l, "\varepsilon"'] \arrow[d, "\nabla_{g,0}"'] & A^{n_1} \arrow[l, "d_0"'] \arrow[d, "\nabla_{g,1}"'] & A^{n_2} \arrow[l, "d_1"'] \arrow[d, "\nabla_{g,2}"'] & \cdots \arrow[l, "d_2"'] \\
0 & M \arrow[l] & A^{n_0} \arrow[l, "\varepsilon"] & A^{n_1} \arrow[l, "d_0"] & A^{n_2} \arrow[l, "d_1"] & \cdots \arrow[l, "d_2"]
\end{tikzcd}$$

The above lemma states that A^{n_i} is projective for all i. □

We end this section with the following conclusion: Let $\mathcal{V} = \{V_i\}$ be the family of orbits. Then $\operatorname{Simp} H(\mathcal{V})$ is a fine moduli for $\mathcal{V}$ in the sense that each orbit is parametrized by $H(\mathcal{V})$, and all relations between the orbits are incorporated in $\operatorname{Simp}(H(\mathcal{V}))$.

When computing the tangent spaces, we frequently use the following:

Lemma 24. *Let $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq A$ be two G-invariant ideals. Then*

$$\operatorname{Ext}^1_{A-G}(A/\mathfrak{q}_1, A/\mathfrak{q}_2) \simeq \operatorname{Hom}_A(\mathfrak{q}_1/\mathfrak{q}_1^2, A/\mathfrak{q}_2)^G, \text{ and}$$

$$\operatorname{Hom}_A(\mathfrak{q}_2/\mathfrak{q}_2^2)^G \twoheadrightarrow \operatorname{Ext}^1_{A_G}(A/\mathfrak{q}_2, A/\mathfrak{q}_1)^G,$$

where $g \cdot \phi = \nabla_g \circ \phi \circ \nabla_{g^{-1}}$.

Proof. First of all,

$$\phi \in \operatorname{Hom}_{A-G}(M, N) \Rightarrow \phi(\nabla_g(m)) = \nabla_g(\phi(m)) \Rightarrow \nabla_{g^{-1}}(\phi(\nabla_g(m))) = \phi(m),$$

so $g \cdot \phi = \nabla_g \circ \phi \circ \nabla_{g^{-1}}$ with the left-to-right notation. The rest of the lemma follows by considering the diagram

$$\begin{tikzcd}
0 & A/\mathfrak{q}_i \arrow[l] & A \arrow[l] \arrow[dr, dashed] & A^{n_0} \arrow[l, "d_0"'] \arrow[d] & A^{n_1} \arrow[l, "d_1"'] \arrow[dl, dashed] & \ldots \arrow[l] \\
 & & & A/\mathfrak{q}_j & &
\end{tikzcd}$$

and using the definition of $\operatorname{Ext}^1_A(A/\mathfrak{q}_i, A/\mathfrak{q}_j)$. □

Chapter 6

Applications of noncommutative GIT

We will give some examples of application of the general theory. The detailed computations can be found in the book [3]. We start with maybe the most central example, $\mathrm{End}_k(V)/\mathsf{GL}(V)$ for which we give the general set-up and refer to the book for the computations for $\dim_k V = 2, 3$. After this, we give an example of a toric variety V defined as its quotient $V/(\mathbb{C}^*)^4$ which we prove to be commutative. Finally we give an example of moduli of $n - \mathrm{Lie}_n$ algebras. The computation of the moduli of 3-dimensional Lie-algebras can be found in [10], and an example with the Kleinian quotient singularity can be found in [11].

6.1 $\mathsf{GL}(n)$-Quotients of $\mathrm{End}_k(k^n)$

Let $G = \mathsf{GL}_k(n)$ act on $M_2(k)$ by conjugacy. This means that G acts on $\mathbb{A}^{n^2}$, and our goal is to classify the orbits, with all possible relations. Let $A = k[\underline{x}] = k[x_{ij}]_{1 \leq i,j \leq n}$. Then for $g \in G$ we have $\nabla_g(x_{ij}) = g(x_{ij})g^{-1}$.

It is a general well known fact in this case that the invariants are the coefficients $s_1, \ldots, s_n$ of the characteristic polynomial $\det((x_{ij}) - \lambda\,\mathrm{id})$, but for other applications, we will also sketch the possible computation of the invariant ring using linear algebra.

Our main observation, is that $\mathsf{GL}(n)$ is generated by the elementary matrices, where any of them comes from applying the corresponding operation on the identity matrix id:

$E_i(c)$: Multiply row i by c.

$E_{ij}(c)$: Add c times row i to row j.

E_{ij} : Exchange row i and row j.

Thus we can write a polynomial as a sum of its homogeneous parts, and find the invariants in each degree under the action of the above three degree-preserving actions. Notice that this indicates that we could have worked with graded algebras, i.e. in the projective setting.

There is no reason for going out of order n. We know that $(s_1, \dots, s_n)$ is the (radical) ideal for the closure of the orbit of the Jordan block with 0 on the diagonal and 1's above. We construct a composition series

$$A/(s_1, \dots, s_n) \to A/\mathfrak{a}_{n-1} \to \cdots \to A/(x_{ij}) \simeq k,$$

where the ideals $\mathfrak{a}_i$ are G-invariant, not necessarily generated by invariant elements. Thus these are our $A[G]$-modules under consideration. Put $V_i = A/\mathfrak{a}_{n+1-i}$. We recall Lemma 24, and can prove:

Lemma 25. *For $1 \leq i \leq n$ we have that*

$$\dim_k \operatorname{Ext}^1_{A-G}(V_1, V_i) = n,$$

also

$$\dim_k \operatorname{Ext}^1_{A-G}(V_n, V_n) = 1.$$

Proof. At first, for $\mathfrak{a}_1 = (s_1, \dots, s_n)$ we consider the action of G on ϕ in the diagram

$$\mathfrak{a}_1 \stackrel{\nabla_g}{\to} \mathfrak{a}_1 \stackrel{\phi}{\to} A/\mathfrak{a}_i \stackrel{\nabla_{g^{-1}}}{\to} A/\mathfrak{a}_i.$$

It then follows that the invariance of ϕ forces it to be $\phi = (\alpha_1, \dots, \alpha_n)$ because it has to map to invariant elements. The only such invariant elements are the $s_1, \dots, s_n$ which are contained in $\mathfrak{a}_i$. Next, from the diagram with $\mathfrak{a}_n = (x_{ij}) = (x_{11}, x_{12}, \cdots, x_{nn})$, we have

$$\mathfrak{a}_n \stackrel{\nabla_g}{\to} \mathfrak{a}_n \stackrel{\phi}{\to} A/\mathfrak{a}_n \stackrel{\nabla_{g^{-1}}}{\to} k = A/\mathfrak{a}_n = k.$$

As ∇_g permutes the elements on the diagonal, and add off the diagonal, it follows that an invariant ϕ must be $\phi = \alpha \cdot \operatorname{id}$. $\square$

Remark 4. *There is a certain indication that in general, the dimension matrix is*

$$D = \begin{pmatrix} n & n & \cdots & n \\ 0 & n-1 & \cdots & n-1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

This is probably relatively hard to prove, but should be accomplished when the interest is high enough.

The setup for $M_3(k)$

A close to complete computation is represented in the book [3]. Here we will only define the orbit closures, to illustrate the complexity of the problem.

We have $A = k[x_{ij}]_{1\le i,j\le 3}$ and $G = \mathsf{GL}_3(k)$ acts by conjugation, defining the similarity classes of matrices $M = (x_{ij})$. As we have already stated, the invariants are the coefficients s_1, s_2, s_3 of the characteristic polynomial:

$$\begin{vmatrix} x_{11}-\lambda & x_{12} & x_{13} \\ x_{21} & x_{22}-\lambda & x_{13} \\ x_{31} & x_{32} & x_{33}-\lambda \end{vmatrix} = -\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3,$$

with $s_1 = \operatorname{tr}(x_{ij})$, $s_2 = -(s_{11}+s_{22}+s_{33})$, $s_{ij} = |C_{ij}(x_{ij})|$, $s_3 = \det(x_{ij})$. This says that $\mathfrak{a}_1 = (s_1, s_2, s_3)$, $V_1 = A/\mathfrak{a}_1$ is the biggest orbit closure in $\mathbb{A}^9$. In fact, we know that this contains the orbits of

$$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}.$$

Now, $\mathfrak{a}_1 = (s_1, s_2, s_3) \subseteq (s_{ij}) = \mathfrak{a}_2$. This is because all $s_{ij} = 0$ implies that the rank of the matrix is $\operatorname{rk} M \le 1$. Finally $(s_{ij}) \subseteq (x_{ij})$, which is the 0-matrix, obviously the smallest invariant, closed set (orbit). So the computation in the book [3] yields

$$A = k[x_{ij}]_{1\le i,j\le 3},\ G = \mathsf{GL}_3(k),\ V_1 = A/(s_1,s_2,s_3),\ V_2 = A/(s_{ij}),\ V_3 = A/(x_{ij}).$$

A fine moduli scheme should contain all relations between these invariants, and there is a lot of relations.

The fine moduli $M_2(k)/\mathsf{GL}_2(k)$

This example is computed completely in the book [3]. With the notation as in the 3×3-example above, we recall the set-up and the result of the computation.

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}],\ G = \mathsf{GL}_2(k).$$

We have $\mathfrak{a}_1 = (x_{11}x_{22} - x_{12}x_{21}, x_{11}+x_{22}) = (s_2, s_1)$, $\mathfrak{a}_2 = \mathfrak{m} = (x_{11}, x_{12}, x_{21}, x_{22})$, $V_1 = A/\mathfrak{a}_1$, $V_2 = A/\mathfrak{a}_2$.

We compute the tangent spaces as follows:

$\operatorname{Ext}^1_{A-G}(V_1, V_1)$:

$$(s_1, s_1) \overset{\nabla_{g^{-1}}}{\to} (s_1, s_2) \overset{\phi}{\to} A/(s_1, s_2) \overset{\nabla_g}{\to} A/(s_1, s_2)$$

from which we see that the invariant ϕ are given by $\phi = \alpha(1,0) + \beta(0,1)$.

Similar computations let us choose bases for the tangent space, and we actually find the dimensions given by

$$\begin{pmatrix} \operatorname{Ext}^1_{A-G}(V_1,V_1) & \operatorname{Ext}^1_{A-G}(V_1,V_2) \\ \operatorname{Ext}^1_{A-G}(V_2,V_1) & \operatorname{Ext}^1_{A-G}(V_2,V_2) \end{pmatrix} = \begin{pmatrix} 2&2\\0&1 \end{pmatrix}$$

We use Singular [9] to choose a free resolution of each of the modules V_1, V_2. With the chosen bases for the tangent spaces, we choose a representation of each basis element in the Yoneda complex. Then we follow the computation in the algorithm given in the proof of 1, and we end up with the final result, the k-algebra A given in section 4.9:

$$A = \begin{pmatrix} k[t_{11}(1), t_{11}(2)] & \langle t_{12}(1), t_{12}(2) \rangle \\ 0 & k[t_{22}] \end{pmatrix} / (f_{12}),$$

$f_{12} = t_{11}(1)t_{12}(2) - t_{11}(2)t_{12}(1) - 2t_{12}(2)t_{22} + t_{12}(1)t_{22}^2$. This is a fine moduli for the orbits:

The simple A-modules are the points on the diagonal: Points (s_1, s_2) on $k[t_{11}(1), t_{11}(2)]$ corresponds to the matrices M with corresponding eigenvalues given by $\mathrm{tr}(M) = s_1, \det(M) = s_2$ which is

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ when } \lambda_1 \neq \lambda_2, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \text{ when } \lambda_1 = \lambda_2 = \lambda.$$

Points λ on $k[t_{22}]$ corresponds to the diagonal matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Consider a point in $k[t_{11}(1), t_{11}(2)]$ and compute the tangent dimensions. These are of dimension 2 exactly when

$$t_{12}(2)(t_{11}(1) - 2t_{22}) - t_{12}(1)(t_{11}(2) - t_{22}^2) = 0,$$

that is when

$$\mathrm{tr}(M) = 2t_{22},\ \det(M) = t_{22}^2.$$

Along the curve $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ the orbit is embedded by $\mathrm{tr}(M)$ and $\det(M)$ and the higher order relations are given by the structure theorem 1.

6.2 Toric varieties

In this section, we consider the quotient of a torus action. The simplest case is the quotient of $\mathbb{A}^2/\mathbb{C}^*$, and this works perfectly well in the commutative scheme theory as $\mathbb{A}^2/\mathbb{C}^* \simeq \mathbb{P}^1_{\mathbb{C}} \coprod \{0\}$. The only uncertainty is about the relations between the closed orbit $\{0\}$ and the generic orbits, but it turns out to be none. This is not surpricing because of the ordinary semistability: $\mathbb{P}^1$ is compact, so that each generic orbit contains 0, and then the scheme-theoretic universal quotient exists in the commutative situation (this is in fact a tautology).

To make the example a bit more interesting, we consider toric varieties. Because it is needed in the computation, we recall their definition.

Definition 31. *A* toric variety *is an irreducible variety X such that*

(i) $(\mathbb{C}^*)^n$ is a Zariski open subset of X,

(ii) The action of $(\mathbb{C}^*)^n$ *on itself extends to an action of* $(\mathbb{C}^*)^n$ *on* X*.*

Let $N \simeq \mathbb{Z}^n$ be a lattice, let $\sigma = \text{Cone}(S)$, $S \subseteq N$ finite, be a rational, strongly convex, polyhedral cone. The dual cone is $\sigma^\vee = \{u \in \mathbb{R}^n | \langle u, v\rangle \geq 0 \text{ for all } v \in S\} \supset M = N^\vee = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

Proposition 5 (Gordan's Lemma.)**.** *If* $\sigma \in N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ *is a rational polyhedral cone, then*

$$S_\sigma = \sigma^\vee \cap M$$

is a finitely generated subgroup.

The affine toric variety associated to σ is

$$V_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$$

and we notice that this is an irreducible variety as S_σ is certainly an integral domain. Put $S_\sigma = \langle \underline{m}_1, \ldots, \underline{m}_r\rangle$, $\underline{m}_i \in \mathbb{Z}^n$, then $\mathbb{C}[S_\sigma] \simeq \mathbb{C}[\underline{t}^{\underline{m}_1}, \ldots, \underline{t}^{\underline{m}_r}]$ and the homomorphism $\mathbb{C}[x_1, \ldots, x_r] \twoheadrightarrow [S_\sigma]$ sending x_i to $\underline{t}^{\underline{m}_i}$ implies that

$$\mathbb{C}[S_\sigma] \simeq \mathbb{C}[x_1, \ldots, x_r]/I(V),$$

V being a variety that is naturally toric: Because σ is strongly convex, $\dim \sigma > 0$, and then $\dim \sigma^\vee = \dim \sigma = n > 0$. Choose $\underline{m}_0 \in \text{int}(\sigma^\vee) \cap M$. Then for all $\underline{m} \in M$, $\underline{m} + l\underline{M}_0 \in \sigma^\vee$ for some l, and so

$$\frac{\underline{t}^{\underline{m}+l\underline{m}_0}}{\underline{t}^{l\underline{m}_0}} \in \mathbb{C}[S_\sigma]_{\underline{t}^{\underline{m}_0}} = \mathbb{C}[M] = \mathbb{C}[\underline{t}^{\mathbb{Z}^n}].$$

This gives us a composition of morphisms

$$\mathbb{C}[x_1, \ldots, x_r] \twoheadrightarrow \mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\sigma]_{\underline{t}^{\underline{m}_0}} \simeq \mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

which on the level of varieties is

$$\mathbb{C}^r \subseteq V(I) \subseteq (\mathbb{X}^*)^n.$$

Considering points in these affine varieties, these inclusions are given for $(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$, by

$$(t_1, \ldots, t_n) \mapsto (\underline{t}^{\underline{m}_1}, \ldots, \underline{t}^{\underline{m}_r}).$$

Because this is the affine composition given by the factorization above, this morphism is injective, $(\mathbb{C}^*)^n \hookrightarrow (\mathbb{C}^*)^r$. The action of $(\mathbb{C}^*)^n$ on itself extends to an action on $(\mathbb{C}^*)^r$ by

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_r) = (\underline{t}^{\underline{m}_1} x_1, \ldots, \underline{t}^{\underline{m}_r} x_r),$$

and as $(\mathbb{C}^*)^n$ is contained densely in V_σ which is closed in $(\mathbb{C}^*)^r$, V_σ is closed and so is a toric variety.

Notice now that we have constructed an affine toric variety corresponding to a cone. A general toric variety can be constructed by gluing a fan of cones, and any toric variety is constructed from its corresponding fan of cones. For any toric variety X, we have the possibility to construct the noncommutative quotient $X/(\mathbb{C}^*)^n$.

A toric example

Remark 5. *We would like to give an example as easy as possible, without being trivial: Let* $\sigma = \mathrm{Cone}\{e_1, e_1+e_2\}$. *Then its dual cone is* $\mathrm{Cone}\{e_1-e_2, e_2\}$, $\sigma^\vee \cap \mathbb{Z}^2 = \{(1,-1),(0,1)\}$. *We have that* $\underline{t}^{(1,-1)} \cdot \underline{t}^{(0,1)} = \underline{t}^{(1,0)}$ *so that this looks free. From this we understand that the best we can do is an example of a cone in* $\mathbb{R}^3$.

We let $\sigma = \mathrm{Cone}\{e_1, e_2, e_1+e_3, e_2+e_3\} \subset \mathbb{R}^3$, giving

$$\sigma^\vee = \mathrm{Cone}\{(1,0,0),(0,1,0),(0,0,1),(1,1,-1)\}.$$

Then we have $\mathbb{C}[x,y,z,w] \to \mathbb{C}[S_\sigma]$ with the relation $\underline{m}_1 + \underline{m}_2 = \underline{m}_3 + \underline{m}_4$ implying that

$$\mathbb{C}[S_\sigma] \simeq \mathbb{C}[x,y,z,w]/(xy-zw) \to \mathbb{C}[\mathbb{Z}^3].$$

On coordinates, the action $(\mathbb{C}^*)^3 \times V(xy-zw) \to V(xy-zw)$ is given by

$$(c_1,c_2,c_3)\cdot(a,b,c,d) = (c_1a, c_2b, c_3c, \frac{c_1c_2}{c_3}d),$$

that is

$$(\mathbb{C}^*)^3 \times \operatorname{Spec} A \to \operatorname{Spec} A,\ A = k[x,y,z,w]/(xy-zw),$$

is given by

$$(c_1,c_2,c_3)\cdot(x,y,z,w) = (c_1x, c_2y, c_3z, \frac{c_1c_2}{c_3}w).$$

When classifying orbits, both in the commutative and the noncommutative setting, we start by computing the ring of invariants. In this situation, it turns out exactly as in the trivial situation $\mathbb{A}^2/\mathbb{C}^*$ that $A^G = \mathbb{C}$.

In this particular case, there are several orbit closures:

$$o(e_1) = Z(y,z,w), ..., o(e_4) = Z(x,y,z)$$

and so on. All orbits contains the closed orbit 0, and using lemma 24, we easily see that $\mathrm{Ext}^1_A(o(\underline{x}), \{0\}) = 0$ as only the zero morphism can be invariant. Also, in all examples computed, it is easy to prove the following conjecture, which we take as a definition:

Definition 32. *Let* A *be a commutative* k*-algebra with* G*-action. Assume that for all pairs of* G*-invariant ideals* $\mathfrak{a} \subsetneq \mathfrak{b}$ *in* A,

$$\mathrm{Ext}^1_{A-G}(A/\mathfrak{b}, A/\mathfrak{a}) = 0.$$

Then A *is called strongly geometric.*

At this point, we conjecture that affine toric varieties are strongly geometric, and together with the computations above, this says that quotients by tori exist and is a commutative, non-affine scheme.

In [10] the noncommutative moduli of 3-dimensional Lie-algebras is computed in all details. We choose to introduce the generalization of this work in the next section, where our main result is just the existence of a noncommutative moduli space which can be computed following the recipe from the computation of 3-dimensional Lie-algebras in [10].

6.3 n-Lie algebras

We assume that all all our vector spaces are over algebraically closed fields k of characteristic 0.

Recall that an m-dimensional Lie-algebra $\mathfrak{g}$ over k is given as an m-dimensional vector space together with a bilinear product, called the bracket, $[\ ,\ ] : \mathfrak{g}^2 \to \mathfrak{g}$ satisfying, for all $x, y, z \in \mathfrak{g}$,

(i) Skew symmetry: $[x, y] = -[y, x]$

(ii) The Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Using the first property, we can write the second property as

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$
$$\Updownarrow$$
$$\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)].$$

This illustrates the relation between the Jacoby identity and the Leibniz rule, and we use this for the n-ary generalization.

Definition 33. *A k-linear endomorphism $f : L \to L$ of a k-algebra L with an n-ary product $[\ , \dots,\ ]$ is called a* derivation *if*

$$[f(x_1), x_2, \dots, x_n] + [x_1, f(x_2), \dots, x_n] + \cdots + [x_1, x_2, \dots, f(x_n)] = f([x_1, \dots, x_n])$$

for all n-tuples $\underline{x} \in L^n$.

Definition 34. *An $n-\mathit{Lie}_m$-algebra, is a k-vector space $\mathfrak{g}$ of dimension $\dim \mathfrak{g} = m$ together with an n-ary product $[\ , \dots,\ ] : \mathfrak{g}^n \to \mathfrak{g}$ satisfying*

(i) $[x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}] = (-1)^\sigma [x_1, x_2, \dots, x_n]$ *for all $\sigma \in S_n$*

(ii) $\mathrm{ad}_{(y_1, \dots, y_{n-1})} = [-, y_1, y_2, \dots, y_{n-1}] : \mathfrak{g} \to \mathfrak{g}$ *is a derivation for all $(n-1)$-tuples $\underline{y} = (y_1, \dots, y_{n-1})$.*

So, in particular, we notice that a $2 - \mathrm{Lie}_m$-algebra is nothing but an m-dimensional Lie-algebra.

The first condition in the lemma states that the bracket defines a linear homomorphism

$$C : \wedge^n \mathfrak{g} \to \mathfrak{g},$$

whereas the second condition states that C satisfy the Jacobian identity. Because of the k-linearity, the Jacobian is satisfied if it is satisfied for the elements in any basis for $\mathfrak{g}$.

Now, we fix a basis $\{e_1, \dots, e_m\}$ for $\mathfrak{g}$. We have that $\dim_k(\wedge^n \mathfrak{g}) = \binom{m}{n} = r$, and we make the convention that the basis for $\wedge^n \mathfrak{g}$ given by $\{\hat{e}_1, \dots, \hat{e}_r\}$ is enumerated by removing tuples by higher to lower degree (lexicographic ordering), more easily explained by the following:

Example 17. *Assume that* $\{e_1, e_2, e_3, e_4\}$ *is a basis for* $\mathfrak{g}$*, and let us write* $e_{ijkl} = e_i \wedge e_j \wedge e_k \wedge e_l \in \wedge^4 \mathfrak{g}$*. We then let the basis for* $\mathfrak{g}$ *be* $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ *where* e_1 *is given by removing* e_4*, that is* $\hat{e}_1 = e_{123}$*,* e_2 *is given by removing* e_3*, that is* $\hat{e}_2 = e_{124}$ *and so on until* $\hat{e}_4 = e_{234}$*.*

To give an $n-$Lie$_m$-algebra is equivalent to choosing a basis for $\mathfrak{g}$ and to give a linear map $C : \wedge^n \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity. With the convention on the basis of $\wedge^n \mathfrak{g}$, the linear map is determined by its $m \times \binom{m}{n}$ coefficient matrix $C = (c_{ij})$.

Lemma 26. *Let* $\mathfrak{g}$ *be an* $n-$ *Lie*$_m$*-algebra and let* C *be its coefficient matrix with respect to a basis* $\{e_i\}_{i=1}^n$*. Then there is a set* $\{C_1, \dots, C_{\binom{m}{n}}\}$ *of column vectors (of size* $m \times 1$*) such that the coefficient matrix* C *satisfies the Jacobi identity if and only if* $C \cdot C_1 = \cdots = C \cdot C_{\binom{m}{n}} = 0$*.*

Proof. The Jacobian gives exactly the correct number of equations which can be factorized as a matrix product as above. □

Definition 35. *A morphism* ϕ *in the category* $n - \mathsf{Lie}_m$ *of* $n-$ *Lie*$_m$*-algebras is a* k*-linear map respecting the bracket.*

It follows automatically that an isomorphism is a morphism with a two-sided inverse, and as the bracket is k-linear, the Jacobi-identity is satisfied if it holds for the elements in any basis for $\mathfrak{g}$.

Lemma 27. *To give an isomorphism of* $n-$ *Lie*$_m$ *algebras, is equivalent to give a base change* $E : \mathfrak{g} \to \mathfrak{g}$*. This induces a commutative diagram*

$$\begin{tikzcd}
\wedge^n \mathfrak{g} \arrow[r, "C"] & \mathfrak{g} \arrow[d, "E"] \\
\wedge^n \mathfrak{g} \arrow[u, "\tilde{E}^{-1} = \wedge^n E^{-1}"] \arrow[r, "E C \tilde{E}^{-1}"'] & \mathfrak{g}
\end{tikzcd}$$

Or main task in all such settings, is to classify, geometrically, the objects up to isomorphism. So in this particular case, we are classifying $\binom{n}{m} \times m$-matrices satisfying the Jacobi identity. We have an action of the linear group $\mathsf{GL}(n)$ given by

$$\nabla_E(C) = E C \tilde{E}^{-1},$$

and the problem is equivalent to finding an orbit space under the group action. This is treated in section 5.

In our particular case, and in all other cases where $\mathsf{GL}(n)$ acts by composition, the rank is invariant for all orbits. Thus we can classify all orbits of given rank (of the coefficient matrix C), find their closures by rank and the Jacobi relations, and then give them the geometry given by $\mathsf{GL}(n)$-equivariant representations of the polynomial algebra $A = k[x_{ij}]$, $1 \le i, j \le \binom{n}{m}$.

Theorem 7. *There exist a noncommutative fine moduli for* $n-$ *Lie*$_m$*-algebras over* k *for any* $m, n \in \mathbb{N}$*.*

Proof. Because the category $n - \mathsf{Lie}_m$ is equivalent to the category $\mathbf{mod}_{A[\mathsf{GL}(n)]}$, it follows from the general construction in Section 5 that a noncommuative quotient exists. This is what is a noncommutative fine moduli. □

6.4 The structure of $3 - \mathsf{Lie}_4$

The implementation of the generalities in the previous sections is perfect for computers. However, when making the computations on a computer, it is often useful to have a hand-computable example to refer to. The example with 3-dimensional Lie algebras is treated in [10], and now we generalize slightly to the next step. As is usual with matrices, a small increase in size makes big differences in computational complexity.

We are in the situation where we classify linear mappings $C : \wedge^3\mathfrak{g} \to \mathfrak{g}$ with respect to the basis $\{e_1, \dots, e_4\} \subset \mathfrak{g}$ for $\mathfrak{g}$ and the corresponding basis $\{e_{123}, e_{124}, e_{134}, e_{234}\} \subseteq \wedge^3\mathfrak{g}$ for $\wedge^3\mathfrak{g}$. The linear group $\mathsf{GL}(3)$ is generated by its elementary matrices, and we recall the following notation:

- E_{ij}: Interchange rows i and j
- $E_{ij}(c)$: Add c times row i to row j
- $E_i(c)$: Multiply row i by c.

Also notice that

$$E_{ij}^{-1} = E_{ij},\ E_{ij}(c)^{-1} = E_{ij}(-c),\ \text{and}\ E_i(c)^{-1} = E_i(\frac{1}{c})$$

so that the inverse of an elementary matrix is itself elementary.

Each linear transformation $E : \mathfrak{g} \to \mathfrak{g}$ induces a linear transformation

$$\tilde{E} : \wedge^3\mathfrak{g} \to \wedge^3\mathfrak{g},$$

and as the orbits under the action of $\mathsf{GL}(3)$ is given by $\tilde{L}^{-1}CL,\ L \in \mathsf{GL}(3)$ we write up (results of computation) the explicit matrix for each of the elementary transformations above:

$$\tilde{E}_{12} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = E_1(-1)E_2(-1)E_{34}, \qquad \tilde{E}_{13} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -E_{24},$$

$$\tilde{E}_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = E_2(-1)E_3(-1)E_{14}, \quad \tilde{E}_{23} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = E_1(-1)E_4(-1)E_{23},$$

$$\tilde{E}_{24} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -E_{13}, \qquad \tilde{E}_{34} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = E_3(-1)E_4(-1)E_{12},$$

$$\tilde{E}_{12}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & 1 \end{pmatrix} = E_{34}(c), \qquad \tilde{E}_{21}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{43}(c),$$

$$\tilde{E}_{13}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -c & 0 & 1 \end{pmatrix} = E_{24}(-c), \qquad \tilde{E}_{31}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{42}(-c),$$

$$\tilde{E}_{14}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix} = E_{14}(c), \qquad \tilde{E}_{41}(c) = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{41}(c),$$

$$\tilde{E}_{23}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{23}(c), \qquad \tilde{E}_{32}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{32}(c),$$

$$\tilde{E}_{24}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{13}(-c), \qquad \tilde{E}_{42}(c) = \begin{pmatrix} 1 & 0 & -c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{31}(-c),$$

$$\tilde{E}_{34}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{12}(c), \qquad \tilde{E}_{43}(c) = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{21}(c),$$

$$\tilde{E}_{1}(c) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_1(c)E_2(c)E_3(c), \quad \tilde{E}_{2}(c) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c \end{pmatrix} = E_1(c)E_2(c)E_4(c),$$

$$\tilde{E}_{3}(c) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} E_1(c)E_3(c)E_4(c), \quad \tilde{E}_{4}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} = E_2(c)E_3(c)E_4(c).$$

Remark 6. *Notice that the listing of elementary operations makes it possible to define several normal forms. One can easily question oneself if there is a general theory of* normal forms, *but that is of course not the case. Or more to say, that is the core of the theory of moduli.*

Our next task is to compute the Jacobian variety: Let

$$C : \wedge^3\mathfrak{g} \to \mathfrak{g}$$

be the coefficient matrix of a fixed basis for a $3 - \mathrm{Lie}_4$-algebra $\mathfrak{g}$. This matrix is a closed point C in $\mathrm{M}\,(3 \times 3) \simeq \mathbb{A}^9$ satisfying the Jacobian identity, i.e., $C \in J \subseteq \mathbb{A}^9$. Here we call J the Jacobian variety.

The coefficient matrix is given by a choice of basis for $\mathfrak{g}$, and the Jacobi-identity is satisfied by linearity if and only if it is satisfied for any selection of 5 elements from $\{e_1, e_2, e_3, e_4\}$. For a $3 - \mathrm{Lie}_4$-algebra the Jacobi identity states that $[-, y_1, y_2]$ is a derivation, for all $y_1, y_2 \in \mathfrak{g}$, which says that for for all $x_1, x_2, x_3, y_1, y_2 \in \mathfrak{g}$ we have

$$[[x_1, x_2, x_3], y_1, y_2] = [[x_1, y_1, y_2], x_2, x_3] + [x_1, [x_2, y_1, y_2], x_3] + [x_1, x_2, [x_1, y_1, y_2]].$$

From this expression we see that if there are two pairs of equal elements, one of the brackets must contain two equal elements (by the pigeon-hole principle), and so identity is automatically satisfied. Because of this, and because the order is taken care of in the wedge product, when we check the Jacobi identity for any selection of 4 out of 3 basis elements, it is sufficient to check it for 2 e_1's, 2 e_2's, 2 e_3's, and 2 e_4's: Totally this gives four equations that can be summarized as follows:

$$C\cdot\begin{pmatrix} C_{22}+C_{33} & C_{34}-C_{12} & -C_{13}-C_{24} & 0 \\ C_{43}-C_{21} & C_{11}+C_{44} & 0 & -C_{13}-C_{24} \\ -C_{31}-C_{42} & 0 & C_{11}+C_{44} & C_{12}-C_{34} \\ 0 & -C_{31}-C_{42} & C_{21}-C_{43} & C_{22}+C_{33} \end{pmatrix} = C\cdot L = 0 \quad (6.1)$$

Lemma 28. *$3-Lie_4$ has two irreducible components, that is*

$$3-Lie_4 = (L_1\cap\cdots\cap L_4)\cup(|C|\cap C\cdot L_1\cap\cdots\cap C\cdot L_4) = J_1\cup J_2$$

We notice that it is obvious that the components are irreducible sub-varieties of $\mathrm{M}(3\times 3)$. Also notice that $J_1 \not\subseteq J_2$.

This description makes it possible to make a rank classification of $3-\mathrm{Lie}_4$. This means that we are classifying the orbits up to rank.

6.5 The orbits of rank 1

Looking at the equation (6.1), we see that the matrix L doesn't contain

$$C_{14},\ C_{23},\ C_{32},\ C_{41}.$$

Thus putting these equal to one gives 4 matrices of rank 1:

$$R1_{14}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ R1_{23}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$
$$R1_{32}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\ R1_{41}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}.$$

Applying the elementary operations explicitly, we find that all the $3-\mathrm{Lie}_4$-algebras R_{ij}, $i,j>0,\ i+j=5$ (the 1's on the skew diagonal), are all isomorphic. As the elementary operations are a combination of row and column operators, we can prove that the E_{ij}-operations sends R_{kl} to itself or another R_{rs}, (just changing the placement of the single 1 to another single 1 on the skew diagonal. Using a sequence of adding multiples of rows to others, i.e., using the $E_{ij}(c)$ we find e.g.

$$R1_{23}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\simeq\begin{pmatrix}0&0&0&0\\c_1&c_2&1&c_3\\0&0&0&0\\0&0&0&0\end{pmatrix}\simeq\begin{pmatrix}0&0&d_1&0\\0&0&1&0\\0&0&d_2&0\\0&0&d_3&0\end{pmatrix}$$

with c_i, d_i arbitrary k-constants, and where the generalization to the other $R1_{ij}$ on the skew diagonal is clear.

The search for a different orbit of rank 1 gives that we can put a 1 in any place off the skew diagonal, and get a $3-\mathrm{Lie}_4$-algebra: $R1_{ij}$ is a $3-\mathrm{Lie}_4$-algebra for each R_{ij} off the skew diagonal. After some small considerations (which in fact is applying the elemenary operations to $R1_{13}$), we suspect that the rank 1 algebras are covered by two different orbit-closures, i.e. $\mathrm{cl}\, o(R1_{13}) \neq \mathrm{cl}\, o(R1_{14})$.

Now we end this example, and leave the computations for the interested reader. We remark that a similar technique can be applied for the classification of finite dimensional associative algebras.

Chapter 7

Pre-Dynamic GIT

7.1 Generalities

In general, given the old, familiar polynomial ring $A = k[x_1, \ldots, x_d]$. What are the variables $x_1, \ldots, x_d$ really? More precise, what do they represent?

We are used to think that they represent a point in d-space, and all polynomials $f \in A$ can be evaluated in the point by $f(x_1, \ldots, x_d)$.

Doing deformation theory as in the preliminaries of these notes, we eventually understand that this is (at best) a bit unprecise. The $x_1, \ldots, x_d$ represents the tangents directions, the invariants called directions for which a polynomial can be differentiated. Of course, if someone choose an origin, the tangent directions gives the position of the point by units in d-space, and the evaluation of the polynomial in that point.

With the above more or less unprecise introduction in mind, when a linear (symmetry-) group G acts on a scheme X, we intuitively get an action on the tangent space of X, so this induces a dynamical action. This is a zero order dynamical action, and not all dynamical actions comes from such a group action. A system of 1'th order partial differential equations on the differential structure is an action of a Lie-algebra $\mathfrak{g}$ on the tangent sheaf θ_X on X, and the orbits under this action are the integral curves, i.e. the solution spaces, of this system. In this section, we consider general actions of Lie algebras on tangent sheaves θ_X, and we generalize this action to general dynamical actions on the phase space $\mathrm{Ph}(X)$ which is defined as the invariant scheme of all systems. In this text we restrict our attention to the affine situation with $\theta_X = \mathrm{Der}_k(A)$ and $\mathrm{Ph}(X) = \mathrm{Ph}(A)$ for $X = \operatorname{Simp} A$ with A an associative k-algebra.

As before, let A be an associative k-algebra, k an algebraically closed field of characteristic 0. We recall that a set of modules $\mathcal{M} = \{M_1, \ldots, M_r\}$ is called a scheme for A if $O_A(\mathcal{M}) = A$. The O_A-construction is a closure operation, i.e. $\mathcal{O}_{\mathcal{O}_A(\mathcal{M})}(\mathcal{M})$ implying that $\mathcal{O}_A(\mathcal{M})$ is a coordinate ring for $\mathcal{M}$, or equivalently an invariant-ring. Notice that under the trivial group action, the coordinates are the set of invariants. In section 5 we considered the action of a linear group

$G \subseteq \mathsf{GL}(n)$ on an affine space $\operatorname{Simp} A$ and we constructed its orbit space. An action of G on a set of modules $\mathcal{M}$ is a group homomorphism

$$G \to \operatorname{Aut}_k(A)$$

where we interpret the action of G on a point M as the composition

$$G \to \operatorname{Aut}_k(A) \to \operatorname{Hom}_k(A, \operatorname{End}_k(M)).$$

This gives the set of orbits $\mathcal{M}/G$ a structure of $\mathcal{A}[G]$-modules, and

$$\mathcal{M}/G \text{ is a scheme for } O_{A[G]}(\mathcal{M}).$$

Such a group action is a symmetry on the set of modules. It also induces a relation on the generators of the invariant ring $A[G]$, and as these generators can be seen as tangent direction, the relation can be considered to be a dynamical action. This is made precise by the following.

Lemma 29. *An action of a group G on* $\operatorname{Rep}(A)$*,*

$$\nabla : G \to \operatorname{Aut}_k(A)$$

induces an action of a Lie algebra

$$\operatorname{Lie}(G) \to \operatorname{Der}_k(A)$$

Proof. This result is a rewriting of the corresponding well known proof of the existence of the functor Lie for a Lie group G. In this case, $\operatorname{Lie}(G)$ is the set of translation invariant vector fields on the Lie group G, and we translate this definition directly into the algebraic version. On a variety $\operatorname{Simp} A$, the tangent bundle is the sheaf $\operatorname{Der}_k(A)$ which to each point M gives the tangent space by the composition

$$\operatorname{Der}_k(A) \overset{\rho_M}{\to} \operatorname{Der}_k(A, \operatorname{End}_k(M)).$$

A vector field is a global section of the sheaf $\operatorname{Der}_k(A)$, i.e. an element

$$v \in \operatorname{Der}_k(A),$$

and a translation of v is $\phi_g(v)$ where

$$\phi_g = \operatorname{Der}_k(\nabla(g)) : \operatorname{Der}_k(A) \to \operatorname{Der}_k(A).$$

We have that $v \in \operatorname{Der}_k(A)$ is translation invariant under the action of G, if for all $g, h \in G$,

$$\phi_g(\phi_h(v)) = \phi_{gh}(v).$$

So now we let $\operatorname{Lie}(G)$ be the subset of translation invariant derivations in $\operatorname{Der}_k(A)$.

$$\operatorname{Lie}(G) = \{v \in \operatorname{Der}_k(A) | \phi_g(\phi_h(v)) = \phi_{gh}(v) \text{ for all } g, h \in G\}.$$

We prove that the bracket $[v,w]$ of two translation invariant derivations v,w are translation invariant. This follows because the composition $v \circ w$ is translation invariant:

$$g(h(v \circ w)(a) = v(w(g(ha))) = v(w((gh)a) = v \circ w((gh)a)$$

because w is translation invariant. By symmetry, $w \circ v$ is translation invariant because v is invariant, and because the sum of translation invariant linear morphisms are translation invariant, the bracket $[v,w]$ is translation invariant. □

Corollary 7.

$$\mathrm{Lie}(\mathrm{Aut}_k(A)) \simeq \mathrm{Der}_k(A)$$

Proof. Let $\phi, \psi \in \mathrm{Aut}_k(A)$. Then for any derivation $\delta \in \mathrm{Der}_k(A)$,

$$\delta(\psi(\phi(a))) = \delta((\psi\phi)(a)).$$

□

The dynamical action defined by a symmetry group as above, is really the zero order dynamics. The dynamics that are forced, i.e., given by change does not always come from a linear group.

Definition 36. *A first order dynamical action on a scheme $\mathcal{M}$ of A-modules is a Lie algebra homomorphism*

$$\mathfrak{g} \to \mathrm{Der}_k(A).$$

Definition 37. *Let A be a fixed k-algebra. We define the category $\mathbf{Alg}_{A/k}$ where the objects are k-algebra homomorphisms $\phi_R : A \to R$ with R an associative k-algebra. A morphism between (R, ϕ_R) and $(R', \phi_{R'})$ is a commutative diagram*

$$\begin{tikzcd} R \arrow[rr, "\psi"] & & R' \\ & A \arrow[ul, "\phi_R"] \arrow[ur, "\phi_{R'}"'] & \end{tikzcd}$$

The following can be found in [3].

Lemma 30. *The functor*

$$\mathrm{Der}_k(A, -) : \mathbf{Alg}_{A/k} \to \mathbf{Mod}_k$$

is represented by a couple $(\mathrm{Ph}(A), d)$.

Writing up what this means, it says that the natural transformation

$$\mathrm{Mor}(\mathrm{Ph}(A), -) \to \mathrm{Der}_k(A, -)$$

given by sending $\psi \in \mathrm{Mor}(\mathrm{Ph}(A), R)$ to $\mathrm{Der}_k(A, \psi)(d) = d \circ \psi$, is an isomorphism. Thus there exists an associative A-algebra $\mathrm{Ph}(A)$ together with a k-derivation

$$d : A \to \mathrm{Ph}(A),$$

i.e. $\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "d"'] & \mathrm{Ph}(A) \\ \mathrm{Ph}(A) & \end{tikzcd}$ such that for any associative A-algebra with a k-derivation $\delta : A \to R$, i.e. $\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\delta"'] & R \\ R & \end{tikzcd}$ there is a unique $\phi : \mathrm{Ph}(A) \to R$ such that $\delta = d \circ \phi$. Notice that this construction is functorial, introducing the *Phase space functor*

$$\mathrm{Ph} : \mathbf{Alg}_{A/k} \to \mathbf{Alg}_{A/k}.$$

Also notice that we have identifications

$$d_* : \mathrm{Der}_k(A) \to \mathrm{Mor}_A(\mathrm{Ph}(A), A) \text{ and } d^* : \mathrm{Der}_k(A, \mathrm{Ph}(A)) \to \mathrm{End}_A(\mathrm{Ph}(A))$$

induced by the universal property.

Definition 38. *Let M be an A-module given by the structure morphism*

$$\rho_M : A \to \mathrm{End}_k(M).$$

The natural k-linear morphism

$$\kappa_M : \mathrm{Der}_k(A) \to \mathrm{Der}_k(A, \mathrm{End}_k(M))/\,\mathrm{Inner} \simeq \mathrm{Ext}^1_A(M, M)$$

given by $\delta \mapsto \delta \circ \rho_M$ is called the Kodaira-Spencer morphism.

To understand the meaning of the Kodaira-Spencer morphism: Consider a dynamical action $\mathfrak{g} \to \mathrm{Der}_k(A)$ on the set of A-modules. Let $f \in A$, and let $\partial \in \mathfrak{g}$ be a derivation. Then $\kappa(\partial) = 0 \Leftrightarrow \partial f(M) = 0$ for all points M. Do not get confused: The "functions" on $\mathrm{Simp}\, A$ are the representations M, and A is the coordinate ring. Thus the set of M such that $\kappa_M(\partial) = 0$ is the integral curve of the system of differential equations $\kappa(\mathfrak{g}) = 0$, and the paralell transports are given by the inner derivations. This explains why we call this a dynamical action, and we will construct higher order dynamical actions by a functorial DeRahm cohomology.

To make this precise, we are in the need of clarifying the parallel transport, i.e. the connections.

We have a sequence

$$A \overset{\rho_M}{\to} \mathrm{End}_k(M) \overset{\mathrm{Hom}(M, M\otimes d)}{\to} \mathrm{Hom}_k(M, M \otimes_A \mathrm{Ph}(A))$$

where we check that the composition μ_M is a k-derivation, that is,

$$\mu_M \in \mathrm{Der}_k(A, \mathrm{Hom}_k(M, M \otimes_A \mathrm{Ph}(A))).$$

Definition 39. *The Kodaira-Spencer class of M is the class $c(M)$ of μ_M,*

$$c(M) = \overline{\mu_M} \in \mathrm{Ext}^1_A(M, M \otimes_A \mathrm{Ph}(A))$$

Lemma 31. *There is a k-linear morphism*

$$r : \mathrm{Der}_k(A) \to \mathrm{Hom}_k(\mathrm{Ext}^1_A(M, M \otimes_A \mathrm{Ph}(A)), \mathrm{Ext}^1_A(M, M))$$

and

$$r(\xi)(\mu_M) = \kappa_M(\xi).$$

Lemma 32. *For an A-module M, $\mathrm{End}_k(M)$ is an object of $\mathbf{Alg}_{A/k}$ and*

$$\mathrm{ad} : A \to \mathrm{End}_k(M)$$

is a k-derivation.

Proof. The structure morphism of M,

$$\rho_M : A \to \mathrm{End}_k(M)$$

is an A-algebra homomorphism and proves that $\mathrm{End}_k(M)$ is an object of $\mathbf{Alg}_{A/k}$. We find that for all $a, b \in A$, $m \in M$

$$\begin{aligned} \mathrm{ad}(ab)(m) &= [ab, m] = abm - mab \\ &= abm - amb + amb - mab \\ &= a(bm - mb) + (am - ma)b \\ &= a[b, m] + [a, m]b \\ &= a\,\mathrm{ad}(b)(m) + \mathrm{ad}(a)(m)b \end{aligned}$$

so that

$$\mathrm{ad}(ab) = a\,\mathrm{ad}(b) + \mathrm{ad}(a)b$$

which proves that $\mathrm{ad} : A \to \mathrm{End}_k(M)$ is a k-derivation. $\square$

Proposition 6. *For every A-module M there is a unique A-algebra homomorphism*

$$i_M : \mathrm{Ph}(A) \to \mathrm{End}_k(M)$$

such that $i_M \delta = \mathrm{ad}$:

$$\begin{tikzcd} \mathrm{Ph}(A) \arrow[rr, "i_M"] & & \mathrm{End}_k(M) \\ & A \arrow[ul, "\delta"] \arrow[ur, "\mathrm{ad}"'] & \end{tikzcd}$$

Now we combine these things:

Theorem 8. *Let $\mathfrak{g}$ be a Lie algebra action (a first order dynamical action)*

$$\mathfrak{g} \to \mathrm{Der}_k(A).$$

Then there is an ideal $\delta_{\mathfrak{g}} \subseteq \mathrm{Ph}(A)$ such that

$$\delta_{\mathfrak{g}} \subseteq \ker i_M \Leftrightarrow \kappa_M(\xi) = 0 \textit{ for all } \xi \in \mathfrak{g}$$

Proof. Given the action

$$\mathfrak{g} \xrightarrow{l} \mathrm{Der}_k(A, \mathrm{Ph}(A)) \to \mathrm{Ext}^1_A(M, M),$$

we let $\delta_{\mathfrak{g}}$ be the two-sided ideal generated by the elements $\mathrm{im}(l(g))$ for $g \in \mathfrak{g}$, that is

$$\delta_{\mathfrak{g}} = \langle \{\mathrm{im}(l(g)) | g \in G\} \rangle.$$

Following the sequence for the Kodaira-Spencer morphism

$$\mathfrak{g} \to \mathrm{Der}_k(A) \to \mathrm{Der}_k(A, \mathrm{Ph}(A)) \to \mathrm{Ext}^1_A(M, M)$$

this sends the a derivation $\delta = l(g)$ to the composition

$$A \xrightarrow{\delta} A \xrightarrow{i} \mathrm{Ph}(A) \xrightarrow{\rho_M} \mathrm{End}_k(M),$$

and

$$\delta_{\mathfrak{g}} \subseteq \ker(\rho_M)$$

□

From this point on, and in the rest of the subsections in this section, we give some results needed for the development of the theory of dynamics.

Let now V be a right A-module, with structure morphism

$$\rho(V) = \rho : A \to \mathrm{End}_k(V).$$

We obtain a universal derivation

$$u(V) = u : A \longrightarrow \mathrm{Hom}_k(V, V \otimes_A \mathrm{Ph}(A))$$

defined by $u(a)(v) = v \otimes d(a)$. Let U and V be right A-modules. Then we have the long exact sequences of Hochschild cohomology,

$$\begin{aligned} 0 \to \mathrm{Hom}_A(U, V) &\to \mathrm{Hom}_k(U, V) \\ &\xrightarrow{\iota} \mathrm{Der}_k(A, Hom_k(U, V)) \xrightarrow{\kappa} \mathrm{Ext}^1_A(U, V) \to 0. \end{aligned}$$

Substituting $U := V$ and $V := V \otimes_A \mathrm{Ph}(A)$ we get

$$\begin{aligned} 0 \to \mathrm{Hom}_A(V, V \otimes_A \mathrm{Ph}(A)) &\to \mathrm{Hom}_k(V, V \otimes_A \mathrm{Ph}(A)) \\ &\xrightarrow{\iota} \mathrm{Der}_k(A, \mathrm{Hom}_k(V, V \otimes_A \mathrm{Ph}(A))) \xrightarrow{\kappa} \mathrm{Ext}^1_A(V, V \otimes_A \mathrm{Ph}(A)) \to 0. \end{aligned}$$

By this, we obtain the non-commutative Kodaira-Spencer class

$$c(V) := \kappa(u(V)) \in \mathrm{Ext}^1_A(V, V \otimes_A \mathrm{Ph}(A))$$

inducing the Kodaira-Spencer morphism

$$g : \Theta_A := \mathrm{Der}_k(A, A) \longrightarrow \mathrm{Ext}^1_A(V, V)$$

via the identity d_*.

If $c(V) = 0$, then the exact sequence above proves that there exist a

$$\nabla \in \mathrm{Hom}_k(V, V \otimes_A \mathrm{Ph}(A))$$

such that $u = \iota(\nabla)$. This is a way of proving that $c(V)$ is the obstruction for the existence of a connection

$$\nabla : \mathrm{Der}_k(A, A) \longrightarrow \mathrm{Hom}_k(V, V)$$

The same exact sequences furnish a proof for the following.

Lemma 33. *Let $\rho : A \to \mathrm{End}_k(V)$ be an A-module and let $\delta \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$ map to* 0 *in $\mathrm{Ext}^1_A(V, V)$, i.e. assume $\kappa(\delta) = 0$. Then there exist an element $Q_\delta \in \mathrm{End}_k(V)$ such that for all $a \in A$*

$$\delta(a) = [Q_\delta, \tilde{\rho}(a)].$$

If V is a simple A-module, $\mathrm{ad}(Q_\delta)$ *is unique.*

Definition 40. *Let $\rho : A \to \mathrm{End}_k(V)$ be a representation. We define the Lie sub-algebra $\mathfrak{g}_V \subset \mathrm{Der}_k(A)$ as*

$$\mathfrak{g}_V := \mathfrak{g}_\rho = \{\gamma \in \mathrm{Der}_k(A) |\ g(\gamma) = \kappa(\delta\rho) = 0\}.$$

This says that there is always a connection,

$$\nabla : \mathfrak{g}_V \to \mathrm{End}_k(V).$$

These simple results will be important in relation to invariant theory, gauge groups in physics, and quantisations, covered in the next Sections.

Any $\mathrm{Ph}(A)$-module W given by its structure map

$$\rho(W)^1 =: \rho^1 : \mathrm{Ph}(A) \longrightarrow \mathrm{End}_k(W)$$

corresponds bijectively to an induced A-module structure $\rho_0 : A \to \mathrm{End}_k(W)$ together with a derivation $\delta_\rho \in \mathrm{Der}_k(A, \mathrm{End}_k(W))$ defining an element

$$[\delta_\rho] \in \mathrm{Ext}^1_A(W, W).$$

Fixing this last element we find that the set of $\mathrm{Ph}(A)$-module structures on the A-module W is in one to one correspondence with

$$\mathrm{End}_k(W)/\,\mathrm{End}_A(W).$$

Conversely, starting with an A-module V and an element $\delta \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$, we obtain a $\mathrm{Ph}(A)$-module V_δ. It is then easy to see that the kernel of the natural map

$$\mathrm{Ext}^1_{\mathrm{Ph}(A)}(V_\delta, V_\delta) \to \mathrm{Ext}^1_A(V, V)$$

induced by the linear map

$$\mathrm{Der}_k(\mathrm{Ph}(A), \mathrm{End}_k(V_\delta)) \to \mathrm{Der}_k(A, \mathrm{End}_k(V))$$

is the quotient

$$\mathrm{Der}_A(\mathrm{Ph}(A), \mathrm{End}_k(V_\delta)) / \mathrm{End}_k(V).$$

The image is a subspace $[\delta_\rho]^\perp \subseteq \mathrm{Ext}^1_A(V, V))$, which is rather easy to compute, see examples below.

Example 18. *Let $A = k[t]$. Obviously, $\mathrm{Ph}(A) = k\langle t, dt\rangle$ and d is given by $d(t) = dt$, such that for $f \in k[t]$, we find $d(f) = J_t(f)$ where $J_t(f)$ denotes the noncommutative derivation of f with respect to t. One should also compare this with the noncommutative Taylor formula from [3]. If $V \simeq k^2$ is an A-module defined by the matrix $X \in \mathrm{M}_2(k)$, and $\delta \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$ is defined in terms of the matrix $Y \in \mathrm{M}_2(k)$, then the $\mathrm{Ph}(A)$-module V_δ is the $k\langle t, dt\rangle$-module defined by the action of the two matrices $X, Y \in \mathrm{M}_2(k)$, and we find*

$$\begin{aligned} e^1_V &:= \dim_k \mathrm{Ext}^1_A(V, V) = \dim_k \mathrm{End}_A(V) = \dim_k\{Z \in \mathrm{M}_2(k) | \ [X, Z] = 0\} \\ e^1_{V_\delta} &:= \dim_k \mathrm{Ext}^1_{\mathrm{Ph}(A)}(V_\delta, V_\delta) = 8 - 4 + \dim\{Z \in \mathrm{M}_2(k) | \ [X, Z] = [Y, Z] = 0\} \end{aligned}$$

We have the following inequalities:

$$2 \leq e^1_V \leq 4 \leq e^1_{V_\delta} \leq 8.$$

Example 19. *Let $A = k[t_1, t_2]$, then we find*

$$\mathrm{Ph}(A) = k\langle t_1, t_2, dt_1, dt_2\rangle / ([t_1, t_2], [dt_1, t_2] + [t_1, dt_2])$$

In particular, we have a surjective homomorphism

$$\mathrm{Ph}(A) \to k\langle t_1, t_2, dt_1, dt_2\rangle / ([t_1, t_2], [dt_1, dt_2], [t_i, dt_i] - 1),$$

the right hand algebra being the Weyl algebra. This homomorphism exists in all dimensions. We also have a surjective homomorphism

$$\mathrm{Ph}(A) \to k[t_1, t_2, \xi_1, \xi_2]$$

i.e. onto the affine algebra of the classical phase-space.

Remark 7. *Since* $\mathrm{Ext}^1_A(V,V)$ *is the tangent space of the miniversal deformation space of* V *as an* A*-module, we see that the non-commutative space* $\mathrm{Ph}(A)$ *also parametrizes the set of* generalized momenta, *i.e. the set of pairs of an* A*-module* V *and a tangent vector of the formal moduli of* V *at that point.*

$\mathrm{Ph}(A)$ is relatively easy to compute. In particular, if $A = k[x_1, .., x_n]$ is the polynomial algebra, we have

$$\mathrm{Ph}(A) = k\langle x_1, .., x_n, dx_1, .., dx_n\rangle/([x_i, x_j], [x_i, dx_j] + [dx_i, x_j]).$$

Notice that any rank 1 representation of $\mathrm{Ph}(A)$ is represented by a pair (q,p) of a closed point $q \in \mathrm{Spec}(k[\underline{x}])$ and a tangent p at that point. We will be in need of the following formulas.

Theorem 9. *Given two points* $(q_i, p_i), i = 1, 2$ *we find*

$$\dim_k \mathrm{Ext}^1_{\mathrm{Ph}(A)}(k(q_1, p_1), k(q_2, p_2)) = 2n, \ \textit{for} \ \ (q_1, p_1) = (q_2, p_2)$$
$$\dim_k \mathrm{Ext}^1_{\mathrm{Ph}(A)}(k(q_1, p_1), k(q_2, p_2)) = n, \ \textit{for} \ \ q_1 = q_2, , p_1 \neq p_2$$
$$\dim_k \mathrm{Ext}^1_{\mathrm{Ph}(A)}(k(q_1, p_1), k(q_2, p_2)) = 1, \ \textit{for} \ \ q_1 \neq q_2.$$

Moreover, there is a generator of,

$$\mathrm{Ext}^1_{\mathrm{Ph}(A)}(k(q_1, p_1), k(q_2, p_2)) = \mathrm{Der}_k(\mathrm{Ph}(A), \mathrm{Hom}_k(k(q_1, p_1)), k(q_2, p_2)))/\,\mathrm{Triv}$$

uniquely characterized by the tangent line defined by the vector $\overline{q_1 q_2}$.

Proof. Assume for convenience that $n = 3$. Put $x_j(q_i, p_i) = q_{i,j},\ dx_j((q_i, p_i) = p_{i,j},\ \alpha_j = q_{1,j} - q_{2,j},\ \beta_j = p_{1,j} - p_{2,j}$.

We see that for any element $\alpha \in \mathrm{Hom}_k(k(q_1, p_1), k(q_2, p_2))$ we have

$$x_j\alpha = q_{1,j}\alpha,\ \alpha x_j = q_{2,j}\alpha,\ dx_j\alpha = p_{1,j}\alpha,\ \alpha dx_j = p_{2,j}\alpha,$$

with the obvious identification. Any derivation

$$\delta \in \mathrm{Der}_k(\mathrm{Ph}(A), \mathrm{Hom}_k(k(q_1, p_1), k(q_2, p_2)))$$

must satisfy the relations

$$\delta([x_i, x_j]) = [\delta(x_i), x_j] + [x_i, \delta(x_j)] = 0$$
$$\delta([dx_i, x_j] + [x_i, dx_j]) = [\delta(dx_i), x_j] + [dx_i, \delta(x_j)] + [\delta(x_i), dx_j] + [x_i, \delta(dx_j)] = 0.$$

Using the above left-right action-rules, the result follows from the long exact sequence computing $\mathrm{Ext}^1_{\mathrm{Ph}(A)}$. The two families of relations above give us two systems of linear equations.

The first in the variables $\delta(x_1), \delta(x_2), \delta(x_3)$ with matrix

$$\begin{pmatrix} -\alpha_2 & \alpha_1 & 0 \\ -\alpha_3 & 0 & \alpha_1 \\ 0 & -\alpha_3 & \alpha_2 \end{pmatrix},$$

and the second in the variables $\delta(x_1), \delta(x_2), \delta(x_3), \delta(dx_1), \delta(dx_2), \delta(dx_3)$ with matrix

$$\begin{pmatrix} -\beta_2 & \beta_1 & 0 & -\alpha_2 & \alpha_1 & 0 \\ -\beta_3 & 0 & \beta_1 & -\alpha_3 & 0 & \alpha_1 \\ 0 & -\beta_3 & \beta_2 & 0 & -\alpha_3 & \alpha_2 \end{pmatrix}.$$

In particular we see that the *trivial* derivation given by

$$\delta(x_i) = \alpha_i,\ \delta(dx_j) = \beta_j,$$

satisfies the relations, and the generator of $\mathrm{Ext}^1_{\mathrm{Ph}\,A}(k(q_1, p_1), k(q_2, p_2))$ is represented by

$$\delta(x_i) = 0,\ \delta(dx_j) = \alpha_j.$$

This is, in an obvious sense, the "tangent vector" $-\overline{q_1, q_2}$

It is easy to extend this result from dimension 3 to any dimension n. $\square$

Notice that this result shows that the "Space" of all rank 1 representations of $\mathrm{Ph}(A)$ is the classical phase space of $\mathrm{Spec}\,A$ of dimension $2n$, but endowed with an extra structure. Between two different points, corresponding to either one point in $\mathrm{Spec}\,A$ and two different tangents, or to two different points in $\mathrm{Spec}\,A$, there is respectively a subspace of dimension n and of dimension 1 of "Ext-tangents". This will be important for the concept of "Ether".

7.2 Blowing Up and Desingularization

The A-algebra $\mathrm{Ph}(A)$ is graded by defining for $a \in A$

$$\deg(a) = 0,\ \deg(d(a)) = 1.$$

By definition, any $\mathrm{Ph}(A)$-representation $\rho : \mathrm{Ph}(A) \to \mathrm{End}_k(V)$, corresponds to a representation $\rho_0 : A \to \mathrm{End}_k(V)$ together with a derivation of A into $\mathrm{End}_k(V)$, which again induces a tangent direction in the moduli space of representations of A at the point ρ. In complete generality we have a map that we shall call *The General Blowing Up Map*

$$\mathfrak{bu} : \mathrm{Simph}_A(\mathrm{Ph}(A)) \to \mathrm{Simp}\,A,$$

where $\mathrm{Simph}_A(\mathrm{Ph}(A))$ is the set of simple graded $\mathrm{Ph}(A)$-modules and the mapping is onto the 0'th. component.

The corresponding morphism in the commutative case is

$$\mathfrak{bu} : \mathrm{Proj}_A(\mathrm{Ph}(A)) \to \mathrm{Spec}\,A,$$

the *Blowing Up Map.* By the universal property of $\mathrm{Ph}(A)$ it is clear that the fibre of $\mathfrak{bu}$ at a k-point $x \in \mathrm{Spec}\,A$ is $\mathrm{Proj}(T(x)) \simeq \mathbb{P}^{n-1}$, where $T(x)$ is the tangent space of Spec A at the point x, supposed to be of embedding dimension n.

Therefore any vector field ξ, on Spec A, i.e. any derivation $\xi \in \mathrm{Der}_k(A, A)$, defines a canonical section of $\mathfrak{bu}$

$$\sigma(\xi) : D(\xi) \to \mathrm{Proj}_A(\mathrm{Ph}(A))$$

defined in the open subscheme $D(\xi)$, where ξ is non-trivial. The blow-up of Spec A defined by ξ is now the closure $\mathrm{Spec}(A, \xi)$ of the image of $\sigma(\xi)$. Thus,

$$D(\xi) \subset \mathrm{Spec}(A, \xi) \subset \mathrm{Proj}_A(\mathrm{Ph}(A)).$$

Blowing up the origin in the affine n-space, would then correspond to the blowing up of $\mathrm{Spec}(k[x_1, ..., x_n])$ defined by the derivation $\xi = \sum x_i \frac{\partial}{\partial x_i}$.

In the commutative general case, if $A = k[x_1, ..., x_n]/(r_1, ..., r_s)$, consider the Jacobian matrix

$$J = (\frac{\partial r_i}{\partial x_j})$$

and let J_α be a maximal sub-determinant such that $J_\alpha \neq 0$ in an open subset $U = \mathrm{Spec}(A) - \mathrm{Sing}(A)$ supposed to be non-empty. Compute the solutions of the linear system of equations

$$\sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j} dx_j = 0,\ i = 1, ..., s.$$

We find solutions of the form

$$dx_l = \sum_{i}^{d} c_i^l / J_\alpha dx_i,\ l = d+1, ..., n.$$

The derivations of A of the form

$$\xi_i := J_\alpha \frac{\partial}{\partial x_i} - \sum_{l=d+1}^{n} c_i^l \frac{\partial}{\partial x_l}$$

are all non-trivial in U. The corresponding blow-ups of A looks like

$$A(\xi_i) = \mathrm{Ph}(A)/(\langle J_\alpha dx_l = \sum_{i}^{d} c_i^l dx_i,\ l = d+1, ..., n\rangle).$$

This gives us a possible easier road to de-singularization since the Ph-operation is canonical and may be iterated.

All this can easily be generalized to perform much more complex blowing ups of noncommutative affine algebraic schemes, defined by the associative k-algebra A.

Denote, generally, $\mathfrak{m}_A = \ker \pi$ where $\pi : \mathrm{Ph}(A) \to A$ corresponds to the trivial derivation of A. Obviously, $\mathfrak{m}_A$ is the ideal of $\mathrm{Ph}(A)$ generated by the elements $\{d(a), a \in A\}$.

Consider now a twosided ideal $\mathfrak{a} \subset A$, and a twosided subideal $\mathfrak{b} \subset \mathfrak{m}$. The twosided ideal $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a}(\mathfrak{m}_A) + (\mathfrak{b}) \subset \mathrm{Ph}(A)$ defines a subscheme

$$\mathfrak{bl}(\mathfrak{a}, \mathfrak{b}) \subset \mathrm{Simph}_A \, \mathrm{Ph}(A))$$

which is our general *blow up* of a noncommutative algebraic Scheme.

The blowing up of a closed subscheme $Y = \mathrm{Spec}(B) \subset X = \mathrm{Spec}\, A$ is gotten by considering the ideal $\mathfrak{a} = \ker \pi$, $\pi : A \to B$, and the twosided subideal $\mathfrak{b} = \ker\{\mathrm{Ph}(\pi) : \mathfrak{m}_A \to \mathfrak{m}_B \subset \mathfrak{m}_A\}$. The fibre over C of this map is the projectivization of the normal bundle $N_B(A)$.

Now, the blowing down of a subscheme $C = \mathrm{Spec}(B) \subset X = \mathrm{Spec}\, A$ is gotten by considering, together with the the points of $X - C$ as simple representations $\rho_x : A \to k(x)$, also the representation $\rho_0 : A \to B$, in the way we construct the general noncommutative algebraic geometry.

7.3 Chern Classes

It is (probably) well known that in the commutative case the Kodaira-Spencer class gives rise to the Chern characters. In this general case, we shall prepare for a result mimicking the Chern-Simons classes. Let us assume given a representation

$\rho_0 : A \to \mathrm{End}_k(V)$ and a momentum at ρ_0, i.e. an extension $\rho_1 : \mathrm{Ph}(A) \to \mathrm{End}_k(V)$ of ρ_0. Consider now the class in $\mathrm{ch}^n(\rho_1) \in \mathrm{HH}^n(A, \mathrm{End}_k(V))$ defined by the following Hochschild cochain, the k-linear map

$$\mathrm{ch}^n : A^{\otimes n} \to \mathrm{End}_k(V)$$

defined by

$$\mathrm{ch}^n(a_1 \otimes a_2 ..., \otimes a_n) = \rho_1(da_1 da_2 ... da_n) \in \mathrm{End}_k(V).$$

It is easy to see that ch^n is a cocycle because

$$\delta(\mathrm{ch}^n)((a_1 \otimes a_2 ... \otimes a_{n+1})) = a_1 \rho_1((da_2 da_3 ... da_{n+1})) \tag{7.1}$$

$$+ \sum_1^n (-1) \rho_1((da_1 ... d(a_i a_{i+1}) ... da_{n+1})) + (-1)^{n+1} \rho_1((da_1 ... da_n) a_{n+1} = 0 \tag{7.2}$$

One may define the *Generalized Chern-Simons Class* of ρ_1 as the class

$$\mathrm{ch}^n(\rho_1) \in \mathrm{HH}^n(A, \mathrm{End}_k(V))$$

defined by $1/n!\ \mathrm{ch}^n$.

7.4 The iterated Phase Space functor Ph^* and the Dirac derivation

The phase-space construction may be iterated. Given the k-algebra A we may form the sequence $\{\mathrm{Ph}^n(A)\}_{0\leq n}$ defined inductively by

$$\mathrm{Ph}^0(A) = A,\ \mathrm{Ph}^1(A) = \mathrm{Ph}(A), ..., \mathrm{Ph}^{n+1}(A) = \mathrm{Ph}(\mathrm{Ph}^n(A)).$$

Let $i_0^n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+1}(A)$ be the canonical embedding and let $d_n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $\mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+2}(A)$ corresponding to the homomorphism

$$\mathrm{Ph}^n(A) \overset{i_0^n}{\to} \mathrm{Ph}^{n+1}(A) \overset{i_0^{n+1}}{\to} \mathrm{Ph}^{n+2}(A)$$

there exist by universality a homomorphism $i_1^{n+1} : \mathrm{Ph}^{n+1}(A) \to \mathrm{Ph}^{n+2}(A)$ such that

$$i_0^n \circ i_1^{n+1} = i_0^n \circ i_0^{n+1}$$

and such that

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

Notice that here we compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms,

$$\{i_j^n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+1}(A)\}_{0\leq j\leq n},$$

such that

$$i_p^n \circ i_0^{n+1} = i_0^n \circ i_{p+1}^{n+1}$$

with the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

We find the following identities,

$$\begin{aligned}
i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1},\ p < q \\
i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\
i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1},\ q < p.
\end{aligned}$$

To see this, compose with i_0^{n-1} and d_{n-1}, and use induction. Thus, the $\mathrm{Ph}^*(A) = \oplus \mathrm{Ph}^n(A)$ is a semi-cosimplicial k-algebra with a cosection h_0, onto A. And it is easy to see that h_0 together with the corresponding cosections

$h_p : \mathrm{Ph}^{p+1}(A) \to \mathrm{Ph}^p(A)$ for $\mathrm{Ph}^p(A)$ replacing A form a trivializing homotopy for $\mathrm{Ph}^*(A)$. Thus we have

$$\mathrm{H}^n(\mathrm{Ph}^*(A)) = 0, \forall n \geq 0,$$

i.e. Ph^{*+1} is a cosimplicial resolution of A. Therefore, for any object

$$\kappa : A \to R \in A/k - \mathbf{alg}$$

the cosimplicial algebra above induces simplicial sets

$$\mathrm{Mor}_k(\mathrm{Ph}^\star(A), R),\ \mathrm{Mor}_A(\mathrm{Ph}^\star(A), R),$$

and one should be interested in the homotopy. See also that this generalises to a canonical functor

$$\mathrm{Spec} : (k - \mathbf{alg}^\Delta)^{op} \longrightarrow \mathbf{SPr}(k)$$

where $(k - \mathbf{alg})^\Delta$ is the category of co-simplicial k-algebras, and $\mathbf{SPr}(k)$ is the category of simplicial presheves on the category of k-schemes enriched by any Grothendieck topology. As usual, the embedding of the category of k-algebras in the category of cosimplicial algebras is defined simply by giving any k-algebra a constant cosimplicial structure. The fact that $\mathrm{Ph}^\star(A)$ is a resolution of A is therefore simply saying that

$$\mathrm{Spec}(\mathrm{Ph}^\star((A)) \to \mathrm{Spec}\, A,$$

is a week equivalence in $\mathbf{SPr}(k)$.

This might be a starting point for a theory of homotopy for k-schemes. We may also consider, for any k-algebra R, the simplicial k-vectorspace

$$\mathrm{Der}_k(\mathrm{Ph}^\star(A), R)$$

Consider this complex for $R = A$, i.e. $\mathrm{Mor}_A(Ph^\star(A), A)$. Clearly

$$\mathrm{Mor}_A(\mathrm{Ph}^{n+1}(A), A) = \mathrm{Der}_k(\mathrm{Ph}^n(A), A), n \geq 0,$$

and we have

$$\mathrm{Mor}_A(\mathrm{Ph}^n(A), A) = \{\xi_0 \circ \xi_{i_1} \circ ... \circ \xi_{i_r} | 0 \leq i_l \leq i_{l+1} \leq n | \xi_0 = \mathrm{id}_A, \xi_i \in \mathrm{Der}_k(A), i \geq 1\}.$$

Since Ph is a functor, and Ph^{*+1} is a cosimplicial resolution of A, we may apply this to any scheme X given in terms of an affine covering $\mathbf{U}$, and obtain an algebraic homology (or cohomology) with converging spectral sequences

$$E^1_{pq} = H_p({H_{\mathbf{U}}}^{-q}(\mathrm{Der}_k(\mathrm{Ph}^\star(A), A))), E_{q,p} = H^{-q}_{\mathbf{U}}(H_p(\mathrm{Der}_k(\mathrm{Ph}^\star(A), A)))$$

If we in $\mathrm{Mor}_A(\mathrm{Ph}^n(A), A)$ identify $\xi \sim \alpha\xi, \alpha \in k^*$, we obtain a rational cohomology with converging spectral sequences

$$E_1^{pq} = H^p(H^q_{\mathbf{U}}(\mathrm{Mor}_A(\mathrm{Ph}^n(A), A), \mathbf{Q})), E_2^{q,p} = H^q_{\mathbf{U}}(H^p(\mathrm{Mor}_A(\mathrm{Ph}^n(A), A), \mathbb{Q}))$$

Remark 8. *The above suggests that we are closing in on Stacks and Motives. Any reasonable cohomology theory defined on the category of k-schemes is now seen to be defined on the image category of* Spec, *so probably extendable to* $\mathbf{SPr}(k)$ *and therefore comes with a homotopy theory attached. Moreover, suppose we instead of the example $R = A$ above, considered the category (actually an ordered set) of morphisms, $\mathfrak{a}(A) := \{A \to A/\mathfrak{p}_i\}$, for some family of irreducible twosided ideals, corresponding in the commutative case to families of subschemes of $X = \operatorname{Spec} A$. By the theorem (4.2.4) of [?] there is for any finite subcategory $\mathfrak{V} \subset \mathfrak{a}(A)$, a formal moduli $H(\mathfrak{V})$ for the deformation functor* $\mathsf{Def}_(\mathfrak{V})$ *of the category of morphisms $\mathfrak{V}$ with the algebra A trivially deformed, provided the corresponding cohomology groups of the deformation theory are countably generated.*

Moreover, we may globalize this to hold for any scheme X and in particular to any projective scheme over k,for which we know that the cohomology groups of the deformation theory will be of finite dimension, implying that the formal moduli $H(\mathfrak{V})$ will be finitely generated formal k-algebras.

Formally the theory will be of the same nature for schemes as for algebras, and so to minimise the place and problems with hanging on to dull dual descriptions, we shall just describe the affine case. Of course in this case the formal moduli $H(\mathfrak{V})$ will not, in general, be finitely generated formal k-algebras, but this should not be of much trouble for most mathematicians.

Obviously, if $\mathfrak{W} \subset \mathfrak{V}$ there is a natural morphism

$$\pi(\mathfrak{V}, \mathfrak{W}) : H(\mathfrak{W}) \to H(\mathfrak{V}),$$

usually just called π or name omitted. Given a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] & \\ R \arrow[rr, "\psi"'] & & R' \end{tikzcd}$$

we have a diagram of canonical morphisms

$$\begin{tikzcd} H(\rho) \arrow[dr] & & H(\rho') \arrow[dl] \\ & H(\psi) & \end{tikzcd}$$

where we have put $H(\psi) := H(\psi : \rho \to \rho')$. And for any diagram like

$$\begin{tikzcd}
& A \arrow[dl, "\rho"'] \arrow[dd, "\rho''"] \arrow[dr, "\rho'"] & \\
R \arrow[dr, "\psi"'] & & R'\psi' \arrow[dl] \\
& R'' &
\end{tikzcd}$$

we find a diagram of canonical morphisms

$$\begin{tikzcd}
H(\psi) & & H(\psi') \\
& H(\rho'') \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

with the same abbreviation as above.

A prime cycle in the motive of A should be any object $(\rho, H(\rho)), \rho \in \mathbf{Irr}(A)$*, the set of which we shall call* $h(A)$*. A* cycle *should then be a linear combination over some abelian group of such prime cycles.*

We should like to define the intersection product *of cycles, as a bilinear product of cycles. If* ρ *and* ρ' *prime cycles, then we define the intersection as the sum*

$$\rho \cap \rho' = \sum \alpha(\rho, \rho')\rho'', \ \alpha(\rho, \rho') := |H(\rho, \psi) \otimes_{H(\rho'')} H(\rho', \psi')|.$$

We would like to compare it to the Serre intersection formula

$$\rho \cap \rho' = \sum_{0}^{\infty} (-1)^i \operatorname{Tor}_i^A(R, R')$$

in the commutative case. But this demands a certain work which will be postponed.

Anyway, the notion of motive over the rationals should be given by the set of finite cycles

$$M(A) = \operatorname{FinMap}(h(A), \mathbb{Q})$$

divided out with some equivalence relation which to be considered in later work.

Consider now the cosimplicial algebra

$$A \overset{i_0^0}{\to} \mathrm{Ph}(A) \overset{i_p^1}{\to} \mathrm{Ph}^2(A) \overset{i_p^2}{\to} \mathrm{Ph}^3(A) \overset{i_p^3}{\to} \cdots$$

where for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $\mathrm{Ph}^n(A)$ and $\mathrm{Ph}^{n+1}(A)$ defined above. The system of k-algebras and homomorphisms of k-algebras $\{\mathrm{Ph}^n(A), i_j^n\}_{n,0\le j\le n}$ has an inductive (direct) limit, defined by,

$$\mathrm{Ph}^\infty(A) = \varinjlim_{n\ge 0}\{\mathrm{Ph}^n(A), i_j^n\}$$

together with homomorphisms

$$i_n : \mathrm{Ph}^n(A) \longrightarrow \mathrm{Ph}^\infty(A)$$

satisfying

$$i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \ldots, n.$$

Moreover, the family of derivations $\{d_n\}_{0\le n}$ define a unique Dirac derivation

$$\delta : \mathrm{Ph}^\infty(A) \longrightarrow \mathrm{Ph}^\infty(A).$$

such that $i_n \circ \delta = d_n \circ i_{n+1}$. Put

$$\mathrm{Ph}^{(n)}(A) := \mathrm{im}\ i_n \subseteq \mathrm{Ph}^\infty(A)$$

The k-algebra $\mathrm{Ph}^\infty(A)$ has a descending filtration of two-sided ideals $\{\mathbf{F}_n\}_{0\le n}$ given inductively by

$$\mathbf{F}_1 = \mathrm{Ph}^\infty(A)\cdot \mathrm{im}(\delta)\cdot \mathrm{Ph}^\infty(A)$$

and

$$\delta\mathbf{F}_n \subseteq \mathbf{F}_{n+1}, \quad \mathbf{F}_{n_1}\mathbf{F}_{n_2}\ldots\mathbf{F}_{n_r} \subseteq \mathbf{F}_n, \quad n_1+\ldots+n_r = n$$

such that the derivation δ induces derivations $\delta_n : \mathbf{F}_n \longrightarrow \mathbf{F}_{n+1}$. Using the canonical homomorphism $i_n : \mathrm{Ph}^n(A) \longrightarrow \mathrm{Ph}^\infty(A)$ we pull the filtration $\{\mathbf{F}_p\}_{0\le p}$ back to $\mathrm{Ph}^n(A)$, obtaining a filtration of each $\mathrm{Ph}^n(A)$ with,

$$\mathbf{F}_1^n = \mathrm{Ph}^n(A)\cdot \mathrm{im}(\delta)\cdot \mathrm{Ph}^n(A)$$

and inductively,

$$\delta\mathbf{F}_p^n \subseteq \mathbf{F}_{p+1}^{n+1}, \quad \mathbf{F}_{p_1}^n\mathbf{F}_{p_2}^n\ldots\mathbf{F}_{p_r}^n \subseteq \mathbf{F}_p^n, \quad p_1+\ldots+p_r = p.$$

Definition 41. *Let* $\mathbf{D}(A) := \varprojlim_{n\ge 1}\mathrm{Ph}^\infty(A)/\mathbf{F}_n$*, the completion of* $\mathrm{Ph}^\infty(A)$ *in the topology given by the filtration* $\{\mathbf{F}_n\}_{0\le n}$*. The* k*-algebra* $\mathrm{Ph}^\infty(A)$ *will be referred to as the* k*-algebra of higher differentials, and* $\mathbf{D}(A)$ *will be called the* k*-algebra of formalized higher differentials. Put*

$$\mathbf{D}_n := \mathbf{D}_n(A) := \mathrm{Ph}^\infty(A)/\mathbf{F}_{n+1}$$

Clearly δ defines a derivation on $\mathbf{D}(A)$, and an isomorphism of k-algebras

$$\epsilon := \exp(\delta) : \mathbf{D}(A) \to \mathbf{D}(A)$$

and in particular an algebra homomorphism

$$\tilde{\eta} := \exp(\delta) : A \to \mathbf{D}(A)$$

inducing the algebra homomorphisms

$$\tilde{\eta}_n : A \to \mathbf{D}_n(A)$$

which, by killing, in the right hand algebra the image of the maximal ideal $\mathfrak{m}(\underline{t})$ of A corresponding to a point $\underline{t} \in \mathrm{Simp}_1(A)$ induces a homomorphism of k-algebras

$$\tilde{\eta}_n(\underline{t}) : A \to \mathbf{D}_n(A)(\underline{t}) := \mathbf{D}_n/(\mathbf{D}_n\mathfrak{m}(\underline{t})\mathbf{D}_n)$$

and an injective homomorphism

$$\tilde{\eta}(\underline{t}) : A \to \varprojlim_{n\geq 1} \mathbf{D}_n(A)(\underline{t})$$

see [**?**].

Remark 9 (Formal curves of representations)**.** *Since* $\mathrm{Ext}^1_A(V,V)$ *is the tangent space of the miniversal deformation space of* V *as an* A*-module, we see that the noncommutative space* $\mathrm{Ph}(A)$ *also parametrizes the set of* generalized momenta*, i.e. the set of pairs of an* A*-module* V *and a tangent vector of the formal moduli of* V *at that point. Therefore the above implies that any representation* $\rho : \mathrm{Ph}^\infty(A) \to \mathrm{End}_k(V)$ *corresponds to a family of* $\mathrm{Ph}^n(A)$*-module structures on* V *for* $n \geq 1$*, i.e. to an* A*-module* $V_0 := V$*, an element* $\xi_0 \in \mathrm{Ext}^1_A(V,V)$*, i.e. a tangent of the deformation functor of* $V_0 = V$ *as* A*-module, an element* $\xi_1 \in \mathrm{Ext}^1_{\mathrm{Ph}(A)}(V,V)$*, i.e. a tangent of the deformation functor of* $V_1 := V$ *as* $\mathrm{Ph}(A)$*-module, an element* $\xi_2 \in \mathrm{Ext}^1_{\mathrm{Ph}^2(A)}(V,V)$*, i.e. a tangent of the deformation functor of* $V_2 := V$ *as* $\mathrm{Ph}^2(A)$*-module, etc.*

All this is just $\rho_0 : A \to \mathrm{End}_k(V)$ *considered as an* A*-module together with a sequence* $\{\xi_n\}, 0 \leq n$*, of a tangent, or a* momentum ξ_0*, an acceleration vector* ξ_1*, and any number of higher order* momenta ξ_n*. Thus, specifying a* $\mathrm{Ph}^\infty(A)$*-representation* V *implies specifying a* formal curve *through* v_0*, the base-point of the* miniversal deformation space *of the* A*-module* V*. Formally, this curve is given by the composition of the homomorphism* $\epsilon(\tau) := \exp(\tau\delta)$ *and* ρ*.*

This is seen as follows. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "i_0^0"] \arrow[d, "\rho_0"'] & \mathrm{Ph}(A) \arrow[r, "i_p^1"] \arrow[dl, "\rho_1"] & \mathrm{Ph}^2(A) \arrow[r, "i_p^2"] & \cdots \arrow[r] & \mathrm{Ph}^\infty(A) \\
\mathrm{End}_k(V) & & & &
\end{tikzcd}$$

where, for each integer n, the symbol i^n_p, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $\mathrm{Ph}^n(A)$ and $\mathrm{Ph}^{n+1}(A)$ defined above. Suppose now that we can extend ρ_0 to a morphism ρ_1 which should be seen as a momentum for the representation ρ_0, and suppose moreover that we can continue finding morphisms $\rho^n_p : \mathrm{Ph}^p(A) \to \mathrm{End}_k(V)$ for $p = 2, ..., \infty$ making the diagram commute and such that,

$$i^{n-1}_p \rho_n = \rho_{n-1}$$

for all $n \geq 2$, and all $n \geq 2$. then it is relatively easy to do the computation and find that the map

$$[\delta] : A \to \mathrm{Ph}^*(A),$$

defined by $[\delta](a) := \sum_{n=0}^{\infty} 1/n! d_n(...(d_0))(a)$, composed with any one of the ρ_p will be an algebra homomorphism.

If we arrange for all the $\rho_p : \mathrm{Ph}^p(A) \to \mathrm{End}_k(V) \otimes k[\tau]/(\tau)^p$ to be (obvious) graded homomorphisms, we have in fact found a formal curve parametrized by τ in the moduli space of Rep(A).

It is, however, impossible to prepare *a physical situation such that a measurement, i.e. an object like ρ_0, is given by an infinite sequence $\{\xi_n\}$, of dynamical data. We shall have to be satisfied with a finite number of data, and normally with just the first one , i.e. the momentum ξ_0, given by ρ_1. This is the problem of* Preparation and of the Time Evolution *of a representation ρ, to be treated in the sequel.*

7.5 The generalized de Rham Complex

Consider now the diagram

$$\begin{tikzcd}
A \arrow[r, "i^0_0"] & \mathrm{Ph}(A) \arrow[r, "i^1_p"] & \mathrm{Ph}^2(A) \arrow[r, "i^2_p"] & \mathrm{Ph}^3(A) \arrow[r, "i^3_p"] & {} \\
 & \mathfrak{m}^1_1 \arrow[u] \arrow[r, "i^1_p"] & \mathfrak{m}^1_2 \arrow[u] \arrow[r, "i^2_p"] & \mathfrak{m}^1_3 \arrow[u] \arrow[r, "i^3_p"] & {}
\end{tikzcd}$$

where for each integer n, the symbol i^n_p, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $\mathrm{Ph}^n(A)$ and $\mathrm{Ph}^{n+1}(A)$ defined above, and where $\mathfrak{m}^1_n$ is the ideal of $\mathrm{Ph}^n(A)$ generated by $\mathrm{im}(d)$ which is the same as the ideal generated by the family $\{i^{n-1}_p(i^{n-2}_p(...(i^1_p(d(A))...))\}$ for all possible p. And, inductively, let $\mathfrak{m}^m_n$ be the ideal generated by $\mathfrak{m}^1_n \mathfrak{m}^{m-1}_n$.

We find an extended diagram

$$\begin{tikzcd}
A \arrow[r, "i_0^0"] \arrow[d] & \mathrm{Ph}(A) \arrow[r, "i_p^1"] \arrow[d] & \mathrm{Ph}^2(A) \arrow[r, "i_p^2"] \arrow[d] & \mathrm{Ph}^3(A) \arrow[r, "i_p^3"] \arrow[d] & ... \\
A \arrow[r, "i_0^0"] \arrow[dr, "d_0"] & A \arrow[r, "i_p^1"] \arrow[dr, "d_1"] & A \arrow[r, "i_p^2"] \arrow[dr, "d_2"] & A \arrow[r, "i_p^3"] \arrow[dr, "d_3"] & ... \\
 & \mathfrak{m}_1^1/\mathfrak{m}_1^2 \arrow[r, "i_p^1"] \arrow[dr, "d_1"] & \mathfrak{m}_2^1/\mathfrak{m}_2^2 \arrow[r, "i_p^2"] \arrow[dr, "d_2"] & \mathfrak{m}_3^1/\mathfrak{m}_3^2 \arrow[r, "i_p^3"] \arrow[dr, "d_3"] & ... \\
 & \mathfrak{m}_1^2/\mathfrak{m}_1^3 \arrow[r, "i_p^1"] \arrow[dr, "d_1"] & \mathfrak{m}_2^2/\mathfrak{m}_2^3 \arrow[r, "i_p^2"] \arrow[dr, "d_2"] & \mathfrak{m}_3^2/\mathfrak{m}_1^3 \arrow[r, "i_p^3"] \arrow[dr, "d_3"] & ... \\
 & & \ddots & \ddots & \ddots
\end{tikzcd}$$

The diagonals are not necessarily complexes, but it suffices to kill d^2, to kill all d^n, $n \geq 2$, and for this it suffices to kill $d_1 d_0$, as one easily see operating with the edge homomorphisms $i_p^n, n \geq 2$ on the elements, $d_1(d_0(a))$ for $a \in A$. Therefore we shall in this general situation make the following definition.

Definition 42. *The curvature $R(A)$ of the associative k algebra A is the k-linear map composition of d_0 and d_1,*

$$R(A) = d_0 d_1 : A \to \mathfrak{m}_2^2/\mathfrak{m}_2^3.$$

Now, kill the curvature $R(A)$ and all the terms under the first diagonal, beginning with $\mathfrak{m}_1^2/\mathfrak{m}_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms and let $\mathbf{\Omega}_n^m$ be the resulting quotient of $\mathfrak{m}_n^m/\mathfrak{m}_n^{m+1}$ for $n \geq 0$. Clearly, $\Omega_n^0 = A$ for all $n \geq 0$, and we have got a graded semi cosimplicial A-module with a k-differential d such that $d^2 = 0$, looking like

$$\begin{tikzcd}
A \arrow[r, "i_0^0"] \arrow[d] & \mathrm{Ph}(A) \arrow[r, "i_p^1"] \arrow[d] & \mathrm{Ph}^2(A) \arrow[r, "i_p^2"] \arrow[d] & Ph^3(A) \arrow[r, "i_p^3"] \arrow[d] & {} \\
A \arrow[r, "i_0^0"] \arrow[dr, "d"] & A \arrow[r, "i_p^1"] \arrow[dr, "d"] & A \arrow[r, "i_p^2"] \arrow[dr, "d"] & A \arrow[r, "i_p^3"] \arrow[dr, "d"] & {} \\
 & \Omega_1^1 \arrow[r, "i_p^1"] \arrow[dr, "d"] & \Omega_2^1 \arrow[r, "i_p^2"] \arrow[dr, "d"] & \Omega_3^1 \arrow[r, "i_p^3"] \arrow[dr, "d"] & {} \\
 & & \Omega_2^2 \arrow[r, "i_p^2"] & \Omega_3^2 \arrow[r, "i_p^3"] & {}
\end{tikzcd}$$

It is a graded complex in two ways. First as a complex induced from the semi-cosimplicial structure with differential of bidegree $(1, 0)$, and second as complex with differential d of bidegree $(1, 1)$.

Lemma 34. *Suppose A is commutative. Then there is a natural morphism of complexes of A-modules*

$$\Omega_A^\star \subset \Omega_\star^\star,$$

with

$$\Omega_A^n := \wedge^r \Omega_A \simeq \Omega_n^n.$$

Proof. Let $a_i \in A, i = 1, ..., r$ and compute in $\Omega_\star^r$ the value of $d^r(a_1 a_2 ... a_r)$. It is clear that this gives the formula

$$\sum d_{i_1}(a_1) d_{i_2}(a_2)..d_{i_r}(a_r) = 0,$$

the sum being over all permutations $(i_1, i_2, ..., i_r)$ of $(0, 1, ..., r-1)$. Here we consider A as a subalgebra of $\mathrm{Ph}^n(A)$ via the unique compositions of the i_0^s : $\mathrm{Ph}^s(A) \subset \mathrm{Ph}^{s+1}(A)$. In particular, we have

$$d_0(a_1)d_1(a_2) + d_1(a_1)d_0(a_2) = 0$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a_2)d_1(a_1) = d_1(a_1)d_0(a_2)$, which follows from commutativity, $d(a_2)a_1 = a_1 d(a_2)$, forces the left and right A-action on Ω_A to be equal. It immediately give us $d_0(a_1)d_1(a_2) = -d_0(a_2)d_1(a_1)$.

Consider now the diagram,

$$\begin{tikzcd}
A \arrow[r, "i_0^0"] \arrow[d] & \mathrm{Ph}(A) \arrow[r, "i_0^1"] \arrow[d] & \mathrm{Ph}^2(A) \arrow[r, "i_0^2"] \arrow[d] & \mathrm{Ph}^3(A) \arrow[r, "i_0^3"] \arrow[d] & ... \\
A \arrow[r, "i_0^0"] & A \oplus \Omega_A^1 \arrow[r, "i_0^1"] & A \oplus \Omega_A^1 \oplus \Omega_A^2 \arrow[r, "i_0^2"] & A \oplus \Omega_A^1 \oplus \Omega_A^2 \oplus \Omega_A^3 \arrow[r, "i_0^3"] & ...,
\end{tikzcd}$$

where the bottom line is a sequence of Nagata-extentions of the k-algebra A, and the vertical homomorphisms correspond to the natural derivations among these, defined by the derivations of the deRham complex, $\Omega^\star$ of A.

The universality of the two systems proves that there is a surjective map

$$\alpha : \Omega_n^n \to \Omega_A^n := \wedge^n \Omega_A.$$

The map that sends the element $da_1 \wedge da_2 \wedge ... \wedge da_n \in \Omega_A^r$ to

$$d_0(a_1)d_1(a_2)..d_{n-1}(a_r) \in \Omega_n^n$$

is an inverse, proving that α is an isomorphism. □

It follows from this that in the commutative case, for any scheme X considered as a covering of affine schemes in some sense, there are two spectral sequences converging to the same cohomology, first

$$E(1)_{p,q}^2 = H^p(H_{dR}^q(X, \Omega_*^*))$$

then

$$E(2)^2_{p,q} = H^q_{dR}(X, H^p(\Omega^*_*)).$$

Let now V be a right A-module, and assume $c(V) = 0$, such that there exist an element $\nabla' \in \mathrm{Hom}_k(V, V \otimes_A \mathrm{Ph}(A))$ with $c = \iota(\nabla')$. This implies that for $a \in A$ and $v \in V$ we have $\nabla'(va) = \nabla'(v)a + v \otimes d_0(a)$. Composing ∇' with the projection, $o : \mathrm{Ph}(A) \to A$, corresponding to the 0-derivation of A, we therefore obtain an A-linear homomorphism $P : V \to V$, *a potential.* Since $i^0_0 : A \to Ph(A)$ is a section of o, we find a k-linear map

$$\nabla_0 := \nabla' - P : V \to V \otimes \mathfrak{m}^1_1.$$

Using the property

$$d_n \circ i^{n+1}_{j+1} = i^n_j \, d_{n+1},$$

we find well defined k-linear maps

$$\nabla_1 : V \to V \otimes \Omega^1_2,\ \nabla_2 : V \to V \otimes \Omega^1_3, ..., \nabla_n : V \to V \otimes \Omega^1_{n+1}\ \forall n \geq 0$$

given by

$$\nabla_{n+1} := \nabla_n \circ i^{n+1}_1,\ n \geq 0,$$

such that for all $v \in V, \omega \in \Omega^n_p$, the formula

$$\nabla_n(v \otimes \omega) = \nabla_n(v)\omega + v \otimes d_n(\omega)$$

makes sense and defines a sequence of *derivations*

$$\nabla_n : V \otimes \Omega^p_n \to V \otimes \Omega^{p+1}_{n+1},$$

sometimes just denoted d_n and called a *connection* ∇ on the A-module V. We obtain a situation just like above,

$$\begin{tikzcd}
V \arrow[r, "i^0_0"] \arrow[d] & V \otimes \mathrm{Ph}(A) \arrow[r, "i^1_p"] \arrow[d] & V \otimes \mathrm{Ph}^2(A) \arrow[r, "i^2_p"] \arrow[d] & V \otimes \mathrm{Ph}^3(A) \arrow[r, "i^3_p"] \arrow[d] & {} \\
V \arrow[r, "i^0_0"] \arrow[dr, "d_0"] & V \arrow[r, "i^1_p"] \arrow[dr, "d_1"] & V \arrow[r, "i^2_p"] \arrow[dr, "d_2"] & V \arrow[r, "i^3_p"] \arrow[dr, "d_3"] & {} \\
& V \otimes \Omega^1_1 \arrow[r, "i^1_p"] \arrow[dr, "d_1"] & V \otimes \Omega^1_2 \arrow[r, "i^2_p"] \arrow[dr, "d_2"] & V \otimes \Omega^1_3 \arrow[r, "i^3_p"] \arrow[dr, "d_3"] & {} \\
& & V \otimes \Omega^2_2 \arrow[r, "i^2_p"] & V \otimes \Omega^2_3 \arrow[r, "i^3_p"] & {}
\end{tikzcd}$$

In general, there are no reasons for these derivations $d_n := \nabla_n,\ n \geq 0$ to define complexes, and we shall make the following definition.

Definition 43. *The curvature $R(V,\nabla)$ of the connection ∇ defined on the right A-module V is the k-linear map, composition of d_0 and d_1,*

$$R(V) = d_0 d_1 : V \to V \otimes \Omega_2^2.$$

The following Lemma is then easily proved,

Lemma 35. *Suppose A is commutative, and assume $c(V)=0$. Let $\nabla : \Theta_A \to \mathrm{End}_k(V)$ be the classical connection corresponding to ∇_0. Suppose moreover that the curvature R of ∇ is 0, then $R(V)=0$, implying that $d^2=0$, and so the diagonals in the diagram above are all complexes.*

Proof. We put

$$\nabla(v_i) = \sum_{j,k,} a_{i,j}^k v_j d_0(x_k)$$

and obtain

$$\nabla_1(\nabla_0(v_i)) = \sum_{j,k,l} \frac{\partial a_{i,j}^k}{\partial x_l} v_j d_1(x_l) d_0(x_k) + \sum_{j,k,l,m} a_{i,j}^k a_{j,m}^l v_m d_1(x_l) d_0(x_k)$$

Now the classical curvature of ∇ may be defined as

$$R_{k,l}^i = \sum_j \frac{\partial a_{i,j}^k}{\partial x_l} v_j + \sum_{j,m} a_{i,j}^k a_{j,m}^l v_m - \sum_j \frac{\partial a_{i,j}^l}{\partial x_k} v_j - \sum_{j,m} a_{i,j}^l a_{j,m}^k v_m,$$

so if $R=0$ and $d_1(x_l)d_0(x_k) = -d_1(x_k)d_0(x_l)$, we find that $\nabla_1(\nabla_0(v_i)) = 0$, from which it follows that $d^2 = 0$ □

7.6 Excursion into the Jacobian Conjecture

The conjecture referred to Jacobi, says that for algebraically closed fields k, any algebraic morphism

$$F : \mathbf{A}^n \to \mathbf{A}^n$$

with everywhere nontrivial Jacobi determinant, det $J(F)$ is an isomorphism. It is usually formulated by considering $F = \{f_i \in k[x_1,...,x_n] : i = 1,...,n\}$ as the algebraic homomorphism

$$F : k[x_1,...,x_n] \to k[x_1,...,x_n], F(x_i) := f_i,$$

where $J(F) = (f_{i,j})$. We have put

$$f_{i,j} = \frac{\partial f_i}{\partial x_j}.$$

Since det $J(F)$ must be a non-zero element of k, it is clear that there exist an inverse $(g_{i,j})$ of the matrix $J(F)$ with $g_{i,j} \in k[x_1, ..., x_n]$ which can be written as

$$\sum_i g_{k,i} \frac{\partial f_i}{\partial x_j} dx_j = \delta_{j,k} dx_j.$$

The integral g_i of $\sum_j g_{i,j} dx_j$ is the polynomial such that

$$dg_i = \sum_j g_{i,j} dx_j.$$

We might try out the following optimistic equation,

$$g_k(f_1, ..., f_n) = x_k?$$

Since $dg_k(f_1, ..., f_n) = \sum_i g_{k,i}(f_1, ..., f_n) \frac{\partial f_i}{\partial x_j} dx_j$, and since we may assume that F has the origin $x_i = 0$ as fixed point, we find that

$$g_k(f_1, ..., f_n) = x_k (mod (\underline{x})^2)$$

and we may reduce the question to the case where

$$f_i(x_1, ..., x_n) = x_i + \alpha_i, \alpha_i \in (\underline{x})^2$$

which is a known result.

We conclude by noting the important fact that this theory works algebraically. We have used the word *dynamical* without rights. There is no *time* in this setup, so it is still pure algebra. To do dynamical (applied) theory, we need something to differentiate with respect to, e.g. time or any other measure of change. There are several attempts to solve this, given by Toën, Quillen, Voevodsky, Connes. One direct way to do this, is to introduce the *all order momenta* on a derived scheme, which is the content of O.A. Laudals contribution in the next section.

Chapter 8

Dynamical Algebraic Structures

8.1 Noncommutative Algebraic Geometry

Mathematics is since the time of Galilei the language of physics. And since Descartes, Newton and Leibnitz, geometry and differential algebra have been our best tools for making the Universe understandable. The last centuries have seen an amazing development in science and technology due to the the parallel achievements in mathematics and physics. The theory of general relativity and the modern theory of quantum physics have transformed our world-view and our daily life in a way almost unimaginable just fifty years ago. And the pace of change is, seemingly, accelerating. And so is the pace of change of the mathematical bases for these two grand theories. The differential geometry and the operator algebra have served these two fundamental sciences well for a century, but the human curiosity does not rest. The feeling that they should somehow be united, has been there for a long time, and has produced a lot of new mathematics. Algebraic geometry is as old as geometry, but has seen a formidable development the last 50 years, starting with the Grothendieck era, including a new fundament for the age-old theory of deformations. Operator algebra, has in the same period due to work of Von Neumann, Gelfand and Connes, been transformed into a fascinating noncommutative geometry.

The physicists have, of course, taken advantage of these developments, and used the new mathematics to construct new models. At the moment the situation is never the less that there are still two theories, the general relativity treating gravitation and to some extent electroweak forces, and the quantum field theory taking care of relativistic quantum theory. The result of the latter is the Standard Model, a wander of an effective theory for most of the forces of nature, but not including gravitation.

The hope has therefore been that by creating some sort of fusion of classical algebraic and the new non-commutative geometry one would be able to create a model fusing the standard model and the theory of gravitation. This is, in our views, what a mature *Noncommutative Algebraic Geometry* should be about.

There are many attempts to create a geometry based on the classical algebraic geometry, modified by Serre, Chevalley and Grothendieck, but where the algebra part is extended from commutative to associative algebras. In this text we give reasons for why we think this effort must include the study of noncommutative deformation of algebraic structures.

The idea is to look at the common goal of quantum theory and Grothendieck's scheme theory, which is the study of the local and global properties of the set of representations of algebras, together with their dynamical structure.

8.2 Moduli of Representations

A point in scheme theory is a representation of a (commutative) ring, i.e. a morphism $\rho : A \to R$ where R is another ring. The scheme is in a general sense the moduli space $\mathrm{Rep}(A)$ of such representations. The object of scheme theory is then to study the properties of these moduli spaces, their categorical relations, and eventually to classify them.

In quantum theory, the objects of interest are also representations $\rho : A \to R$ of a ring of observables A, but here $R = \mathrm{End}_k(V)$, where k is a field we may use for *measuring* for any observable $a \in A$, the eigenvalues of $\rho(a)$ as operator on the k-vector space V. The aim is to study the structure of the moduli space of such representations $\mathrm{Rep}(A)$ and in particular to understand the dynamical properties of this space.

In both cases the local structure of the moduli spaces are defined via deformation theory. But here is where noncommutative deformation theory enters, not only because the rings we must work with are non-commutative, but also because the local structure we are interested in is no longer given by commutative algebra. In fact, the local structure of $\mathrm{Rep}(A)$ in a finite family $\mathfrak{B} \subset \mathrm{Rep}(A)$ is not the superposition of the local structure of each one of the representations. Noncommutative deformation of the family $\mathfrak{B}$ produces a homomorphism

$$\eta : A \to O(\mathfrak{V}),$$

the O-construction, which is the localisation process in noncommutative algebra.

8.3 Blowing down subschemes

Let us first take an easy example. Given an affine scheme defined by a k-algebra A, then a subscheme is given locally by a quotient C of A and can be considered as a representation of A. Then the Blow Down of $\mathrm{Spec}\, C$ in $\mathrm{Spec}\, A$ is given by

$$\eta : A \to O(\mathfrak{V})$$

where $\mathfrak{V}$ is the family of points outside $\mathrm{Spec}\, C$ in $\mathrm{Spec}\, A$ plus the representation C.

8.4 Moduli of Simple Modules

The basic notions of affine noncommutative algebraic geometry related to a (not necessarily commutative) associative k-algebra, for k an arbitrary field have been treated by many authors in several texts, see e.g.[**?**], [**?**], [**?**], [**?**], [**?**]. Given a finitely generated algebra A, let $\mathrm{Simp}_{<\infty}(A)$ be the category of simple finite dimensional representations, i.e. right modules of A. We show in [**?**] that any *geometric k-algebra A*, see also [**?**], may be recovered from the (non-commutative) structure of $\mathrm{Simp}_{<\infty}(A)$, and that there is an underlying quasi-affine (commutative) scheme-structure on each component $\mathrm{Simp}_n(A) \subset \mathrm{Simp}_{<\infty}(A)$ parametrizing the simple representations of dimension n. In fact, we have shown the following.

Theorem 10. *There is a commutative k-algebra $C(n)$ with an open subvariety $U(n) \subseteq \mathrm{Simp}_1(C(n))$, an étale covering of* $\mathrm{Simp}_n(A)$ *over which there exists a* versal representation $\tilde{V} \simeq C(n) \otimes_k V$*, a vector bundle of rank n defined on* $\mathrm{Simp}_1(C(n))$ *and a* versal family*, i.e. a morphism of algebras,*

$$\tilde{\rho} : A \longrightarrow \mathrm{End}_{C(n)}(\tilde{V}) \to \mathrm{End}_{U(n)}(\tilde{V}),$$

inducing all isoclasses of simple n-dimensional A-modules.

$\mathrm{End}_{C(n)}(\tilde{V})$ induces also a bundle of *operators* on the étale covering $U(n)$ of $\mathrm{Simp}_n(A)$.

8.5 Evolution in the Moduli of Simple Modules

Assume given a derivation $\xi \in \mathrm{Der}_k(A)$. Pick any $v \in \mathrm{Simp}_n(A)$ corresponding to the right A-module V, with structure homomorphism $\rho_V : A \to \mathrm{End}_k(V)$, then ξ composed with ρ_v gives us an element

$$\xi_v \in \mathrm{Ext}^1_A(V, V).$$

Therefore, ξ defines a unique one-dimensional distribution in $\Theta_{\mathrm{Simp}_n(A)}$, which once we have fixed a versal family, defines a vector field

$$[\xi] \in \Theta_{\mathrm{Simp}_n(A)},$$

and in good cases, a (rational) derivation

$$[\xi] \in \mathrm{Der}_k(C(n)).$$

Moreover, O.A. Laudal has proved in [**?**] the following.

Theorem 11. *Formally, at any point $v \in U(n) \subset \mathrm{Simp}(C(n))$ with local ring $\hat{C}(n)_v$ there is a derivation $[\xi] \in \mathrm{Der}_k(\hat{C}(n)_v)$ and a Hamiltonian $Q_\xi \in \mathrm{End}_{\hat{C}(n)_v}(\hat{V}_v)$ such that, as operators on $\hat{V}_v$, we have*

$$\xi = [\xi] + [Q_\xi, -].$$

This means that for every $a \in A$, considered as an element $\tilde{\rho}(a) \in \mathrm{M}_n(\hat{C}(n)_v)$, $\xi(a)$ acts on $\hat{V}_v$ as

$$\tilde{\rho}(\xi(a)) = [\xi](\tilde{\rho}(a)) + [Q_\xi, \tilde{\rho}(a)].$$

This result will turn out to be a very general version of the Dirac equation. Notice also that we have the canonical isomorphism

$$\mathrm{Der}_k(A, A) \simeq \mathrm{Mor}_A(\mathrm{Ph}(A), A).$$

Therefore the derivation ξ and the A-module V correspond to a $\mathrm{Ph}(A)$-module V_ξ.

The Schrødinger equation, where time is ξ is then

$$\xi(\phi) = Q(\phi)$$

and the solutions are given by the following.

Theorem 12. *The* evolution operator $u(\tau_0, \tau_1)$ *that changes the state* $\psi(\tau_0) \in \tilde{V}(v_0)$ *into the state* $\phi(\tau_1) \in \tilde{V}(v_1)$*, where* τ *is a parameter of the integral curve* $\mathbf{c}$ *connecting the two points* v_0 *and* v_1*, i.e. the* time passed *is given by*

$$\phi(\tau_1) = u(\tau_0, \tau_1)(\phi(\tau_0)) = \exp[\int_{\mathbf{c}} Q(\tau) d\tau] \ (\phi(\tau_0)),$$

where $\exp \int_{\mathbf{c}}$ *is the noncommutative version of the ordinary action integral, essentially defined by the equation*

$$\exp[\int_{\mathbf{c}} Q(\tau) dt] = \exp[\int_{\mathbf{c}_2} Q(\tau) d\tau] \circ \exp[\int_{\mathbf{c}_1} Q(\tau) d\tau]$$

where $\mathbf{c}$ *is* $\mathbf{c}_1$ *followed by* $\mathbf{c}_2$*.*

see again [**?**].

There is an important special case of the above results that we shall refer to as the *Singular Case*. Suppose $[\xi]_v = 0$, then the derivation $\xi \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$ maps to $0 \in \mathrm{Ext}^1_A(V, V)$. This situation deserves the status as a Corollary.

Corollary 8. *In the general case, let*

$$\rho : A \to \mathrm{End}_k(V)$$

be any representation and suppose given a derivation $\xi \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$*, corresponding to a* 0*- tangent vector of* V*, meaning that* ξ *maps to* $0 \in \mathrm{Ext}^1_A(V, V)$*.*

Then there exists an operator, $Q_\xi \in \mathrm{End}_k(V)$ *such that for any* $a \in A$*,* $\xi(a) = [Q_\xi, a]$*, and the* evolution operator *in this case is reduced to its first order term given by the* Hamiltonian Q_ξ *in the state-space* V *as the map*

$$\forall v \in V,\ \frac{\partial v}{\partial t} := \xi(v) = Q_\xi(v).$$

This implies that the corresponding 0*'th order changes of the A-module structure of V, can be considered equivalently as a* Heisenberg-process*, or as a* Schrødinger-process.

Proof. The fact that the condition implies that there exists an operator $Q_\xi \in \operatorname{End}_k(V)$ such that for any $a \in A$, $\xi(a) = [Q_\xi, a]$, means that the lifting of the A-module V to the $A \otimes k[\epsilon]$-module $V_\xi := V \otimes k[\epsilon]$ with the action of A defined by ξ is trivial. The automorphism $E_\xi := (id + Q_\xi \epsilon)$ of $V_0 := V \otimes k[\epsilon]$ induces an isomorphism between the trivial lifting V_0 and V_ξ. In fact, for any $a \in A$, as operator in $V \otimes k[\epsilon]$, we have the formula for left operators

$$(id + Q_\xi \epsilon)(a(id - Q_\xi \epsilon)) = (\mathrm{id} + Q_\xi \epsilon)(a - aQ_\xi \epsilon) = a + (Q_\xi a - aQ_\xi)\epsilon = a + \xi(a)\epsilon.$$

Thus, the infinitesimal action of ξ in V is the endomorphism Q_ξ which again induces the infinitesimal action (i.e.the derivation) of $\operatorname{End}_k(V)$,

$$\operatorname{ad}(Q_\xi) \in \operatorname{Der}_k(\operatorname{End}_k(V))$$

since for any $\psi \in \operatorname{End}_k(V)$ we find as above

$$(\mathrm{id} + Q_\xi \epsilon)(\psi(\mathrm{id} - Q_\xi \epsilon)) = (\mathrm{id} + Q_\xi \epsilon)(\psi - \psi Q_\xi \epsilon) = \psi + (Q_\xi \psi - \psi Q_\xi)\epsilon = \psi + [Q_\xi, \psi]\epsilon.$$

□

8.6 Dynamical Structures

As we have seen, in Subsection 8.1 the dynamics of the space of representations of our algebra A, i.e. the dynamics of the space of measurements of the family of observables that A is assumed to represent can be encoded in the category of representations of the k-algebra $\mathrm{Ph}^\infty(A)$, and the universal Dirac derivation δ. We would therefore like to use the tools developed above for the k-algebra $\mathrm{Ph}^\infty(A)$, and with $\xi = \delta$.

However, $\mathrm{Ph}^\infty(A)$ is rarely of finite type, and so the moduli space of simple modules does not have a classical algebraic geometric structure.

We shall therefore introduce the notion of *dynamical structure* to reduce the problem to a situation we can handle. This is also what physicists do. They invoke a *parsimony principle*, or an *action principle*, originally proposed by Fermat, and later by Maupertuis, with the purpose of reducing the preparation needed to be able to see ahead.

Definition 44. *A* dynamical structure *σ is a two-sided δ-stable ideal $(\sigma) \subset Ph^\infty(A)$ such that*

$$A(\sigma) = \mathrm{Ph}^\infty(A)/(\sigma),$$

the corresponding dynamical system*, is of finite type. A dynamical structure, or system, is* of order $\leq n$ *if the canonical morphism*

$$\sigma : \mathrm{Ph}^{(n-1)}(A) \to A(\sigma)$$

is surjective. If A is generated by the coordinate functions*, $\{t_i\}_{i=1,2,...,d}$ a dynamical system of order n may be defined by a* force law*, i.e. by a system of equations*

$$\delta^n t_p = \Gamma^p(\underline{t}_i, \underline{d}t_j, \underline{d}^2 t_k, .., \underline{d}^{n-1} t_l),\ p = 1, 2, ..., d.$$

Put

$$A(\sigma) := \mathrm{Ph}^\infty(A)/(\delta^n t_p - \Gamma^p)$$

where $\sigma := (\delta^n t_p - \Gamma^p)$ is the twosided δ-ideal generated by the defining equations of σ. Obviously δ induces a derivation $\delta_\sigma \in \mathrm{Der}_k(A(\sigma), A(\sigma))$, also called the Dirac derivation, and usually just denoted δ.

Notice that if $\sigma_i,\ i = 1, 2$, are two different order n dynamical systems, then we may well have

$$A(\sigma_1) \simeq A(\sigma_2) \simeq Ph^{(n-1)}(A)/(\sigma_*),$$

as k-algebras.

Assuming that the k-algebra A is finitely generated, and given a dynamical structure σ, then by definition $A(\sigma)$ is finitely generated and we can use the machinery of Subsection 8.1, with $A = A(\sigma)$ and $\xi = \delta$, the Dirac derivation. We obtain, as above, see Theorem 12.

Theorem 13. *There exists a* versal family*, i.e. a morphism of algebras,*

$$\tilde{\rho} : A(\sigma) \longrightarrow \mathrm{End}_{C(n)}(\tilde{V}) \to \mathrm{End}_{U(n)}(\tilde{V}),$$

inducing all isoclasses of simple n-dimensional $A(\sigma)$-modules.

Moreover, formally at any point $v \in U(n) \subset \mathrm{Simp}(C(n))$, with local ring $\hat{C}(n)_v$, there is a derivation $[\delta] \in \mathrm{Der}_k(\hat{C}(n)_v)$ and a Hamiltonian $Q \in \mathrm{End}_{\hat{C}(n)_v}(\tilde{V}_v)$ such that, as operators on $\tilde{V}_v$, we have

$$\delta = [\delta] + [Q, -].$$

This means that for every $a \in A(\sigma)$, considered as an element $\tilde{\rho}(a) \in \mathrm{M}_n(\hat{C}(n)_v)$, $\delta(a)$ acts on $\tilde{V}_v$ as

$$\tilde{\rho}(\delta(a)) = [\delta](\tilde{\rho}(a)) + [Q, \tilde{\rho}(a)].$$

In line with our general philosophy where time is a metric on an appropriate moduli space, the Dirac derivation δ is the time-propagator for representations. We shall consider $[\delta]$ as measuring *time* in $\mathrm{Simp}_n(\mathbf{A}(\sigma))$, respectively in $\mathrm{Spec}(C(n))$. This is reasonable since the last equation is equivalent to the following statement: The derivation δ induces an extension of $\tilde{V}_v$ as A-module, which modulo the derivation $[\delta]$ of $\hat{C}(n)_v$ is trivial. This is formally true for any derivation δ by the definition of the versal family, i.e. the $\hat{C}(n)_v$-module $\hat{V}_v$.

If $\hat{C}(n)_v$ had not been the versal base space, then we would have had to be careful. See also Theorem 18, where the relationship between the Dirac derivation and metrics is explained.

Notice also that $\mathrm{End}_{C(n)}(\tilde{V}) \simeq \mathrm{M}_n(C(n))$, and be prepared in what follows, to see this used without further warning. There are local (and even global) extensions of this result, where $[\delta]$ and Q may be assumed to be defined (rationally) on $C(n)$, see [**?**]. In this case, we may see that, provided the field k is (sufficiently) algebraically closed, any *quantum field* $\psi \in \mathrm{End}_{C(n)}(\tilde{V})$ can be expressed as a (finite) rational polynomial of generalized *creation* and *annihilation* operators.

Assume for a while that $k = \mathbb{R}$ and that our constructions go through as if k were algebraically closed. Let $v(\underline{\tau}_0) \in \mathrm{Simp}_n(\mathbf{A}(\sigma))$ be an element, an *event*. Suppose there exist an integral curve $\mathbf{c}$ of $[\delta]$ through $v(\tau_0) \in \mathrm{Simp}_1(C(n))$, ending at $v(\tau_1) \in \mathrm{Simp}_1(C(n))$, given by the automorphisms $e(\tau) := \exp(\tau[\delta])$, for $\tau \in [\tau_0, \tau_1] \subset \mathbb{R}$. The supremum of τ for which the corresponding point, $v(\tau)$, of $\mathbf{c}$ is in $\mathrm{Simp}_n(\mathbf{A}(\sigma))$ should be called the *lifetime* of the particle. It is relatively easy to compute these lifetimes, and so to be able to talk about *decay* when the fundamental vector field $[\delta]$ has been computed. In [**?**], O.A. Laudal has also proposed a mathematically sound way of treating interaction purely in terms of noncommutative deformation theory.

Let $\phi(\tau_0) \in \tilde{V}(v_0) \simeq V$ be a (classically considered) state of our *quantum system* at the time τ_0 and consider the (uni-)versal family

$$\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow \mathrm{End}_{C(n)}(\tilde{V})$$

restricted to $U(n) \subseteq \mathrm{Simp}_1(C(n))$, the étale covering of $\mathrm{Simp}_n(\mathbf{A}(\sigma))$. We shall consider $\mathbf{A}(\sigma)$ as our *ring of observables.* What happens to $\phi(\tau_0) \in \tilde{V}(v_0)$ when *time* passes from τ_0 to τ, along $\mathbf{c}$? This leads to a solution of the Schrödinger equation

$$\frac{d\phi}{d\tau} = Q(\phi),$$

along $\mathbf{c}$ applying the Theorem 12, proving that ϕ is completely determined by the value of $\phi(\tau_0)$, for any $\tau_0 \in \mathbf{c}$. Here, we shall not go into the problem of *preparing* $\phi(\tau_0) \in V(\tau_0)$, i.e. of how to exactly determine *where we are*, at some chosen *clock-time* τ, see [**?**].

In the situation of Theorem 10 we observe that, since $\delta = [\delta] + [Q, -]$. is a derivation defined in the algebra $\mathrm{End}_{C(n)}(\tilde{V})$, the eigenvalues $\Lambda := \{\lambda\}$ of the eigenvectors a_λ of δ, will have a structure as an additive sub-monoide of the reals. Assume now that $[[\delta], Q] = 0$ and suppose that $[\delta](\psi) = \nu\psi$ and $[Q, \psi] = \epsilon\psi$, then

$$\exp(\delta)(\psi) = \exp(\mathrm{ad}(Q))\exp([\delta])(\psi) = \exp(\mathrm{ad}(Q))(\psi(t+\nu)) = \exp(\epsilon)\exp(\nu)(\psi),$$

which means that if Λ has a generator h_0, then we have a Heisenberg relation

$$\Delta E \times \Delta t \geq h := \exp(h_0).$$

where $\exp(\epsilon) = \Delta E$, $\exp(\nu) = \Delta t$. Compare with [**?**], (4.4), where the the singular situation corresponding to $\delta = \mathrm{ad}(Q)$. is considered.

Remark 10. *Let A be any associative k-algebra, finitely generated by $\{t_i, i = 1, ..., d\}$, and let σ be a dynamical structure. Given any representation, $\rho : A(\sigma) \to \mathrm{End}_k(V)$, we must have $g^{i,j} := \rho([dt_i, t_j]) = \rho([dt_j, t_i]) = g^{j,i}$, and moreover,*

$$\rho(\delta(g^{i,j})) = \rho([d^2(t_i), t_j] + [dt_i, dt_j]).$$

This looks, superficially, like a very generalized field equation, where $g^{i,j}$ is the inverse of a metric $[d^2 t_i, t_j]$ is the action of a force and $[dt_i, dt_j]$ is the curvature of ρ. We shall return to this, but first we must introduced the third main ingredient in this story, the relations induced by "non-observable" infinitesimal automorphisms, the gauge groups.

8.7 Gauge Groups and Invariant Theory

We may use the above in an attempt to make precise the notion of *gauge group*, *gauge fields*, and *gauge invariance*, and thus to be able to understand why the physicists define their objects, the *fields* and *particles* the way they do.

Suppose, in line with our philosophy, that we have uncovered the moduli space $\mathbf{M}$ of the mathematical model X of our phenomena $\mathbf{P}$, and that A is the *affine k-algebra* of (an affine open subset of) this space, assumed to contain all the parameters of our interest of the states of X. Now we consider the global gauge group and invariant theory.

Suppose furthermore that we have identified a k-Lie algebra $\mathfrak{g}_0 \subset \mathrm{Der}_k(A)$, of infinitesimal automorphisms, i.e. of derivations of A, a *global gauge group*, leaving invariant the physical properties of our phenomena $\mathbf{P}$. We would then be led to consider the quotient space $\mathbf{M}/\mathfrak{g}_0$ which in our noncommutative algebraic geometry is equivalent to restricting our representations $\rho : A \to \mathrm{End}_k(V)$ to those representations V for which $\mathfrak{g}_0 \subset \mathfrak{g}_V$. This would then imply that the corresponding Hamiltonians Q_γ define a $\mathfrak{g}_0$-connection on V,

$$Q : \mathfrak{g}_0 \longrightarrow \mathrm{End}_k(V),$$

such that for all $a \in A$ and for all $\xi \in \mathfrak{g}_0$, $\rho(\xi(a)) = [Q_\xi, \rho(a)]$. This is usually written

$$\rho(\xi(a)) = [\xi, \rho(a)].$$

The curvature, i.e. the obstruction for Q to be a Lie- algebra homomorphism

$$R(\xi_1, \xi_2) := [Q_{\xi_1}, Q_{\xi_2}] - Q_{[\xi_1, \xi_2]} \in \mathrm{End}_A(V)$$

corresponds to a *global force* acting on the representation ρ. These forces, *mediated* by the *gauge-particles* $\xi \in \mathfrak{g}_0$ will be the first to be studied in some details. Put

$$\mathrm{Rep}(A, \mathfrak{g}_0) := \{\rho \in \mathrm{Rep}(A) | \kappa(\xi\rho) = 0, \forall \xi \in \mathfrak{g}_0\} = \{\rho \in \mathrm{Rep}(A) | \mathfrak{g}_0 \subset \mathfrak{g}_\rho\},$$

where $\mathrm{Rep}(A)$ is the category of all representations of A and notice that, in the commutative situation, if we consider the case where the gauge group $\mathfrak{g}_0 = \mathrm{Der}_k(A)$ then $\mathrm{Rep}(A, \mathfrak{g}_0)$ is the category of *A-Connections* for which the space of isomorphism classes is discrete with respect to time. Notice that this is also the situation in the classical quantum theory where the *Hilbert Space* is always considered as the unique state space of interest.

Definition 45. *An object* $V \in \mathrm{Rep}(A, \mathfrak{g}_0)$ *is called simple if there are no non-trivial subobjects of* V *in* $\mathrm{Rep}(A, \mathfrak{g}_0)$. *The generalised quotient* $\mathrm{Simp}(A)/\mathfrak{g}_0$ *is by definition the set* $\mathrm{Simp}(A, \mathfrak{g}_0)$ *of iso-classes of simple objects in* $\mathrm{Rep}(A, \mathfrak{g}_0)$.

If the curvature also vanish, there is a canonical homomorphism

$$\phi : U(\mathfrak{g}_0) \to \mathrm{End}_k(V).$$

where $U(\mathfrak{g}_0)$ is the universal algebra of the Lie algebra $\mathfrak{g}_0$.

In the general case let

$$A'(\mathfrak{g}_0) \subset \mathrm{End}_k(A)$$

be the sub-algebra generated by A and $\mathfrak{g}_0$. Then we put, for all $a \in A,\ \xi \in \mathfrak{g}_0$,

$$A(\mathfrak{g}_0) = A'(\mathfrak{g}_0)/(\xi a - a\xi - \xi(a))$$

and we have an identification between the set of $\mathfrak{g}_0$-connections on V and the set of k-algebra homomorphisms

$$\rho_{\mathfrak{g}} : A(\mathfrak{g}_0) \to \mathrm{End}_k(V)$$

since any such would respect the relation above, such that, for $a \in A, \xi \in \mathfrak{g}_0$,

$$\rho_{\mathfrak{g}_0}(\xi a) = \rho_{\mathfrak{g}_0}(\xi)\rho_{\mathfrak{g}}(a) = \rho_{\mathfrak{g}_0}(a)\rho_{\mathfrak{g}_0}(\xi) + \rho_{\mathfrak{g}_0}(\xi(a)).$$

Therefore $\mathrm{Rep}(A)/\mathfrak{g}_0 := \mathrm{Rep}(A, \mathfrak{g}_0) \simeq \mathrm{Rep}(A(\mathfrak{g}_0))$, and we note for memory the trivial

Lemma 36. *In the above situation, we have the following isomorphisms:*

$$\begin{aligned} \mathrm{Rep}(A)/\mathfrak{g}_0 :&= Rep(A, \mathfrak{g}_0) \simeq \mathrm{Rep}(A(\mathfrak{g}_0)) \\ \mathrm{Simp}(A)/\mathfrak{g}_0 :&= \mathrm{Simp}(A, \mathfrak{g}_0) \simeq \mathrm{Simp}(A(\mathfrak{g}_0)). \end{aligned}$$

Notice that the commutator in $A(\mathfrak{g}_0)$, of A and $\mathfrak{g}_0$ is the subring

$$A^{\mathfrak{g}_0} := \{a \in A | \forall \xi \in \mathfrak{g}_0, \xi(a) = 0\} \subset A.$$

Notice also that the commutativisation $A(\mathfrak{g}_0)^{\mathrm{com}}$, of $A(\mathfrak{g}_0)$ is the quotient of $A(\mathfrak{g}_0)$ by an ideal containing $\{\xi(a) | a \in A, \xi \in \mathfrak{g}_0\}$. Therefore there is a natural map

$$A^{\mathfrak{g}_0} \to A(\mathfrak{g}_0)^{\mathrm{com}}.$$

However, this map may not be injective, so we cannot in general identify the rank 1 points of $\mathrm{Simp}(A, \mathfrak{g}_0)$ with $\mathrm{Simp}_1(A^{\mathfrak{g}_0})$.

If the k-algebra C is assumed commutative, the classical invariant theory identifies the two schemes $\mathrm{Spec}(C)/\mathfrak{g}_0$ and $\mathrm{Spec}(C^{\mathfrak{g}_0})$ which in the above light is not entirely kosher. However, if $\mathfrak{a} \subset C$ is an ideal, stable under the action of $\mathfrak{g}_0$, then since any derivation γ of C acts on the multiplicative operators $a \in C$ as $\gamma(c) = \gamma c - c\gamma$, it is clear that the quotient $C/\mathfrak{a}$ is a $C(\mathfrak{g}_0)$-representation. Moreover, as representation of C, we have

$$C/\mathfrak{a} \in \mathrm{Simp}_1(C)/\mathfrak{g}_0$$

if and only if the subset $\mathrm{Simp}_1(C/\mathfrak{a}) \subset \mathrm{Simp}_1(C)$ is the closure of a *maximal integral subvariety* for $\mathfrak{g}_0$. The *space* of such integral subvarieties is what Laudal in [**?**] have termed the *non-commutative quotient*, $\mathrm{Spec}(C)/\mathfrak{g}_0$.

Now we consider the local gauge group. Suppose, in the general case, that there is an A-Lie algebra $\mathfrak{g}_1$ acting A-linearly on those A-modules V which we would consider of *physical* interest. Then $\mathfrak{g}_1$ should be called a *local gauge group.* One may then want to know whether the given action of $\mathfrak{g}_0$ moves $\mathfrak{g}_1$ in its formal moduli as an A-Lie algebra. If so, the action of $\mathfrak{g}_1$ would not be invariant under the the gauge transformations induced by $\mathfrak{g}_0$, and we should not consider $(\rho, \mathfrak{g}_1)$ as physically kosher.

If, on the other hand, the action of $\mathfrak{g}_0$ does not move $\mathfrak{g}_1$ in its formal moduli, it should follow that there is a relation between the $\mathfrak{g}_0$-action (i.e. the connection) on V, and the action of $\mathfrak{g}_1$. Now, since $\mathfrak{g}_0 \subset \mathrm{Der}_k(A)$, it follows from the Kodaira-Spencer map

$$\mathfrak{ks} : \mathrm{Der}_k(A) \to A^1(A, \mathfrak{g}_1 : \mathfrak{g}_1)$$

see [**?**], Lemma(2.3), that we have the following result,

Lemma 37. *Let $c_{i,j}^k \in A$ be the structural constants of $\mathfrak{g}_1$ with respect to some A-basis $\{x_i\}$, and let $\pi : \mathfrak{F} \to \mathfrak{g}_1$ be a surjective morphism of a free A-Lie algebra $\mathfrak{F}$ onto $\mathfrak{g}_1$ mapping the generators $\mathfrak{x}_i$ of $\mathfrak{F}$ onto x_i. Let $\mathfrak{F}_{i,j} = [\mathfrak{x}_i, \mathfrak{x}_j] - \sum_k c_{i,j}^k \mathfrak{x}_k \in$* $\ker(\pi)$*, and let $\xi \in$* $\mathrm{Der}_k(A)$*. Then, $\mathfrak{ks}(\xi)$ is the element of $A^1(A, \mathfrak{g}_1 : \mathfrak{g}_1)$ determined by the element of* $\mathrm{Hom}_{\mathfrak{F}}(\ker(\pi), \mathfrak{g}_1)$ *given by the map*

$$\mathfrak{F}_{i,j} \to -\sum_k \xi(c_{i,j}^k) x_k.$$

For $\mathfrak{ks}(\xi)$ to be 0, there must exist an A- derivation $D_\xi : \mathfrak{F} \to \mathfrak{g}_1$ such that

$$D_\xi(\mathfrak{F}_{i,j}) = D_\xi([\mathfrak{x}_i, \mathfrak{x}_j] - \sum_k c^k_{i,j}\mathfrak{x}_k) = -\sum_k \xi(c^k_{i,j})x_k.$$

Assume $\mathfrak{g}_1$ as an A-module is such that $\mathfrak{g}_0 \subset \mathfrak{g}_{\mathfrak{g}_1}$, see Definition 40. Let now $\nabla : \mathfrak{g}_0 \to \mathrm{End}_k(\mathfrak{g}_1)$ be a $\mathfrak{g}_0$-connection of the A-module $\mathfrak{g}_1$, and let $\xi \in \mathfrak{g}_0$. Then $\mathfrak{ks}(\xi) = 0$ since we may let

$$\mathfrak{D} : \mathfrak{g}_0 \to \mathrm{Der}_k(\mathfrak{g}_1)$$

be defined by

$$\mathfrak{D}_\xi(\gamma) = \mathrm{ad}(\nabla_\xi)(\gamma)$$

such that, for $\xi \in \mathfrak{g}_0,\ a \in A,\ \gamma \in \mathfrak{g}_1$,

$$\mathfrak{D}_\xi(a \cdot \gamma) = a\mathfrak{D}_\xi(\gamma) + \xi(a) \cdot \gamma.$$

If the curvature

$$R(\xi_1, \xi_2) := [\mathfrak{D}_{\xi_1}, \mathfrak{D}_{\xi_2}] - \mathfrak{D}_{[\xi_1,\xi_2]}$$

representing the 2. order action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ vanish, the map

$$\mathfrak{D} : \mathfrak{g}_0 \to \mathrm{Der}_k(\mathfrak{g}_1)$$

is a Lie-algebra morphism, and $\mathfrak{D}$ defines a Lie algebra structure on the sum

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

The Lie-products of the sum is defined as the product in each Lie algebra, with the cross-products defined for $\xi \in \mathfrak{g}_0, \gamma \in \mathfrak{g}_1$ as

$$[\xi, \gamma] = \mathfrak{D}_\xi(\gamma).$$

A structure like this, a Lie-Cartan pair, is now often called a Lie algebroid.

The situation above comes up when we have chosen a dynamical structure σ for A with Dirac derivation δ. Assume that there exist, as above, a global gauge group $\mathfrak{g}_0 \subset \mathrm{Der}_k(A)$, and suppose moreover that there is an A-Lie algebra $\mathfrak{g}_1$ that acts as a local gauge group on the A-module V. Then we would be lead to consider the quotient of A by both $\mathfrak{g}_0$ and $\mathfrak{g}_1$, i.e. the representations

$$\rho : A \to \mathrm{End}_k(V)$$

for which there exists a diagram

$$\begin{tikzcd}
\mathfrak{g}_0 \arrow[r, "\mathfrak{D}"] \arrow[d, "\nabla"] & \mathrm{Der}_k(\mathfrak{g}_1) & \mathfrak{g}_1 \arrow[l] \arrow[d, "\nabla_1"] \\
\mathrm{End}_k(V) & & \mathrm{End}_A(V) \arrow[ll]
\end{tikzcd}$$

Here $\mathfrak{D}$, which we might call a *generalised spin structure*, and ∇ are connections, and ∇_1 is the action of $\mathfrak{g}_1$ on V. If $\mathfrak{D} = \mathrm{ad}(\nabla)$ has vanishing curvature then there is a connection of the Lie algebroid $\mathfrak{g}$

$$\nabla_2 : \mathfrak{g} \to \mathrm{End}_k(V).$$

The category of representations $\rho : A \to \mathrm{End}_k(V)$ with this property vis a vis the Lie algebra $\mathfrak{g}$, and simple as such, will be written

$$\mathrm{Simp}(A)/\mathfrak{g}.$$

According to our philosophy, this should be the object of study in mathematical physics.

Notice for later use that if $A = k[t_1, ..., t_d]$ is a polynomial algebra and

$$\rho : A \to \mathrm{End}_k(V)$$

is an object of $\mathrm{Simp}(A)/\mathfrak{g}$ with an extension to

$$\rho_\xi : \mathrm{Ph}(A) \to \mathrm{End}_A(V)$$

corresponding to a derivation $\xi \in \mathrm{Der}_k(A, \mathrm{End}_k(V)$, then, since in $\mathrm{Ph}(A)$ we have the relations $[dt_i, t_j] = [dt_j, t_i]$, we must have

$$\rho_\xi(dt_i) \in \mathrm{End}_A(V), \rho_\xi([dt_i, t_j]) =: g_\xi^{i,j} = g_\xi^{j,i} \in \mathrm{End}_k(V).$$

Any element $\xi \in \mathfrak{g}_0$ composed with ρ defines a morphism

$$\rho_\xi : \mathrm{Ph}(A) \to \mathrm{End}_k(V)$$

with $\rho_\xi(dt_i) = \xi(t_i) \in A$, and so

$$g_\xi^{i,j} = [\rho_\xi(dt_i), t_j] = 0$$

for all $\xi \in \mathfrak{g}_0$.

Notice also that physicists have a way of classifying, or naming *states*, i.e. the elements of the representation vector space V, according to certain numbers associated to them, like spin, charge, hyperspin, etc. We find this in the situation above, as follows.

Consider the Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{g}_1$. It will operate on the above representation space V as diagonal matrices, and the eigenvectors may be labelled by the corresponding eigenvalues.

Notice also that if $V_i \in \mathrm{Rep}(A, \mathfrak{g}), i = 1, 2$, then it follows from (1.25) that an extension of the $A(\mathfrak{g})$-module V_1 with V_2 will also sit in $\mathrm{Rep}(A, \mathfrak{g})$.

Notice, finally that, given an action of the Lie algebra $\mathfrak{g}_0$ on A then, since $\mathrm{Ph}(-)$ is a functor in the category of algebras and algebra morphisms, the action of $\mathfrak{g}_0$ extends to $\mathrm{Ph}(A)$, but not necessarily to a dynamical system of the type $A(\sigma)$.

This will turn out to be important for our version of the Standard Model. There we will also meet the following extension of the situation above.

8.8 The Generic Dynamical Structures associated to a Metric

We start with the commutative case, with *metrics* and *gravitation.* Let $k = \mathbb{R}$ be the real numbers and consider a commutative polynomial k-algebra $C = k[t_1, ..., t_d]$. Let

$$g = 1/2 \sum_{i,j=1,..,r} g_{i,j} dt_i dt_j \in \mathrm{Ph}(C),$$

correspond to a (non-degenerate) Riemannian metric, i.e. such that $g_{i,j} = g_{j,i}$ The determinant $\mathfrak{m}$ of the matrix $(g_{i,j})$ is non-zero on an affine open subspace of $\mathbf{A}^d := \mathrm{Spec}\, C$, and there we put

$$(g^{i,j}) := (g_{i,j})^{-1}.$$

We shall consider the localization $C \to C_{\mathfrak{m}}$ of C and the diagram of canonical morphisms

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_{\mathfrak{m}} \arrow[d] \\ \mathrm{Ph}(C) \arrow[r] & \mathrm{Ph}(C_{\mathfrak{m}}) \end{tikzcd}$$

Also, we work with C instead of $C_{\mathfrak{m}}$, making sure that every construction made for $\mathrm{Ph}^*(C)$ goes through for $\mathrm{Ph}^*(C_{\mathfrak{m}})$. In particular, all representations $\rho : C \to \mathrm{End}_k(V)$, are supposed to be extendible to $C_{\mathfrak{m}}$, which simply means that $\rho(\mathfrak{m})$ is invertible and $V = V_{\mathfrak{m}}$.

Recall that the Levi-Civita connection

$$D : \Theta_C \to \mathrm{End}_k(\Theta_C)$$

is defined to be killing the metric , i.e. it satisfy the equation,

$$Dg = 0.$$

Moreover it is without torsion, i.e.

$$D_\xi(\kappa) - D_\kappa(\xi) = [\xi, \kappa], \forall \xi, \kappa \in \Theta_C.$$

It is given in terms of the Christoffel symbols Γ defined by

$$\sum_l g_{l,k} \Gamma^l_{j,i} = 1/2 \left(\frac{\partial g_{k,i}}{\partial t_j} + \frac{\partial g_{j,k}}{\partial t_i} - \frac{\partial g_{i,j}}{\partial t_k} \right),$$

as

$$D_{\delta_i}(\delta_j) = \sum_k \Gamma^k_{i,j} \delta_k$$

where we have put $\delta_i := \frac{\partial}{\partial t_i}$.

The dual action of D,

$$D^* : \Theta_C \to \mathrm{End}_k(\Omega_C),$$

normally just called D, comes out as

$$D_{\delta_i}(dt_j) = \sum_k \Gamma^j_{i,k} dt_k$$

A short computation then proves that

$$D_\xi(g) = \sum_{i,j} D_\xi(g_{i,j} dt_i dt_j) = \sum_{i,j} \xi(g_{i,j}) dt_i dt_j + \sum_{i,j,k,l} g_{l,j} \Gamma^l_{k,i} \xi_k dt_i dt_j + \sum_{i,j,k,l} g_{i,l} \Gamma^l_{k,j} \xi_k dt_i dt_j.$$

Since in $\mathrm{Ph}^\infty(C)$ we have

$$\delta(g) = \sum_{i,j,k=1,..,r} \frac{\partial g_{i,j}}{\partial t_k} dt_k dt_i dt_j + \sum_{i,j,=1,..,r} g_{i,j}(\delta^2 t_i dt_j + dt_i \delta^2 t_j),$$

we may plug in the formula

$$\delta^2 t_l = -\Gamma^l := -\sum \Gamma^l_{i,j} dt_i dt_j$$

on the right hand side, and see that we in the commutative situation, i.e. for the dynamical structure generated by

$$(\sigma_0) := \{[dt_i, t_j], [dt_i, dt_j] | t_i, t_j \in C\}$$

have got a solution of the *Lagrange equation*

$$\delta(g) = 0.$$

This solution has the form of a *force law*

$$d^2 t_l = -\Gamma^l := -\sum \Gamma^l_{i,j} dt_i dt_j,$$

of the dynamical structure (σ_0). The dynamical system is then, of course, the commutative algebra

$$C(\sigma_0) = k[\underline{t}, \underline{\xi}]$$

where ξ_j is the class of dt_j. The Dirac derivation now takes the form

$$[\delta] = \sum_l (\xi_l \frac{\partial}{\partial t_l} - \Gamma^l \frac{\partial}{\partial \xi_l}),$$

coinciding with the fundamental vector field $[\delta]$ in $\mathrm{Simp}_1(C(\sigma_0)) = \mathrm{Spec}(k[t_i, \xi_j])$.

The equation

$$[\delta](g) = 0$$

imply that g is constant along the integral curves of $[\delta]$ in $\mathrm{Simp}_1(\mathrm{Ph}(C))$, and these integral curves projects into $\mathrm{Simp}_1(C)$ to give the geodesics of the metric g with equations

$$\ddot{t}_l = -\sum_{i,j} \Gamma^l_{i,j} \dot{t}_i \dot{t}_j.$$

We may also look at this from another point of view. Suppose given any dynamical structure with Dirac derivation δ on $\mathrm{Ph}(C)$. Consider $\mathrm{Simp}_1(\mathrm{Ph}(C))$. It is obviously represented by $C(1) := k[\underline{t}, \underline{\xi}]$, and the Dirac derivation induces a derivation $[\delta] \in \mathrm{Der}_k(C(1))$ and the Hamiltonian must vanish. Therefore we have two options for the same notion of time in the picture, g and $[\delta]$. The last derivation must therefore be a *Killing vector field*, i.e. we must have a solution of the Lagrange equation,

$$[\delta](g) = 0,$$

and we are left with the above solution for δ.

Since the metric is related to the gravitational force, the group of isometries, $O(g)$ of the metric g, i.e. the group of algebraic automorphisms of C leaving the metric g invariant would, in line with our philosophy, be an obvious global gauge group. We shall refer to its Lie algebra as $\mathfrak{o}(g)$. Since $\mathrm{Ph}(-)$ is a functor, $O(g)$ would also act on $\mathrm{Ph}(C)$ and would induce an action of $\mathfrak{o}(g)$ on $\mathrm{Ph}(C)$, and so also on $\mathrm{Ph}^\infty(C)$.

Remark 11 (The Lie Algebra of Isometries)**.** *Consider the metric*

$$g = 1/2 \sum_{i=1}^{d} g_{i,j} dt_i dt_j \in \mathrm{Ph}(C)$$

and a derivation $\eta = \sum_i \eta_i \frac{\partial}{\partial t_i} \in \mathrm{Der}_k(C)$ *acting on* C*, and so on* $\mathrm{Der}_k(C)$ *by the Lie product, and by functoriality on* $\mathrm{Ph}(C)$*. In particular* η *acts on* Ω_C *such that* $\eta(dt_i)$ *is defined by*

$$\eta(dt_j)(\delta_k) = dt_j([\eta, \delta_k]) = -\delta_k(\eta_j).$$

The Lie algebra of Killing vectors *is the the Lie algebra of derivations*

$$\mathfrak{o}(g) = \{\eta \in \mathrm{Der}_k(C) | \eta(g) = 0\}$$

where the equation $\eta(g) = 0$ *is equivalent to*

$$\sum_{i,j} \eta(g_{i,j}) dt_i dt_j - \sum_{i,j,k} g_{i,j} \frac{\partial \eta_i}{\partial t_k} dt_k dt_j - \sum_{i,j,k} g_{i,j} \frac{\partial \eta_j}{\partial t_k} dt_i dt_k = 0,$$

implying, for all $i, j = 1, ..., d$,

$$\eta(g_{i,j}) - \sum_k \frac{\partial \eta_k}{t_i} g_{k,j} - \sum_k \frac{\partial \eta_k}{\partial t_j} g_{i,k} = 0.$$

Compare this with the formulas defining the Levi-Civita connection,

$$D : \Theta_C \to \mathrm{End}_k(\Theta_C)$$

given as above by

$$\sum_{i,j} D_\eta(g_{i,j} dt_i dt_j) = \sum_{i,j} \eta(g_{i,j}) dt_i dt_j + \sum_{i,j,k,l} g_{l,j} \Gamma^l_{k,i} \eta_k dt_i dt_j + \sum_{i,j,k,l} g_{i,l} \Gamma^l_{k,j} \eta_k dt_i dt_j = 0$$

from which we see that in this case the condition for $\eta \in \mathfrak{o}(g)$ is the usual

$$g(D_{\delta_i}(\eta), \delta_j) + g(D_{\delta_j}(\eta), \delta_i) = 0.$$

There are two fundamental examples, the Euclidean and the Minkowski metrics. First, suppose all $g_{i,j}$ are constants. We are interested in the linear derivations. We find that the derivations are given in terms of matrices $(\gamma_{i,j}) := (\frac{\partial \gamma_j}{\partial t_i})$, where $\gamma_{i,j} g_{j,j} = -\gamma_{j,i} g_{i,i}$. This gives in dimension 2, for the Euclidean respectively for the Minkowski metric,

$$(\gamma_{i,j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (\gamma_{i,j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding 1-dimensional (rotation) Lie groups acting on C, with coordinates (t_1, t_2), are given by the exponential

$$O(g) = \exp(\tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix},$$

respectively,

$$O(g) = \exp(\tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} \cosh(\tau) & \sinh(\tau) \\ \sinh(\tau) & \cosh(\tau) \end{pmatrix}.$$

as we know.

We will now consider metrics, gravitation, and Energy.

In the commutative case, (and for the corresponding 1-dimensional representation), treated above, the Hamiltonian was trivial, and the notion of time was taken care of by a vector field $[\delta]$. Let us now consider representations ρ for which the Dirac derivation $[\delta]$ vanish, and the notion of time is taken care of by the Hamiltonian Q.

This is accomplished by introducing another dynamical structure related to the metric g. Notice first that a non-degenerate metric $g = 1/2 \sum_{i,j=1}^d g_{i,j} dt_i dt_j \in \mathrm{Ph}(C)$ induces a duality, i.e. an isomorphism of C-modules

$$\Theta_C = \mathrm{Hom}_C(\Omega_C, C) \simeq \Omega_C,$$

such that

$$g(\delta_i, -) = dt_i.$$

Recall the relations $[dt_i, t_j] = [dt_j, t_i]$ in $\mathrm{Ph}(C)$, and consider the twosided ideal (σ_g) of $\mathrm{Ph}(C)$ generated by

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j})$$

and put

$$C(\sigma_g) := \mathrm{Ph}(C)/(\sigma_g).$$

Let moreover

$$T := \sum_j T_j dt_j = -1/2 \sum_{i,j,l} \frac{\partial g_{i,j}}{\partial t_l} g^{l,i} dt_j.$$

An easy computation shows that

$$T_l = -1/2(\sum_k \Gamma^k_{k,l} + \sum_{k,p,q} g^{k,q}\Gamma^p_{k,q} g_{p,l}) = -1/2(\sum_j \Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l})$$

where $\bar{\Gamma}^j_{j,l} := \sum_{j,p,q} g^{j,q}\Gamma^p_{j,q} g_{p,l}$.
Consider the inner derivation of $C(\sigma_g)$, defined by

$$\delta := \mathrm{ad}(Q),\ \ Q = g - T.$$

After a dull computation we obtain, in $C(\sigma_g)$,

$$\delta(t_i) = [Q, t_i] = dt_i,\ \ \delta^2(t_i) = [Q, dt_i],\ \ i = 1, ..., d.$$

Therefore, by universality we have a well-defined dynamical structure (σ_g) with Dirac derivation $\delta = \mathrm{ad}(g - T)$.

It is clear that if $\phi : C \to C'$ is an isomorphism of k-algebras, and $g' = \phi(g)$, then $C(\sigma_g)$ is isomorphic to $C'(\sigma_{g'})$, so the construction is *covariant* in the language of the physicists. In particular, (σ_g) is invariant with respect to isometries, implying that $\mathfrak{o}(g)$ is a sub-Lie algebra of the global gauge group $\mathfrak{g}_0$ of $C(\sigma_g)$.

Any representation,

$$\rho : C(\sigma_g) \to \mathrm{End}_k(V)$$

is defined by

$$\rho(t_i) = t_i,\ \ \rho(dt_i) = \rho([Q, t_i]),$$

and since $\rho([dt_i, t_j]) = [\rho(dt_i), t_j] = g^{i,j}$, we find that if we put

$$\delta_i := \frac{\partial}{\partial t_i},\ \xi_i := \sum_{j=1}^{d} g^{i,j}\delta_j,$$

then

$$\rho(dt_i) = \nabla_{\xi_i}$$

defines a connection,

$$\nabla : \mathrm{Der}_k(C) =: \Theta_C \to \mathrm{End}_k(E)$$

when the ξ_i's generate $\mathrm{Der}_k(C)$ as a C-module.
Since any connection on a free C-module E is given as

$$\nabla_{\delta_{t_i}} = \delta_{t_i} + \nabla_i$$

where $\nabla_i \in \mathrm{End}_C(E)$, we find in this case that

$$\rho_1(dt_i) = \nabla_{\xi_i} = \xi + \psi_i$$

for some (arbitrary) *potential* $\psi = \{\psi_i\} \in \mathrm{End}_C(V)^d$.
In the general case, the metric is non-degenerate in $C_{\mathfrak{m}}$, and the derivations $\{\xi_i, i = 1, ..., d\}$ forms a C-basis for $\mathrm{Der}_k(C_{\mathfrak{m}})$. Therefore, any representation $\rho : C(\sigma_g) \to \mathrm{End}_k(V)$ induces a C-connection

$$\nabla : \mathrm{Der}_k(C_{\mathfrak{m}}) \to \mathrm{End}_k(V).$$

Fixing one, then any other connection is given by

$$\nabla'_{\xi_i} = \nabla_{\xi_i} + \psi_i.$$

For an arbitrary potential ψ, any other morphism $\rho'_1 : C(\sigma_g) \to \mathrm{End}_k(V)$ extending ρ, may be defined by

$$\rho'_1(dt_i) = \rho(dt_i) + \psi_i.$$

Put $\rho_\psi := \rho' = \rho + \psi$. Since the derivation $\eta \in \mathrm{Der}_k(C, \mathrm{End}_k(V))$ induced by ρ_ψ is given as $\eta(t_i) = [\rho_\psi(g - T), t_i]$, its image in $\mathrm{Ext}^1_C(V, V)$ must be 0. It follows that η induces a derivation $\mathrm{ad}(Q_\eta) \in \mathrm{Der}_k(\mathrm{End}_k(V))$ where

$$Q_\eta = \rho(g - T) + [\psi] + 1/2 g_{i,j}\psi_i\psi_j + 1/2(\sum_l \Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l})\psi_l,$$

and where

$$[\psi] = \sum_i \psi_i \nabla_{\delta_{t_i}} =: \sum_i \psi_i \frac{\partial}{\partial t_i}.$$

Notice that if we consider, for a given ψ, the representation

$$\psi : \mathrm{Ph}(C) \to \mathrm{End}_C(V)$$

given by $\psi(t_i) = t_i, \psi(dt_i) = \psi_i$, the formula above would look like

$$Q_\eta = \rho(g-T) + [\psi] + \psi(g-T).$$

We may consider this as a momentum operation in the state space V, a sum of a *horizontal*, a *vertical*, and between them a *mixed* component $[\psi]$. Denote the *horizontal* and the *vertical* terms

$$Q_h := Q = \rho(g-T),\ Q_v := \psi(g-T).$$

We obtain the following formula for the *time development* of ρ_ψ, i.e. for the action $[Q] := [Q, -]$, of the Dirac derivation δ on $\mathrm{End}_k(V)$

$$[Q] := \mathrm{ad}(Q_h + [\psi] + Q_v), i.e.\ \rho_\psi(d^{n+1}t_i) = [Q_h + [\psi] + Q_v, \rho_\psi(d^n t_i)].$$

What we termed $[\delta]$ vanish here. Therefore we might be tempted to write

$$\delta = [Q].$$

However, the Dirac derivation acts on the algebra of observables, $C(\sigma_g)$, and our $[Q]$ acts trivially on the moduli space of representations, but as the time development in each representation. We have seen that it is reasonable to write $\delta = [\delta] + [Q, -]$, since for geometric algebras $A(\sigma)$ that are determined by its simple finite dimensional representations, one might expect $[\delta] + [Q, -]$ to determine δ.

At the end of this Section, in 17, we shall see that we may consider the algebras $C(\sigma_g)$ as fibres of a family of algebras parametrized by the metrics of C, then the distinction between δ, $[\delta]$ and $[Q] = [Q, -]$ become more serious, and we shall therefore choose to reserve the notation δ for the derivation of the algebra of observables, and fuse $[\delta]$ and $[Q] = [Q, -]$

The formula above then, finally, reads,

$$[\delta] := \mathrm{ad}(Q_h + [\psi] + Q_v),$$

meaning that for any $n \geq 1$ we have

$$\rho_\psi(d^{n+1}t_i) = [Q_h + [\psi] + Q_v, \rho_\psi(d^n t_i)].$$

Remark 12 (The Global Gauge Group of $C(\sigma)$)**.** *Since any representation (ρ_1, V) of $C(\sigma_g)$ induces a C-connection on V, it is reasonable to accept $\mathfrak{g}_0 = \mathrm{Der}_k(C)$ as the global gauge group for $C(\sigma_g)$. We should therefore be interested in the invariant theory of $C(\sigma_g)$ modulo $\mathfrak{g}_0$. Now, $\mathrm{Der}_k(C)$, is generated by the derivations $\{\xi_i\}, i = 1, ..., d$, and clearly $\mathfrak{o}(g) \subset \mathfrak{g}_0$. We should therefore also try to express $\mathfrak{o}(g)$ in terms of the $\xi_i's$.*

Any Killing vector must have the form

$$\eta = \sum_i \alpha_i \xi_i$$

Put this into the earlier equations,

$$\eta(g_{i,j}) - \sum_k \frac{\partial \eta_k}{\partial t_i} g_{k,j} - \sum_k \frac{\partial \eta_k}{\partial t_j} g_{i,k} = 0,$$

and use the well known formulas,

$$\frac{\partial g_{i,k}}{\partial t_l} = \sum_p (g_{p,k}\Gamma^p_{i,l} + g_{i,p}\Gamma^p_{k,l}),$$

$$\frac{\partial g^{r,m}}{\partial t_q} = -\sum_k (g^{r,k}\Gamma^m_{k,q} + g^{k,m}\Gamma^r_{k,q}).$$

We find that $\eta \in \mathfrak{o}(g)$ *if and only if,*

$$\frac{\partial \alpha_i}{\partial t_j} + \frac{\partial \alpha_j}{\partial t_i} = 2\sum_k \Gamma^k_{i,j}\alpha_k, \forall i,j = 1,...,d.$$

Moreover the Lie structure is given by,

$$[\xi_i, \xi_j] = \sum_k c^k_{!,j}\xi_k,\ c^k_{!,j} = \Gamma^{j,i}_k - \Gamma^{i,j}_k,$$

where we have put,

$$\Gamma^{j,i}_p := \sum_k g^{j,k}\Gamma^i_{k,p},\ \Gamma^{i,j}_p = \sum_k g^{i,k}\Gamma^j_{k,p}.$$

Notice that the representation, $\rho = \rho_\Theta$ *of* $C(\sigma_g)$*, defined on* Θ_C*, by the Levi-Civita connection, has a Hamiltonian*

$$Q := \rho(g - T) = 1/2 \sum_{i,j} g^{ij}\nabla_{\delta_i}\nabla_{\delta_j},$$

i.e. the generalized Laplace-Beltrami operator, which is also invariant with respect to isometries, although the proof demands some algebra.

This might have been considered a quantum version of the Einstein Field Equation, g *is the metric,* Q *is the quantum Mass-Energy-Stress Operator, and*

$$T = T_l dt_l,\ T_l = -1/2(\sum_j (\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2(\operatorname{tr}\nabla_l + \operatorname{tr}\bar{\nabla}_l),$$

see below for the computation. This might have been our replacement of the Ric. *However this analogy does not fit into the noncommutative algebraic geometry that we have chosen to be our basis. The analogy of Einstein's Field Equations will have to wait.*

For the Levi-Civita connection we shall, as above, denote by

$$\rho_\Theta : C(\sigma_{g_\Theta}) \to \mathrm{End}_k(\Theta_C)$$

the representation of $C(\sigma_{g_\Theta})$, and by

$$D_- : \Theta_C \to \mathrm{End}_k(\Theta_C)$$

the corresponding connection.

For the representation ρ_Θ and for an element (a state) $\phi \in \Theta_C$, we would in line with classical Quantum Theory, assume the dynamics given by the Schrödinger equation

$$\frac{d\phi}{d\tau} = Q(\phi),$$

where τ would be an ad hoc chosen time parameter. But, again see the discussion above, we have two, and only two options for the notion of time, namely ϕ itself or the metric g measuring the time t.

Since we have

$$D_\phi(\phi) = \mu\frac{d\phi}{dt},$$

where D_ϕ is the Levi-Civita connection applied to ϕ, and $\mu = g(\phi, \phi)^{1/2}$, it seems reasonable to replace the classical Schrödinger equation in our situation by the following

$$\frac{d\phi}{dt} = Q(\phi), \phi \in \Theta_C.$$

We find a general equation of motion for the representations of $C(\sigma_g)$ formulated as in the already proved statement.

Theorem 14 (The Generic Equation of Motion)**.** *Assume the metric g is non-degenerate. Then we know that the derivations ξ_i generate $\mathrm{Der}_k(C)$. Put $\delta_i := \frac{\partial}{\partial t_i}$.*

Let $\rho_0 : C \to \mathrm{End}_k(V)$ be a representation, and let $\rho : C(\sigma_g) \to \mathrm{End}_k(V)$ be an extension considered as a preparation, i.e. as fixing a momentum *of ρ. Put*

$$\rho(dt_i) = \nabla_{\xi_i} \in \mathrm{End}_C(V)$$

and let ρ_ψ be the representation given by

$$\rho_\psi(dt_i) = \nabla_{\xi_i} + \psi_i,\ \ \psi_i \in \mathrm{End}_C(V)$$

The corresponding ρ_0-derivation

$$\eta \in \mathrm{Der}_k(C, \mathrm{End}_k(V))$$

maps to $0 \in \mathrm{Ext}^1_C(V, V)$ since we have

$$\eta(t_i) = \rho_\psi(dt_i) = [\rho_\psi(g - T), t_i].$$

This implies that the time development in $\mathrm{End}_k(V)$ *is given by the derivation*

$$[\delta] = \mathrm{ad}(Q_h + [\psi] + Q_v)$$

where

$$\begin{aligned} Q_h &: = \rho(g-T) = Q := 1/2 \sum_{i,j} g^{i,j} \nabla_{\delta_i} \nabla_{\delta_j} \\ [\psi] &: = \sum_i \psi_i \nabla_{\delta_i} \\ Q_v &: = \psi(g-T) = 1/2 \sum_{i,j} g_{i,j} \psi_i \psi_j + 1/2 (\sum_{j,l} \Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) \psi_l \end{aligned}$$

In particular,

$$\eta(t_i) = [Q_\eta, t_i] = [Q + [\xi], t_i].$$

Therefore the time development induced in $\mathrm{End}_k(V)$ *by* ρ_ψ *is infinitesimally given by*

$$[\delta] = \mathrm{ad}([\psi]) + \mathrm{ad}(Q),$$

and the corresponding first order time development in the state-space is given by the operator

$$P := Q + [\psi] : V \to V.$$

The total energies measured by the representation should be the eigenvalues of P.

This turns out to be a general version of the Dirac equation that we shall return to frequently in the sequel. Notice that the potentials ψ will pop up as the tangent directions in the space of connections $\mathbf{P}$ in the following. The "classical" Dirac equation appears when we put $\psi_k = \sum_i \gamma_i g^{i,k}$ with $\gamma_i \in \mathfrak{g}_1$ and find

$$[\psi] = \sum_i \gamma_i \xi_i =: [\delta].$$

This choice of ψ_i and in particular, the choice of the γ_i's, will follow from our final model to follow in later work.

Remark 13. *Since*

$$T = \sum_l T_l dt_l, \quad \textit{with} \;\; T_l = -1/2 (\sum_j (\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2 (\mathrm{tr}\, \nabla_l + \mathrm{tr}\, \bar{\nabla}_l),$$

and since

$$Q = \rho(g-T) = 1/2 \sum g^{i,j} \nabla_{\delta_i} \nabla_{\delta_j},$$

we find that the purely GR-Schrödinger energy equation

$$Q(\phi) = E\phi$$

takes the form in C,

$$1/2 \sum g^{i,j} \nabla_{\delta_i} \nabla_{\delta_j}(\phi) = E\phi$$

picking out the energy E, *of "states"* $\phi \in C$. *We find solutions of the form*

$$\phi = \exp(-\underline{e}.\underline{t}) \text{ where } \underline{e} := (e_1, ..., e_d), \underline{t} := (t_1, ..., t_d)$$

with energy given by

$$E = 1/2 \sum_{i,j} e_i g^{i,j} e_j.$$

Notice that the energy E *is* $1/2$ *of the square length of the momentum* $\omega = \rho(\sum_l e_l dt_l) = \sum_l e_l \xi_l$, *as one might have expected, since we classically have*

$$E = 1/2m|\omega|^2,$$

where m *=mass. Notice also that the classical curvature in this case is*

$$\rho(F_{i,j}) = [\rho(dt_i), \rho(dt_j)] - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j})\rho(dt_p) = [\nabla_{\xi_i}, \nabla_{\xi_j}] - \nabla_{[\xi_i,\xi_j]},$$

and that we usually write

$$R_{i,j} := [dt_i, dt_j], F_{i,j} := R_{i,j} - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j}) dt_p.$$

Notice also that we shall, when it is clear what we are talking about, write $F_{i,j}$ *for* $\rho(F_{i,j})$. *Obviously,* $F_{i,j}$ *is the obstruction for* ρ *inducing a representation of the Lie algebra* $\Theta_C = \mathrm{Der}_k(C)$.

Now, to be able to handle this time-development, we need to know formulas, for $d^l t_i,\ l \geq 1,\ i = 1, ..., n$, in $C(\sigma_g)$. To this end, put

$$\bar{\nabla}_l := (\bar{\Gamma}_{i,l}^j)$$
$$\bar{\Gamma}_{p,q}^i := \sum_{l,r} g^{r,i} \Gamma_{r,p}^l g_{l,q},$$

Computing, we find, see [?] for a proof,

Theorem 15 (The Generic Force Laws). *In $C(\sigma_g)$ where the Dirac derivation δ is defined, we have the following force law expressed in two different ways in* $\mathrm{Ph}(C)$,

$$
\begin{aligned}
(1)\ d^2t_i := & \delta^2(t_i) = [g - T, dt_i] = -1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_p dt_q \\
& +1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_p R_{q,i}) + [dt_i, T] \\
(2)\ d^2t_i = & -\sum_{p,q}\Gamma^i_{p,q}dt_p dt_q - 1/2\sum_{p,q} g_{p,q}(F_{i,p}dt_q + dt_p F_{i,q}) \\
& + 1/2\sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]dt_l + [dt_i, T] \\
= & -\sum_{p,q}\Gamma^i_{p,q}dt_p dt_q - \sum_{p,q} g_{p,q}F_{i,p}dt_q + 1/2\sum_{p,q} g_{p,q}[F_{i,q}, dt_p] \\
& + 1/2\sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]dt_l + [dt_i, T].
\end{aligned}
$$

Remark 14. *We shall consider the above formulas as general Force Laws in* $\mathrm{Ph}(C)$ *induced by the metric g. This means the following:*

First, assume given a representation

$$\rho_0 : C \to \mathrm{End}_k(V)$$

and pick any tangent vector (momentum) of the formal moduli of the C-module V, i.e. an extension of ρ_0,

$$\rho_1 : \mathrm{Ph}(C) \to \mathrm{End}_k(V).$$

Then, if ρ_1 can be extended to a representation

$$\rho_2 : \mathrm{Ph}^2(C) \to \mathrm{End}_k(V)$$

with $\rho(d_2(d_1t_i)) = \rho_1(d^2t_i)$ given by the formula of the Force Law, this means that the force law has induced a second order momentum in the formal moduli space of the representation ρ_1, usually called $E \cdot \mathbf{a}(\rho_0)$ where E is the energy of the object in movement and **a** *is the acceleration, explaining the name Force Law.*

We might also consider $(\mathfrak{c}_g)$, the δ-stable ideal generated by any one of these equation in $\mathrm{Ph}^\infty(C)$. *Since the force laws above hold in the dynamical system defined by (σ_g), we obviously have $(\mathfrak{c}_g) \subset (\sigma_g)$, and we might hope these new dynamical systems might lead to new Quantum Field Theories as defined above, with equally new and interesting properties.*

*One immediate result is that the restriction of the force law , in the commutative case, reduces to General Relativity as we have seen above, since we find the same geodesics, see [**?**],* For a connection ∇, on a free C-module E, the second Force Law above will now take the form in $\mathrm{End}_C(E)$,

$$\rho_E(d^2t_i)+\sum_{p,q}\Gamma^i_{p,q}\nabla_{\xi_p}\nabla_{\xi_q}$$
$$=1/2\sum_p F_{p,i}\nabla_{\delta_p}+1/2\sum_p\nabla_{\delta_p}F_{p,i}+1/2\sum_{l,q}\delta_q(\Gamma^{i,q}_l-\Gamma^{q,i}_l)\nabla_{\xi_l}+[\nabla_{\xi_i},\rho_E(T)],$$

where we, as above, have put $\rho(dt_i)=\nabla_{\xi_i}=\sum_j g^{i,j}\nabla_{\delta_j}$.

Notice that considering the representation ρ_Θ, corresponding to the Levi-Civita connection, the above translate into,

$$\rho(dt_i)=[Q,t_i],\quad \rho(d^2t_i)=\sum_{j=1}^d[Q,\rho(dt_i)],$$

where Q is the Laplace-Beltrami operator.

Given any *observable* $a\in\mathrm{Ph}(C)$, we would expect that the dynamics of the future *values* of a to be the spectrum of the operator,

$$f(\tau):=\exp(\tau\cdot\mathrm{ad}(Q))(a).$$

Collecting the above results and definitions, we may in the light of the general philosophy of this work, express what we have done as follows. Given the moduli space of "models", our "universe", assumed to be given as an affine space $\underline{C}=\mathrm{Spec}\,C$, "time" is defined by a metric g. The "furniture", i.e. the non-gravitational material content in the universe, is identified with the category of representations $\mathrm{Rep}(C)$. The dynamical properties of any such representation ρ_0 is "controlled" by the possible extensions of ρ_0 to $\mathrm{Ph}^\infty(C)$, and the time operator given by the Dirac derivation δ. The results of this subsection fuses the two notions of time. This fusion is given by the generic dynamical structure, $C(\sigma_g)$ of the k-algebra C, induced by a metric g, the Dirac Derivation δ in $C(\sigma_g)$, the time-evaluation, and Energy operator, $Q=g-T$. This we see is implying General Relativity and Quantum Theory, nicely related.

Remark 15. *What happens if we have a subspace of a space with a given metric, and want to compare the two possible dynamical structures? This is a problem that has been central to the development both of general relativity and cosmology, and we shall take a look at the general problem, but stick to the case where the spaces are affine algebraic varieties, and in fact, affine spaces,* $\underline{C}=\mathrm{Spec}(C)\subset\underline{B}=\mathrm{Spec}\,B$ *with*

$$\Phi:B=k[\underline{x}]\to C=k[\underline{t}],$$

the morphism of k-algebras corresponding to the inclusion. We then have polynomial functions, defining Φ,

$$x_i=x_i(t_1,...,t_d),‘;i=1,...,n.$$

Assume there is a metric $h = 1/2\sum h_{i,j}dx_idx_j \in \mathrm{Ph}(B)$, *so that we may consider the dynamical system* $B(\sigma_h)$. *A natural problem would be to try to find a metric* $g = 1/2\sum g_{i,j}dt_idt_j \in \mathrm{Ph}(C)$ *such that* Φ *induces a homomorphism of dynamical systems*

$$\phi : B(\sigma_h) \to C(\sigma_g).$$

It is easy to see that in $C(\sigma_g)$, *we have the formulas*

$$dx_k = \sum_j \frac{\partial x_k}{\partial t_j} dt_j + 1/2\sum_{i,j} \frac{\partial}{\partial t_i}(\frac{\partial x_k}{\partial t_j})g^{i,j}$$

such that,

$$[dx_k, x_l] = \sum_{i,j} \frac{\partial x_k}{\partial t_i}\frac{\partial x_l}{\partial t_j} g^{i,j}.$$

Therefore, the condition for, given the metric h *of* $\underline{B}$, *the existence of a compatible metric* g *in* $\underline{C}$ *is*

$$h^{k,l} = \sum_{i,j} \frac{\partial x_k}{\partial t_i}\frac{\partial x_l}{\partial t_j} g^{i,j}.$$

The analogy with the structure of the three fundamental forms in the study of unique time coordinates in general relativity, associated to a Cauchy hypersurface, is obvious.

8.9 The classical Gauge Invariance

The space of representations ρ of $C(\sigma_g)$ on a free (or projective) C-module V is given as above, by

$$\rho_\psi(t_i) = t_i,\ \rho_\psi(dt_i) = \sum_{l=1}^n g^{il}\delta_l + \psi_i,$$

where $\psi_i \in \mathrm{End}_C(V)$. The set of iso-classes is identified with the space of equivalence classes of the corresponding *potentials* $\psi := (\psi_1, \phi_2, ..., \psi_n)$. It does not form an algebraic variety, but it has a nice structure.

The set of *potentials* is not isomorphic to, but a *torsor* under

$$\mathcal{P} := (\mathrm{End}_C(V))^n.$$

If, however, $V = \Theta_C$, the Levi-Civita connection provides a natural isomorphism. The *tangent space* $\mathbf{T}_{(\rho_1,\rho_2)}$, between any two representations, $\rho_l : C(\sigma_g) \to \mathrm{End}_C(V_{\rho_l}),\ l = 1,2$, represented by elements $\psi(l) \in \mathcal{P},\ l = 1,2$, may also be identified with a quotient of $\mathcal{P}$. In fact,

$$\mathrm{Ext}^1_{C(\sigma_g)}(\rho_1, \rho_2) = \mathrm{Der}_k(C(\sigma_g), \mathrm{End}_k(V))/\,\mathrm{Triv}\,.$$

Any derivation $\xi \in \mathrm{Der}_k(C(\sigma_g), \mathrm{End}_k(V))$, maps the relations of $C(\sigma_g)$ to zero, so we shall have

$$[\xi(dt_i), t_j] + \rho_1(dt_i)\xi(t_j) - \xi(t_j)\rho_2(dt_i) = \xi(g^{i,j}).$$

Since V is a free C-module such that $\mathrm{Ext}^1_C(V, V) = 0$, there exists a linear map $\Phi_0 \in \mathrm{End}_k(V)$ such that $\xi(t_j) = t_j\Phi_0 - \Phi_0 t_j$, for all j. We may therefore, for a chosen ξ, assume all $\xi(t_i) = 0$, and it follows from the above equation that the derivation ξ is determined by the family of elements $\xi(dt_i) \in \mathrm{End}_C(V))$, $i = 1, ..., n$.

In case $\rho_1 = \rho_2 = \rho$, corresponding to $\psi \in \mathcal{P}$, we see that

$$\mathbf{T}_\rho = \mathrm{Ext}^1_{C(\sigma_g)}(\rho, \rho) = \mathrm{Hom}(\mathrm{Ph}(C), \mathrm{End}_C(V))/\,\mathrm{Triv}$$

where the the trivial derivations, mapping t_i to 0, are exactly those given by the n-tuples

$$((\sum_j^n g^{1,j}(\frac{\partial\Phi}{\partial t_j}) + [\psi_1, \Phi]), ..., (\sum_{j=1}^n g^{n,j}(\frac{\partial\Phi}{\partial t_j}) + [\psi_n, \Phi]))$$

for some $\Phi \in \mathrm{End}_C(V)$ by

$$W(dt_i) = (\sum_j^n g^{i,j}(\frac{\partial\Phi}{\partial t_j}) + [\psi_i, \Phi]).$$

The expression

$$\Phi(\psi) := (\xi_1(\Phi) + [\psi_1, \Phi], ..., \xi_n(\Phi) + [\psi_n, \Phi])$$

therefore corresponds to an infinitesimal gauge transformation

$$\Phi \in \mathrm{Der}_k(\mathcal{P})$$

of the space $\mathcal{P}$ of representations of $C(\sigma_g)$, acting linearly like

$$\Phi(\rho_0 + \psi) = (\rho_0 + \psi) + \Phi(\psi)$$

The *physical* relevant space is therefore the quotient

$$\mathbf{P} = \mathcal{P}/\mathfrak{h}$$

of $\mathcal{P}$ with respect to the action of the abelian Lie algebra $\mathfrak{h} := \mathrm{End}_C(V)$.

As in the finite dimensional situation, the Dirac derivation, here $\delta = \mathrm{ad}(g - T)$, induces a vector field

$$[\delta] \in \Theta_{\mathbf{P}},$$

so long as we by vector field understand any map which to an element ψ in $\mathbf{P}$ associates an element in its tangent space, i.e. in $\mathrm{Ext}^1_{C(\sigma)}(V_\rho, V_\rho)$ for $\rho = \rho_0 = \psi$. It must however vanish at ρ, since the Dirac derivation $\delta = \mathrm{ad}(g - T)$, necessarily must be mapped to a trivial derivation in $\mathrm{Der}_k(C(\sigma_g), \mathrm{End}_k(V))$, therefore to 0 in $\mathrm{Ext}^1_{C(\sigma_g)}(V_\rho, V_\rho)$. But then it corresponds to an infinitesimal transformation of V as we have seen,

$$[\delta] = \mathrm{ad}(Q_h + [\psi] + Q_v),$$

meaning that

$$\rho_\psi(d^{n+1}t_i) = [Q_h + [\psi] + Q_v, \rho_\psi(d^n t_i)].$$

This may be interpreted as saying that time, defined by the dynamical structure (σ_g), acts in all orders, within each representation $\rho_\psi : C(\sigma_g) \to \mathrm{End}_k(V)$.

Remark 16. *The physicists usually write $\delta\phi := \Phi(\phi)$, not caring to mention Φ, taking for granted that $\delta\phi := \delta_\Phi(\phi)$ stands for an infinitesimal movement of ϕ in the direction of Φ, and call the transformation above an* infinitesimal gauge transformation. *The literature on gauge theory and its relation to noncommutativity of space and to quantization of gravity, is huge. We think that the introduction of the* noncommutative phase space, *and in the metric case the* generic *dynamical system*

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j}),$$

*can to some degree, elucidate the philosophy behind this effort. See e.g. the papers, [**?**], and [**?**], where the authors initially introduce noncommutativity in the ring of* observables *generated by* coordinates $\hat{x}^\nu$ *by imposing*

$$[\hat{x}^\nu, \hat{x}^\mu] = \Theta^{\nu,\mu},$$

where $\Theta^{i,j}$ are constants.

The above treatment of the notion of gauge groups and gauge transformations may also explain why, in physics, one considers potentials *as* interaction carriers, *thus as particles* mediating force *upon other particles. And maybe one can also see why the notion of* Ghost Fields or Particles *of Faddeev and Popov, comes in. It seems to me that the introduction of ghost particles is linked to working with a particular section of the quotient map $\mathcal{P} \to \mathbf{P}$.*

The Dirac derivation, which is entirely dependent upon the notion of a noncommutative phase space, is not (explicitely) found in present day physics. The parsimony principle is therefore normally introduced via the construction of a Lagrangian and an Action Principle, i.e. a function of the (assumed physically significant) variables, the fields and their derivatives, defined in $\mathcal{P}$, assumed to to be invariant under the gauge transformations so really defined in $\mathbf{P}$ and supposed to stay stable during time *development, see [**?**]. A non trivial element in the toolbox of the physicists helping them to guess the Lagrangian is the Chern-Simons functional, and that we now spell out in some generality. Since everything we have done above is functorially (natural), we may work on nonsingular schemes instead of commutative k-algebras.*

Let us as above assume given a a metric g on some scheme X, and that we have an affine covering given in terms of a family of commutative k-algebras

C_α and a bundle V defined on X corresponding to a set of representations $\rho_0 : C = C_\alpha \to \mathrm{End}_k(V)$. Let $\rho : O_X(\sigma_g) \to \mathrm{End}_k(V)$ be a momentum at ρ_0, i.e. an extension of ρ_0 to $O_X(\sigma_g)$. Then we know that ρ induces a connection on V, and we have denoted by $\mathcal{P} := \mathrm{End}_{O_X}(V)^d$, the set of such connections, or if we want to, the set of representations $\rho : O_X(\sigma_g) \to \mathrm{End}_k(V)$. Recall that there is a unique 0-object *in $\mathcal{P}$, given the metric, by $\rho_1(dt_i) = \xi_i$ such that all other representations ρ is ρ_1 plus a potential $\psi \in \mathrm{End}_{O_X}(V)^d$.*

Consider now the class in $\mathrm{ch}^n(\rho) \in \mathrm{HH}^n(O_X, \mathrm{End}_k(V))$, the Chern-classes defined by the Hochschild co-chain, the k-linear map

$$\mathrm{ch}^n : C_\alpha^{\otimes n} \to \mathrm{End}_k(V)$$

defined by

$$\mathrm{ch}^n(c_1 \otimes c_2 ..., \otimes c_n) = \rho_1(dc_1 dc_2 ... dc_n) \in \mathrm{End}_k(V).$$

We know that ch^n *is a co-cycle since*

$$\delta(\mathrm{ch}^n)((c_1 \otimes c_2 ... \otimes c_{n+1})) = c_1\rho_1((dc_2dc_3...dc_{n+1})) \tag{8.1}$$

$$+ \sum_1^n (-1)\rho_1((dc_1...d(c_ic_{i+1})...dc_{n+1})) + (-1)^{n+1}\rho_1(dc_1...dc_n)c_{n+1} = 0 \tag{8.2}$$

The Generalized Chern-Simons Class *of ρ, is then the class in the obvious double complex, defined by the covering $\{C_\alpha\}$, and the classes*

$$\mathrm{ch}^n(\rho) \in \mathrm{HH}^n(C, \mathrm{End}_k(V))$$

defined by $1/n!\ \mathrm{ch}^n$.

Let $\Phi \in \mathfrak{h} := \mathrm{End}_C(V)$ and consider Φ as a Hochschild 0*-cocycle*

$$\Phi : k = C^0 \to \mathrm{End}_k(V),\ \Phi(\alpha) = \alpha\Phi,$$

then

$$\delta\Phi(c) = \rho(dc)\Phi - \Phi\rho(dc).$$

In particular,

$$\delta\Phi(t_i) = [\xi_i + \psi_i, \Phi] = \xi_i(\Phi) + [\psi_i, \Phi].$$

This proves that

$$\mathrm{ch}^1(\rho_1 + \psi) = \mathrm{ch}^1(\rho_1 + \psi + \Phi(\psi))$$

i.e. that the Ghost Fields *have well defined Chern-Simons class, or that the Chern-Simons class is a well defined functional*

$$\mathrm{ch}^* : \mathbf{P} \to \mathrm{HH}^*(C, \mathrm{End}_k(V)).$$

8.10 A Generalized Yang-Mills Theory

Given a metric $g \in \mathrm{Ph}(C)$ as above, and assume there is a *gauge group*, i.e. a k- Lie algebra $\mathfrak{g}_0$ operating on $C = k[t_1, .., t_d]$ with extension to $C(\sigma_g)$, where σ is a dynamical structure. Normally σ would be σ_g, and $\mathfrak{g}_0 = \mathrm{Der}_k(C)$ since all representations of $C(\sigma_g)$ have a C-connection. Clearly therefore, the Lie algebra of Killing vectors $\mathfrak{o}(g)$ is contained in $\mathfrak{g}_0$.

In the general situation we have, together with the global gauge group $\mathfrak{g}_0$, also a local gauge group, i.e. a C-Lie algebra $\mathfrak{g}_1$ acting insensitively upon the representations V that pops up in our theory, and also insensitive to the action of $\mathfrak{g}_0$ see subsection (1.8). We therefore have a $C(\sigma)$-connection

$$\mathfrak{D} : \mathfrak{g}_0 \to \mathrm{Der}_k(\mathfrak{g}_1)$$

which is a kind of a general spin structure, or *Coupling Morphism.* Recall that when the curvature of $\mathfrak{D}$ vanish, $\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie-algebroid.

Let us pause to consider a non-trivial example of this situation.

Theorem 16. *Assume $\mathfrak{g}_1^0$ is a finite dimensional k-Lie algebra and consider the C-Lie algebra $\mathfrak{g}_1 := \mathfrak{g}_1^0 \otimes_k C$. Consider a representation*

$$\rho : C(\sigma_g) \to \mathrm{End}_k(V)$$

with gauge global group $\mathfrak{g}_0 = \mathrm{Der}_k(C)$ and local gauge group $\mathfrak{g}_1 \subset \mathrm{End}_C(V)$. Let the $\mathfrak{g}_0$-connection

$$\mathfrak{D} : \mathfrak{g}_0 \to \mathrm{Der}_k(\mathfrak{g}_1)$$

be defined as

$$\mathfrak{D}(\xi_i) = \mathrm{ad}(\nabla_{\xi_i} + \psi_i), \textit{with } \psi_i = g^{i,l}\gamma_l,\ \ \gamma_l \in \mathfrak{g}_1^0.$$

Then we find that the curvature of $\mathfrak{D}$ is

$$F_{i,j}^{\psi} = \mathrm{ad}([\psi_i, \psi_j]).$$

It vanishes if and only if

$$[[\gamma_i \gamma_j], \gamma] = 0$$

for all $\gamma \in \mathfrak{g}_1, \forall i, j = 1, ..., n$.
Put now

$$\rho_{\psi,A} : C(\sigma_g) \to \mathrm{End}_k(V),$$

defined by $\rho_1(dt_i) = \nabla_\xi + \psi_i + A_i$ where $A_i \in C$ is a usual potential. Then the curvature of $\rho_{\psi,A}$ is equal to

$$F_{i,j} = \mathrm{ad}([\psi_i, \psi_j]) + \xi_i(A_j) - \xi_j(A_i).$$

Proof. By definition, the curvature of $\mathfrak{D}$ is

$$F_{i,j} := [\mathfrak{D}(\xi_i), \mathfrak{D}(\xi_j)] - \mathfrak{D}([\xi_i, \xi_j]),$$

where

$$[\mathfrak{D}(\xi_i), \mathfrak{D}(\xi_j)] = [\mathrm{ad}(\xi_i) + \mathrm{ad}(\psi_i), \mathrm{ad}(\xi_j) + \mathrm{ad}(\psi_j)]$$
$$= \mathrm{ad}([\xi_i, \xi_j]) + \mathrm{ad}([\xi_i, \psi_j]) + \mathrm{ad}([\psi_i, \xi_j]) + \mathrm{ad}([\psi_i, \psi_j])$$

and

$$\mathfrak{D}([\xi_i, \xi_j]) = \sum_k c_{i,j}^k \mathfrak{D}(\xi_k) = \sum_k c_{i,j}^k ad(\xi_k) + \sum_k c_{i,j}^k ad(\psi_k),$$

where $c_{i,j}^k = (\Gamma_k^{j,i} - \Gamma_k^{i,j})$. Moreover,

$$\mathrm{ad}([\xi_i, \psi_j]) + \mathrm{ad}([\psi_i, \xi_j]) = \mathrm{ad}(\xi_i(\psi_j)) - \mathrm{ad}(\xi_j(\psi_i)).$$
$$\xi_i(\psi_j) - \xi_j(\psi_i) = (g^{i,l}\frac{\partial g^{j,k}}{\partial t_l} - g^{j,l}\frac{\partial g^{i,k}}{\partial t_l})\gamma_k$$
$$= (-g^{i,l}\sum(g^{j,t}\Gamma_{t,l}^k + g^{t,k}\Gamma_{t,l}^j) + g^{j,l}\sum(g^{i,t}\Gamma_{t,l}^k + g^{t,k}\Gamma_{t,l}^i))\gamma_k$$
$$= (\Gamma_t^{j,i} - \Gamma_t^{i,j})g^{t,k}\gamma_k$$
$$= \sum_k c_{i,j}^k \psi_k,$$

where we have used the standard formulas for the derivatives of the $g^{i,j}$. Therefore,

$$\begin{aligned}F_{i,j} :&= [\mathfrak{D}(\xi_i), \mathfrak{D}(\xi_j)] - \mathfrak{D}([\xi_i, \xi_j])\\ &= \mathrm{ad}([\xi_i, \xi_j]) + \sum_k c_{i,j}^k \,\mathrm{ad}(\psi_k) + \mathrm{ad}([\psi_i, \psi_j] - \sum_k c_{i,j}^k ad(\xi_k) - \sum_k c_{i,j}^k ad(\psi_k)\\ &= \mathrm{ad}([\psi_i, \psi_j].\end{aligned}$$

The rest is easily seen. □

Notice the intuitive link to Dirac spinors.

Anyway, in case $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebroid, the action of $\mathfrak{g}_1$ and the connection $\mathfrak{D}$ extends to a connection

$$\mathfrak{D} : \mathfrak{g} \to \mathrm{End}_k(V).$$

According to our philosophy, we should therefore consider (as moduli space, for our models), the invariant (or quotient) space

$$\mathrm{Simp}(C)/\mathfrak{g} \simeq \mathrm{Simp}(C(\mathfrak{g})).$$

Now we have,

Theorem 17 (Scholie)**.** *Consider a representation*

$$\rho_0 : C \to \mathrm{End}_k(V)$$

with gauge group $\mathfrak{g}$ *as above, and an extension* $\rho : C(\sigma_g) \to \mathrm{End}_k(V)$ *considered as a* momentum *of* ρ_0*, and as a reference point in the set of connections* $\mathcal{P}$*, on* (V, ρ_0)*.*

The tangent space of $\mathcal{P}$*, at any* ρ *is of the same form, i.e. a quotient* $\mathbf{P}$ *of* $\mathcal{P}$ *by the action of the Lie algebra* $\mathrm{End}_C(V)$*. However, we know that the first order time-development, the Dirac derivation,* $\delta = \mathrm{ad}(g - T)$ *induces the trivial vector field on* $\mathcal{P}$*, so the first order time-development of the* ρ_0 *given by any* $\rho_\psi := \rho + \psi$*,* $\psi \in \mathcal{P}$*, does not have a time-development as a representation given by the Dirac derivative* δ *of* $C(\sigma_g)$*. We know that the time development on* V *is given by the formula*

$$[\delta] = \mathrm{ad}(Q_h + [\psi] + Q_v),$$

meaning that

$$\rho_\psi(d^{n+1}t_i) = [(Q_h + [\psi] + Q_v), \rho_\psi(d^n t_i)].$$

This takes care of the complete time development of an endomorphism of V*, as well as for a state vector* $\phi \in V$*.*

The second order time-development is also given in terms of the Force Law in $\mathrm{Ph}(C)$*,*

$$\begin{aligned} d^2 t_i = &- \sum_{p,q} \Gamma^{\mathrm{id}}_{p,q} t_p dt_q - \sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q} [F_{i,q}, dt_p] \\ &+ 1/2 \sum_{l,p,q} g_{p,q} [dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)] dt_l + [dt_i, T]. \end{aligned}$$

Given the gauge group $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$*, there are tangent directions* $\mathfrak{v} = \{\mathfrak{v}_i\}$ *with* $\mathfrak{v}_i = \sum_j \gamma_j g^{j,i}$*, and* $\gamma_i \in \mathfrak{g}_1$*, "normal" to* C *in* V*, naturally related to derivations* $\mathfrak{D} : \mathfrak{g}_0 := \mathrm{Der}_k(C) \to \mathfrak{g}_1 := \mathfrak{g}_1^0 \otimes C$*. Put*

$$[\psi] := \sum_i \gamma_i \nabla_{\xi_i} = \sum_{i,j} \gamma_j g^{j,i} \frac{\partial}{\partial t_i} = \sum_i \mathfrak{v}_i \frac{\partial}{\partial t_i} = \mathfrak{v}.$$

Then we may formally write the time operator in the same form as in our general Quantum Field Theory, see **??**

$$[\delta] = \mathrm{ad}(Q_h + [\psi] + Q_v) \in \mathrm{Der}(\mathrm{End}_k(\tilde{V}))$$

where

$$Q_h := \rho(g-T) = Q := 1/2\sum_{i,j} g^{i,j}\nabla_{\delta_i}\nabla_{\delta_j}$$
$$[\psi] := \sum_i \psi_i\nabla_{\delta_i}$$
$$Q_v := \psi(g-T) = 1/2\sum_{i,j} g_{i,j}\psi_i\psi_j + 1/2(\sum_{j,l}\Gamma^j_{j,l}+\bar{\Gamma}^j_{j,l})\psi_l$$

with the corresponding first order time-action in the state-space defined by

$$[\delta] = [\mathfrak{v}] + Q \in \mathrm{End}_k(V).$$

Given a classical potential $A = \{A_i\}, A_i \in C$, *put*

$$\rho_{\psi,A} : C(\sigma_g) \to \mathrm{End}_k(V),$$

defined by $\rho_1(dt_i) = \nabla_\xi + \psi_i + A_i$. *Then the curvature of* $\rho_{\psi,A}$ *is equal to*

$$F_{i,j} = \mathrm{ad}([\psi_i,\psi_j]) + \xi_i(A_j) - \xi_j(A_i).$$

Remark 17. *As we shall see, the gauge groups of the standard model, i.e. the Lie algebras*

$$\mathfrak{g}' := \mathfrak{u}(1)\times\mathfrak{sl}(2) \subset \mathfrak{g}_1' := \mathfrak{u}(1)\times\mathfrak{su}(2)\times\mathfrak{su}(3),$$

which is part of our toy model, *see [?], also pops up in our cosmological model, and now as a real local gauge group in the above sense. The elementary particles in that model should therefore, in line with the usage of present quantum theory, be the points of* $\mathrm{Simp}(\mathrm{Ph}(\tilde{H}))/\mathfrak{g} = \mathrm{Simp}(\mathrm{Ph}(\tilde{H})(\mathfrak{g}))$. *In fact, we shall see that an impressive part of the structure of the Standard Model is contained in the structure of this* non-commutative quotient (or invariant) space.

The above is a generalization of the physicists treatment of the type of representations,

$$\rho_W : \mathrm{Ph}(C) \to \mathrm{End}_k(V),$$

parametrized by what they call *Gauge Fields*, the $W_i^l \in C$, in the following formula,

$$\rho_W(t_i) = t_i,\ \rho_W(dt_i) = \sum_{l=1}^n g^{il}\frac{\partial}{\partial t_l} + \sum_{l=1}^r W_i^l\gamma_l.$$

Put $W_i := \sum_{l=1}^r W_i^l\gamma_l$ and let as above $\xi_i = \sum_{l=1}^r g^{il}\frac{\partial}{\partial t_l}$, and denote by $C^p_{l,m}$ the structural constants of $\mathfrak{g}_1$ such that

$$[\gamma_l,\gamma_m] = \sum_{p=1}^r C^p_{l,m}\gamma_p.$$

Now, recall that $[\xi_i, \xi_j] = \sum_k (\Gamma_k^{j,i} - \Gamma_k^{i,j})\xi_k$, and consider the curvature of this representation,

$$F_{i,j} = [\rho(dt_i), \rho(dt_j)] - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j})\rho(dt_p) = [\nabla_{\xi_i}, \nabla_{\xi_j}] - \nabla_{[\xi_i,\xi_j]}.$$

$$\begin{aligned} F_{i,j} &= \rho_W([dt_i, dt_j]) - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j})\rho_W(dt_p) \\ &= \sum_{l=1}^{r} (\nabla_{\xi_i}(W_j^l \gamma_l) - \nabla_{\xi_j}(W_i^l \gamma_l) + \sum_{l,m,p=1}^{r} C_{l,m}^p W_i^l W_j^m \gamma_p \\ &- \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j}) W_p^l \gamma_l. \end{aligned}$$

Put $F_{i,j} = \sum F_{i,j}^l \gamma_l$, then we obtain the equation

$$F_{i,j}^l = (\xi_i(W_j^l) - \xi_j(W_i^l)) + \sum_{p,m=1}^{r} C_{p,m}^l W_i^p W_j^m - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j}) W_p^l$$

which, if $\sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j}) W_p^l = 0$, is the classical expression for the curvature in this case.

The Euler-Lagrange equations of the Lagrangian

$$\mathbf{L}_{gf} = -1/4 F^{\mu\nu\alpha} F_{\mu\nu}^{\alpha},$$

used by the physicists gives us the corresponding equation of motion,

$$\xi_\mu F_{\mu\nu}^a + c_{ab}^c W^{\mu b} F_{\mu\nu}^c = 0.$$

The *Yang-Mills equation* corresponding to the vanishing of

$$1/2 \sum_{j=1}^{n} [F_{i,j}, \xi_j + W_j].$$

With a *Source* added, it looks like

$$\xi_\mu F_{\mu\nu}^a + c_{ab}^c W^{\mu b} F_{\mu\nu}^c = -J_\nu^a.$$

In the general metric case with non-abelian gauge group it is difficult to find gauge invariant Lagrangians of reasonable physical relevance, so we have to operate differently. Here is where the Generic Equation of Motion above comes in and give us equations of motion quite generally. Recall that the Force Law is given by

$$d^2t_i = -\sum_{p,q}\Gamma^i_{p,q}dt_pdt_q - 1/2\sum_{p,q}g_{p,q}(F_{i,p}dt_q + dt_pF_{i,q})$$
$$+ 1/2\sum_{l,p,q}g_{p,q}[dt_p,(\Gamma^{i,q}_l - \Gamma^{q,i}_l)]dt_l + [dt_i,T].$$

When the metric is Euclidean, or Minkowski, this reduces to

$$d^2t_i = -1/2\sum_p g_{p,p}(F_{i,p}dt_p + dt_pF_{i,p}) = -\sum_p g_{p,p}F_{i,p}dt_p - 1/2\sum_p g_{p,p}[dt_p,F_{i,p}].$$

Therefore,

$$\rho(d^2t_i) = -\sum_{p,m}g_{p,p}F_{i,p}\rho(dt_p) - 1/2\sum_{p,m}g_{p,p}\frac{\partial}{\partial t_p}(F^m_{i,p})\gamma_m - 1/2\sum_{p,l,m}g_{p,p}W^l_pF^m_{i,p}c^q_{l,m}\gamma_q,$$

where we let the curvature conserve its name, $F_{i,j} := \rho(F_{i,j})$. The Yang-Mills equation above is now seen to imply,

$$\rho(d^2t_i) = -\sum_p g_{p,p}F_{i,p}\rho(dt_p).$$

This, however indicates that the tangent to the representation $\dot{\rho}$ given by

$$\psi = (\psi_i) = (-1/2\sum_{p,m}g_{p,p}\frac{\partial}{\partial t_p}(F^m_{i,p})\gamma_m - 1/2\sum_{p,l,m}g_{p,p}W^l_pF^m_{i,p}c^q_{l,m}\gamma_q),$$

is a classical 0-tangent, so the Yang-Mills equation should just tell us that there exists a potential $\Phi \in \mathrm{End}_k(V)$, such that

$$(\psi_i) = ((\sum_j^n g^{1,j}(\frac{\partial\Phi}{\partial t_j}) + [\phi_1,\Phi]),...,(\sum_{j=1}^n g^{n,j}(\frac{\partial\Phi}{\partial t_j}) + [\phi_n,\Phi])).$$

Compare with the Lorentz Force Law, classically and for an electric field,

$$\mathbf{a}_i = -\sum_{p=1}^n F_{i,p}v_p.$$

Interpreting $\rho(d^2t_i) = m\mathbf{a}_i$ and $\rho(dt_p) = mv_p$, we recover the classical equation of movement in a field, see [?], p. 115, and as we shall see in Example **??**, we may use this to deduce Maxwell and Bloch's equations for the time development of spin, including the Seiberg-Witten monopole equation.

8.11 Reuniting GR, Y-M and General Quantum Field Theory

Let us have a look at the significance of the conclusion of the earlier sections, and the last theorems.

We have, for every polynomial algebra C, outfitted with some metric g, proved that there exist a derivation

$$\delta := \mathrm{ad}(g - T) \in \mathrm{Der}_k(\mathrm{Ph}(C)),$$

such that it coincides with the canonical derivation $d : C \to C(\sigma_g)$ in the generic dynamical system (GDS). The corresponding force laws in $\mathrm{Ph}(C)$ generates equations of motions in General Relativity (GR), as well as in the generalised Yang-Mills (YM) theory introduced above. In fact, we find a very satisfying identity between the notions of *Time* in GDS, GR and in YM. The *Dirac derivation* $\delta = \mathrm{ad}(g-T)$ in GDS, inducing a quantum field theory (QFT) on the space **P** where $[\delta]$ and the Hamiltonian Q are deduced from the general Laplace-Beltrami operator, $Q = \mathrm{ad}(g - T)$.

Since T vanish in $\mathrm{Ph}(C)^{\mathrm{com}}$, the time in GR reduces to the Dirac derivation

$$[\delta] = \sum_l (\xi_l \frac{\partial}{\partial t_l} - \Gamma^l \frac{\partial}{\partial \xi_l}),$$

and a trivial Hamiltonian. The Schrödinger equation in GDS is given as

$$(Q - E)(\psi) = 0.$$

and we shall come back to our generalisation of QFT.

To unify Quantum Field Theory (QFT) and (GDS) and GR, we might have started with the dynamical system $C(g)$ generated by, say, the Force Laws of type (1) or (2). Unluckily the structure of this system in general, seems to be very complicated. It is, for example not easy to decide whether or not $C(g)$ has finite dimensional representations at all. Anyway, it seems that we may be content with the above structures, all deduced from GDS, since physicists probably do not know how to include the second order momentum in the preparation of their experiments. That shortcoming in the DGT is therefore not yet a big theoretical problem.

However, we may study an obvious unification of GR, GDS, and GQT, where the algebra of observables is $\mathrm{Ph}^\infty(C)$, and we shall show that we are able to classify, i.e. compute the moduli space of, the finite dimensional representations of $\mathrm{Ph}^\infty(C)$, even though this space will turn out to be of infinite dimension. Thus we may hope to extend our earlier methods, and obtain a unified theory. There are however lots of problems involved in this scheme, one is the action of the gauge groups that turns up. Another is the philosophically maybe reasonable, but very unpopular, consequence of this restriction of the theory to just the finitely defined measurable entities: Our Space, and everything else modelled by such a theory, would be discrete, simple objects would have point-like structures, etc.

The computation of the moduli space of finite dimensional representations of $\mathrm{Ph}^\infty(C)$ mentioned above, and the further analysis of the resulting "Quantum Field Theory" will for economical reasons be postponed and fused with the results above on generic action of equation of time in later work.

To go further, we should have to go back to our philosophy and ask ourselves why we are capable of identifying and communicate our sense of natural objects, or rather the impressions that we have about such objects, with their relations like distances between them and the like. One obvious answer which is at the base of this paper, is that we assume we have the notion of Time as a metric on our moduli space of the events we think we have identified and that this clock is running smoothly such that coupled with the universal constancy of the velocity of light, this makes most objects look the same today as yesterday up to obvious shifts corresponding to symmetries that we have called gauge symmetries. This is the reason for studying the dynamic structure (σ_g) corresponding to one fixed metric g. In a sense we assume that most of the Furniture that we have identified as representations of the algebra (or if one wish, of the scheme) of observables. stays constant up to an understandable gauge. This is tantamount to the assumption that our world stays reasonably constant even though we know, and clearly see, that there are cataclysms in the Universe, completely changing objects .

Now this is also the reason why we have to go into a more technical relationship between our Time g, and the Furniture, i.e. the representations, V. First, let us admire a commutative diagram that will help us through the arguments.

$$
\begin{tikzcd}
C \arrow[r, "i"] & \mathrm{Ph}(C) \arrow[r] \arrow[d] \arrow[ddr, "\rho_1"] & \mathrm{Ph}^2(C) \arrow[r] \arrow[dd, "\rho_2"] & \mathrm{Ph}^3(C)\ldots \arrow[r] \arrow[ddl, "\rho_3"] & \mathrm{Ph}^\infty(C)\circlearrowright^{\delta} \\
C \arrow[u] \arrow[d] \arrow[r, "i"] & C(\sigma_g)\circlearrowright^{\delta_g} \arrow[dr, "\rho"] & & & \\
C \arrow[d] \arrow[r, "i"] & \mathrm{Ph}(C)(\mathrm{com}) \arrow[r] \arrow[d, "\pi"] & \mathrm{End}_k(V) & & \\
C \arrow[r, "i"] & \mathrm{End}_C(C) & & &
\end{tikzcd}
$$

Here $\delta_g = \mathrm{ad}(g-T)$, and π an is arbitrary representation,

$$\mathrm{Ph}(C)(\mathrm{com}) := \mathrm{Ph}(C)/([dt_i, t_j], [dt_i, dt_j]) \to \mathrm{End}_C(C).$$

The morphisms $\rho_p,\ p = 1, 2, ..$ are uniquely defined by

$$\begin{aligned}
\mathrm{ad}(g-T)\circ\rho_1 =& d\circ\rho_2 : \mathrm{Ph}(C)\to \mathrm{End}_k(V)\\
\mathrm{ad}(g-T)\circ\rho_2 =& d\circ\rho_3 : \mathrm{Ph}^2(C)\to \mathrm{End}_k(V)\\
\mathrm{ad}(g-T)\circ\rho_n =& d\circ\rho_{n+1} : \mathrm{Ph}^n(C)\to \mathrm{End}_k(V), n\geq 1
\end{aligned}$$

We shall now be able to prove a theorem that makes a mathematically reasonable relationship between Time and Furniture, or rather between the infinitesimal changes of one, and the infinitesimal changes of the other. Until now we have had essentially two different notions of time, the metric of our moduli space of our models, and the Dirac derivative of $\mathrm{Ph}^\infty(-)$ of the same. Now we find that the relationship between these notions is tight.

But first let us recall that we have taken the liberty of working with two notions of metric. First the classical metric $\overline{g} := (g_{i,j})$, and then the element $g := 1/2\sum_{i,j} g_{i,j}dt_i dt_j \in \mathrm{Ph}(C),$. Obviously $\overline{g} = \mathrm{sym}(g)$, the symmetrization of g. The context will clearly show which one we are talking about, so we shall continue talking about the metric g.

Theorem 18. *Let $\mathbf{M}$ be the space (of isomorphism classes) of metrics on C. For every point $g\in\mathbf{M}$ consider the diagram*

$$\begin{tikzcd}
 & \mathrm{Ph}(C) \arrow[r] \arrow[d] \arrow[dr, "\rho_1"] & \mathrm{Ph}^2(C) \arrow[r] \arrow[d, "\rho_2"] & \mathrm{Ph}^3(C)\dots \arrow[r] \arrow[dl] & \mathrm{Ph}^\infty(C)\circlearrowright^{\delta} \\
C \arrow[ur, "d"] \arrow[r, "i"'] & C(\sigma_g) \arrow[r, "\rho"'] & \mathrm{End}_k(V) & &
\end{tikzcd}$$

Where $\rho_0 := i\circ\rho$ is a representation of C and ρ a momentum of ρ_0. Let ρ_1 be the induced representation of $\mathrm{Ph}(C)$. *Consider the family of k-algebras*

$$\mu : \mathbf{C}(\sigma)\to\mathbf{M}$$

indexed by the possible metrics of C such that $C(\sigma_g)$ corresponds to $g\in\mathbf{M}$. Let

$$\mathbf{T}_{\mathbf{M},g} = \{(h_{i,j})\},\ h_{i,j} = h_{j,i}\in C$$

be the tangent space to $\mathbf{M}$ at g. Define $h^{i,j}$ by

$$h_{i,j} = -\sum_{p,q} g_{i,p}h^{p,q}g_{q,j},\ h=\{h_{i,j}\}\in\mathbf{T}_{\mathbf{M}}.$$

Consider now the first order deformation of the metric $g\in\mathrm{Ph}(C)$, $g+\epsilon h\in\mathrm{Ph}_{k[\epsilon]}(C\otimes k[\epsilon]) = \mathrm{Ph}(C)\otimes k[\epsilon]$ and the corresponding Dirac derivation $\mathrm{ad}(g+\epsilon h-T')$ in $\mathrm{Ph}_{k[\epsilon]}(C\otimes k[\epsilon])$. Put $d't_i := \mathrm{ad}(g+\epsilon h-T')(t_i)$.

Then we find, in $C(\sigma_g)\otimes k[\epsilon]$,

$$[d't_i, t_j] = g^{i,j} - h^{i,j}\epsilon.$$

Moreover, the ρ_1-derivation $\eta : \mathrm{Ph}(C) \to \mathrm{End}_k(V)$ defined by

$$\eta(t_i) = 0,\ \eta(dt_i) = \sum_{l,q} h^{i,l}\rho_1(g_{l,q}dt_q) = \sum_l h^{i,l}\nabla_{\delta_l} \in \mathrm{Diff}^1(V,V)$$

corresponds to a 1.-order derivative of ρ_1, i.e. to the morphism

$$\eta(\rho_1) := \rho_2 : \mathrm{Ph}^2(C) \to \mathrm{End}_k(V),$$

for which $\rho_2(d^2t_i) = \eta(dt_i)$. This induces an element

$$\eta(h) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)$$

where V is the representation ρ_1 producing an injective map

$$\eta : \mathbf{T}_{\mathbf{M},g} \to \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)$$

onto the linear subspace

$$\mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)^{(1)} \subset \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)$$

of first order *non-trivial tangent space of the* Ph(C)*-representation (ρ_1,V), defined by the derivations $\eta : \mathrm{Ph}(C) \to \mathrm{End}_k(V)$ where for all i, $\eta(dt_i) \in \mathrm{Diff}^1(V,V)$.*

Thus, any non-trivial deformation of the metric g induces a non-trivial deformation of the Ph(C)*-representation (ρ_1,V), and any* first order *non-trivial deformation of the* Ph(C)*-representation (ρ_1,V) induces a non-trivial deformation of the metric.*

Proof. Consider the first order deformation $g+\epsilon h$ of the metric g. The corresponding derivation $\mathrm{ad}(g-T)$ of $\mathrm{Ph}(C)$ defines the Dirac derivation in $C(\sigma_g)$ and the derivation $(\mathrm{ad}(g+\epsilon h - T')$ defines a derivation in $\mathrm{Ph}(C)\otimes k[\epsilon]$. Here

$$T = -1/2\sum_{i,j,l}\frac{\partial g_{i,j}}{\partial t_l}g^{l,i}dt_j,$$

and

$$T' = -1/2\sum_{i,j,l}(\frac{\partial(g_{i,j}+h_{i,j}\epsilon)}{\partial t_l})(g^{l,i}+h^{l;i}\epsilon)dt_j$$

where $h^{i;i}$ is defined by

$$h_{i,j} = -\sum_{p,q} g_{i,p}h^{p,q}g_{q,j}.$$

We find in the quotient, $C(\sigma_g)\otimes k[\epsilon]$,

$$
\begin{aligned}
d't_i &= [(g+h\epsilon)-T',t_i] \\
&= [g-T+1/2\epsilon\sum_{p,q,l} h_{p,q}dt_p dt_q + 1/2\epsilon\sum_{p,q,l}(\frac{\partial g_{p,q}}{\partial t_l}h^{l,p}dt_q + \frac{\partial h_{p,q}}{\partial t_l}g^{l,p}dt_q),t_i]) \\
&= dt_i + 1/2\epsilon\sum_{p,q,l}(h_{p,q}g^{p,i}dt_q + h_{p,q}dt_p g^{q,i} + \frac{\partial g_{p,q}}{\partial t_l}h^{l,p}g^{q,i} + \frac{\partial h_{p,q}}{\partial t_l}g^{l,p}g^{q,i}) \\
&= dt_i + \epsilon\sum_{l,q}(-h^{i,l}g_{l,q})dt_q + \epsilon P,\ P\in C.
\end{aligned}
$$

where

$$P = 1/2(h_{p,q}\xi_p(g^{q,i}) + \partial\partial g_{p,q}\partial t_l h^{l,p}g^{q,i} + \frac{\partial h_{p,q}}{\partial t_l}g^{l,p}g^{q,i}).$$

Here we have used

$$
\begin{aligned}
h_{p,q}dt_p g^{q,i} &= h_{p,q}g^{q,i}dt_p + h_{p,q}[dt_p,g^{q,i}], \\
h_{p,q}[dt_p,g^{q,i}] &= h_{p,q}\frac{\partial g^{q,i}}{\partial t_p}.
\end{aligned}
$$

From this follows

$$[d't_i,t_j] = g^{i,j} - h^{i,j}\epsilon.$$

Now, consider the derivation

$$\eta\in \mathrm{Der}_k(\mathrm{Ph}(C),\mathrm{End}_k(V))$$

defined in $\mathrm{End}_k(V)$ by

$$\eta(t_i)=0,\ \eta(dt_i)=\sum_j h^{i,j}\nabla_{\delta_j}\in \mathrm{Diff}^1(V)\subset \mathrm{End}_k(V)$$

It is well defined since

$$\eta[dt_i,t_j] = [\eta(dt_i),t_j] = h^{i,j} = h^{j,i} = [\eta(dt_j),t_i] = \eta[dt_j,t_i].[dt_i,t_j]\neq g^{i,j}$$

and therefore defines an element

$$\eta(h)\in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)^{(1)}.$$

Moreover, $\eta(h)\in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)$, is 0 only if there exists an $S\in \mathrm{End}_k(V)$ such that

$$0=\eta(t_i)=[S,t_i]\ \forall t_i\in C$$

implying that $S \in \mathrm{End}_C(V)$, so $S = (S_{r,s})$ is a matrix with entries in C. Furthermore, we must have

$$\eta(dt_i) = [S, \rho_1(dt_i)] = (-\xi_i(S_{k,l})) \in \mathrm{End}_C(V).$$

Since $\eta(dt_i) = \sum_l h^{i,l}\nabla_{\delta_l} \in \mathrm{Diff}^1(V,V)$, with $h^{i,j} = h^{j,i} \in C$ not all vanishing, this contradiction proves that $\eta(h) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)$, defined by the tangent h at $g \in \mathbf{M}$, is not zero. □

Remark 18. *Given any non-zero tangent $h \in \mathbf{T}_{\mathbf{M},g}$, we have seen that there corresponds to any representation ρ_0 with momentum ρ in terms of g, a non-zero second-order tangent ρ_2 of ρ_0, given by the element $\eta(h) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V,V)$, determined by the ρ_1-derivation η. The corresponding extension of V with itself as $\mathrm{Ph}(C)$-module is given by the way t_i, dt_j operates on $V \oplus \epsilon V$ where*

$$t_i(v_1,v_2) = (t_iv_1, t_iv_2),\ dt_j(v_1,v_2) = (\rho_1(dt_j)(v_1), (\rho_1(dt_j)(v_2) - ad(\eta(dt_j))(v_1))$$

resulting in

$$[dt_i, t_j] = g^{i,j} - \epsilon h^{i,j} \in \mathrm{End}_{k[\epsilon]}(V \oplus \epsilon V)$$

so that this extension of ρ_1 is really a representation of

$$\mathrm{Ph}(C) \otimes k[\epsilon]/(\sigma_{g+\epsilon h}) = C(\sigma_g) \otimes k[\epsilon]/(\sigma_{g+\epsilon h}).$$

This $\eta(h)$ is the measure of an acceleration of the representation ρ_0 as representation, and should correspond to a "cataclysmic change" of any massy "particle" $\rho_0 : C \to \mathrm{End}_k(V)$ with given momentum ρ, given by $\ddot{\rho}_0 = \rho_2$. The solutions of the corresponding Hamiltonian equations $Q(\phi) = E\phi$ in an appropriate one-dimensional deformation $\tilde{V}_\tau$ of ρ_1 should be a "wave" $\phi(\tau) \in V$.

This fits well with the present understanding of gravitational waves. It also fit reasonably well with the present cosmological theory. Since a "cataclysmic change" of any massy "particle $\rho_0 : C \to \mathrm{End}_k(V)$" will influence the metric, our Time, and vice versa, we find a mathematical reason for taking Mach's principle, that "everything depends upon everything" seriously.

Moreover, let us, for every metric $g \in \mathbf{M}$ consider the scalar curvature R as a function on the space $\underline{C} := \mathrm{Spec}\, C$, and fix a "compact" subset $\Omega \subset \underline{C}$. Since $R \sim 1/r^d$, where r is the "radius of curvature" of Ω at the corresponding point, it is not unreasonable to consider the Hilbert action

$$S(g) := \int_\Omega R dv_g,$$

where dv_g is the volume element in $\underline{C}$ defined by the metric g, as related to the "gravitational mass" content of the part of space, Ω. But then one could look at S as a functional

$$S : \mathbf{M} \to \mathbf{R},$$

the stability of which would give us a unique vector field on $\mathbf{M}$*, the Hilbert-Einstein tensor*

$$\mathfrak{G} \in \Theta_{\mathbf{M}},\ \mathfrak{G}(g) = \{\mathfrak{G}_{i,j}\} := \{\mathrm{Ric}_{i,j} - 1/2Rg_{i,j}\} \in T_{\mathbf{M},g}.$$

Recall also that we have put, $g = 1/2\sum_{i,j} g_{i,j}$ *and* $\overline{g} = \mathrm{sym}(g)$*, so one should have written,* $\mathfrak{G} = \mathrm{Ric} - 1/2R\overline{g}$*.*

For any representation (ρ_0, V) *with momentum* $\rho : C(\sigma_g) \to \mathrm{End}_k(V)$*, the injective map*

$\eta : \mathbf{T}_{\mathbf{M},g} \to \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V, V)$ *would then give us a unique element*

$$[\mathfrak{G}] := \eta(\mathfrak{G}) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V, V),$$

which would be a kind of Field Equation *of Einstein-Hilbert type, for the universe, with respect to the furniture* (V, ρ_1)*.*

Any way, an "increment" h of the metric g in $\Theta_{\mathbf{M}}$ *would correspond to the element of* $\eta(h) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V, V)$ *of first order, which corresponds to an increment of energy of the furniture* (V, ρ_1) *given by the representation* ρ *of* $C(\sigma_g)$*. So, again we find a Schrödinger type equation equating derivation with respect to time, i.e. with the action of* $[\mathfrak{G}]$*,*

$$\frac{\partial}{\partial t}(\rho_1) = \rho_2.$$

where ρ_2 *corresponds to an Hamiltonian operator* Q *on* V*.*

8.12 Family of representations versus family of metrics

Consider again the diagram,

$$\begin{tikzcd}
 & \mathrm{Ph}(C) \arrow[r] \arrow[d] \arrow[dr, "\rho_1"] & \mathrm{Ph}^2(C) \arrow[r] \arrow[d, "\rho_2"] & \mathrm{Ph}^3(C)\ldots \arrow[r] \arrow[dl] & \mathrm{Ph}^\infty(C)\circlearrowright^\delta \\
C \arrow[ur, "d"] \arrow[r, "i"'] & C(\sigma_g) \arrow[r, "\rho"'] & \mathrm{End}_k(V) & &
\end{tikzcd}$$

Where $\rho_0 := i \circ \rho$ is a representation of a commutative polynomial k-algebra C, g a metric, and ρ a momentum of ρ_0 defined on the corresponding $C(\sigma_g)$.

Let ρ_1 be the induced representation of $\mathrm{Ph}(C)$ and consider a deformation of ρ_1 over the 1-dimensional polynomial algebra $k[\tau]$,

$$\tilde{\rho}_1 : \mathrm{Ph}(C) \to \mathrm{End}_{k[\tau]}(\tilde{V}),$$

with $\rho_1 = \tilde{\rho}(0)$.

For any τ_0, we may look at the composition,

$$k[\tau] \to k[\epsilon] \simeq k[\tau]/(\tau - \tau_0)^2 \to k(\tau_0) := k[\tau]/(\tau - \tau_0).$$

The corresponding extensions

$$\mathrm{Ph}(C) \stackrel{\tilde{\rho}_1}{\to} \mathrm{End}_{k[\tau]}(\tilde{V}) \to \mathrm{End}_{k[\epsilon]}(V \otimes k[\epsilon]) \to \mathrm{End}_k(V)$$

defines the following notation,

$$\frac{\partial}{\partial \tau}(\tilde{\rho}_1)(\tau_0) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V, V).$$

Given a curve g_τ in $\mathbf{M}$ and a representation $C(\sigma_{g_{\tau_0}})$, we know what to understand by

$$\frac{\partial}{\partial \tau}(g_\tau)(\tau_0) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V, V)^{(1)}.$$

We may formulate at least three problems:

(1?) Given a curve g_τ in $\mathbf{M}$, and a representation $\rho_{\tau_0} : C(\sigma_{g_{\tau_0}}) \to \mathrm{End}_k(V)$, does there exist a 1-dimensional family of representations

$$\tilde{\rho}_1 : \mathrm{Ph}(C) \to \mathrm{End}_{k[\tau]}(\tilde{V})$$

with $\rho_{\tau_0 1} = \tilde{\rho}_1(\tau_0)$, such that for all τ_1,

$$\frac{\partial}{\partial \tau}(g_\tau)(\tau_1) = \frac{\partial}{\partial \tau}(\tilde{\rho}_1)(\tau_1) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V_{\tau_1}, V_{\tau_1})^{(1)}.$$

(2?) Given a metric $g := g_0$ and a representation $\rho_0 : C(\sigma_g) \to \mathrm{End}_k(V)$, inducing a representation ρ_1 of $\mathrm{Ph}(C)$. Let

$$\tilde{\rho}_1 : \mathrm{Ph}(C) \to \mathrm{End}_{k[\tau]}(\tilde{V}),$$

be a 1-dimensional family of representations with $\rho_1 = \tilde{\rho}_1(0)$ and such that for all τ_0,

$$\frac{\partial}{\partial \tau}(\tilde{\rho}_1)(\tau_0) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V_{\tau_0}, V_{\tau_0})^{(1)}.$$

Does there exist a curve of metrics $g_\tau \in \mathbf{M}$ such that $g_0 = g_0$ and

$$\frac{\partial}{\partial \tau}(\tilde{\rho}_1)(\tau_0) = \frac{\partial}{\partial \tau}(g_\tau)(\tau_0) \in \mathrm{Ext}^1_{\mathrm{Ph}(C)}(V_{\tau_0}, V_{\tau_0})^{(1)}.$$

(3?) Does there exist an (algebraic) integral curve g_τ in $\mathbf{M}$ of the vector field $\mathfrak{G}$ such that

$$\frac{\partial}{\partial \tau}(g_\tau)(\tau) = [\mathfrak{G}](\tau).$$

Recall that in our model, the Dirac derivation δ induces the dynamical structure $\delta := \delta_g = \mathrm{ad}(g - T)$ for every metric, i.e. for every point in the space of metrics $\mathbf{M}$. It also creates the time-like vector field $[\mathfrak{G}]$ in $\mathbf{M}$ together with the observable time in our space $\mathrm{Spec}\, C$. So it is reasonable to promote δ to our Chronos =TIME.

If we may answer affirmatively on (1?), and (3?), it seems that this takes care of the problem of relations between the metrics and the furniture of our space $\operatorname{Spec} C$. However, suppose we find that some interesting representation in our furniture is the iterated extensions of sub representations of Θ_C, and that the local gauge group $\mathfrak{g}_1$ acts non-trivially on Θ_C. Then we obviously find new conditions for our time model, the metric g. It is reasonable to think that g must then be $\mathfrak{g}$-invariant. This would restrict our choice of metrics, and thereby also the stock of furniture satisfying the conditions above.

We notice that distances in space are, today, measured by the time light needs to connect points. If space is empty, i.e. if the only representations we consider are the trivial representation $\mathrm{Simp}_1(C)$, then space becomes continuous. However, if we consider some furniture, given by the representation V, the eigenvalues of the time operator $\mathrm{ad}(g-T)$ operating as a derivation on $\mathrm{End}_k(V)$, may be discrete. The semigroup of real eigenvalues may then have a least positive element, the Planck constant h see [?], section 1.7. If the representation V is related to light, say photons, then measurable space must also be discontinuous, or quantised as it is called in the physics literature.

8.13 Relations to Clifford Algebras

In the situation above where we are given a polynomial algebra $C = k[t_1, ..., t_d]$ and a non-singular metric g, we know that the C-module of differentials Ω_C generated by the dt_i, is provided with the metric g^{-1}. Therefore we find that for every point in $\underline{t} \in \underline{C}$, there is a quadratic form on $T_{\underline{t}} := \Omega_{C,\underline{t}}$, given by $g^{-1}(\underline{t})$. We might then consider the Clifford algebra $\mathrm{Cliff}(T_{\underline{t}}, g^{-1})$. This can now be generalised to construct a generalised Clifford algebra

$$\mathrm{Cliff}(C, g) := \mathrm{Ph}(C)/(dt_i dt_j + dt_j dt_i - 2g^{i,j}).$$

Given a point $\underline{t} \in \underline{C}$, we have canonical homomorphisms of k-algebras

$$\mathrm{Ph}(C) \to \mathrm{Cliff}(C, g) \to \mathrm{Cliff}(T_{\underline{t}}, g^{-1}).$$

However, there are no decompositions of this composed morphism into something like

$$\mathrm{Ph}(C) \to C(\sigma_g) \to \mathrm{Cliff}(T_{\underline{t}}, g^{-1}).$$

Never the less, as some physicists have remarked, the algebra $\mathrm{Cliff}(T_{\underline{t}}, g^{-1})$ may be of interest in quantum theory, in particular in relation to the notion of rotation (in quantum mechanics). Its use in the theory of gravity seems to be very unnatural.

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