



# Diagrammatic Algebra

J. Scott Carter

Tokyo 2019

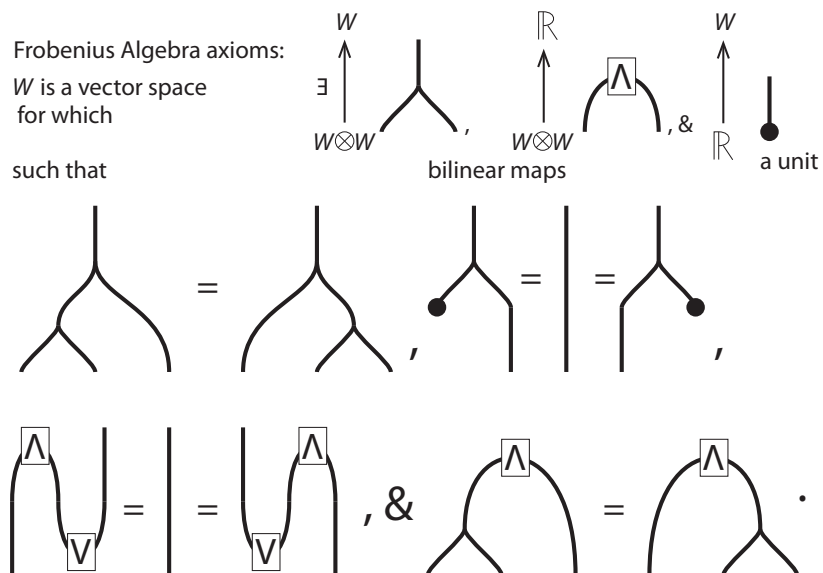
The bulk of this talk is based on Joint work with Seiichi Kamada. Other contributors include Victoria Lebed, Masahico Saito, and Seung Yeop Yang. This handout mostly contains drawings. Many of these will appear on the board. Others are too complicated to present in real time. Some will invite you to consider them at your leisure.

## Outline

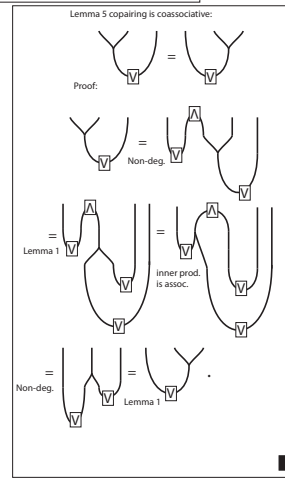
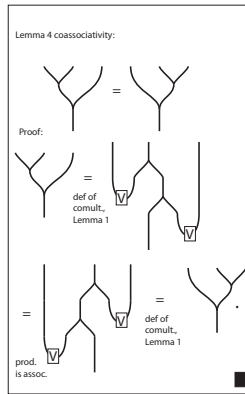
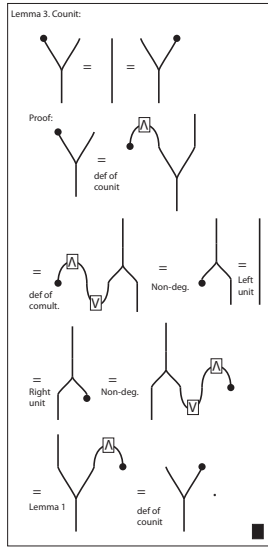
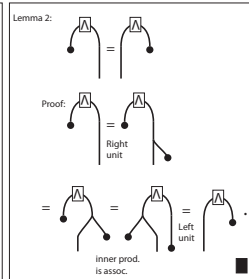
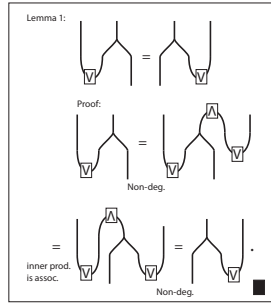
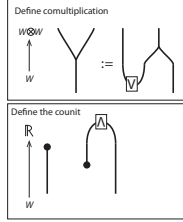
0. Metaphorical overview
1. Multi-categorical analogue of a Frobenius Algebra
  - (a)  $M(n, n)$
  - (b) Frobenius Alg. axioms as diagrams
  - (c) Objects, multiple arrows and exchangers
2. An associated category as a 4-category
3. Adding a braiding
  - (a) Category of tangles.
  - (b) Multi-category of 2-tangles
  - (c) Qualgebras, knotted handle-bodies
  - (d) Foams and foam moves
  - (e) abstract tensors

# 1. Frobenius Algebras

Example:  $M(n, n)$ .

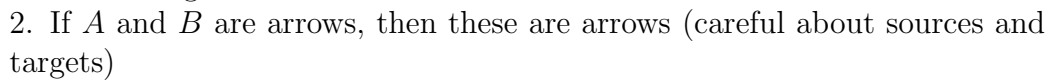


Thm: A Frobenius algebra is also a coalgebra.



Objects of **FA**:  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Here  $1 \Leftarrow \bullet$ .

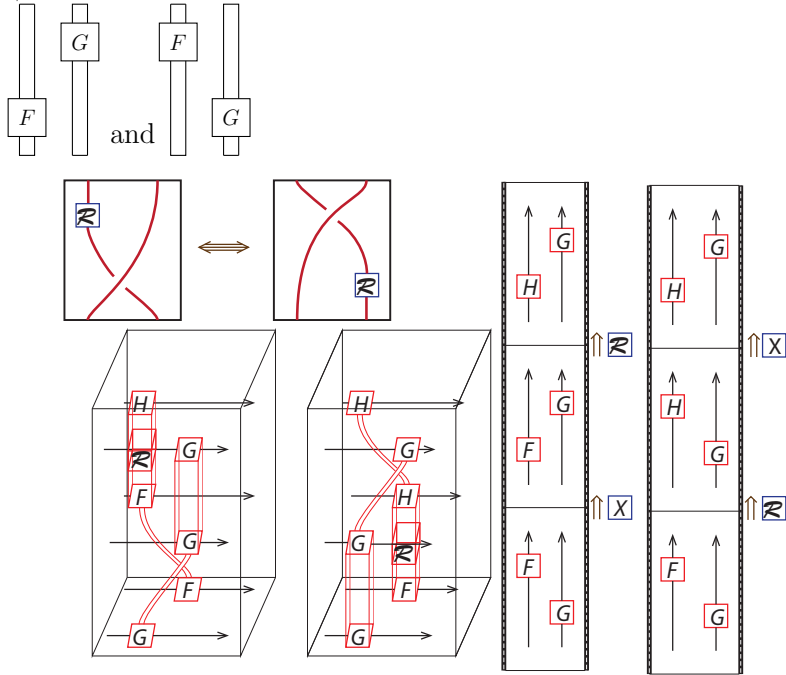
A diagrammatic equation. On the left is a single vertical rectangle labeled  $\alpha$  below it. This is followed by an equals sign. To the right of the equals sign are three vertical lines, each with a dot at the top and a dot at the bottom, labeled  $\alpha$  strings below them. An ellipsis  $\dots$  is placed between the first and last of these vertical lines, indicating a sequence of  $\alpha$  such lines.

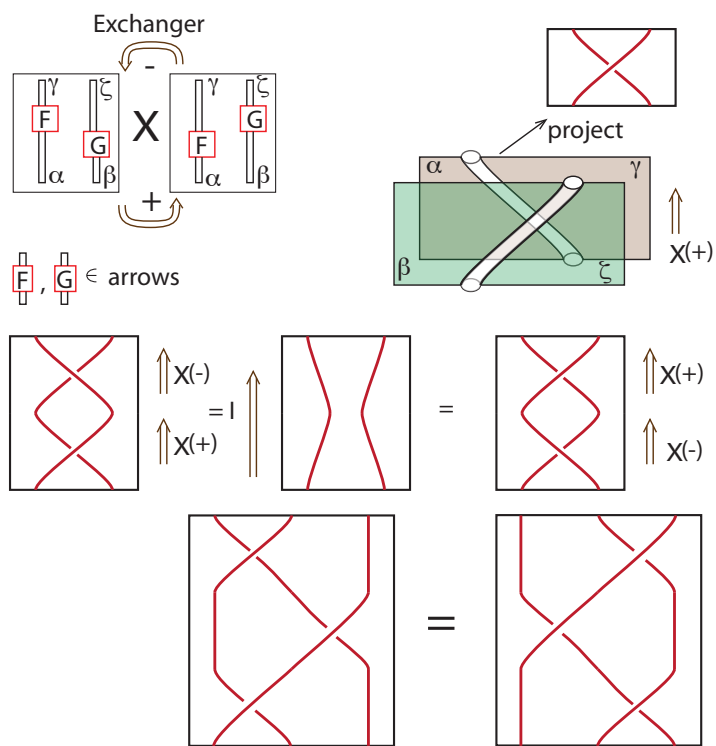


**Exchanger axiom.** Suppose that  $\gamma \xleftarrow{F} \alpha$  and  $\zeta \xleftarrow{G} \beta$  are arrows. There is a natural family  $X$  of 2-arrows

$$X : (F \otimes l_\zeta) \circ (l_\alpha \otimes G) \Rightarrow (l_\gamma \otimes G) \circ (F \otimes l_\beta)$$

which are 2-isomorphisms. Here  $(F \otimes l_\zeta) \circ (l_\alpha \otimes G)$  and  $(l_\gamma \otimes G) \circ (F \otimes l_\beta)$  are algebraic expressions of the graphic (right and left respectively) :



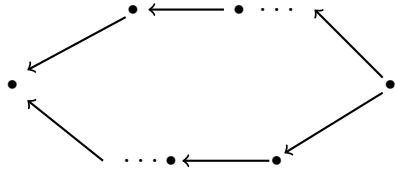


“Change followed by exchange” is the same thing as “exchange followed by change.”

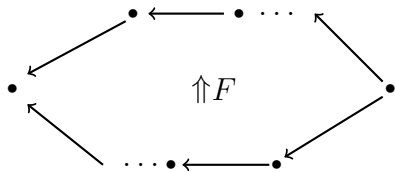
Four principles:

1. Arrows are used to make comparisons.
2. Different things are not equal.
3. Simultaneity is illusory.
4. “Doing” and then “undoing” may or may not be the same as “not doing.”

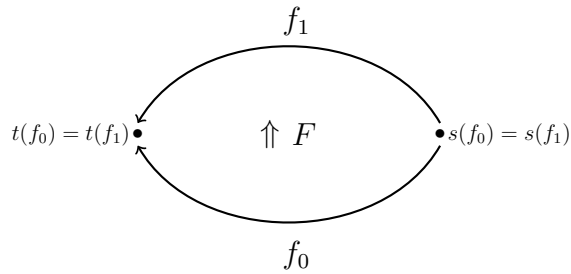
Typical diagram in a category:



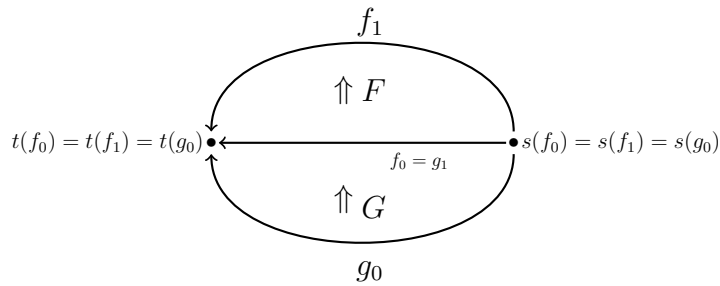
Replace by a double arrow



When the set of 1-arrows is a category, we have double arrows:

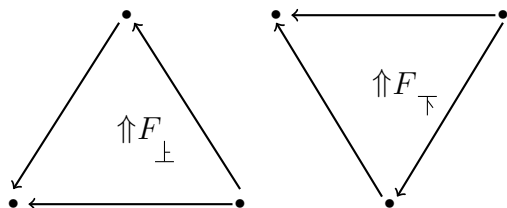


These are composed as follows:

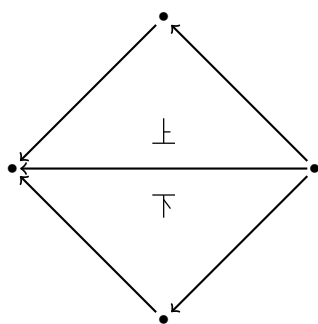




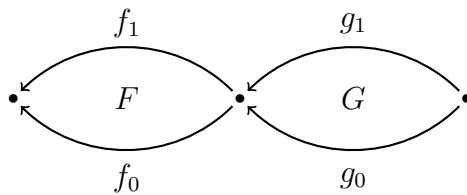
Compose all source arrows, but leave two target arrows (or vice versa):



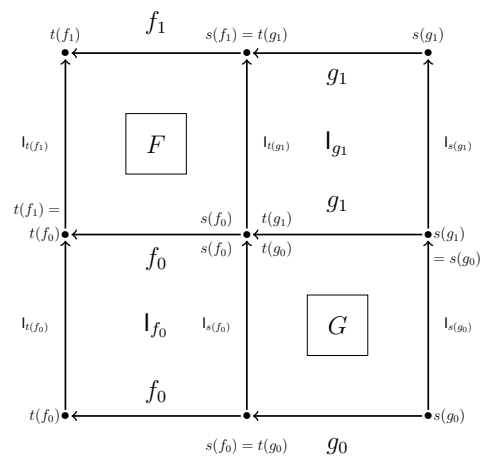
Globular composition:



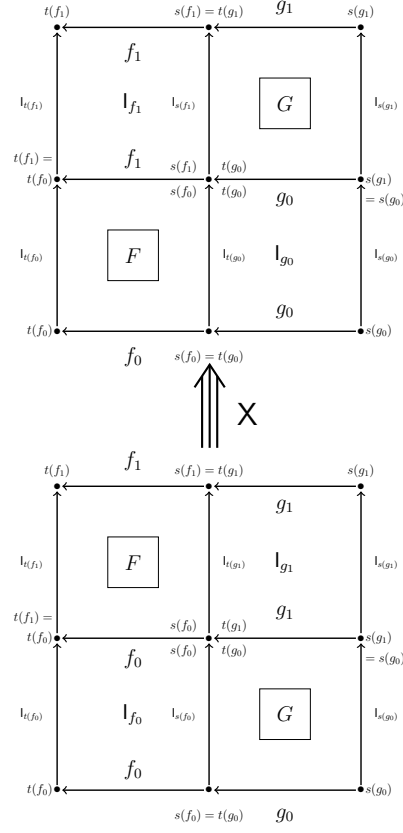
Don't allow:



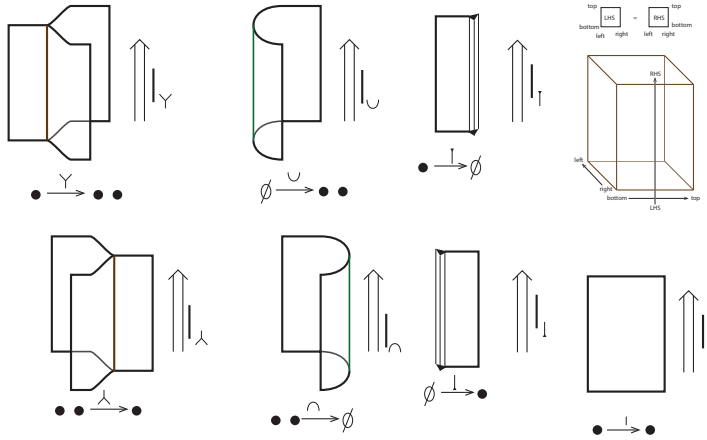
But replace with



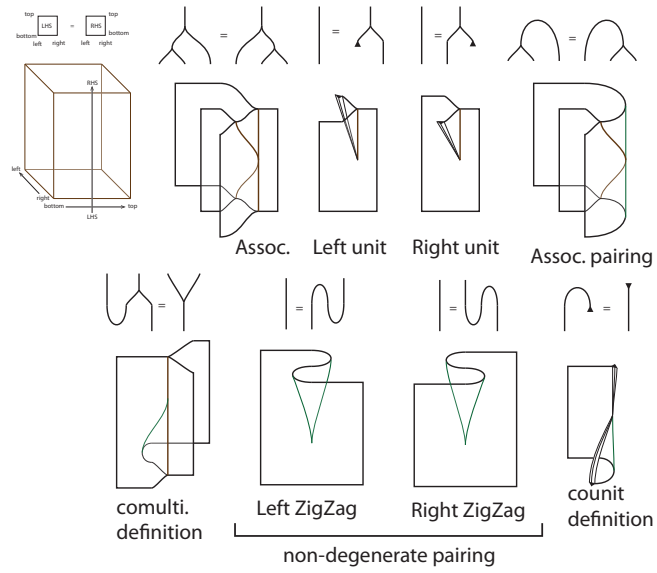
And assume a natural family of triple arrow isomorphisms:



Identity double arrows:



Generating double arrows:



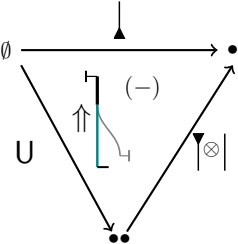
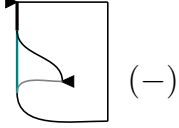
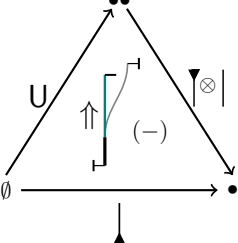
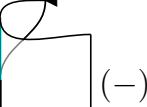
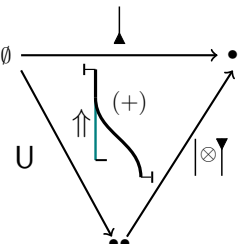
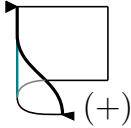
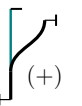
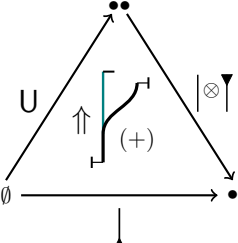
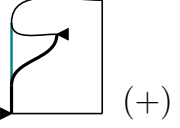
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|-------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------|
|  <p>unit via counit (left)</p>         |    |  <p>Fold lower left corner back</p>      |
|  <p>unit via counit inv. (left)</p>    |   |  <p>Fold upper left corner back</p>      |
|  <p>unit via counit (right)</p>      |  |  <p>Fold lower left corner forward</p> |
|  <p>unit via counit inv. (right)</p> |  |  <p>Fold upper left corner forward</p> |

Table 1: Writing units in terms of counits

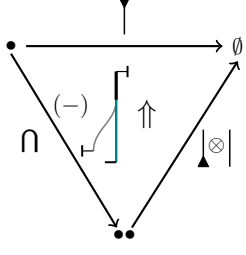
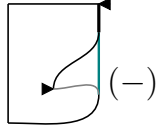
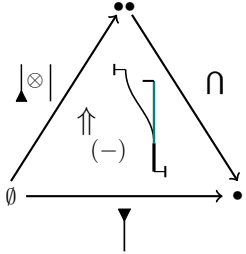
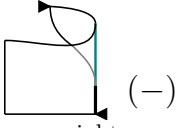
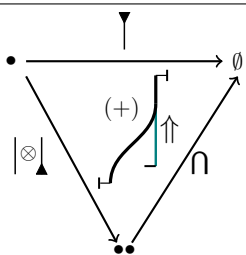
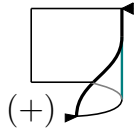
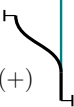
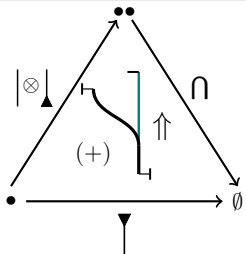
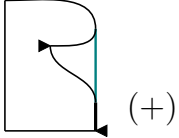
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|-------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------|
|  <p>counit via unit (left)</p>         |    |  <p>Fold lower right corner back</p>      |
|  <p>counit via unit inv. (left)</p>    |   |  <p>Fold upper right corner back</p>      |
|  <p>counit via unit (right)</p>      |  |  <p>Fold lower right corner forward</p> |
|  <p>counit via unit inv. (right)</p> |  |  <p>Fold upper right corner forward</p> |

Table 2: Writing counits in terms of units

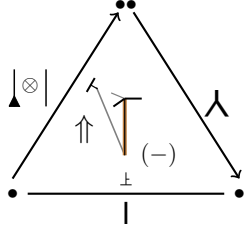
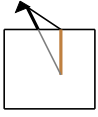
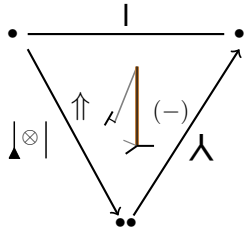
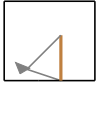
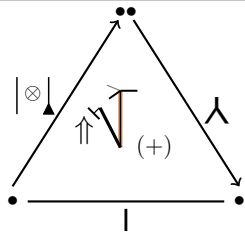
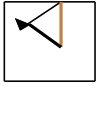
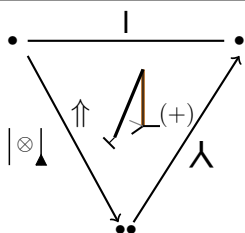
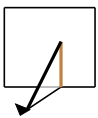
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|--------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------|
|  <p>Left unit axiom <math>(-)</math></p>            |    |  <p>upper left triangular flap back <math>(-)</math></p>      |
|  <p>Left unit axiom inverse <math>(-)</math></p>    |   |  <p>lower left triangular flap back <math>(-)</math></p>      |
|  <p>Right unit axiom <math>(+)</math></p>         |  |  <p>upper left triangular flap forward <math>(+)</math></p> |
|  <p>Right unit axiom inverse <math>(+)</math></p> |  |  <p>lower left triangular flap forward <math>(+)</math></p> |

Table 3: Units

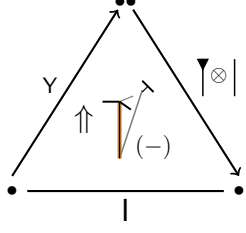
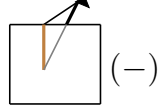
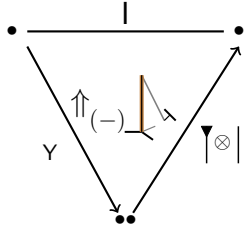
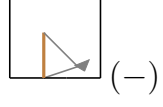
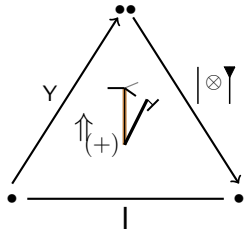
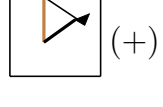
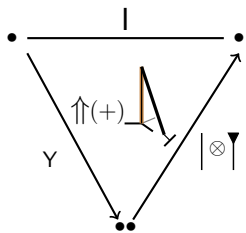
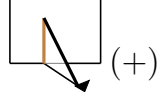
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|-----------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
|  <p>Left counit axiom</p>            |    |  <p>upper right triangular flap back</p>      |
|  <p>Left counit axiom inverse</p>    |   |  <p>lower right triangular flap back</p>      |
|  <p>Right counit axiom</p>         |  |  <p>upper right triangular flap forward</p> |
|  <p>Right counit axiom inverse</p> |  |  <p>lower right triangular flap forward</p> |

Table 4: Counits

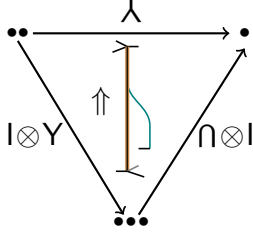
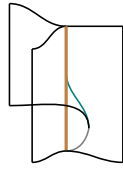
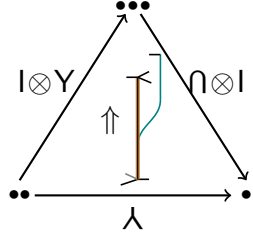
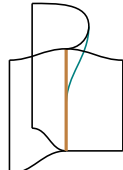
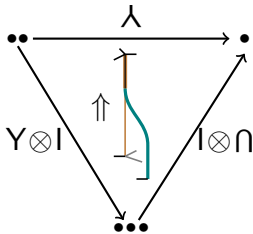
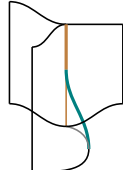
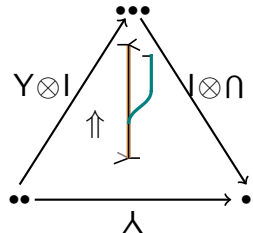
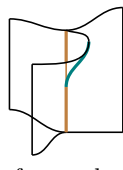
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|--------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------|
|  <p>Multi. via comult. left</p>         |    |  <p>Right, back, below fold<br/>ends on seam</p>    |
|  <p>Multi via comult. inv. left</p>     |   |  <p>Right, back, above fold<br/>ends on seam</p>    |
|  <p>Multi. via comult. right</p>      |  |  <p>Right, front, below fold<br/>ends on seam</p> |
|  <p>Multi. via comult. inv. right</p> |  |  <p>Right, front, above fold<br/>ends on seam</p> |

Table 5: Writing multiplication in terms of comultiplication



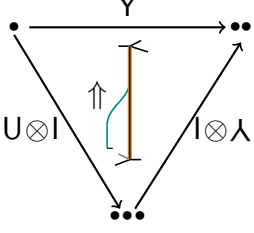
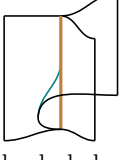
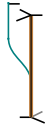
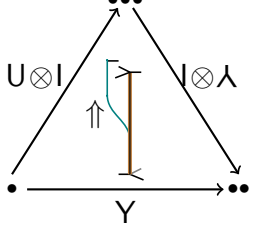
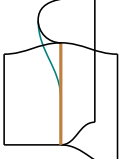
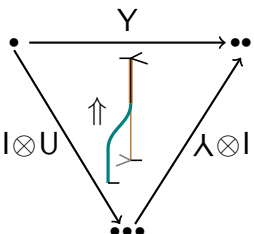
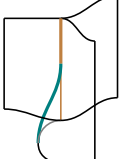
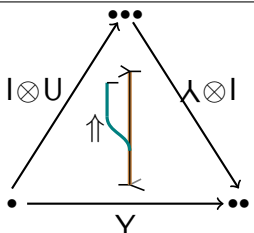
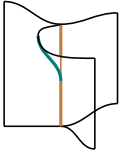
|                                                                                                                          |                                                                                     |                                                                                                                                       |
|--------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------|
|  <p>Comult. via mult. left</p>          |    |  <p>Left, back, below fold<br/>ends on seam</p>    |
|  <p>Comulti via mult. inv. left</p>     |   |  <p>Left, back, above fold<br/>ends on seam</p>    |
|  <p>Comulti. via mult. right</p>      |  |  <p>Left, front, below fold<br/>ends on seam</p> |
|  <p>Multi. via comult. inv. right</p> |  |  <p>left back above fold<br/>ends on seam</p>    |

Table 6: Writing comultiplication in terms of multiplication

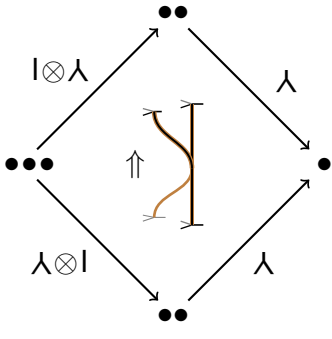
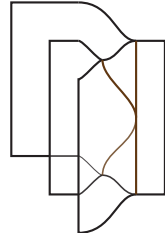
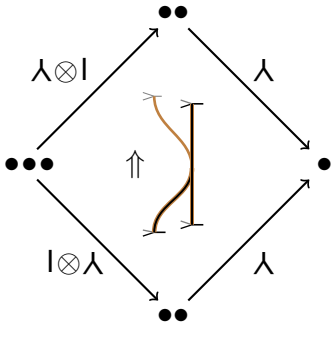
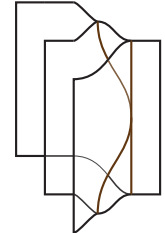
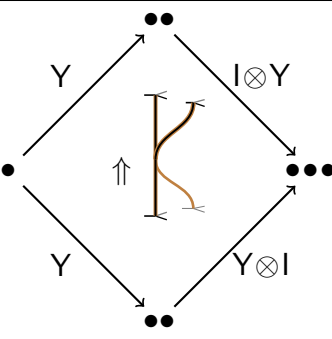
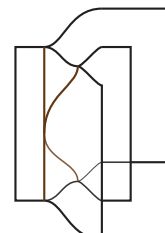
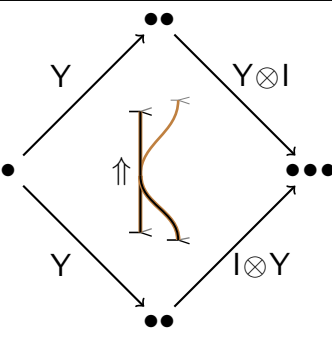
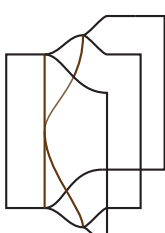
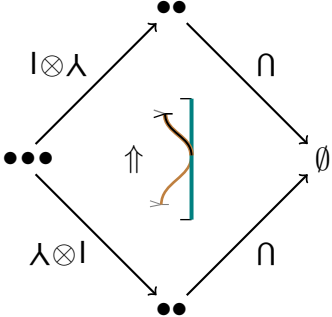
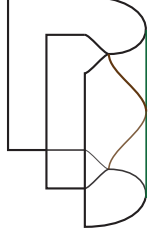
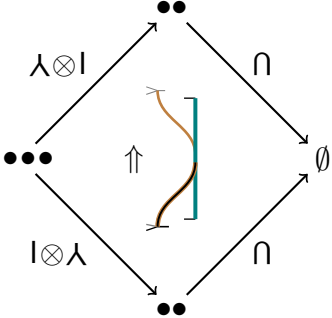
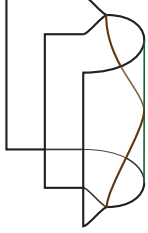
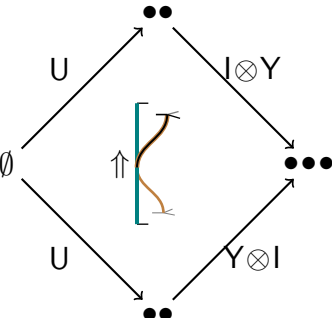
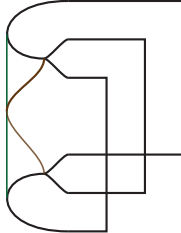
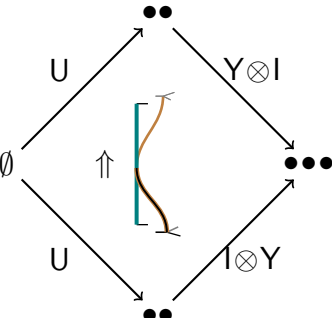
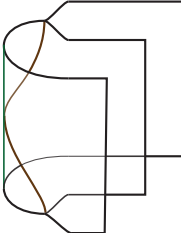
|                                                                                                                          |                                                                                      |                                                                                                                                   |
|--------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------|
|  <p>Associativity axiom</p>             |    |  <p>Left seam crosses<br/>top to bottom</p>    |
|  <p>Associativity axiom inverse</p>     |   |  <p>Left seam crosses<br/>bottom to top</p>    |
|  <p>Coassociativity axiom</p>         |  |  <p>Right seam crosses<br/>top to bottom</p> |
|  <p>Coassociativity axiom inverse</p> |  |  <p>Right seam crosses<br/>bottom to top</p> |

Table 7: Associativity and coassociativity

|                                                                                                                         |                                                                                      |                                                                                                                                                           |
|-------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|
|  <p>Associative pairing</p>            |    |  <p>Left seam crosses<br/>a fold on the right<br/>top to bottom</p>    |
|  <p>Associativity pairing inv.</p>    |   |  <p>Left seam crosses<br/>a fold on the right<br/>bottom to top</p>    |
|  <p>Coassociative copairing</p>      |  |  <p>Right seam crosses<br/>a fold on the left<br/>top to bottom</p>  |
|  <p>Coassociative copairing inv.</p> |  |  <p>Right seam crosses<br/>a fold on the right<br/>bottom to top</p> |

19  
Table 8: Associativity and coassociativity of the pairing

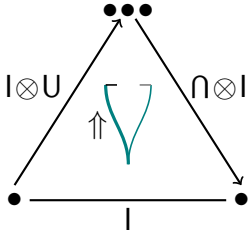
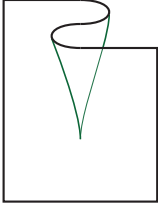
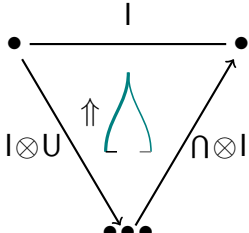
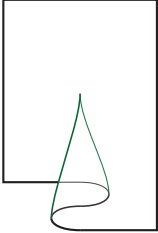
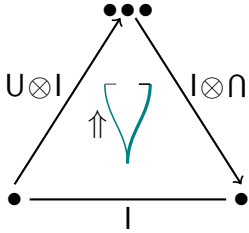
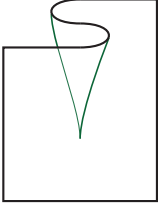
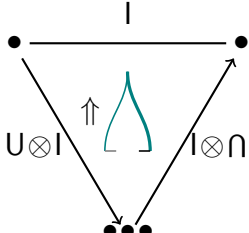
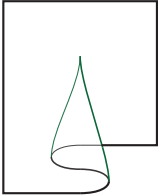
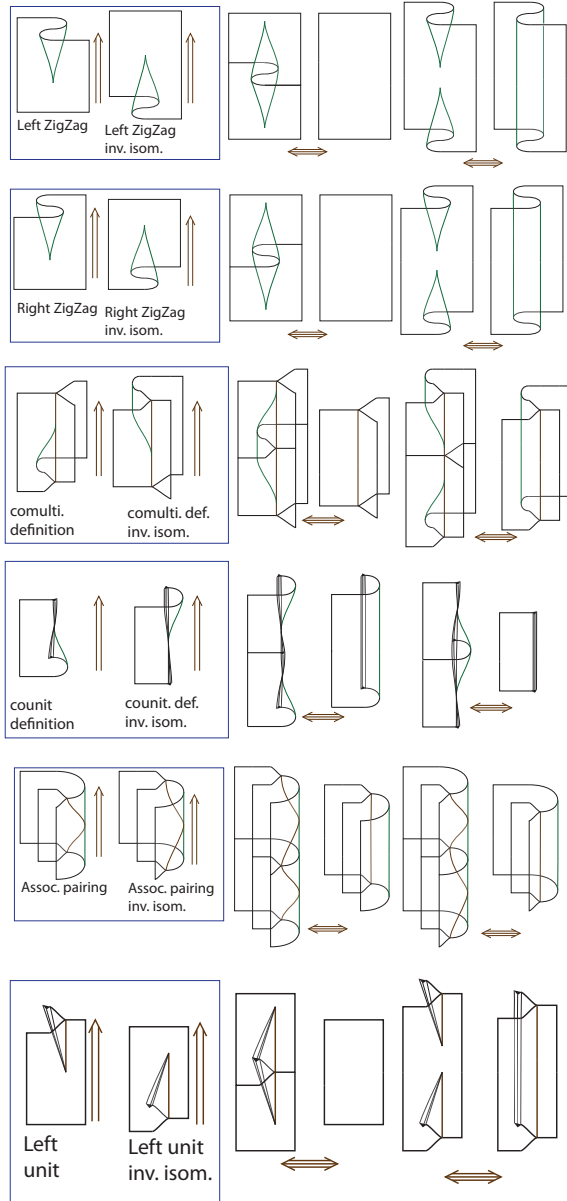
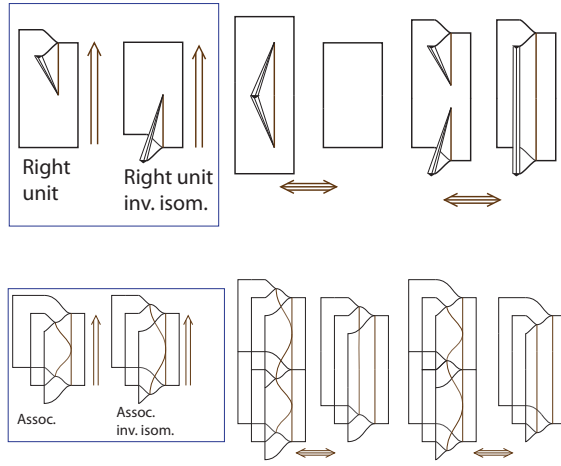
|                                                                                                            |                                                                                     |                                                                                                                    |
|------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------|
|  <p>Left cusp down</p>    |    |  <p>left zig zag (top)</p>       |
|  <p>Left cusp up</p>      |    |  <p>left zig zag (bottom)</p>    |
|  <p>Right cusp down</p> |  |  <p>Right zig zag (top)</p>    |
|  <p>Right cusp up</p>   |  |  <p>Right zig zag (bottom)</p> |

Table 9: Non-degeneracy of the pairing and copairing

Note: There are also triple arrows that can be seen as identities among relations. Here are some examples. These next few represent the third principle.

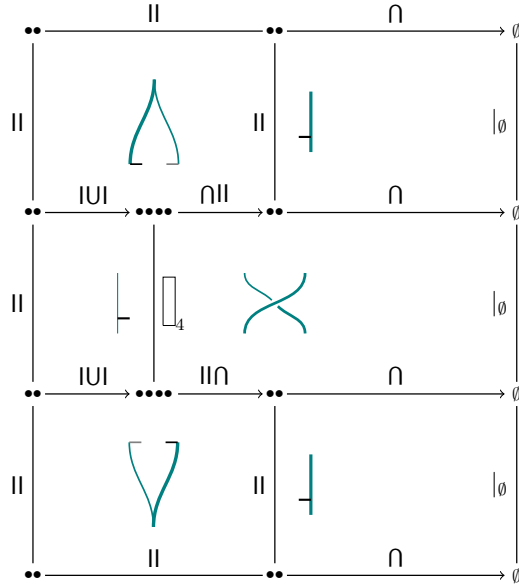




Let's consider this sequence of double arrows:

$$n \xRightarrow{\quad \begin{array}{c} \text{V} \\ \text{I} \end{array} \quad} [n \circ (n \parallel n) \circ (n \parallel n)] \xRightarrow{\quad \begin{array}{c} \text{X} \\ \text{I} \end{array} \quad} [n \circ (n \parallel n) \circ (n \parallel n)] \xRightarrow{\quad \begin{array}{c} \text{A} \\ \text{I} \end{array} \quad} n$$

Rewrite as this:



Compare with the identity on:

$$\bullet \bullet \xrightarrow{\cap} \emptyset.$$

More on this later.

---

Additional double arrows. Consider the generating 1-arrows:

|                                                |                                                |
|------------------------------------------------|------------------------------------------------|
| $\emptyset \xrightarrow{\cap} \bullet$         | $\bullet \xrightarrow{\cup} \emptyset$         |
| $\bullet \bullet \xrightarrow{\cap} \bullet$   | $\bullet \xrightarrow{\cup} \bullet \bullet$   |
| $\bullet \bullet \xrightarrow{\cap} \emptyset$ | $\emptyset \xrightarrow{\cup} \bullet \bullet$ |

Compare compositions in each row to identity arrows:

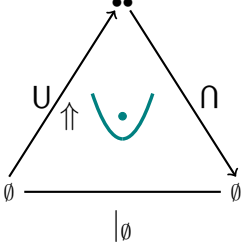
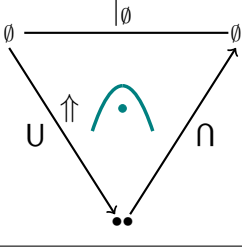
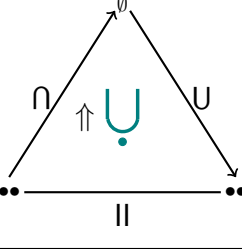
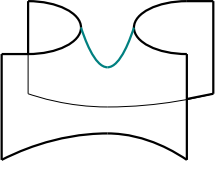
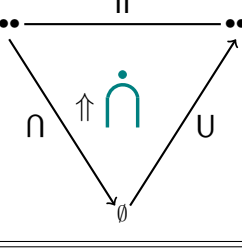
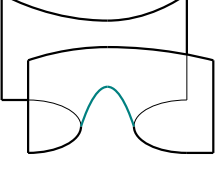
|                                                                                                                      |                                                                                                           |                                                                                                            |
|----------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------|
|  <p>birth glyph double arrow</p>    |  <p>birth surface</p>    |  <p>birth surface</p>    |
|  <p>death glyph double arrow</p>    |  <p>death surface</p>   |  <p>death surface</p>    |
|  <p>saddle glyph double arrow</p> |  <p>saddle surface</p> |  <p>saddle surface</p> |
|  <p>fork glyph double arrow</p>   |  <p>fork surface</p>   |  <p>fork surface</p>   |

Table 10: Birth, death, saddle, and fork double arrows



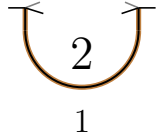
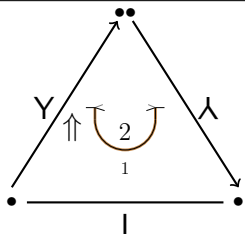
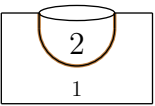
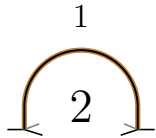
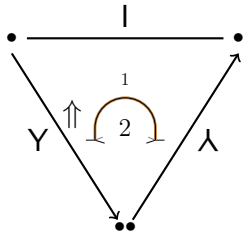
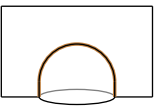
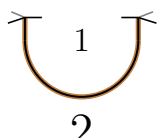
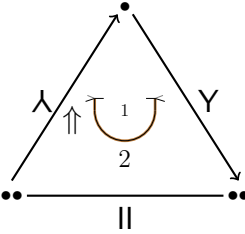
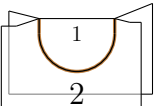
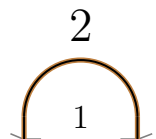
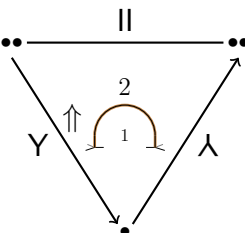
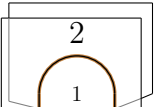
|                                                                                                                   |                                                                                     |                                                                                                                       |
|-------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|
|  <p>zip close bottom, glyph</p>  |    |  <p>zip close bottom, surface</p>  |
|  <p>zip close top, glyph</p>     |   |  <p>zip close top, surface</p>     |
|  <p>zip open bottom, glyph</p> |  |  <p>zip open bottom, surface</p> |
|  <p>zip open top, glyph</p>    |  |  <p>zip open top, surface</p>    |

Table 11: Optimal points on seams

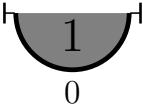
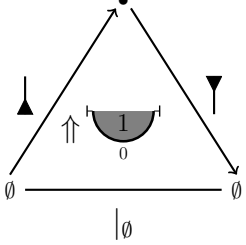
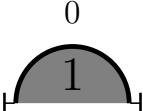
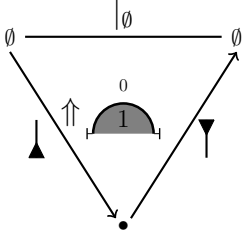
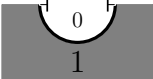
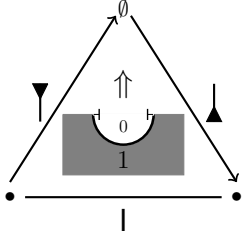
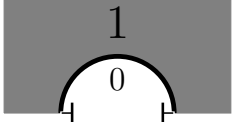
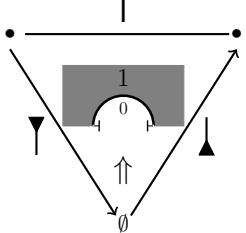
|                                                                                                                          |                                                                                      |
|--------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
|  <p>Birth of unit/counit</p>            |    |
|  <p>death of unit/counit</p>            |    |
|  <p>canal between counit and unit</p> |  |
|  <p>Arch from counit to unit</p>      |  |

Table 12: Birth, death, arch, and canal for unit/counit pairs

**Summary Section 1:** Frobenius Algebra axioms  $\longrightarrow$  multiple category with  $\text{obj} = \mathbb{N}$ , arrows multi, comulti, unit, counit, pairing and copairing. Double arrows correspond to foams embedded in 3-space, and triple arrows correspond to isotopy of foams.

**Thm.** There is a multi-functor  $b/2$  the multi-category gen by

|                                               |                                               |
|-----------------------------------------------|-----------------------------------------------|
| $\emptyset \xrightarrow{\triangle} \bullet$   | $\bullet \xrightarrow{\nabla} \emptyset$      |
| $\bullet\bullet \xrightarrow{\wedge} \bullet$ | $\bullet \xrightarrow{\vee} \bullet\bullet$   |
| $\bullet\bullet \xrightarrow{\cap} \emptyset$ | $\emptyset \xrightarrow{\cup} \bullet\bullet$ |

and isotopy classes of embedded foams.

## 2. Promote to a multi-category with quadruple arrows

Obj:  $t$  and  $f$ .

Generating 1-arrows:

$f \longrightarrow f$

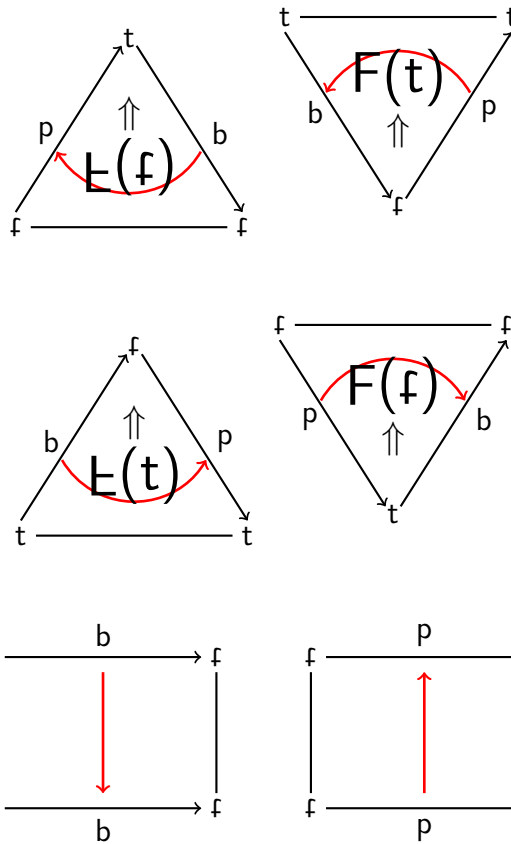
$t \longrightarrow t$

$p : f \rightarrow t$

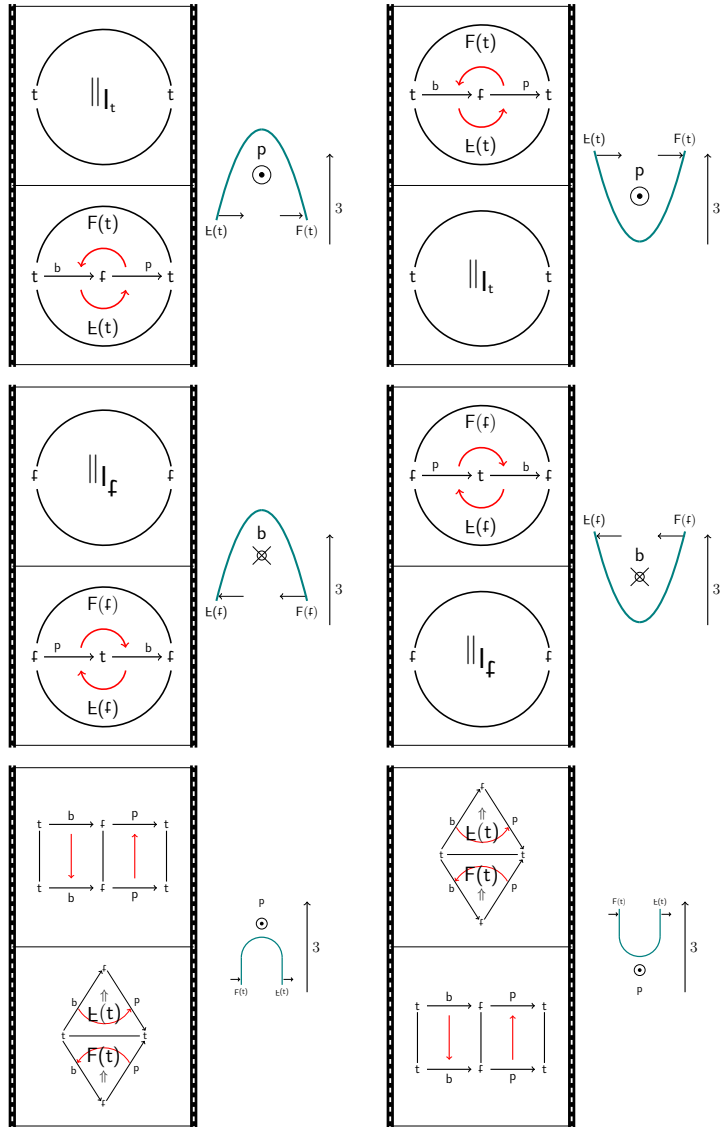
$b : t \rightarrow f$

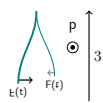
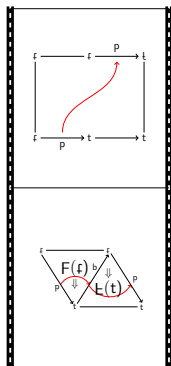
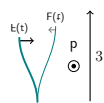
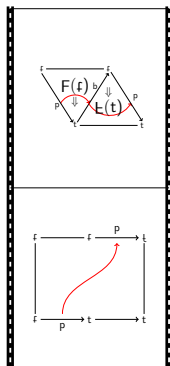
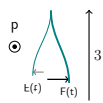
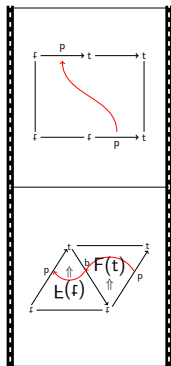
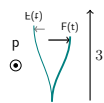
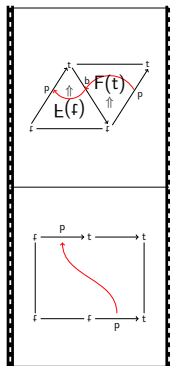
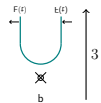
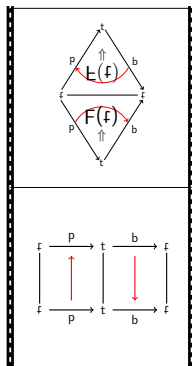
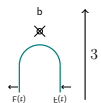
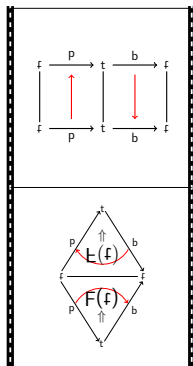
In general, a reduced non-identity arrow is a finite sequence  $pbpb \dots$  or  $bpbp \dots$ .

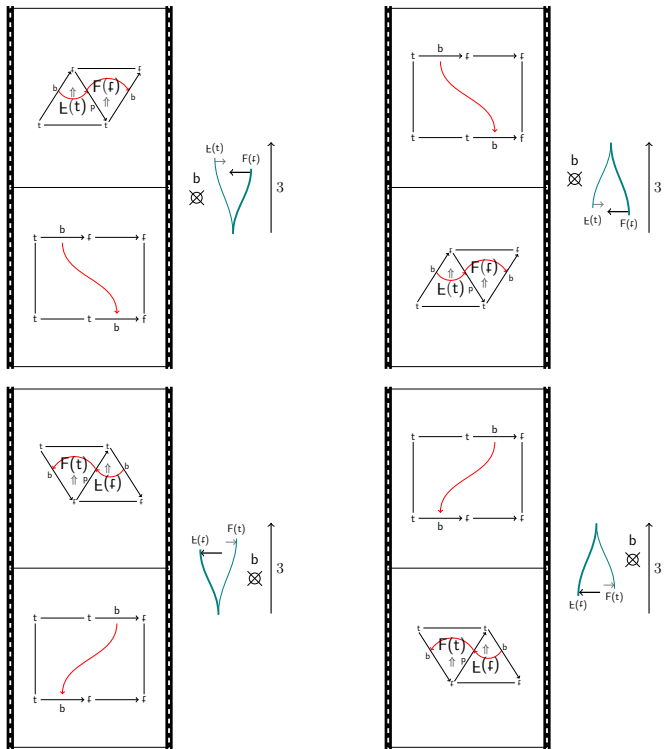
Generating double arrows:



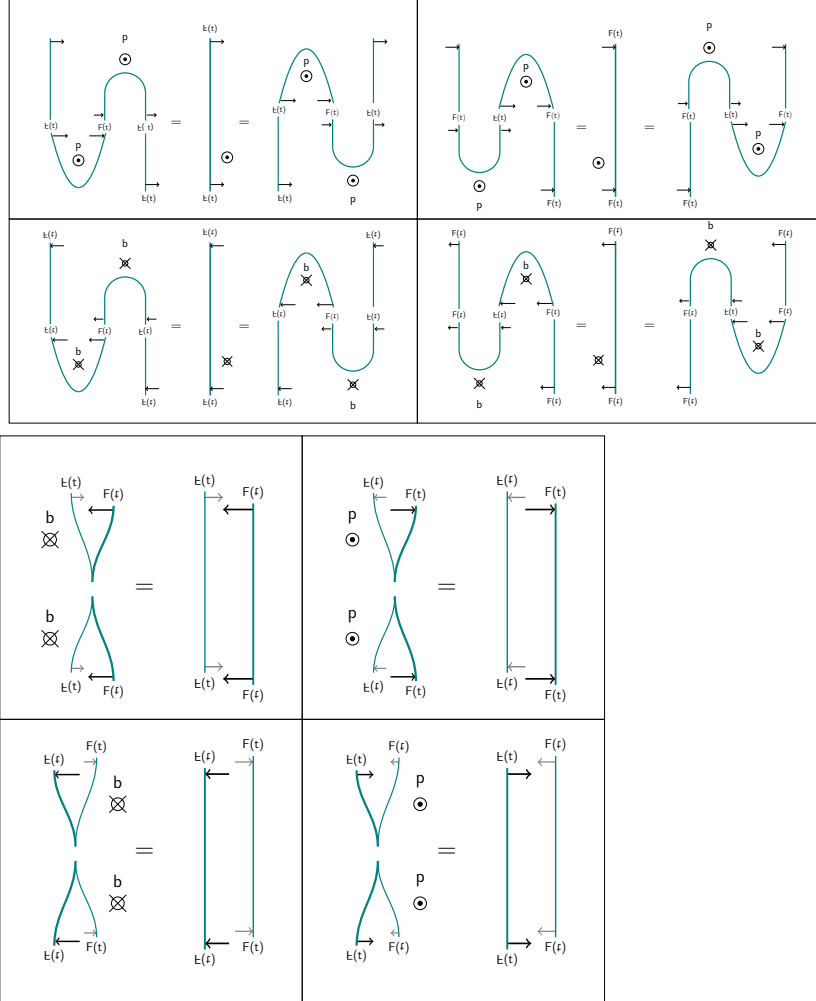
Generating triple arrows:



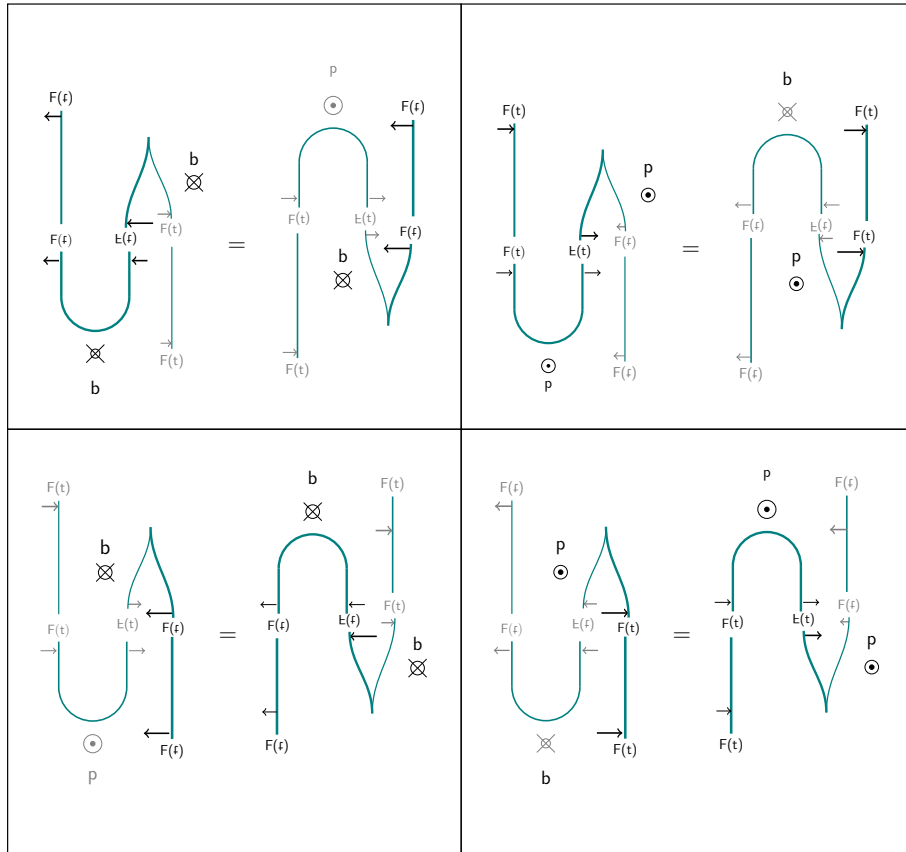
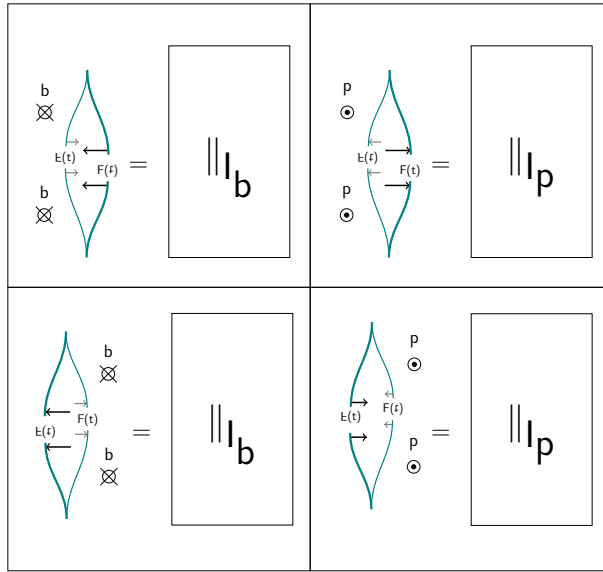


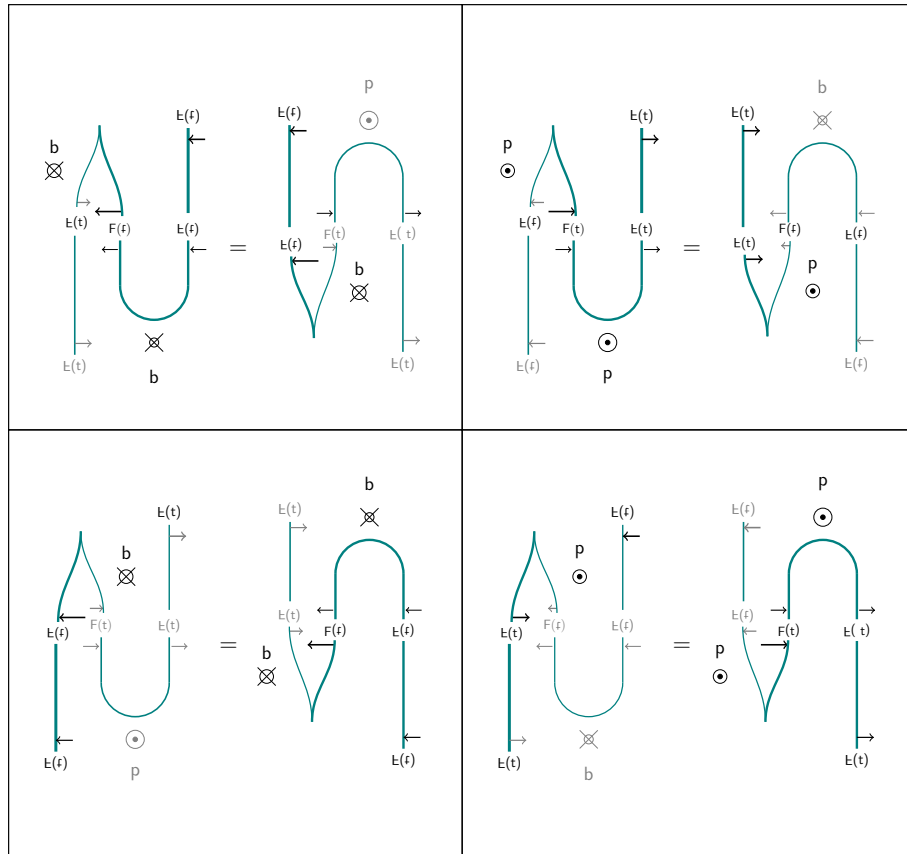


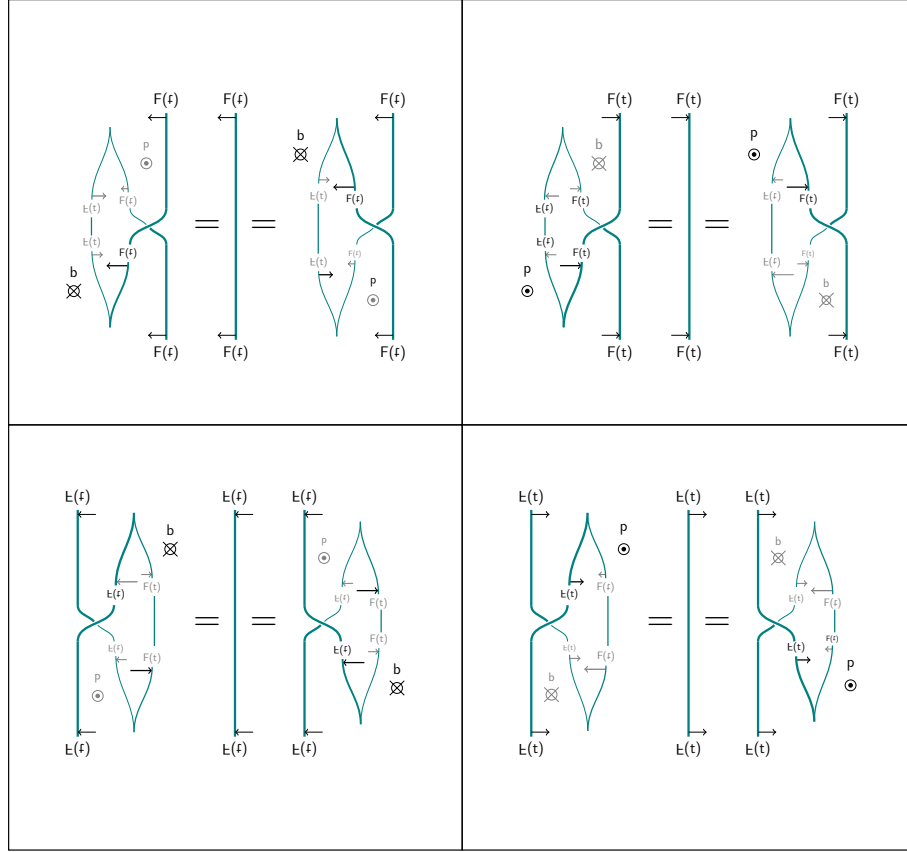
Quadruple arrows are equalities:











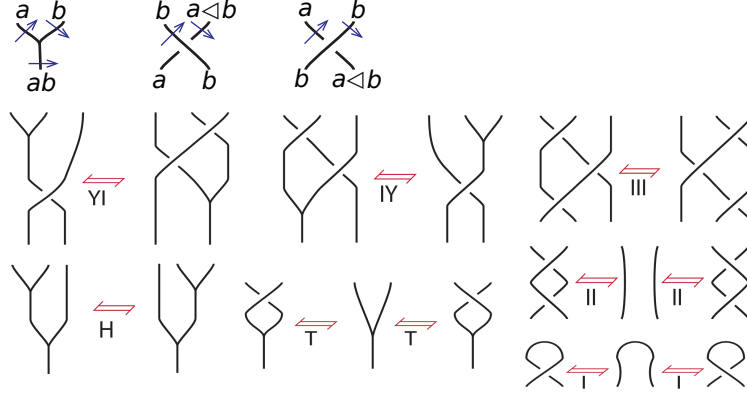
Add exchanger axioms.

**Thm.** There is a multi-functor from the multi-category that is described above to the multi-category of isotopy classes of embedded oriented surfaces in 3-space.

Such a surface separates space into regions. Checkerboard color these. The two objects correspond to the black and white regions. The arrows correspond to arcs that intersect the surface transversely. The double arrows are embedded disks, and the triple arrows are chunks of space.

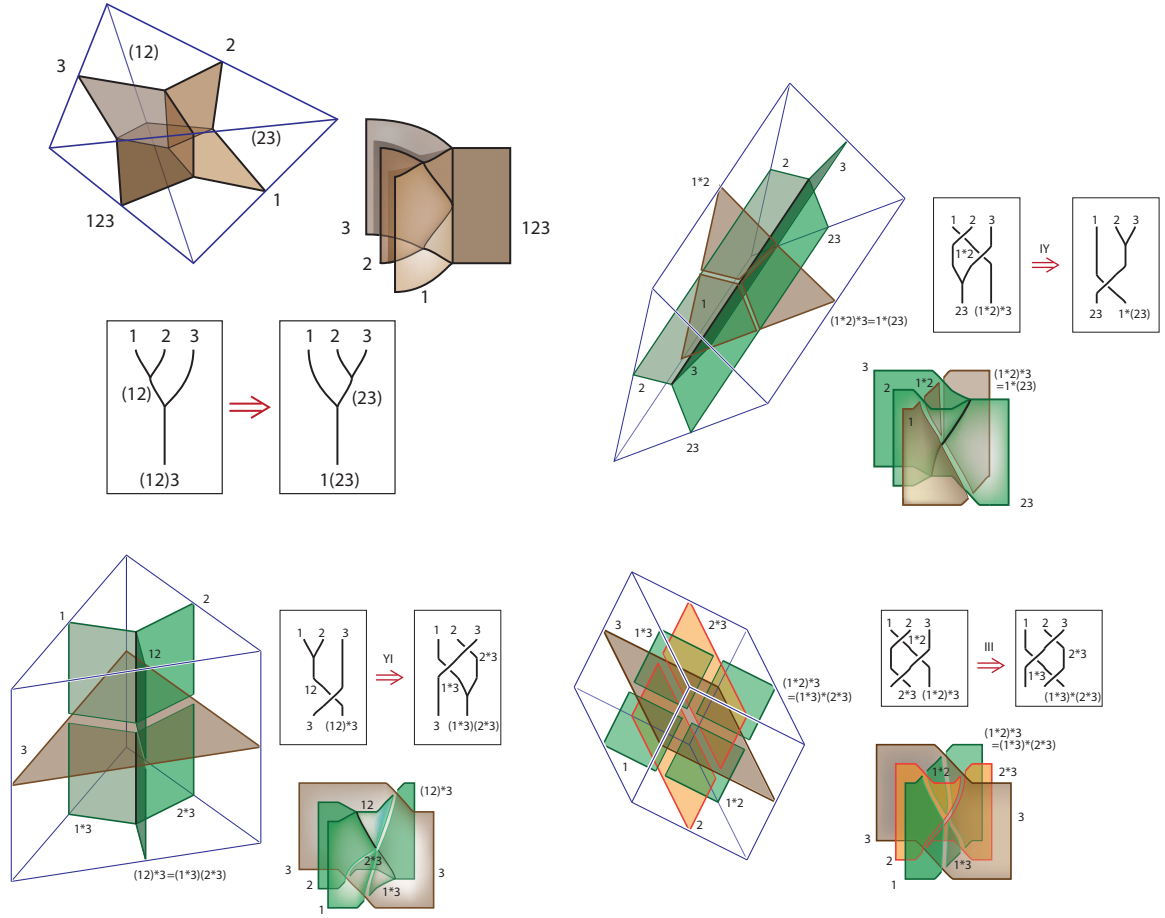
### 3. Adding a braiding

Start with one object and only the identity arrow defined upon it. Then create a generating non-identity double arrow from the identity arrow to itself. The exchanger axiom allows for braiding.



Qualgebra Axioms:

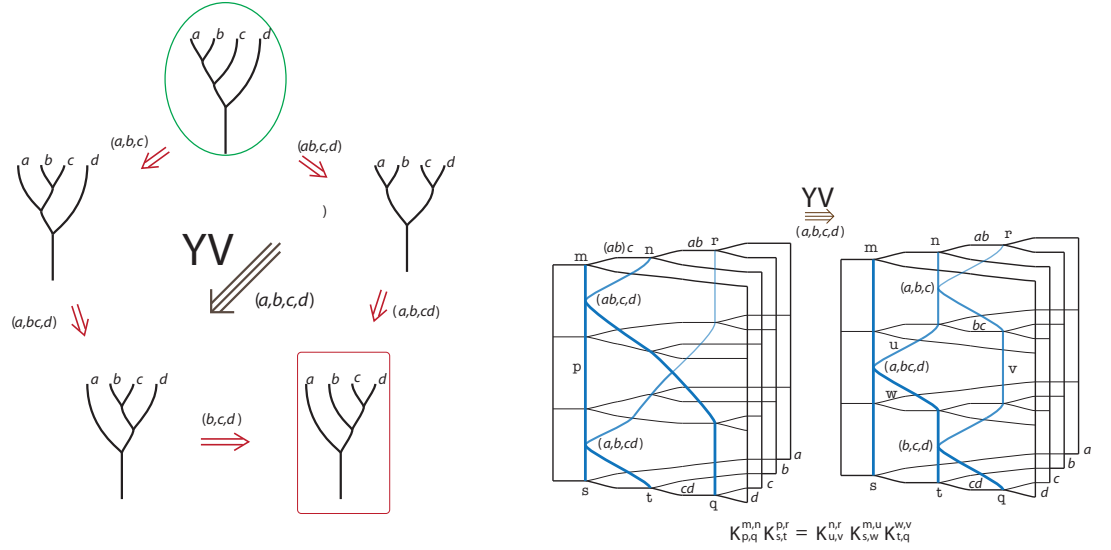
- |     |                                                                                               |                          |
|-----|-----------------------------------------------------------------------------------------------|--------------------------|
| H   | $(ab)c = a(bc)$                                                                               | (associativity);         |
| YI  | $(ab) \triangleleft c = (a \triangleleft c)(b \triangleleft c)$                               | (distributivity);        |
| IY  | $(a \triangleleft b) \triangleleft c = a \triangleleft (bc)$                                  | (exponential law);       |
| III | $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ | (self-distributivity);   |
| II  | $\forall a, b \exists ! c \text{ such that } c \triangleleft b = a$                           | (right invertibility);   |
| I   | $a \triangleleft a = a$                                                                       | (idempotence);           |
| T   | $a \cdot b = b \cdot (a \triangleleft b)$                                                     | (twisted commutativity). |

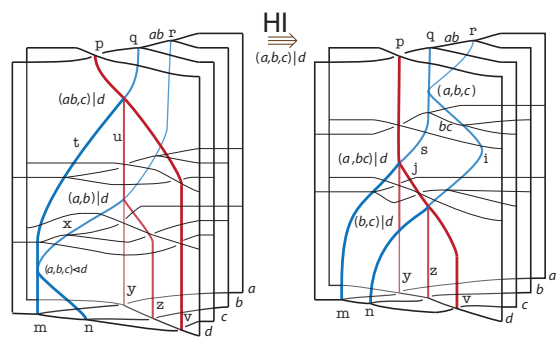


|     |                                                                                          |
|-----|------------------------------------------------------------------------------------------|
| H   | $Y_j^{\ell,n} Y_i^{j,k} = Y_p^{n,k} Y_i^{\ell,p}.$                                       |
| YI  | $Y_d^{a,b} X_{e,f}^{d,c} = X_{d,e}^{b,c} X_{e,g}^{a,d} Y_f^{g,e}.$                       |
| IY  | $X_{d,g}^{a,b} X_{h,f}^{g,c} Y_e^{d,h} = Y_i^{b,c} X_{e,f}^{a,i}.$                       |
| III | $X_{d,e}^{a,b} X_{i,h}^{e,c} X_{f,g}^{d,i} = X_{j,k}^{b,c} X_{f,e}^{a,j} X_{g,h}^{e,k}.$ |

| Partition       | Foam move name | Associated chain | Prismatic structure                                        |
|-----------------|----------------|------------------|------------------------------------------------------------|
| $(1, 2, 3, 4)$  | YV             | $(a, b, c, d)$   | $\Delta^4$                                                 |
| $(1, 2, 3) 4$   | HI             | $(a, b, c) d$    | $\Delta^3 \times \Delta^1$                                 |
| $(1, 2) (3, 4)$ | YY             | $(a, b) (c, d)$  | $\Delta^2 \times \Delta^2$                                 |
| $(1, 2) 3 4$    | YII            | $(a, b) c d$     | $\Delta^2 \times \Delta^1 \times \Delta^1$                 |
| $1 (2, 3, 4)$   | IH             | $a (b, c, d)$    | $\Delta^1 \times \Delta^3$                                 |
| $1 (2, 3) 4$    | IYI            | $a (b, c) d$     | $\Delta^1 \times \Delta^2 \times \Delta^1$                 |
| $1 2 (3, 4)$    | IYY            | $a b (c, d)$     | $\Delta^1 \times \Delta^1 \times \Delta^2$                 |
| $1 2 3 4$       | IIII           | $a b c d$        | $\Delta^1 \times \Delta^1 \times \Delta^1 \times \Delta^1$ |

Table 13: The foam moves, corresponding chains, partitions, and prismatic sets

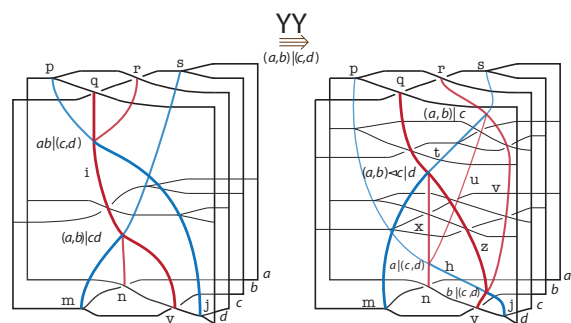




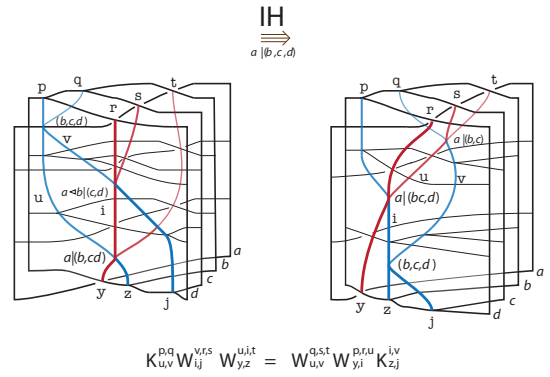
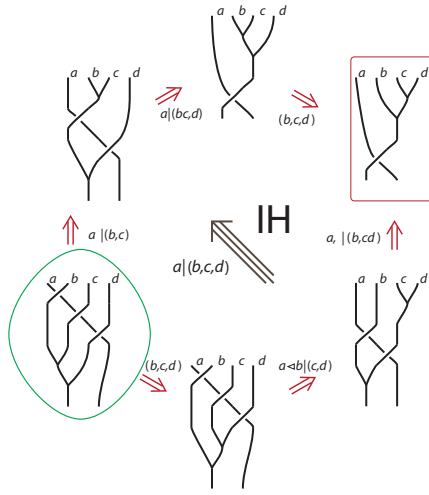
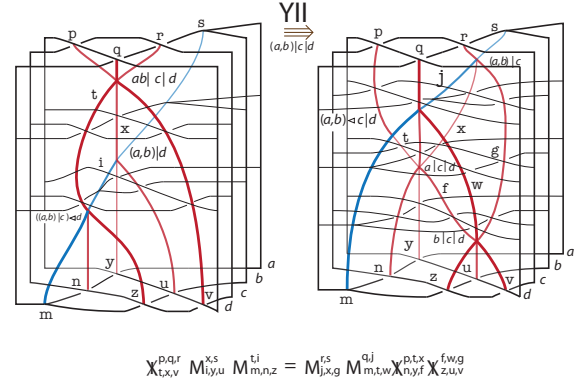
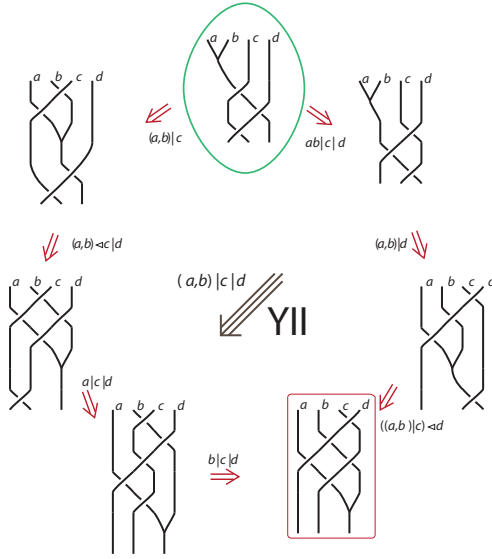
The diagram illustrates the relationship between different types of recombination events (crossover, gene conversion, and double crossover) and their corresponding patterns of allelic variation. The diagram is divided into three main sections:

- Top Section:** Shows a crossover event between two homologous chromosomes. The resulting patterns of allelic variation are shown as  $(a,b)|cd$  and  $ab|(c,d)$ .
- Middle Section:** Shows a double crossover event between two homologous chromosomes. The resulting patterns of allelic variation are shown as  $(a,b)|c,d$  and  $(a,b)|c$ .
- Bottom Section:** Shows a gene conversion event between two homologous chromosomes. The resulting patterns of allelic variation are shown as  $a|(c,d)$  and  $(a,b)-ac|d$ .

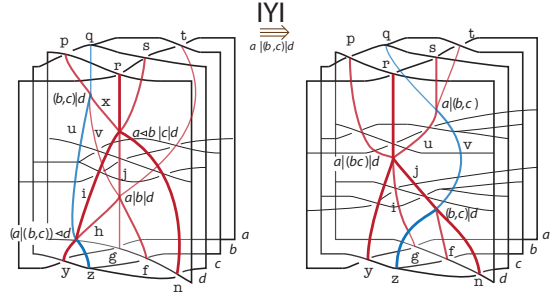
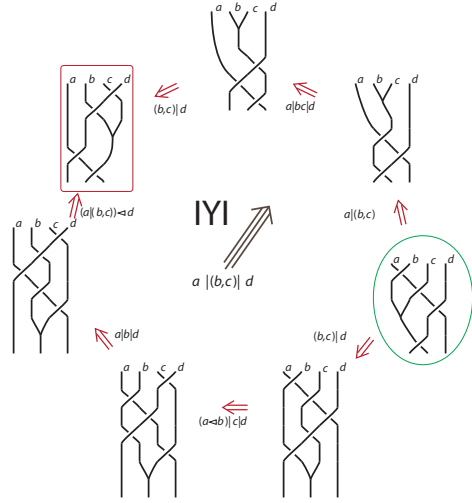
The diagram uses various symbols and labels to represent different types of recombination events and their resulting patterns of allelic variation. The labels  $(a,b)|cd$ ,  $ab|(c,d)$ ,  $(a,b)|c,d$ ,  $(a,b)|c$ ,  $a|(c,d)$ , and  $(a,b)-ac|d$  represent different patterns of allelic variation resulting from the recombination events shown.



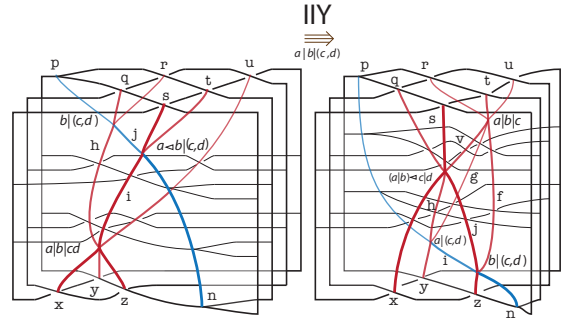
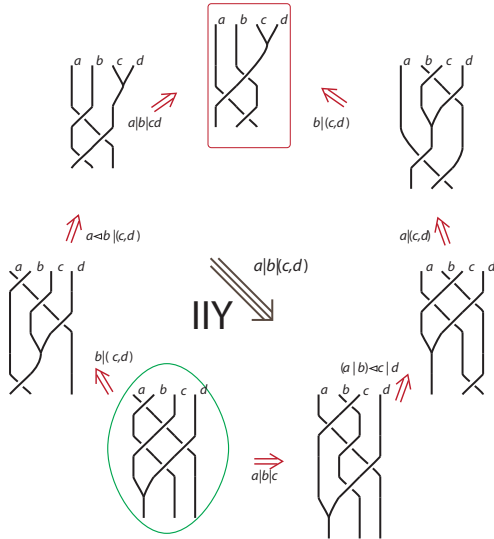
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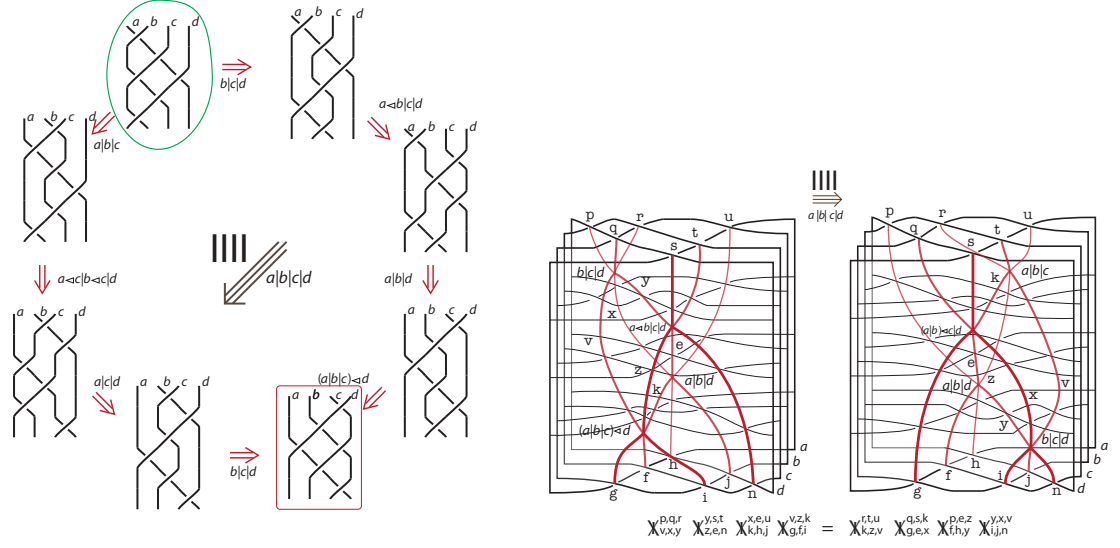




$$M_{u,v,x}^{p,q} \mathbb{X}_{i,j,n}^{x,r,s} \mathbb{X}_{h,g,f}^{v,i,t} W_{y,z}^{u,i,h} = W_{u,v}^{q,s,t} \mathbb{X}_{y,i,j}^{p,r,u} M_{z,f,n}^{i,v}$$



$$W_{h,j}^{p,q,r} W_{i,n}^{j,s,t} \mathbb{X}_{x,y,z}^{h,i,u} = \mathbb{X}_{v,g,f}^{r,t,u} \mathbb{X}_{x,h,j}^{q,s,v} W_{y,i}^{p,h,q} W_{z,n}^{i,f}$$



|      |                                                                                                                                                     |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------|
| YV   | $K_{p,q}^{m,n} K_{s,t}^{p,r} = K_{u,v}^{n,r} K_{s,w}^{m,u} K_{t,q}^{w,v}$                                                                           |
| HI   | $M_{t,u,v}^{p,q} M_{x,y,z}^{u,r} K_{m,n}^{t,x} = K_{s,i}^{q,r} M_{m,y,j}^{p,s} M_{n,z,v}^{j,i}$                                                     |
| YY   | $W_{i,j}^{p,q,r} M_{m,n,y}^{i,s} = M_{t,u,v}^{r,s} M_{m,x,z}^{q,t} W_{n,h}^{p,x,u} W_{y,i}^{h,z,v}$                                                 |
| YII  | $X_{t,x,v}^{p,q,r} M_{i,y,u}^{x,s} M_{m,n,z}^{t,i} = M_{j,x,g}^{r,s} M_{m,t,x}^{q,j} X_{n,y,f}^{p,t,x} X_{z,u,v}^{f,w,g}$                           |
| IH   | $K_{u,v}^{p,q} W_{i,j}^{v,r,s} W_{y,z}^{u,i,t} = W_{u,v}^{q,s,t} W_{y,i}^{p,r,u} K_{z,j}^{i,v}$                                                     |
| IYI  | $M_{u,v,x}^{p,q} X_{i,j,n}^{x,r,s} X_{h,g,f}^{v,j,t} W_{y,z}^{u,i,h} = W_{u,v}^{q,s,t} X_{y,i,j}^{p,r,u} M_{z,f,n}^{j,v}$                           |
| IY   | $W_{h,j}^{p,q,r} W_{i,n}^{j,s,t} X_{x,y,z}^{h,i,u} = X_{v,g,f}^{r,t,u} X_{x,h,j}^{q,s,v} W_{y,i}^{p,h,g} W_{z,n}^{t,j,f}$                           |
| IIII | $X_{v,x,y}^{p,q,r} X_{z,e,n}^{y,s,t} X_{k,h,j}^{x,e,u} X_{g,f,i}^{v,z,k} = X_{k,z,v}^{r,t,u} X_{g,e,x}^{q,s,k} X_{f,h,y}^{p,e,z} X_{i,j,n}^{y,x,v}$ |