Theory of Rigidity & Force Transmission in Granular Systems

W.M. KECK FOUNDATION

NSF

TRUTH EVEN UNTO ITS INNERMOST PARTS BRANDEIS

Dapeng Bi

Sumantra Sarkar

Kabir Ramola

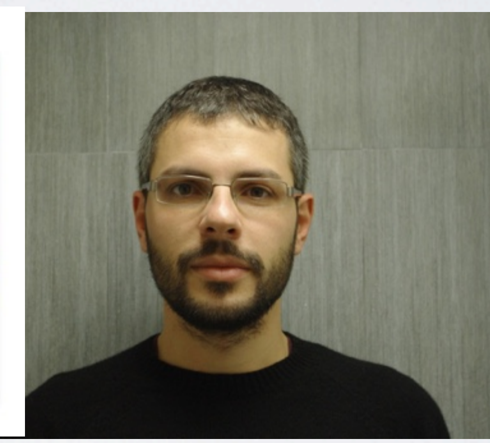
Jetin Thomas

Bob Behringer,

Jie Ren

Dong Wang

Jeff Morris



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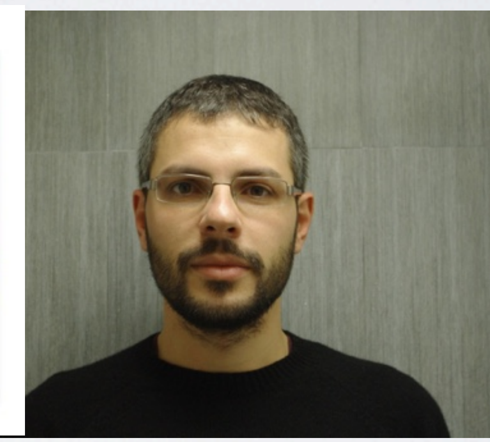

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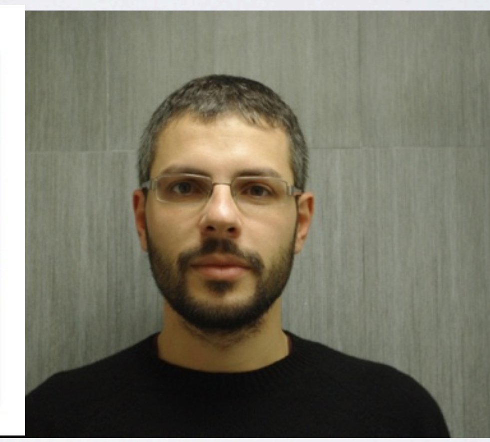

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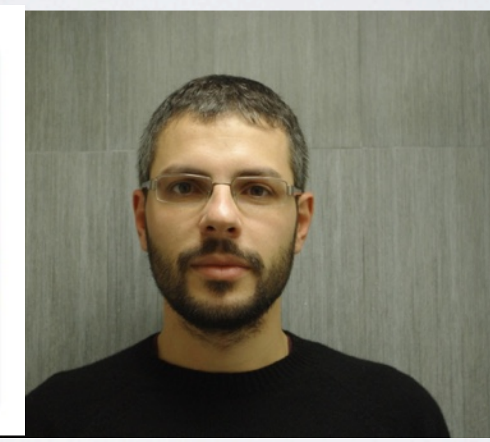
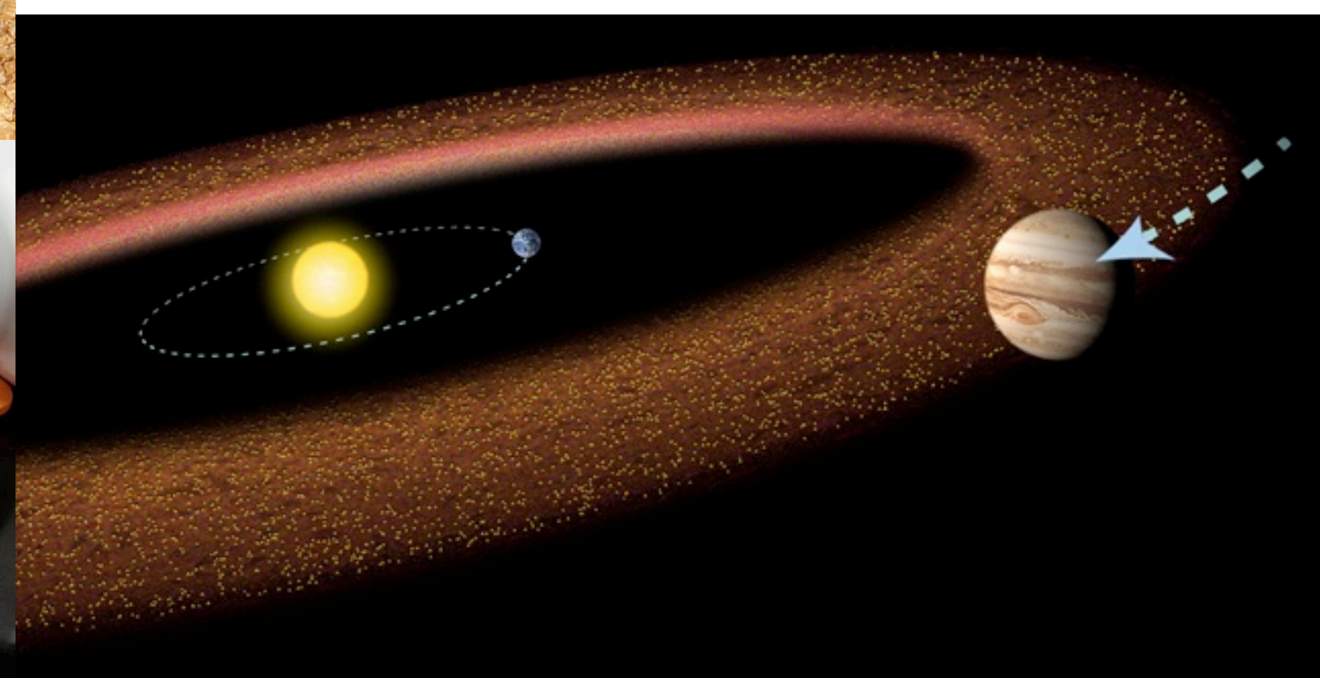


Romain Mari

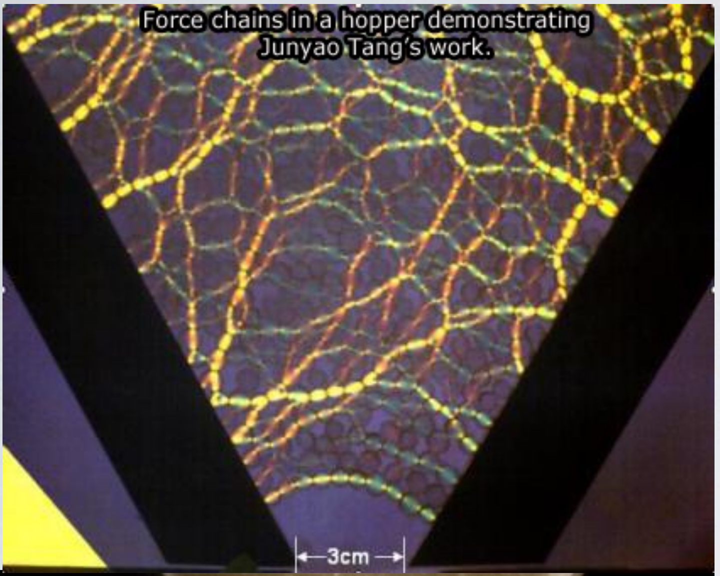
Abhi Singh

GRANULAR MATTER

- **Collection of macroscopic objects**
- **Purely repulsive, contact interactions. No thermal fluctuations to restore or create contacts**
- **Friction: Forces are independent degrees of freedom**
- **States controlled by driving at the boundaries or body forces: shear, gravity**
- **Non-ergodic in the extreme sense: stays in one configuration unless driven**



LOOKING INSIDE SAND



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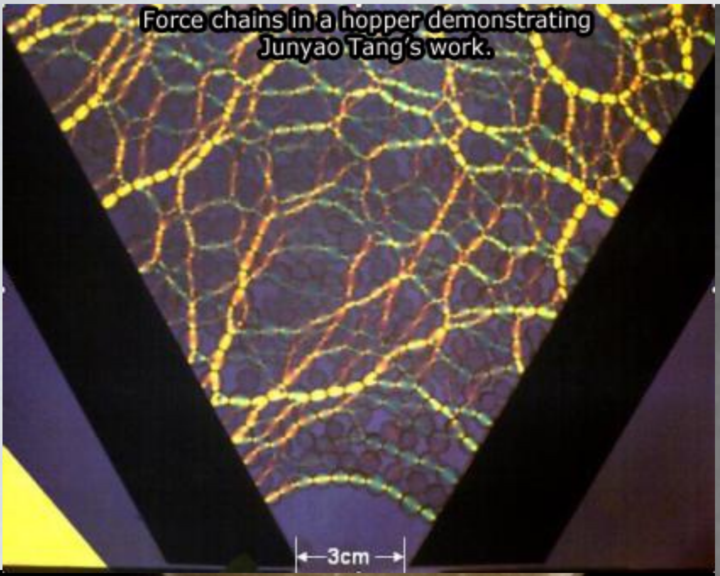
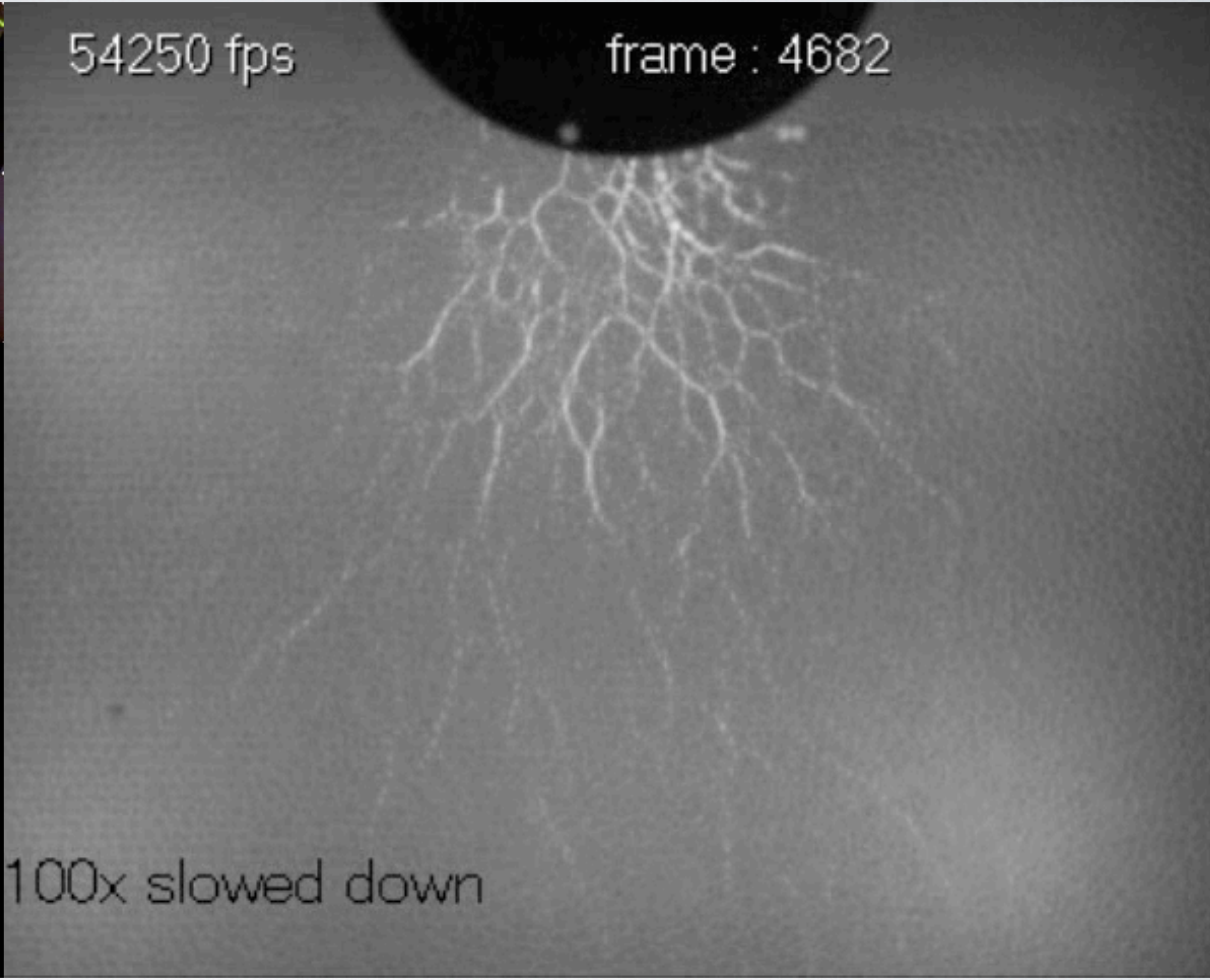
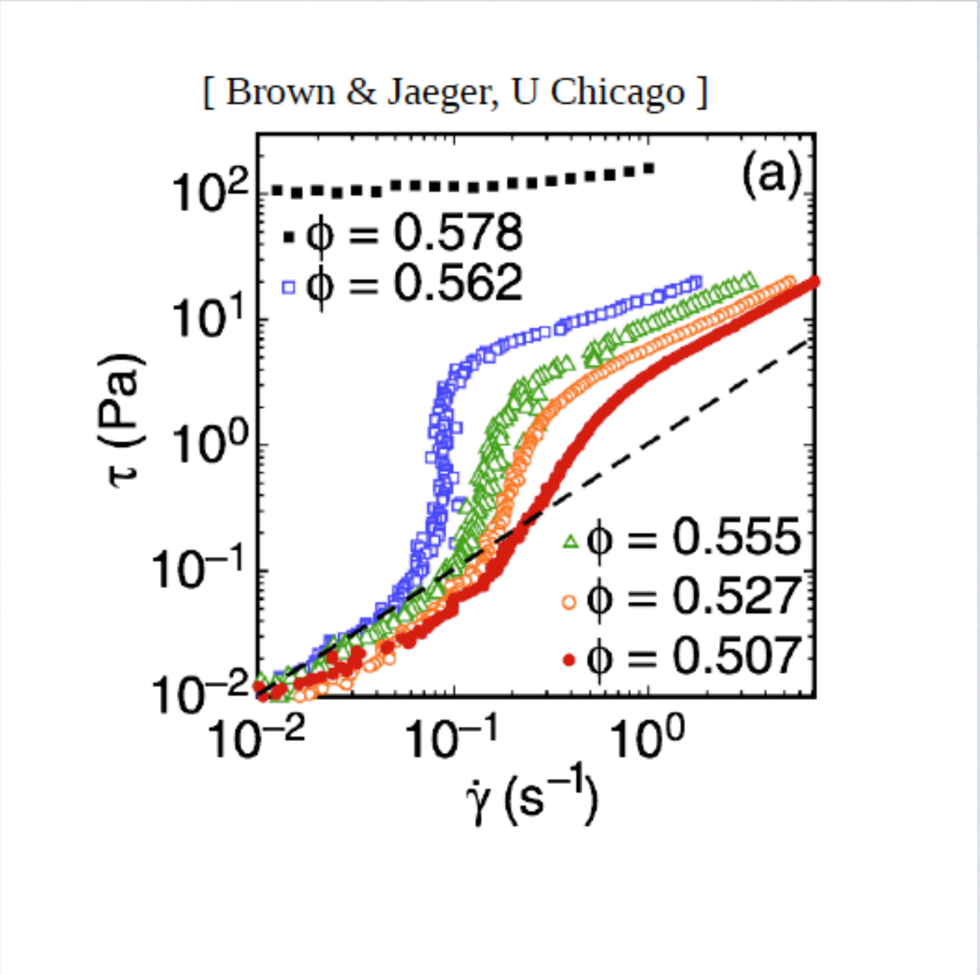


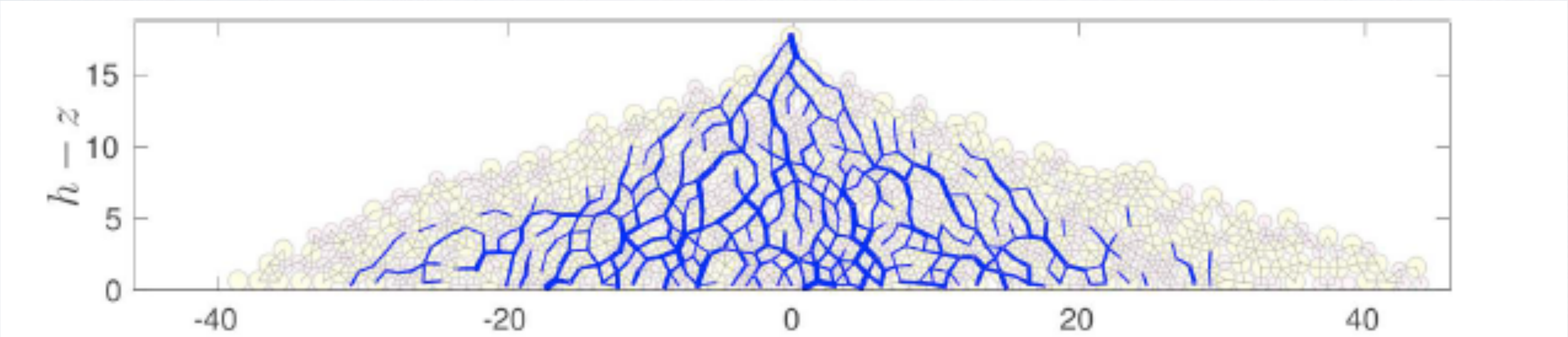


Two problems:

- **Discontinuous Shear Thickening in Dense Suspensions**



- **Stress Transmission in Static Granular Aggregates**



Procaccia group: Numerical Simulations (2016)

Statics of Granular Media: Constraints of mechanical equilibrium determine collective behavior

Imposed stresses determine sum of stresses over all grains

Local force balance satisfied for every grain

Positivity of all forces $f_N \geq 0$

Local torque balance satisfied for every grain

Friction law on each contact $f_t \leq \mu f_N$

Stress Metric

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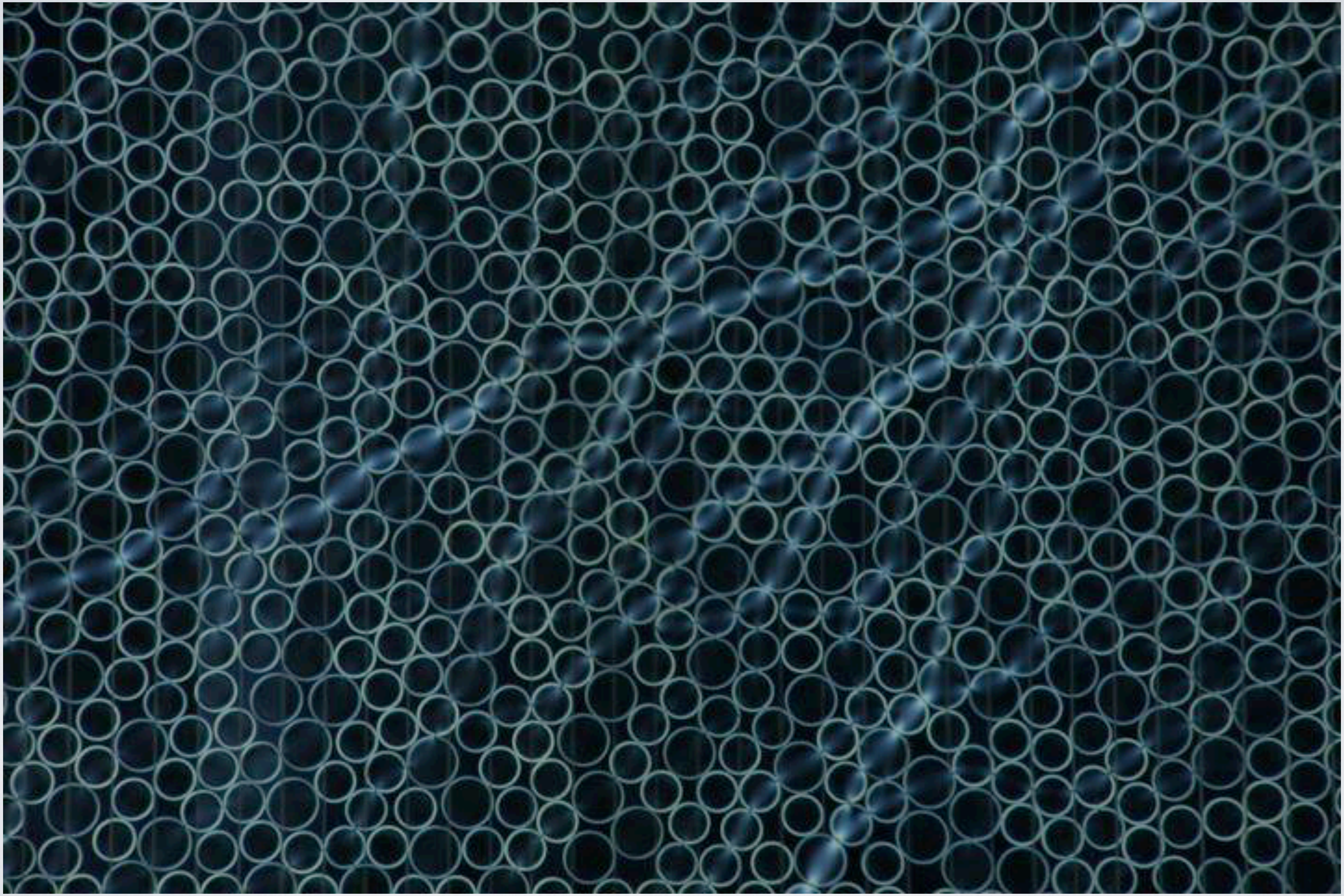
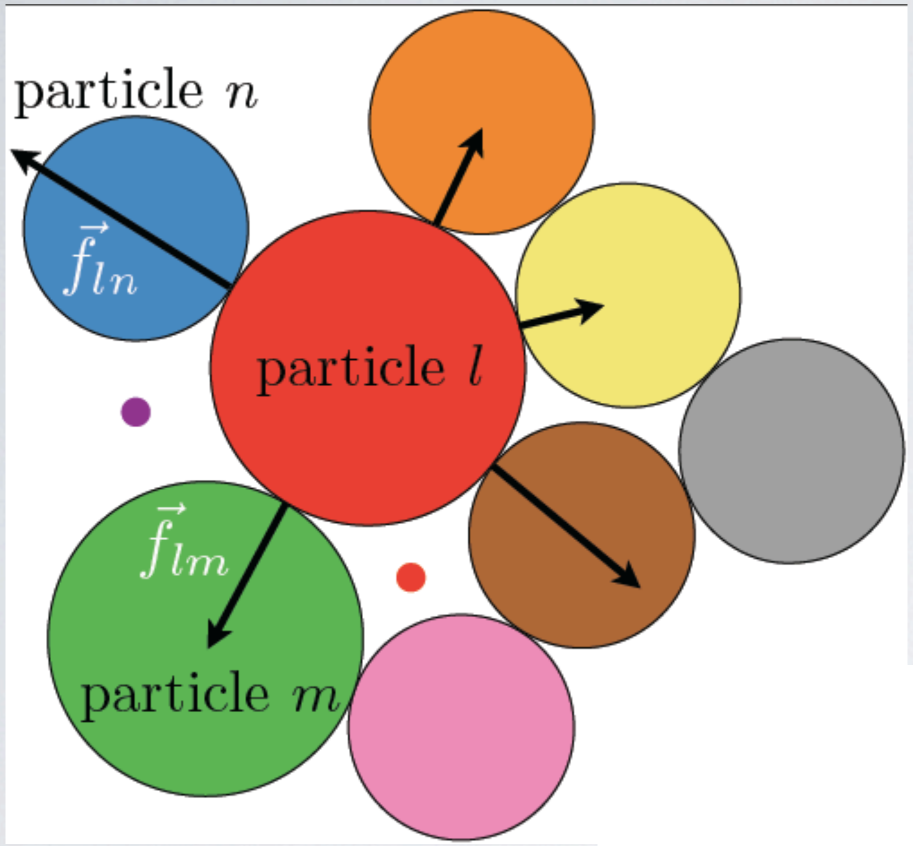
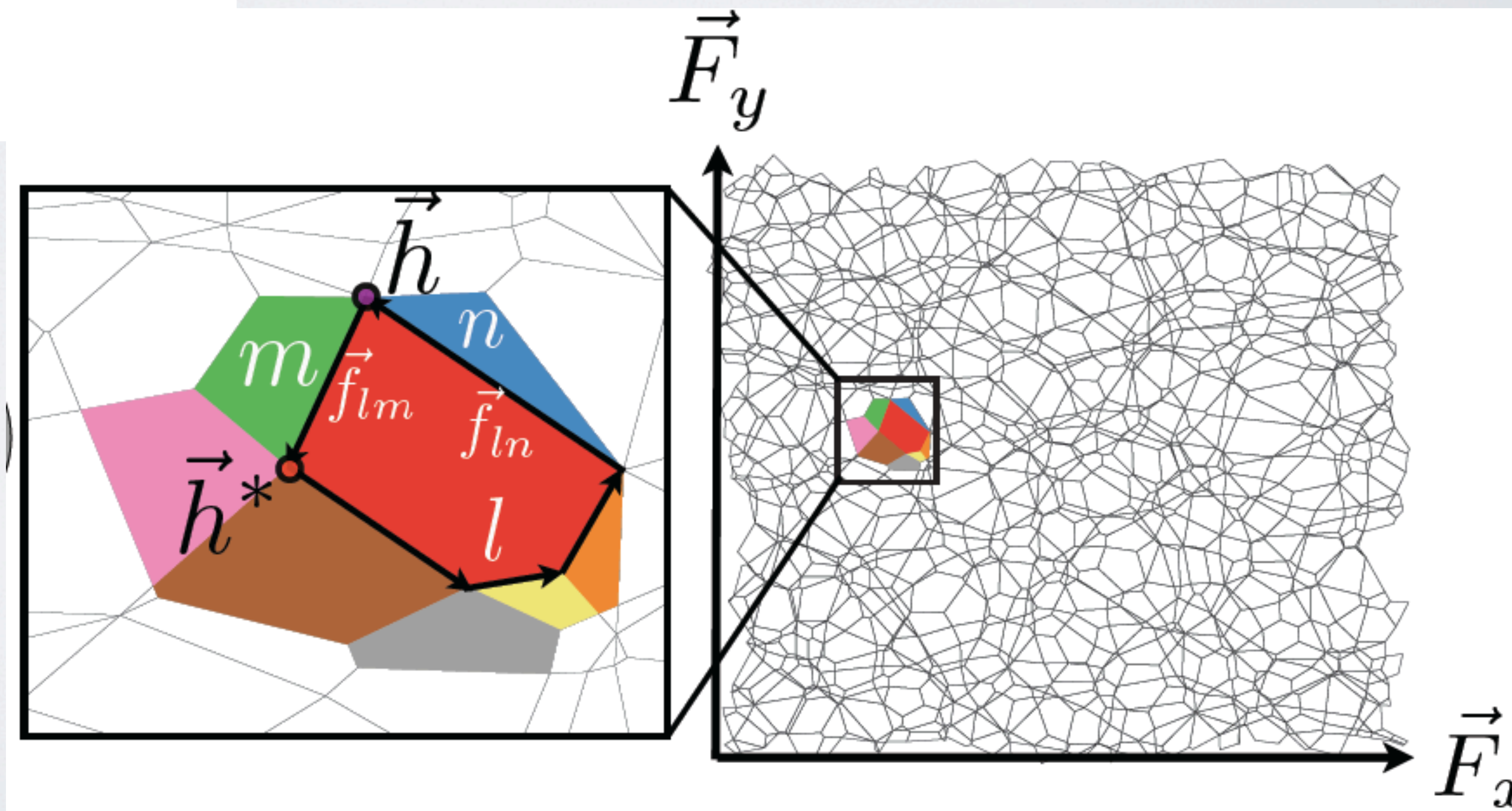


Force Tilings





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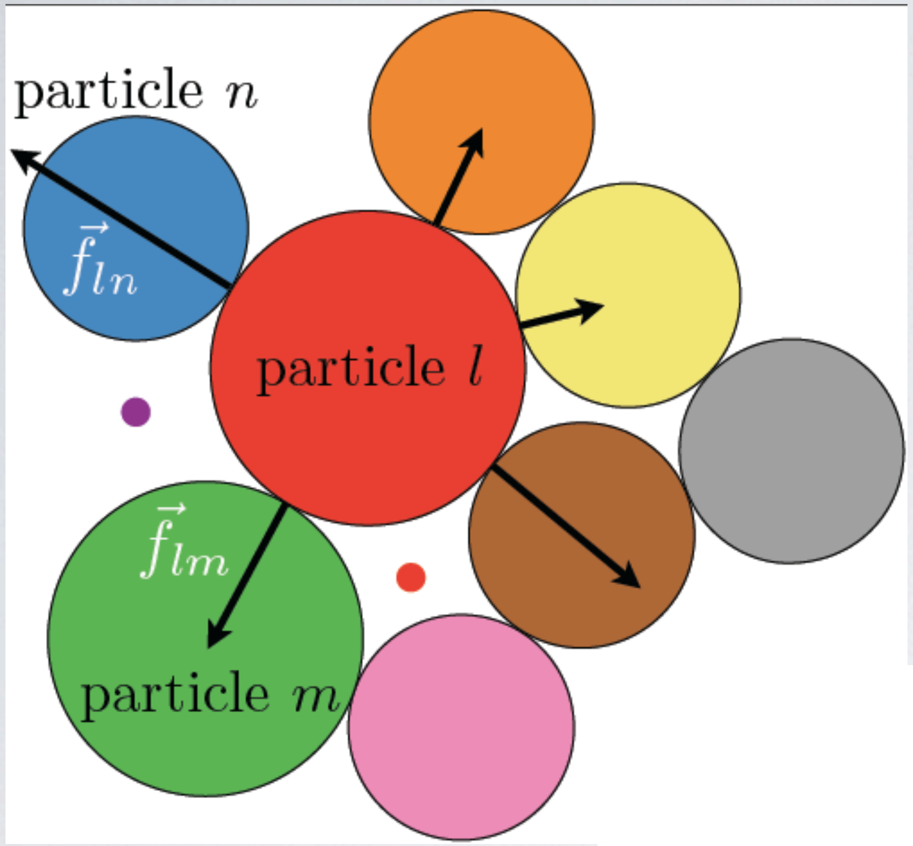
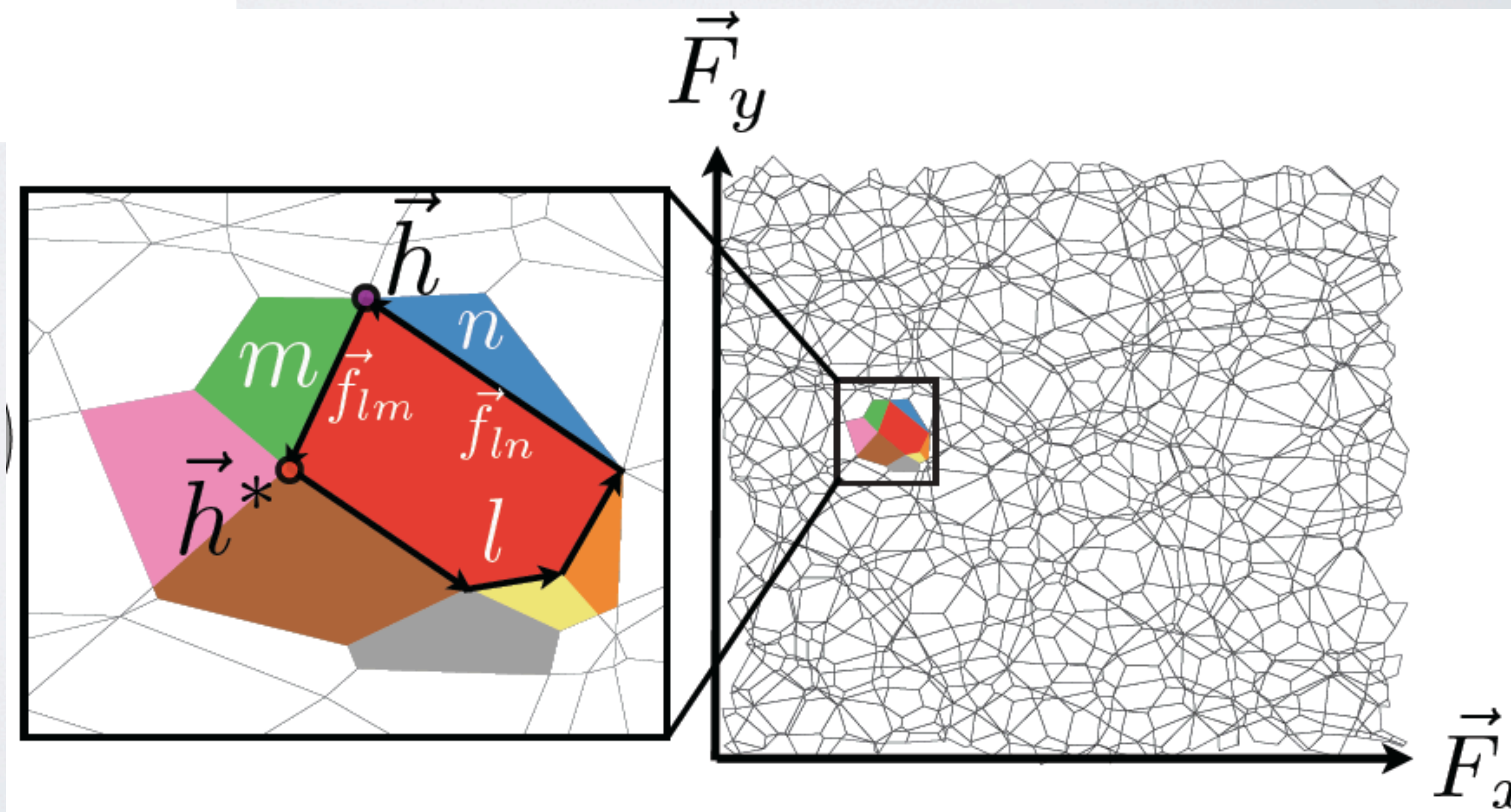


For systems where all normal forces are repulsive, we have a single sheet



TWO REPRESENTATIONS

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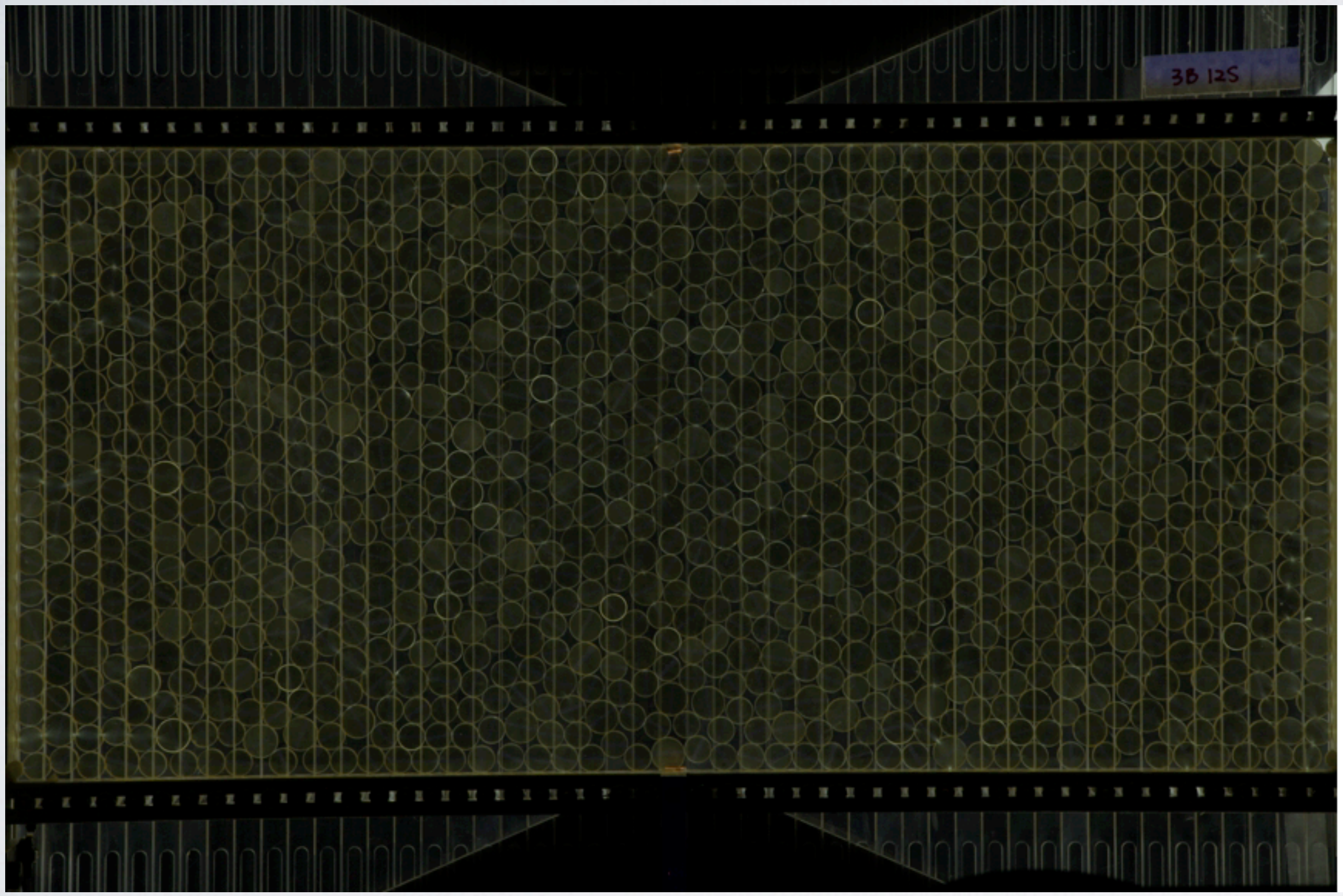

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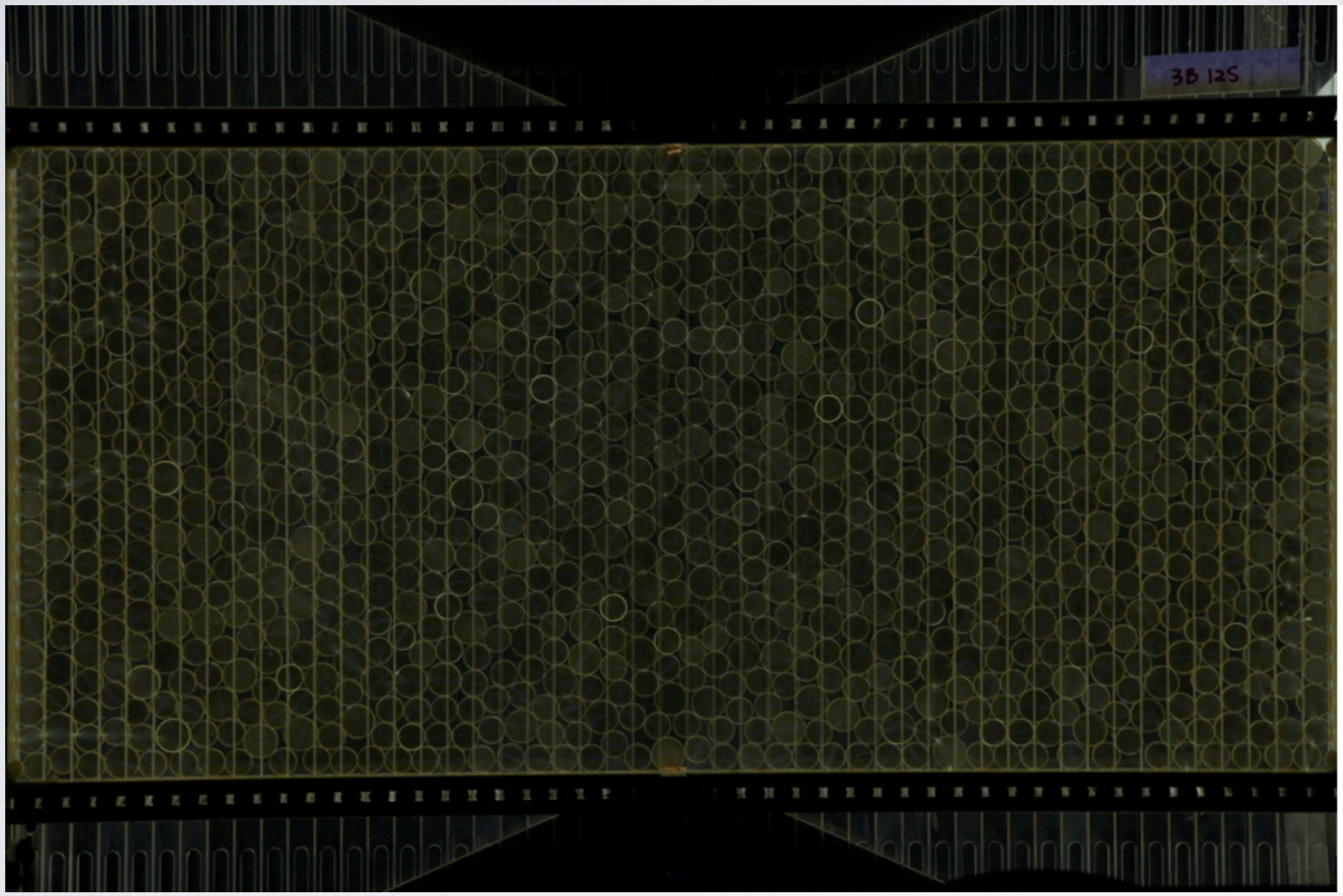
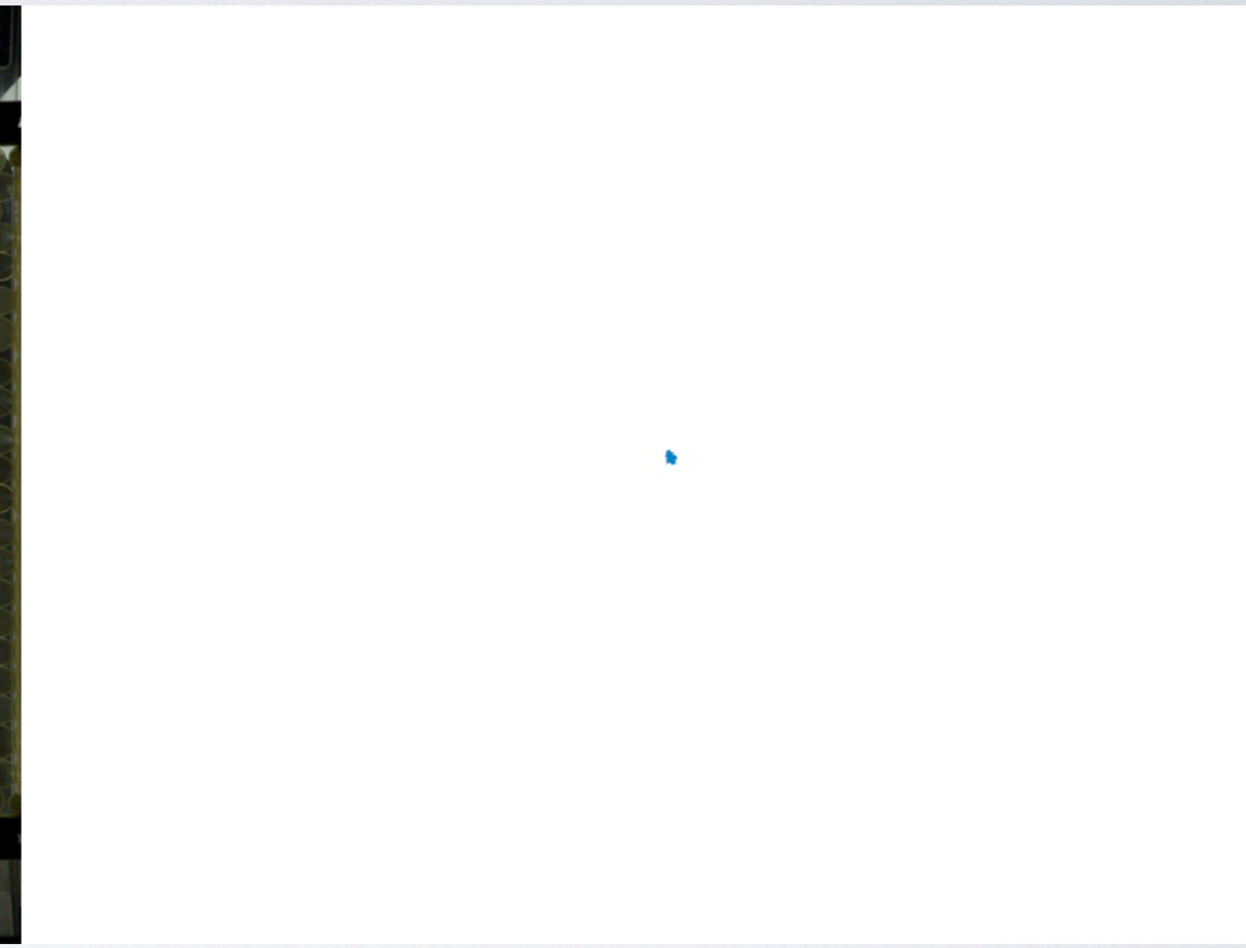
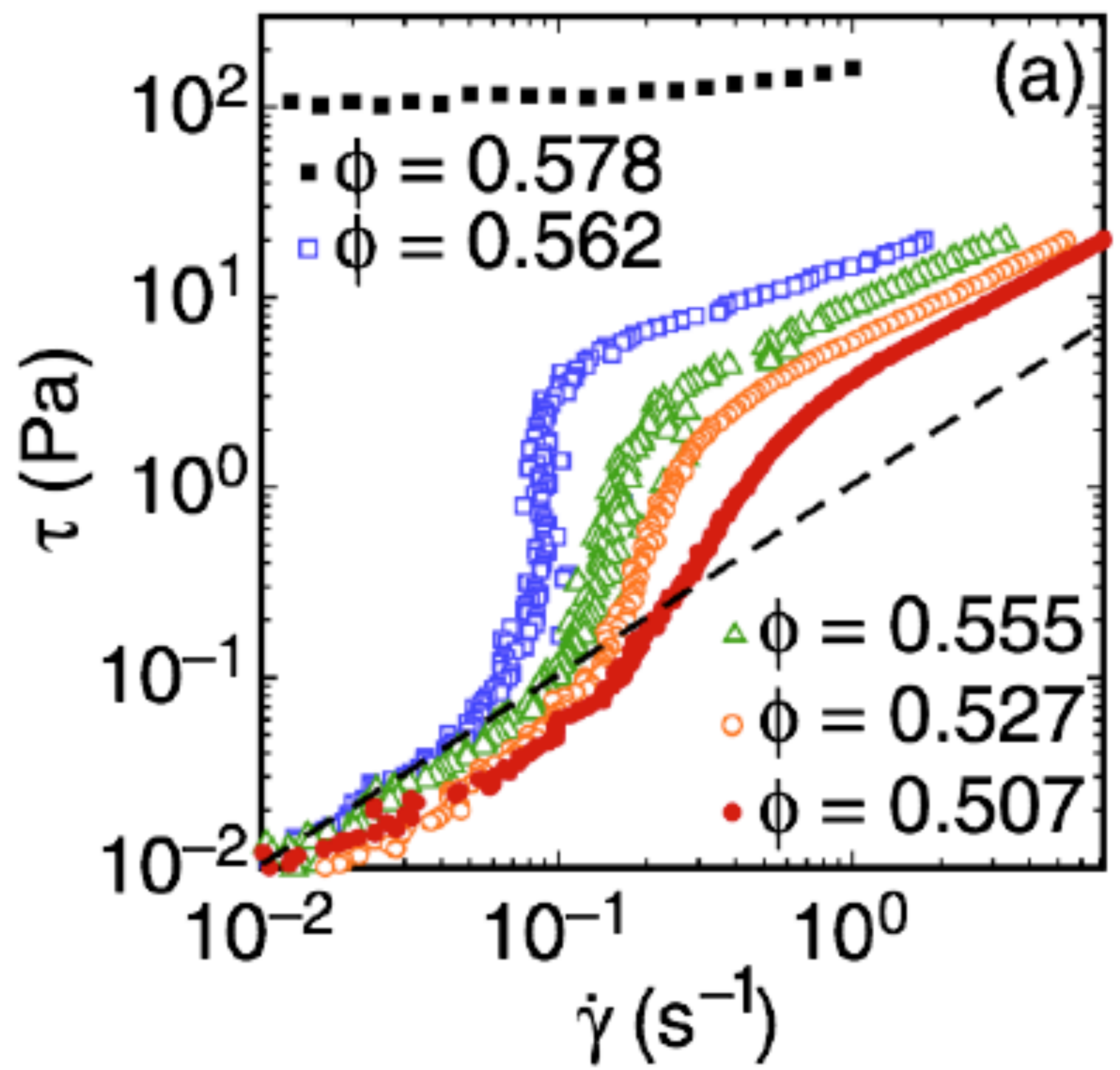




http://www.aps.org/meetings/march/vpr/2015/videogallery/index.cfm

Discontinuous Shear Thickening



Discontinuous Shear Thickening

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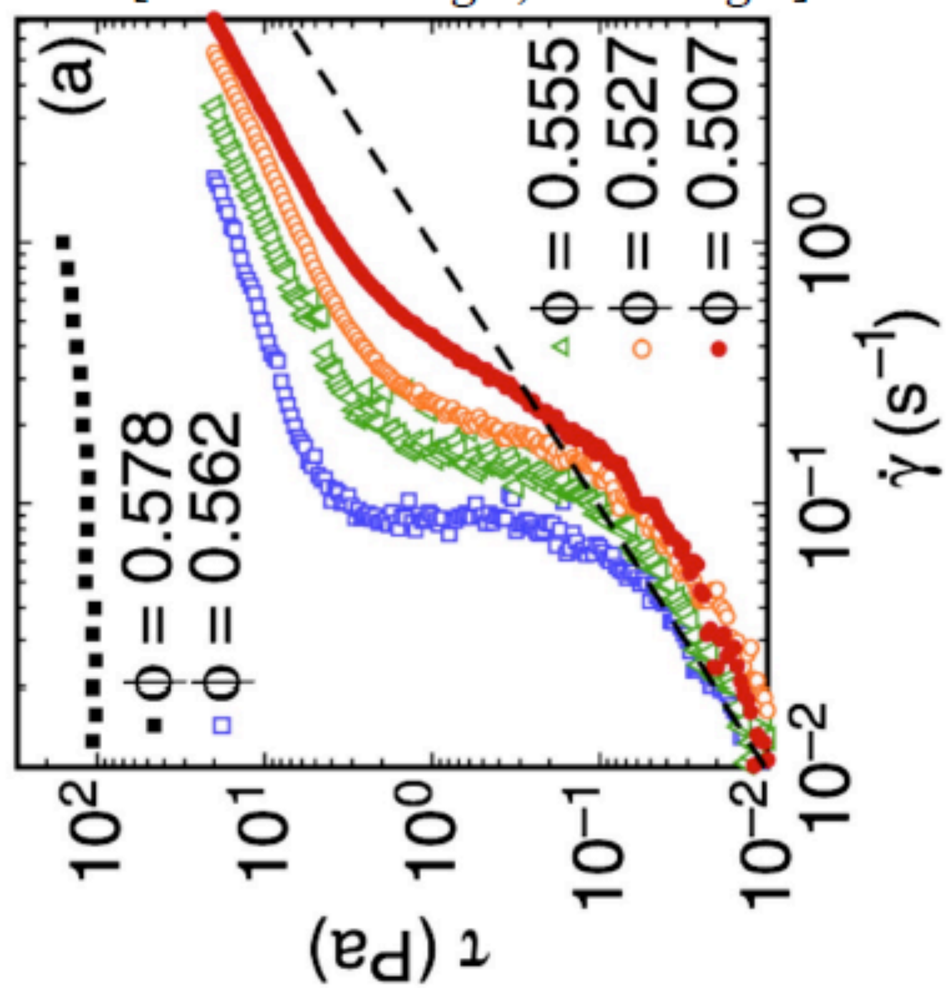
[Brown & Jaeger, U Chicago]



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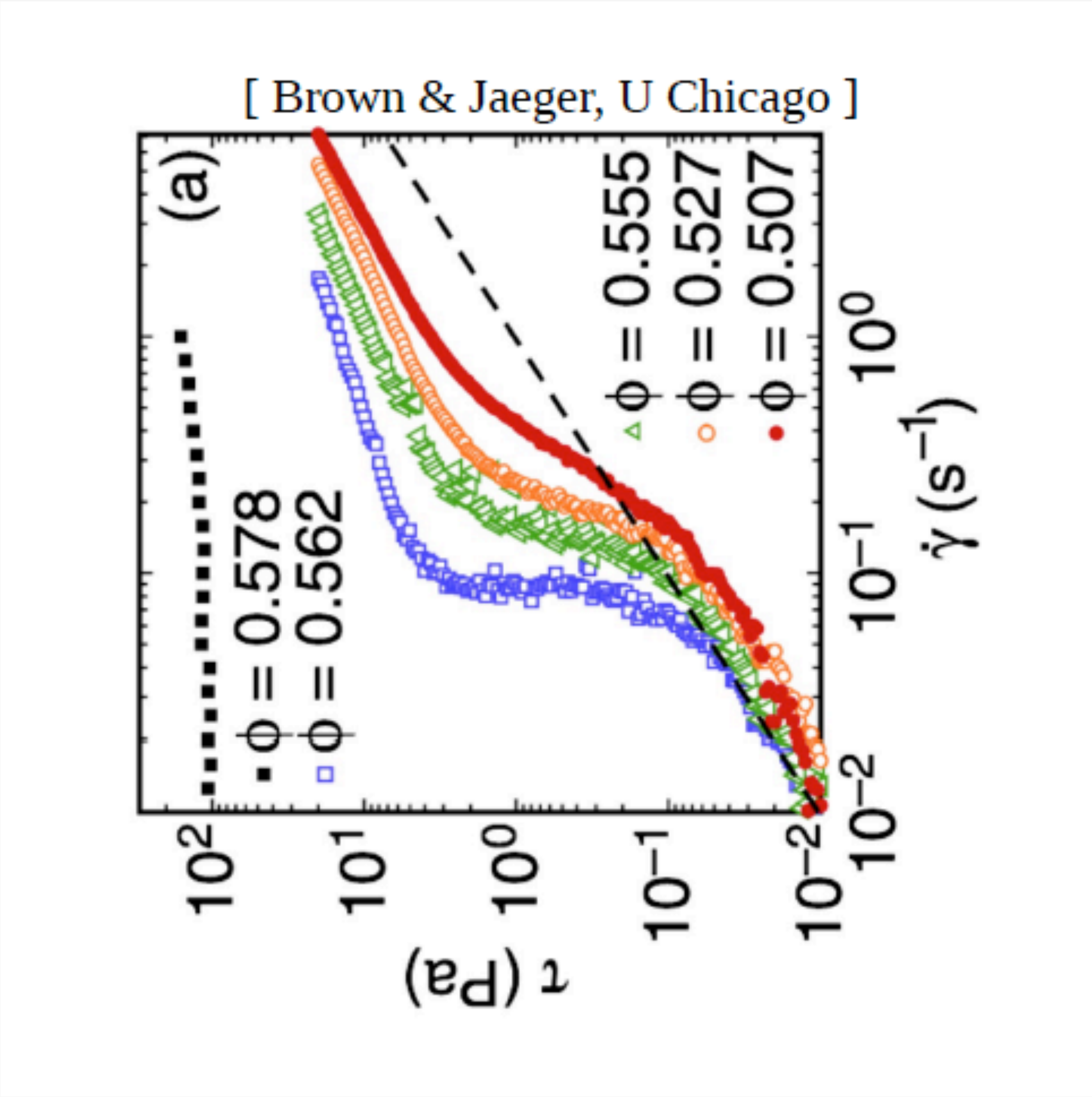

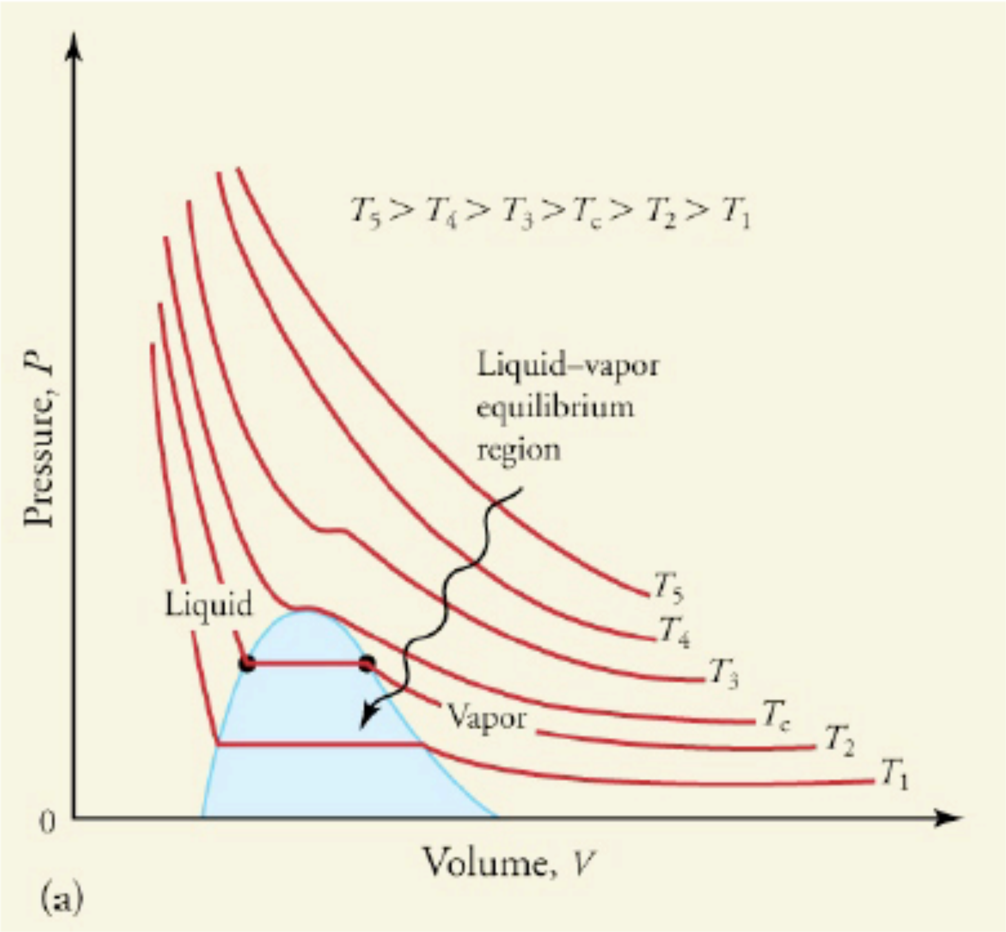

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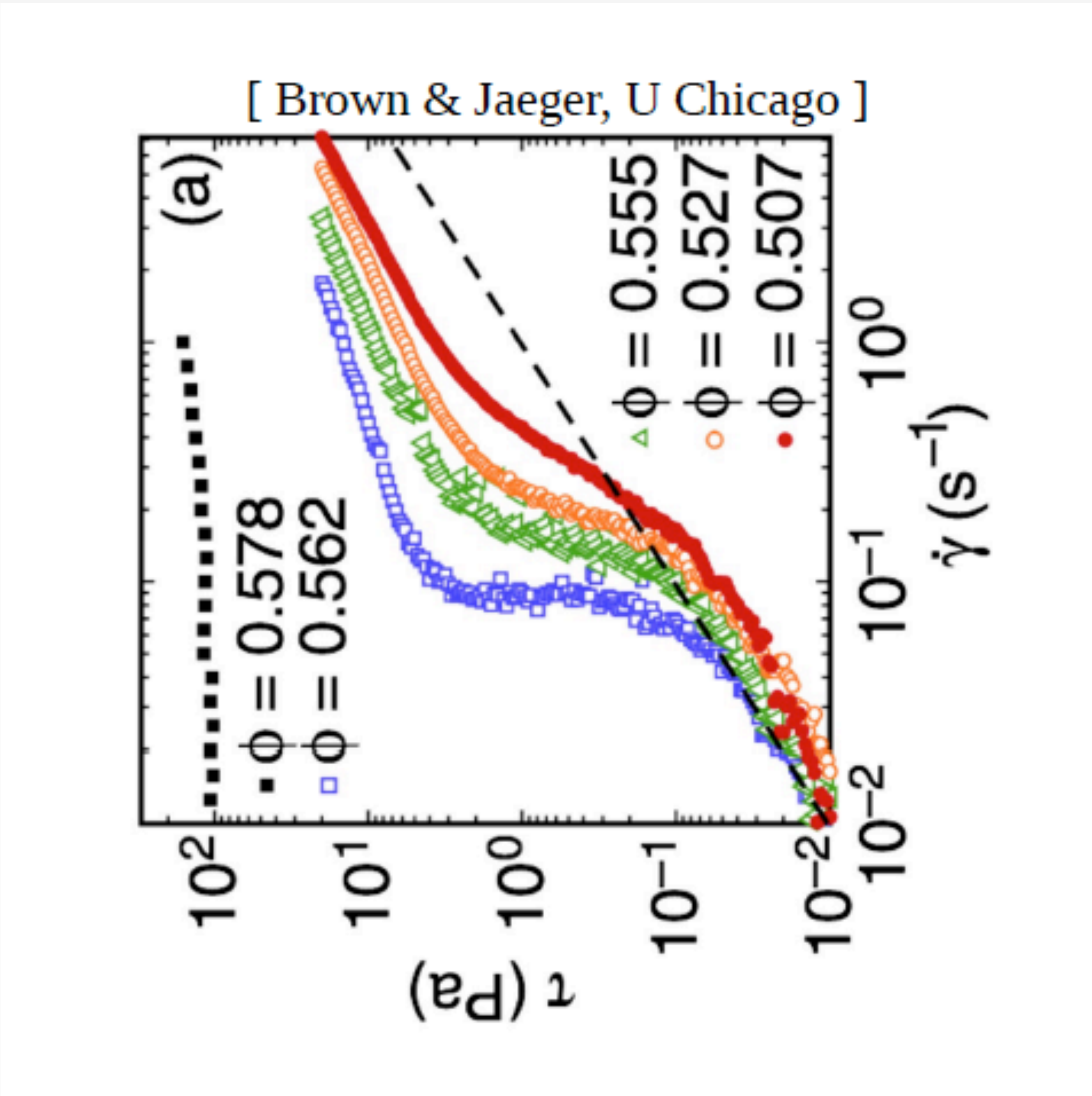
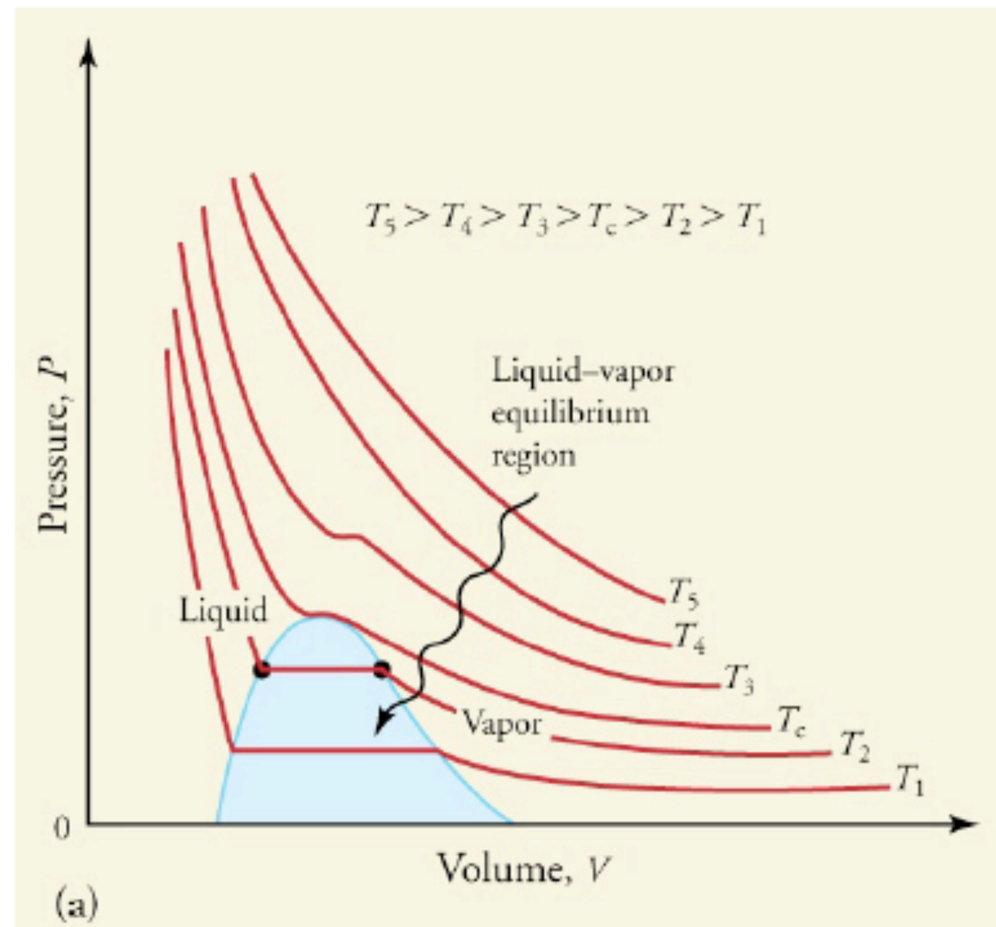
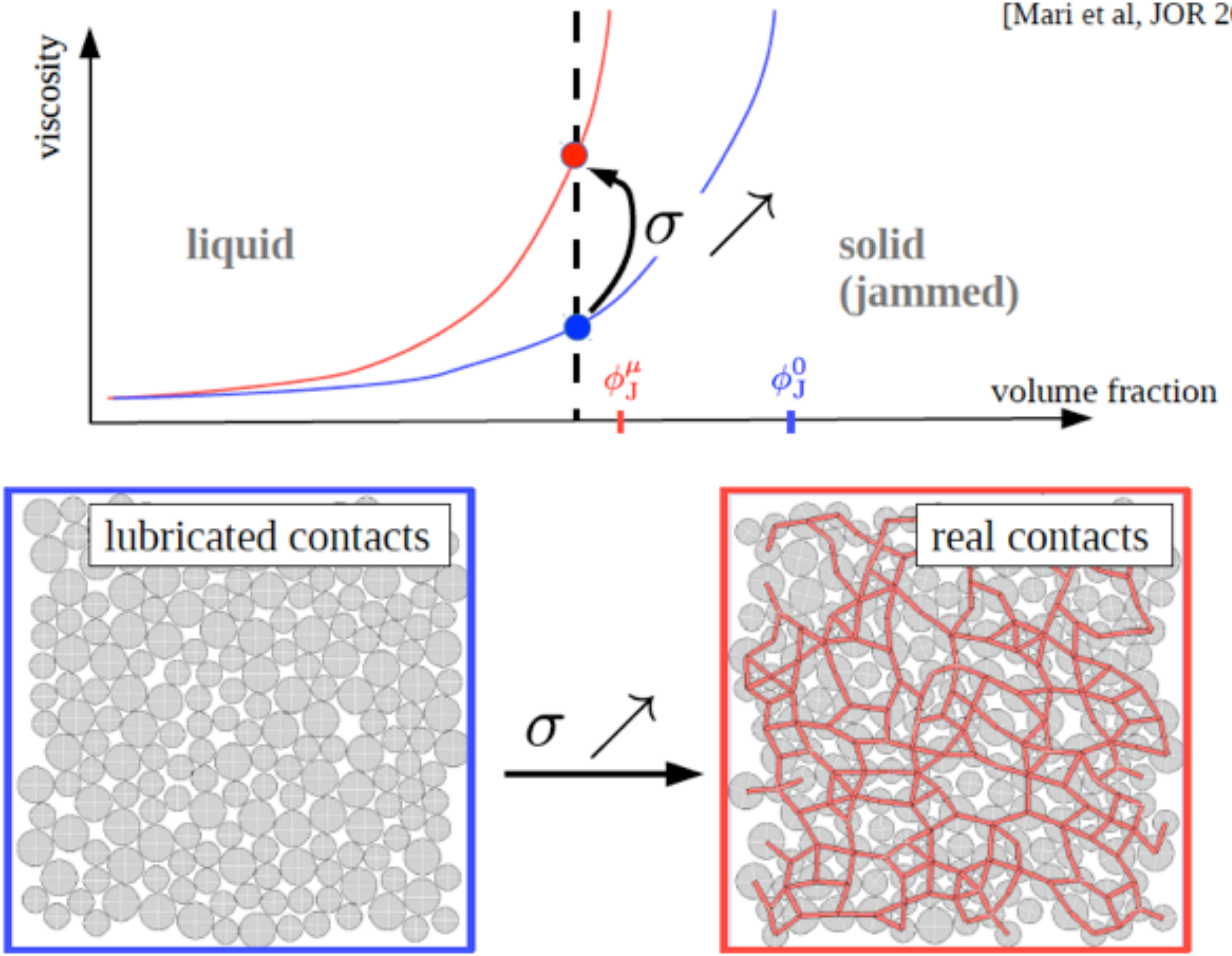




Shear thickening as a (out-of-equilibrium) phase transition?

A Thickening Scenario

[Fernandez et al, PRL 2013]
[Seto et al, PRL 2013]
[Heussinger, PRE 2013]
[Wyart and Cates PRL 2013]
[Mari et al, JOR 2014]



Phase Diagram from Rheology: Abhi Singh & Jeff Morris

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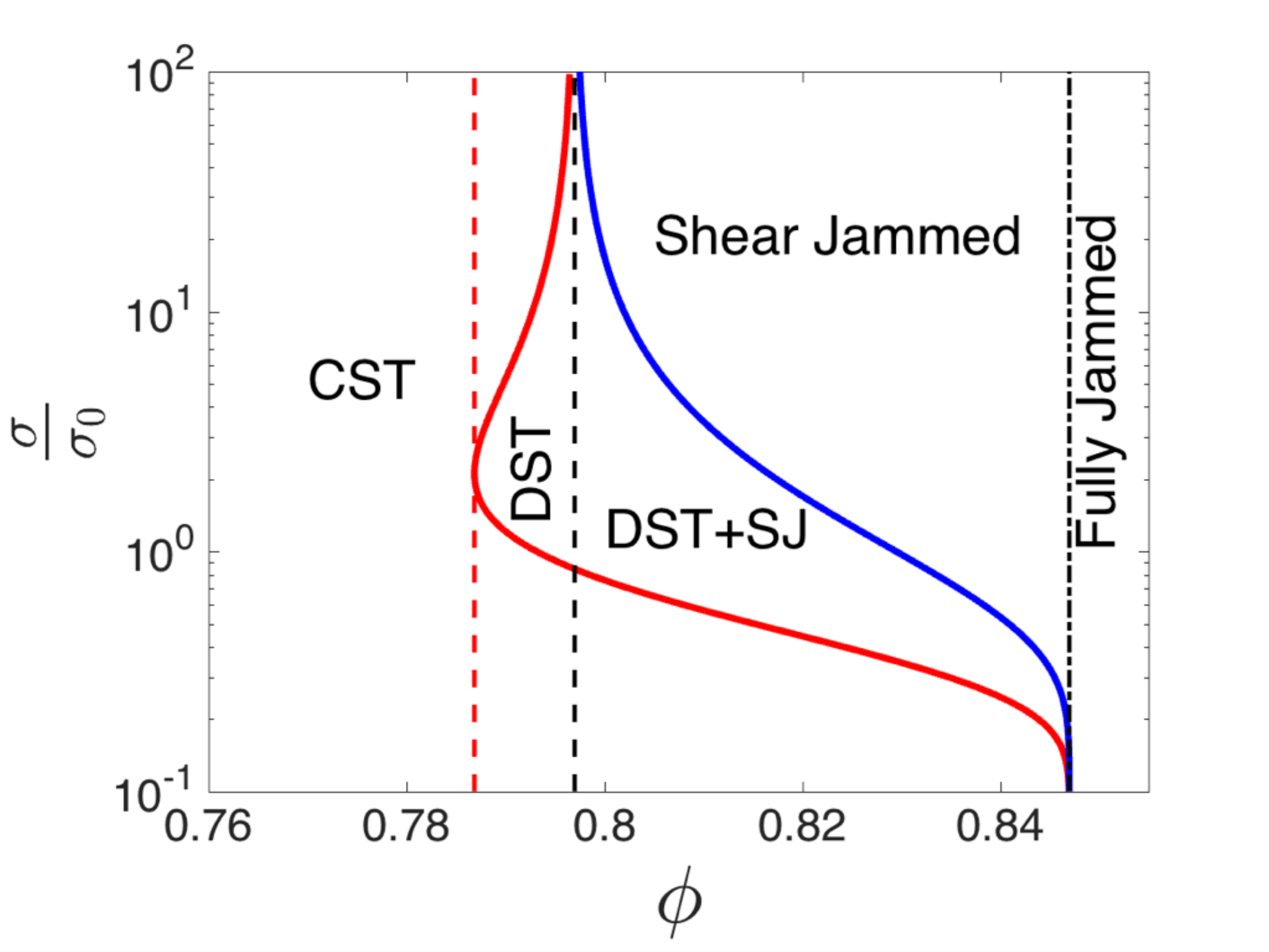

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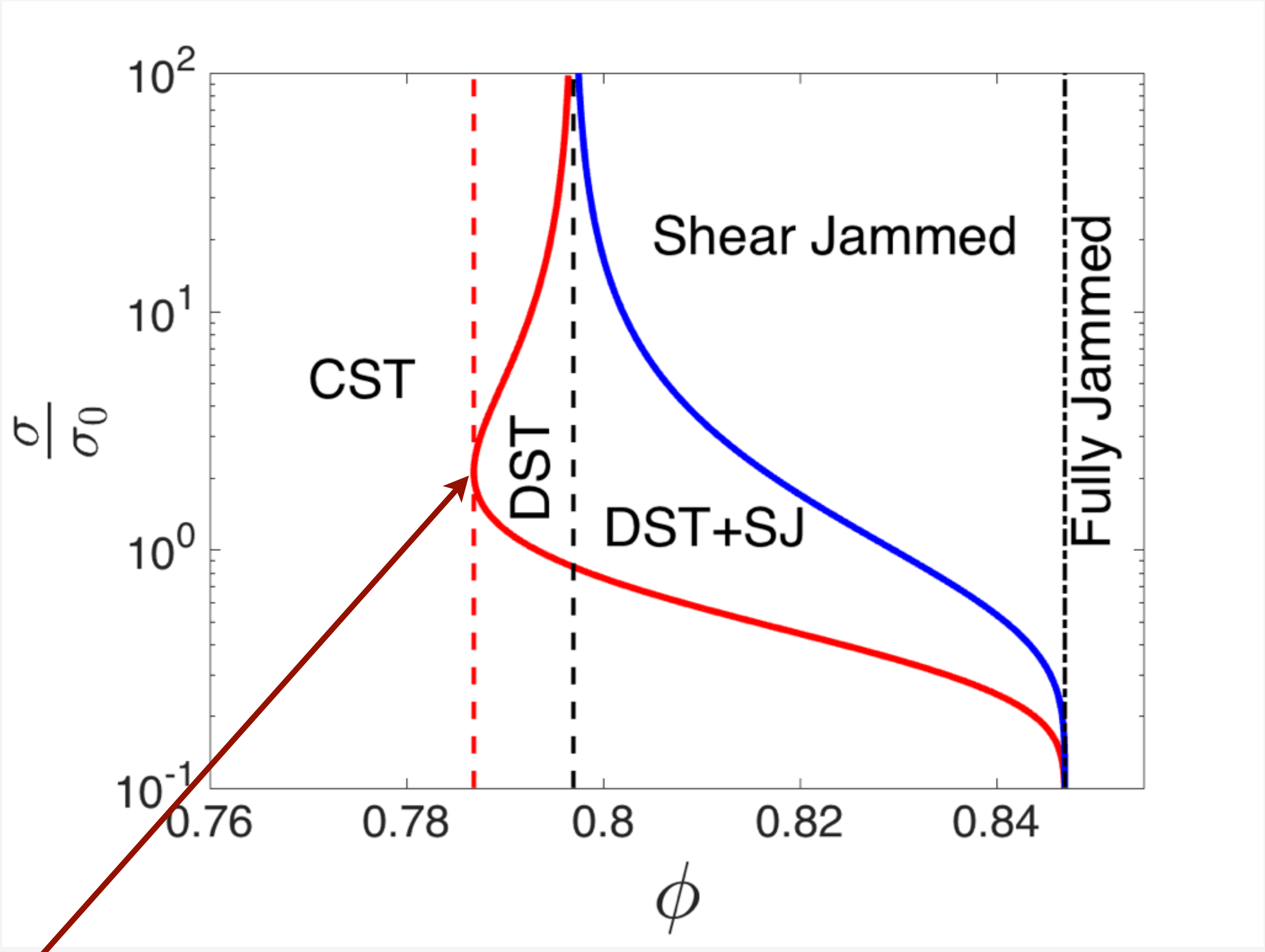
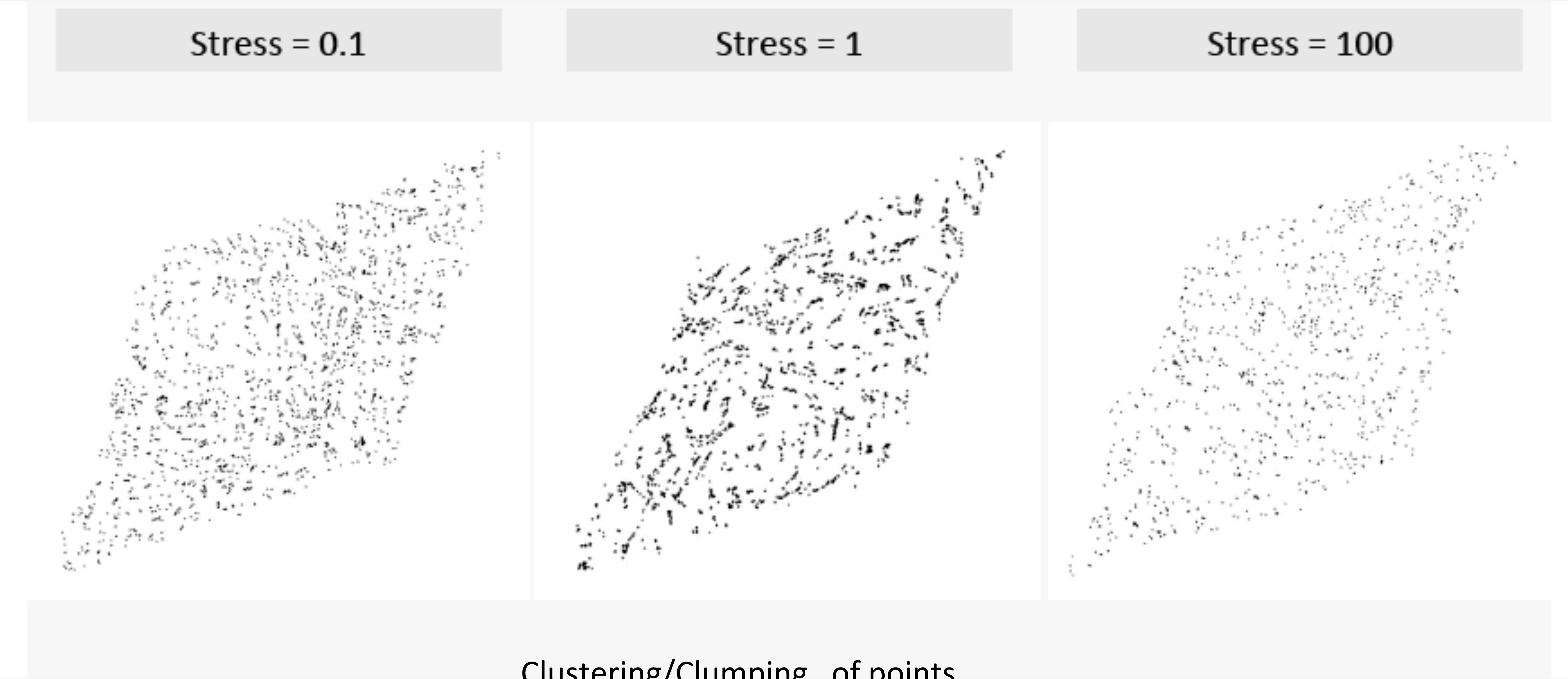


Is this a critical point ?

Clustering of points

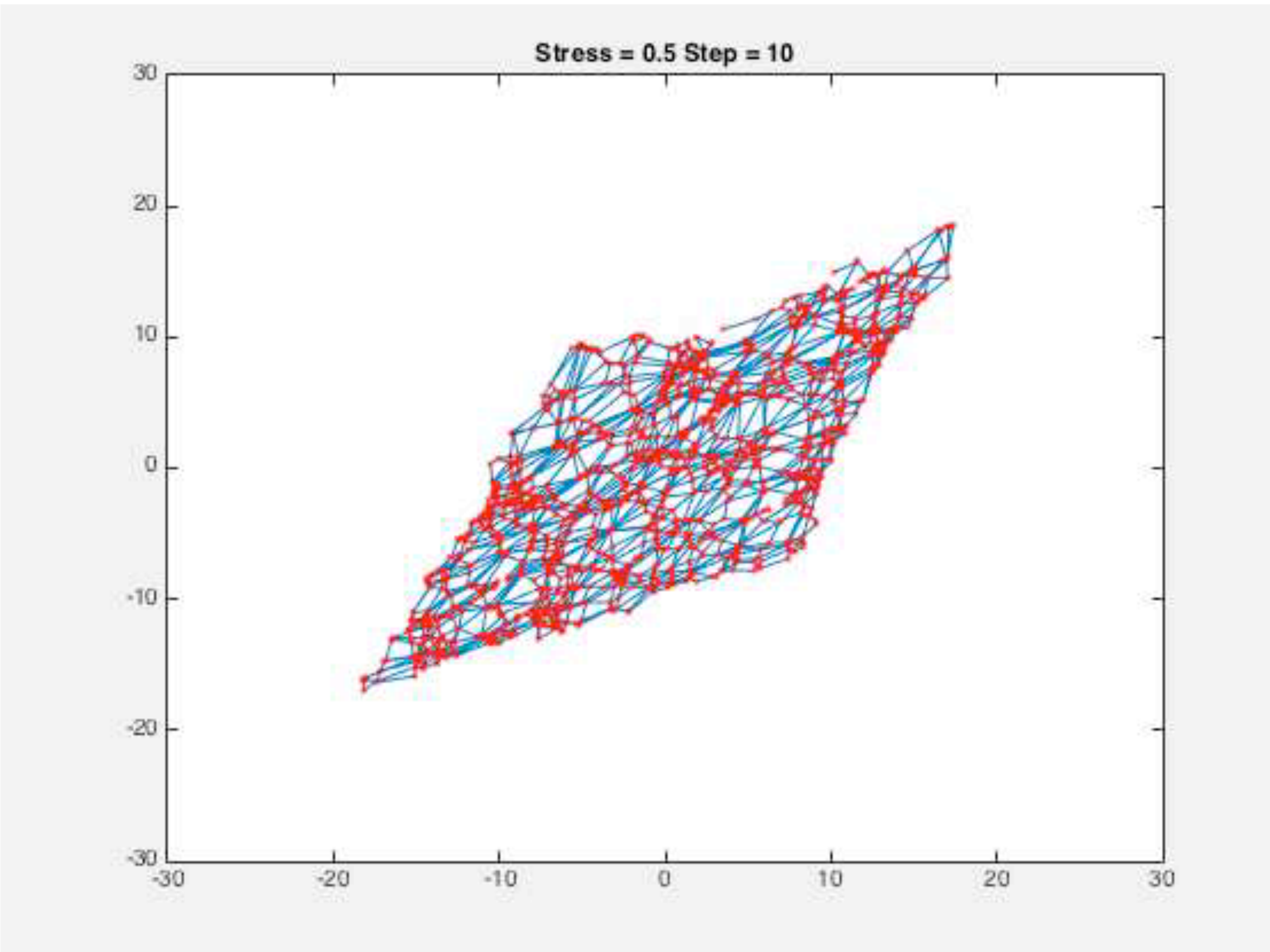


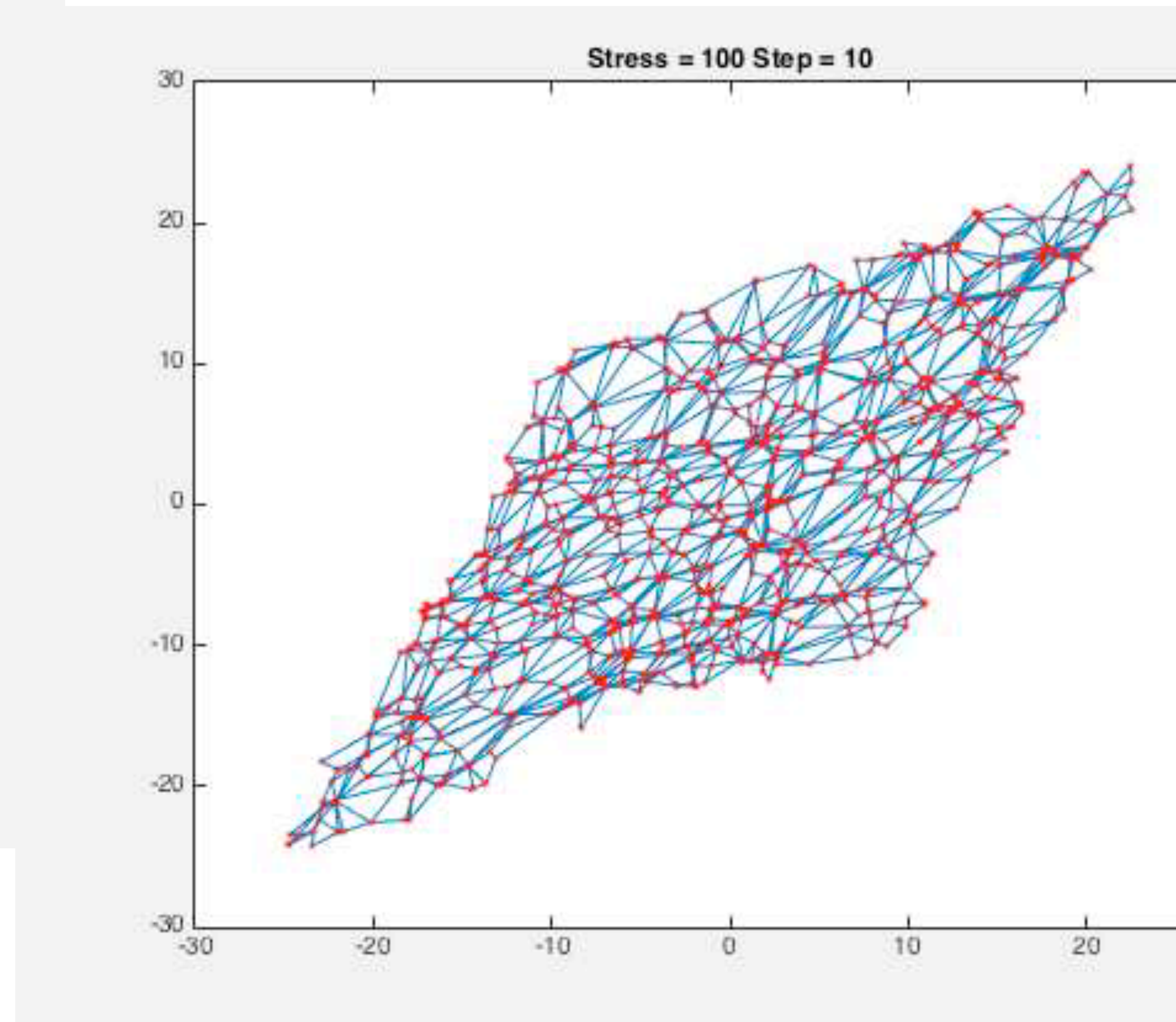
Clustering/Clumping of points

Height map: constructed from simulations

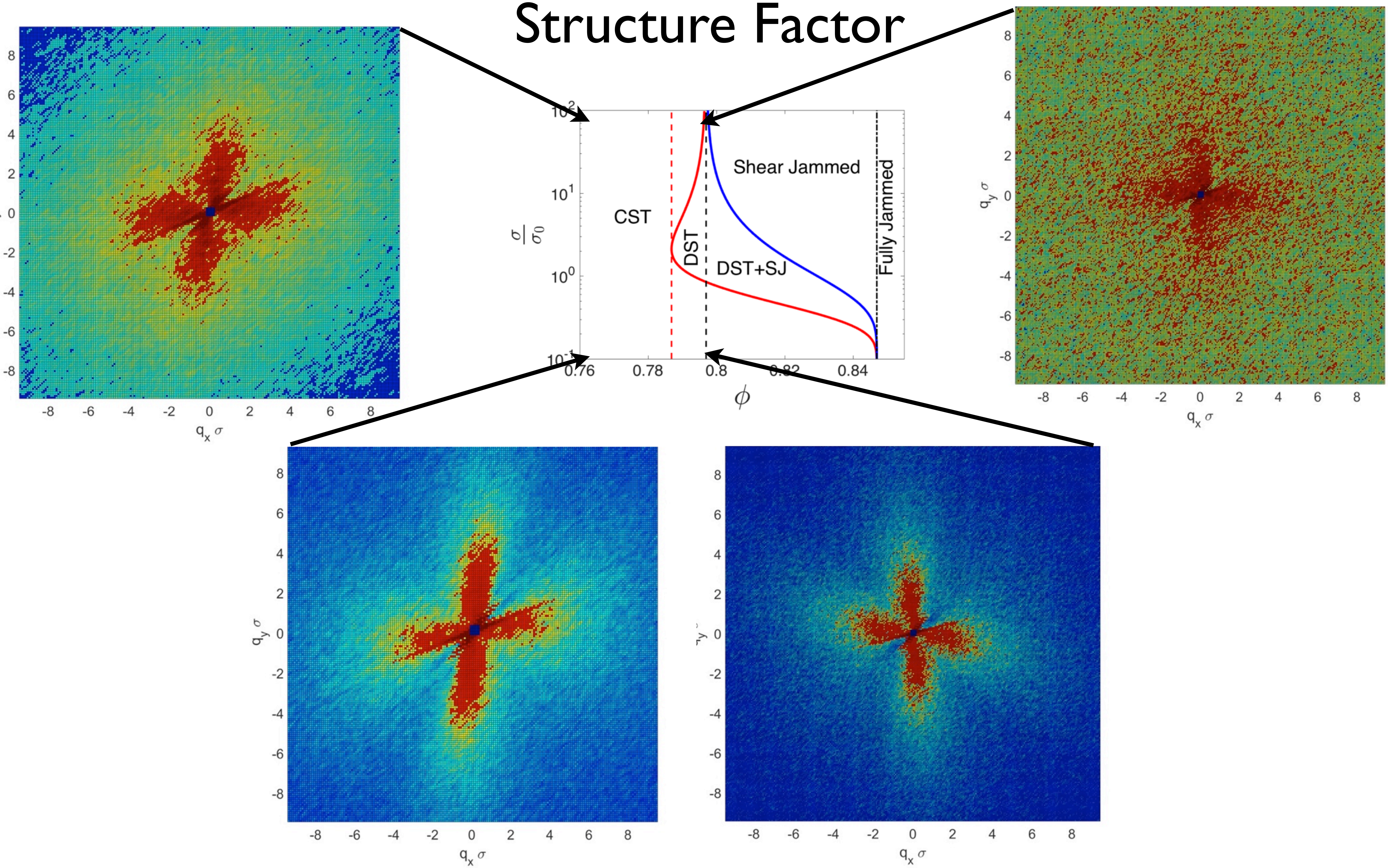
Point Patterns: Vertices of Force tilings

0.76

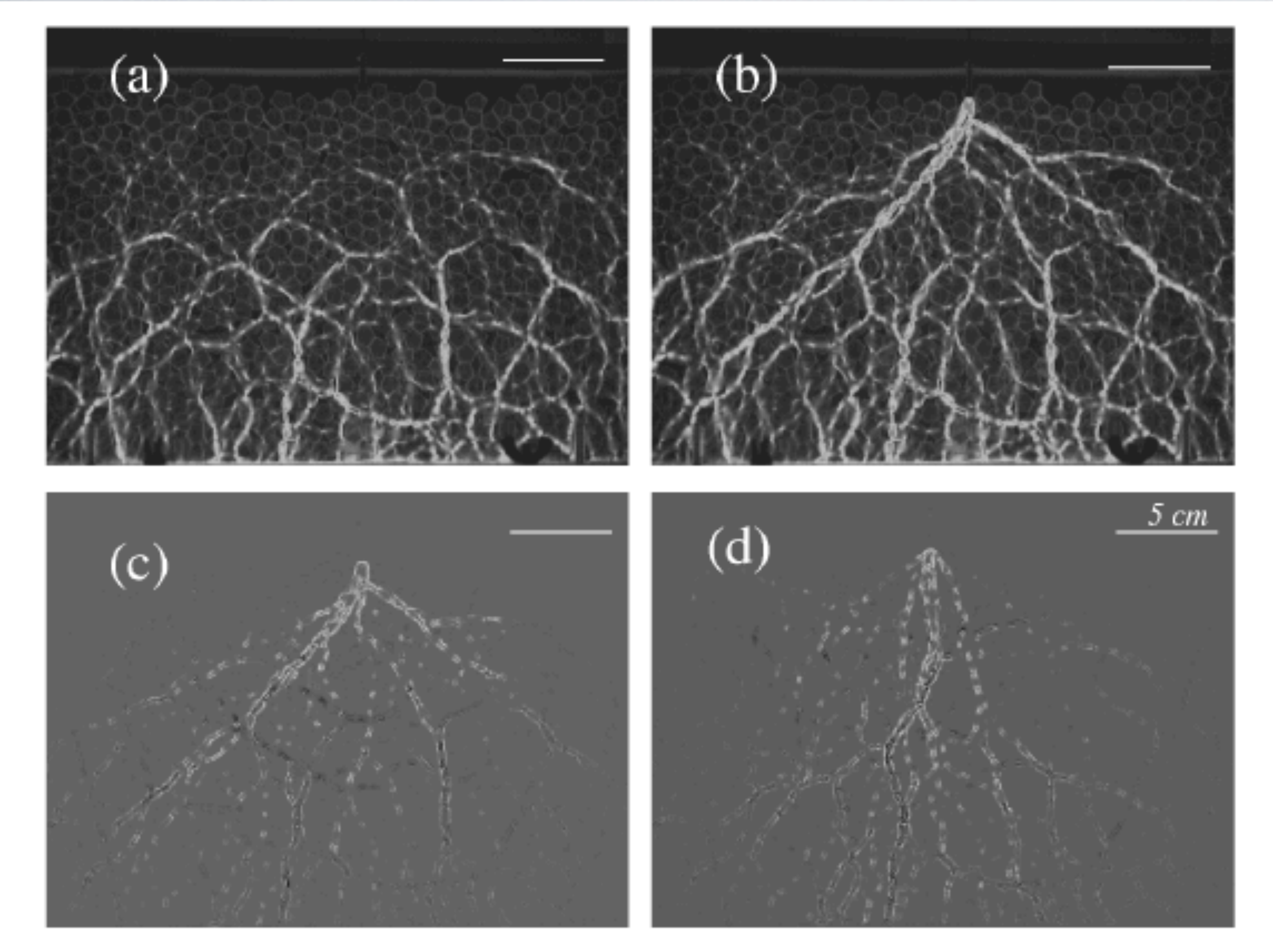




Point Patterns: Vertices of Force tilings
Structure Factor



How do granular materials respond to applied forces?



“Forces are carried primarily by a tenuous network that is a fraction of the total number of grains” Geng et al, PRL (2001)

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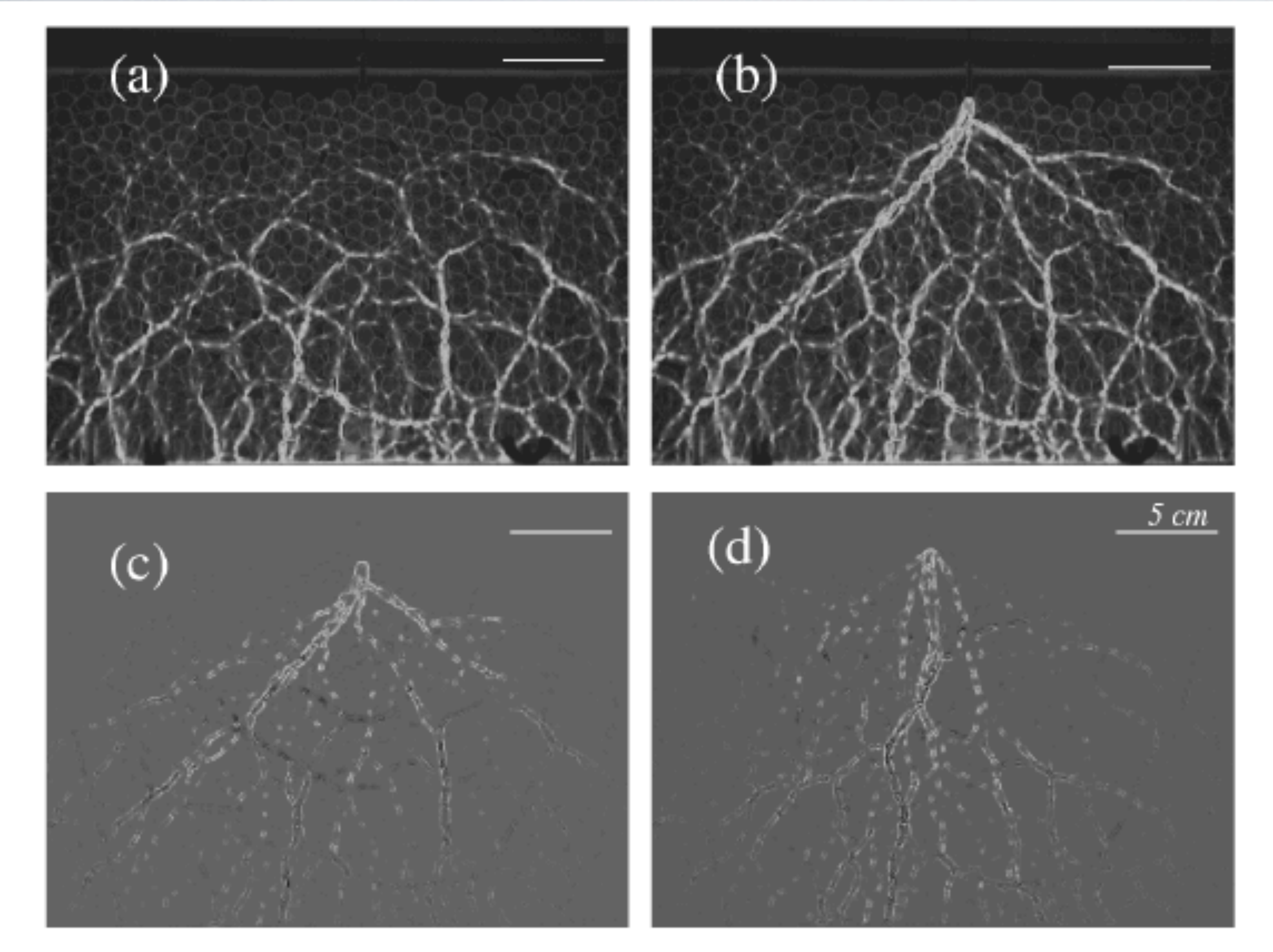

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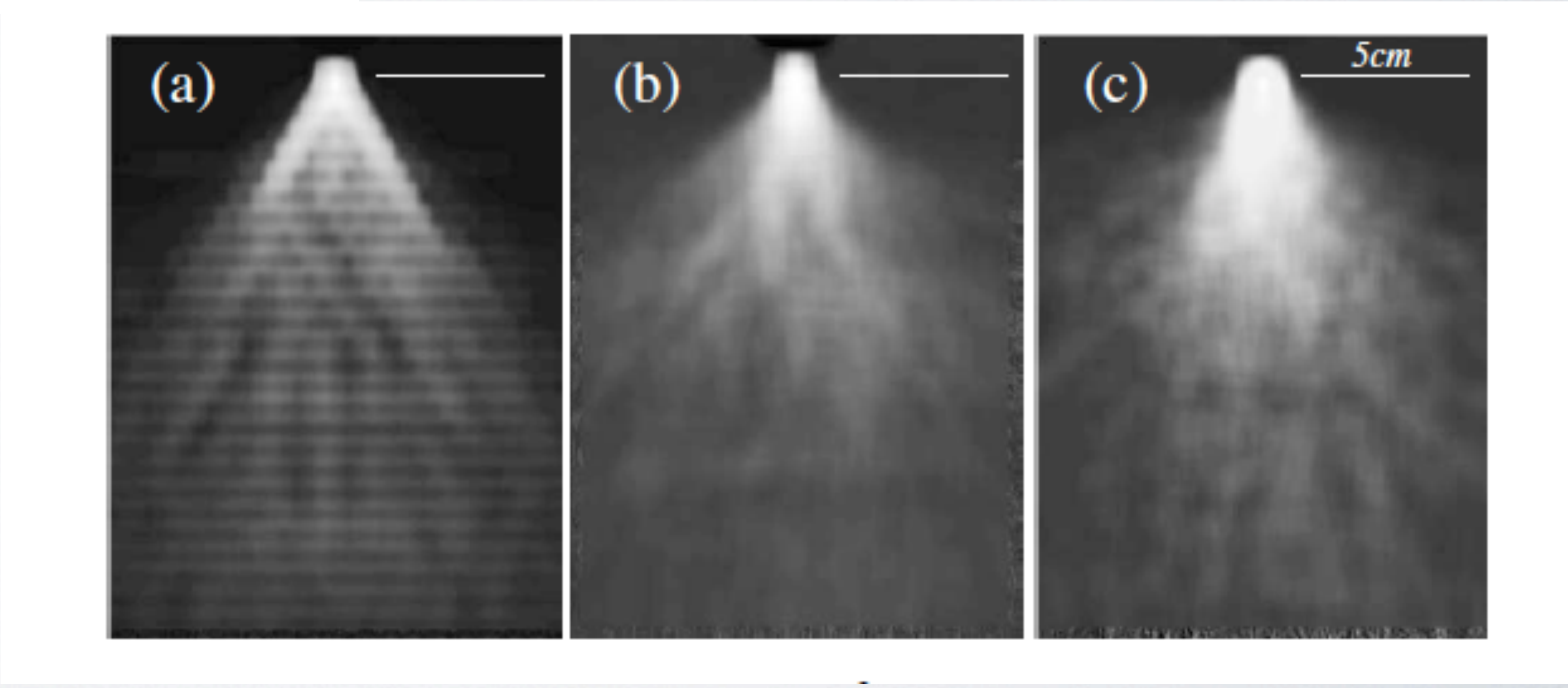
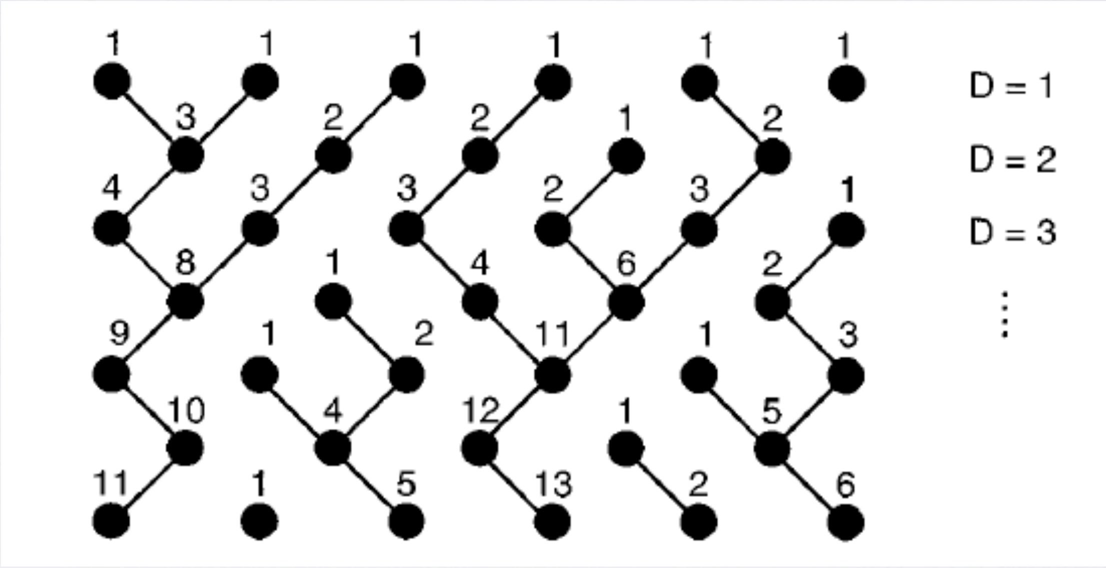


Ensemble averaged patterns are sensitive to nature of underlying spatial disorder

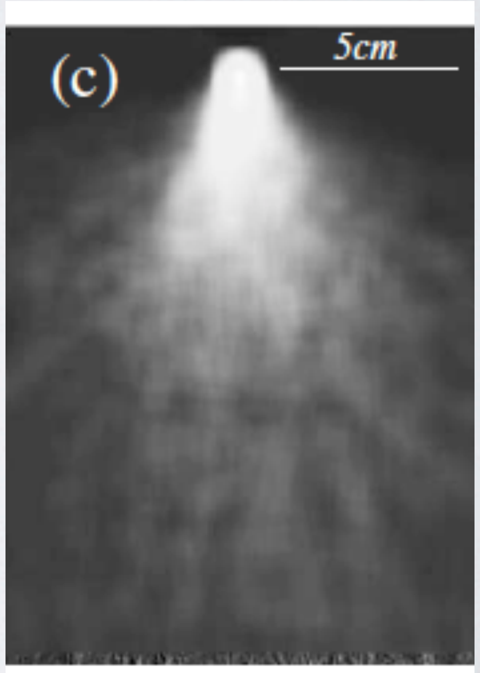
Theoretical Models

q-model (Coppersmith et al (1996)): Scalar force balance on an ordered network. Disorder incorporated at contacts: how forces get transmitted at contacts. In continuum, reduces to the diffusion equation.

Broad distribution of forces.



In response to a localized force at the top of a pile, the pressure profile at the bottom has a peak with width proportional to the square root of the height.



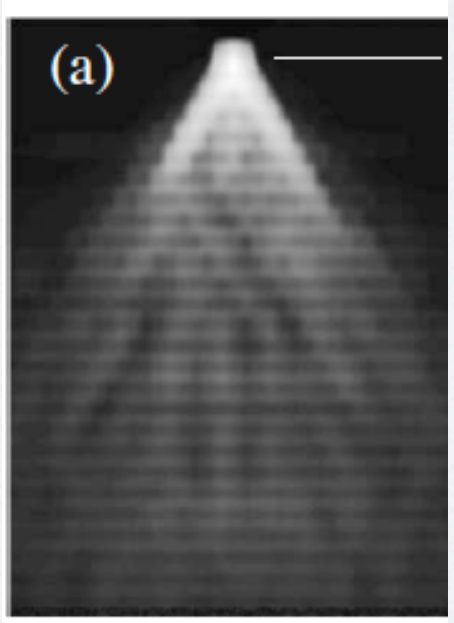
Theoretical Models

Missing stress-geometry equation: no well defined strain field/compatibility relations

Continuum models with prescribed constitutive law relating stress components.
determined by history of preparation. For example,

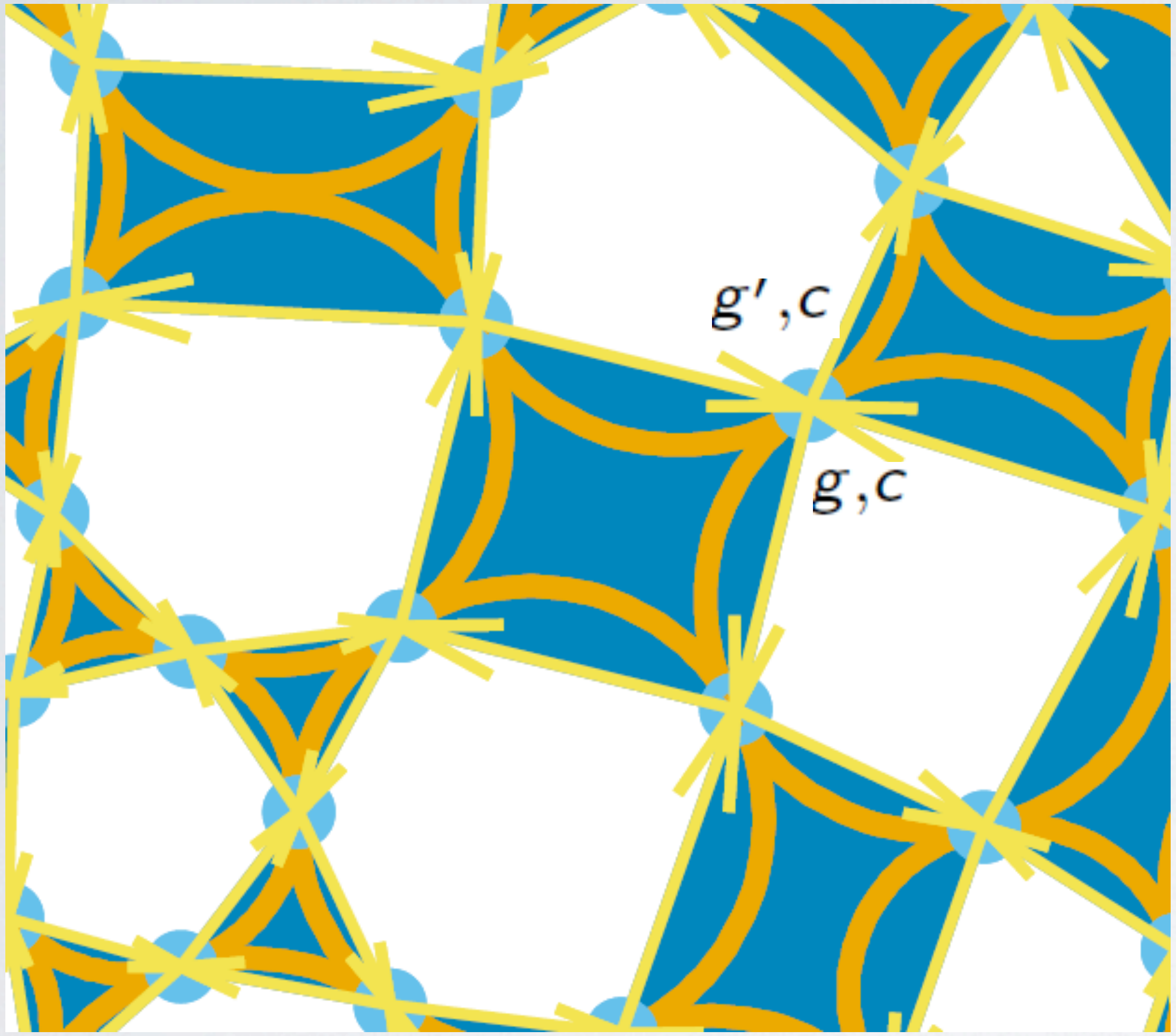
$$\sigma_{zz} = c_0^2 \sigma_{xx}$$

More elaborate closure relations: (Review: J.-P. Bouchaud Les Houches Lectures)



Stresses propagate/ get transmitted along lines

Imposing vectorial force balance (2D)



Forces on every grain sum to zero:

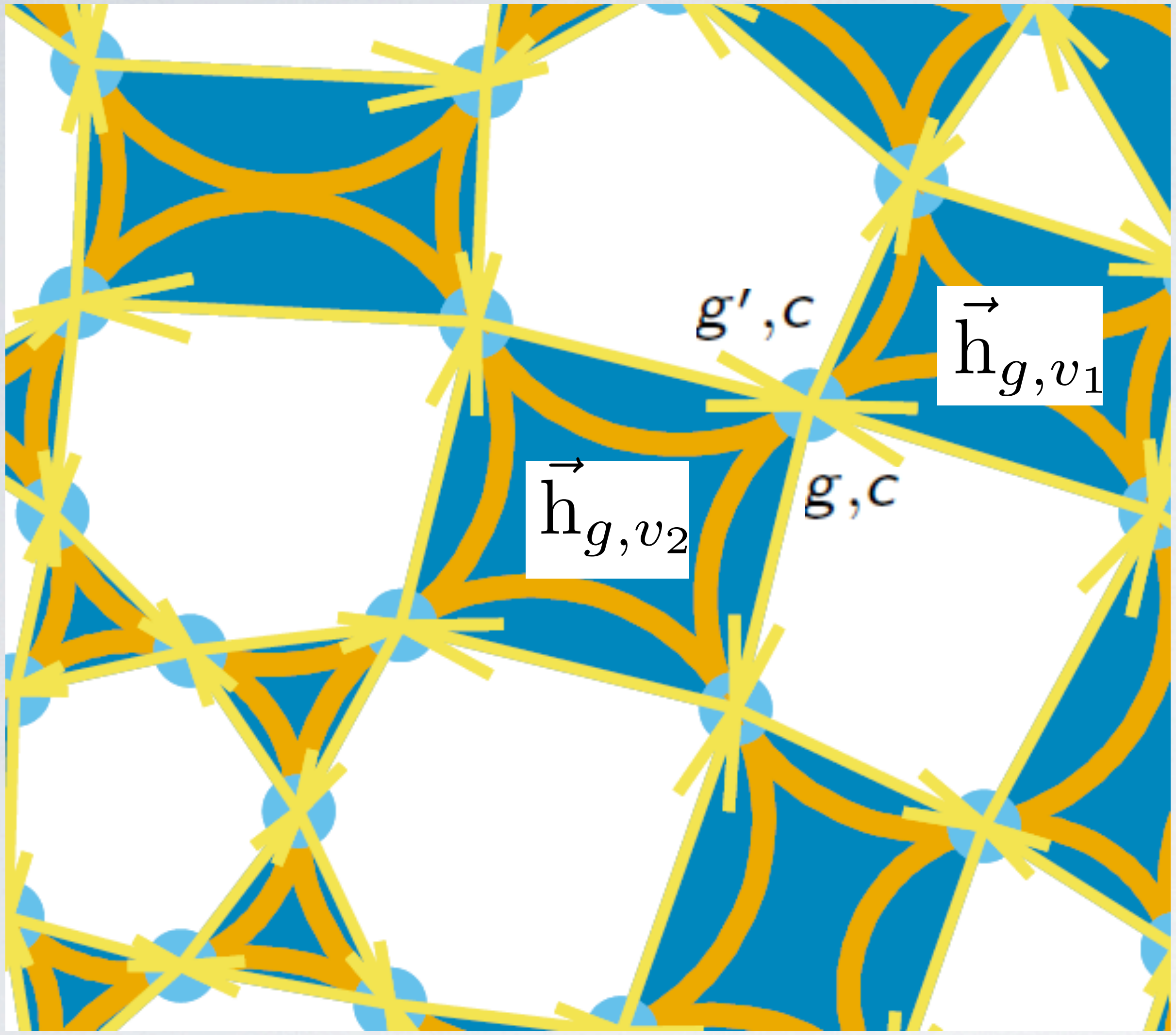
$$\sum_c \vec{f}_{g,c} = 0.$$

Newton's third law dictates:

$$\vec{f}_{g,c} = -\vec{f}_{g',c}$$

Height Representation

Ball & Blumenfeld, 2003



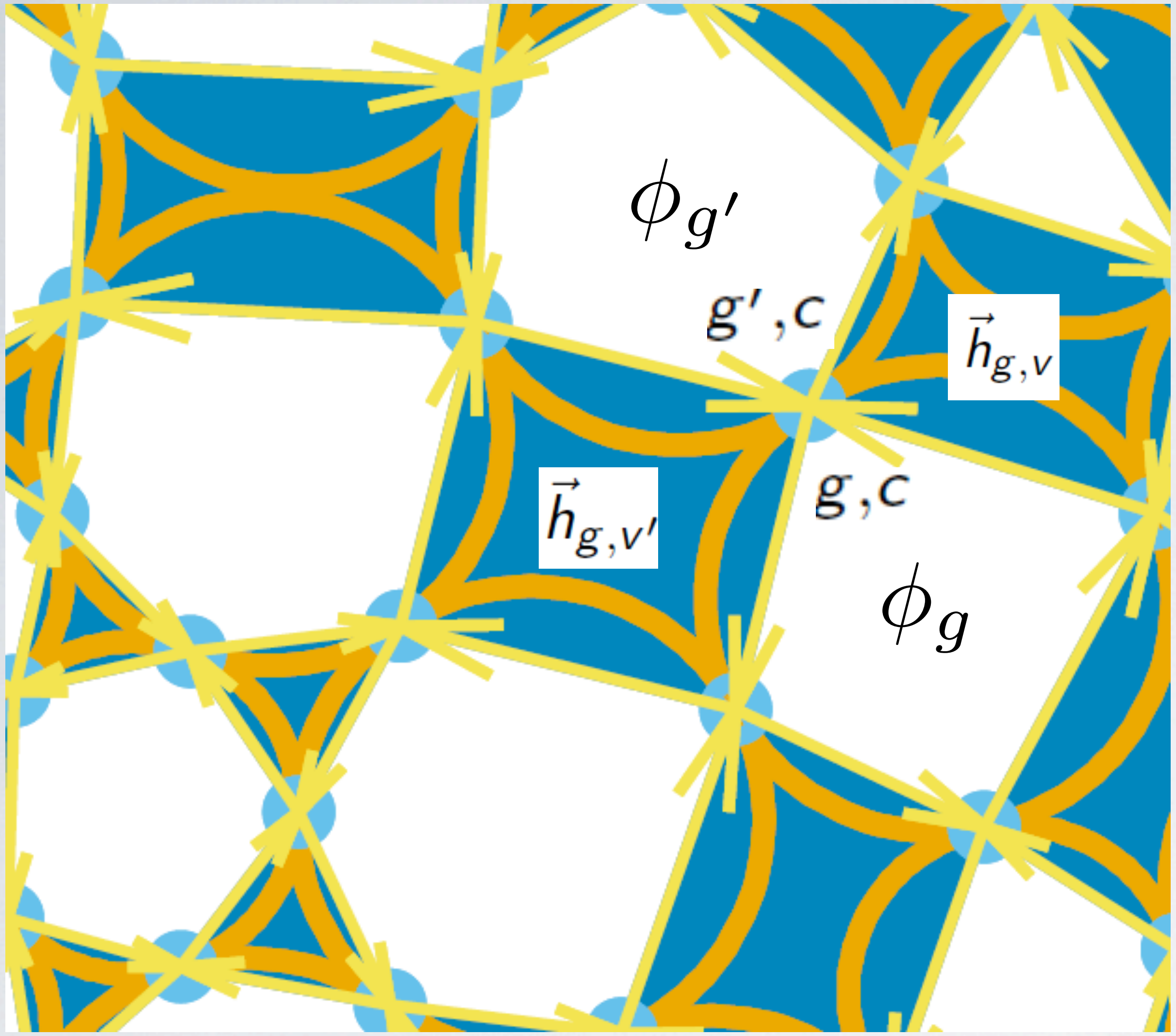
$$
\begin{aligned}
\vec{f}_{g,c_1} &= \vec{h}_{g,v_1} - \vec{h}_{g,v_2}, \\
\vec{f}_{g,c_2} &= \vec{h}_{g,v_2} - \vec{h}_{g,v_3}, \\
\vec{f}_{g,c_3} &= \vec{h}_{g,v_3} - \vec{h}_{g,v_4}, \\
\underbrace{\vec{f}_{g,c_4}}_{0} &= \underbrace{\vec{h}_{g,v_4} - \vec{h}_{g,v_1}}_{0}.
\end{aligned}
$$

The conditions of mechanical equilibrium ensure the uniqueness of the height representation

$$\nabla \cdot \hat{\sigma} = 0 \rightarrow \text{Vector potential}$$

The heights live on a random network

Force Response of a network to a perturbation



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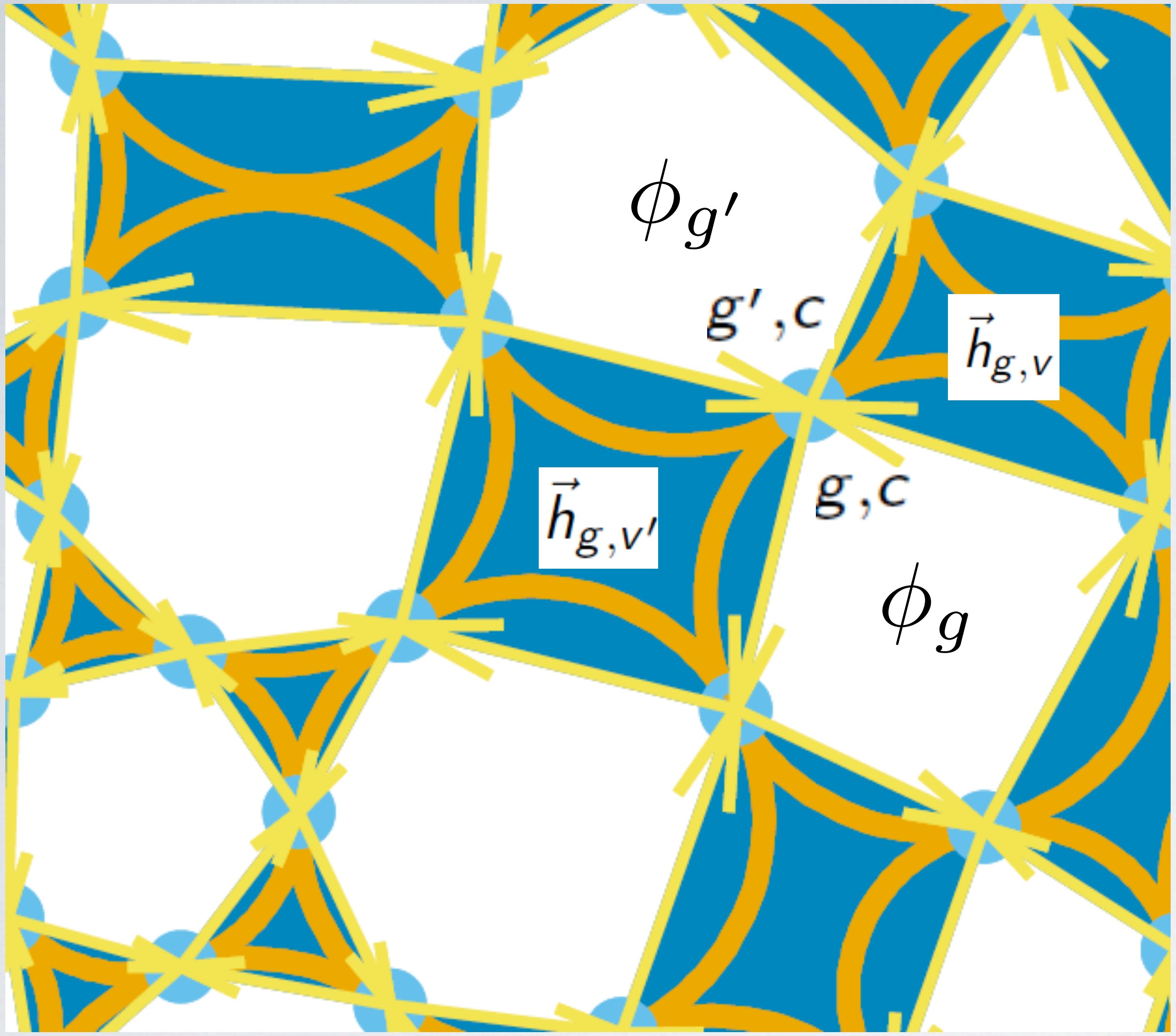

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\vec{f}_{g,c_1} &= \vec{h}_{g,v_1} - \vec{h}_{g,v_2} + \vec{\phi}_{g_1} - \vec{\phi}_{g_0}, \\
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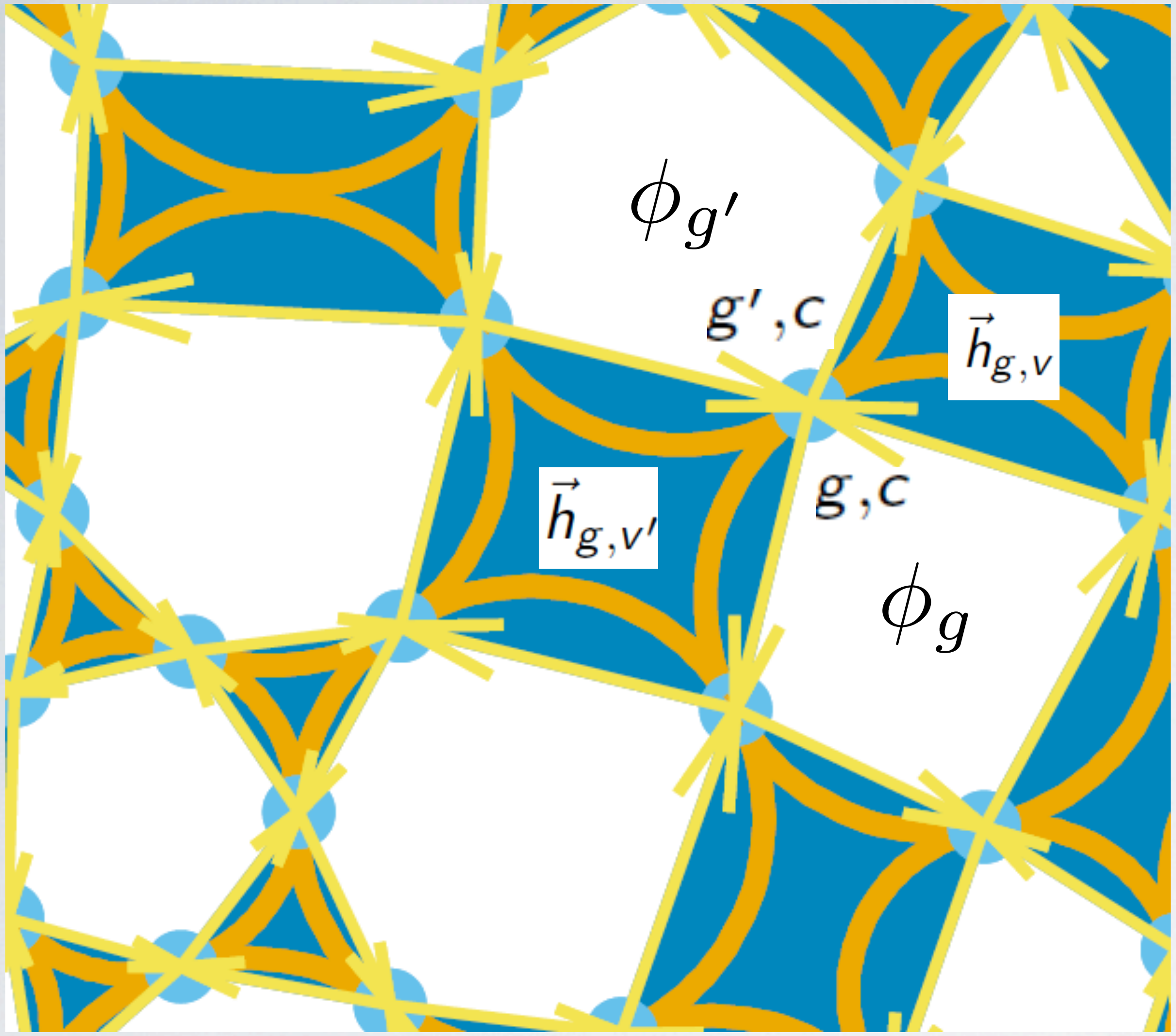

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\end{aligned}
$$

Geometry of contact network represented by the network Laplacian

Matrix whose diagonal elements contain the number of contacts, otherwise the adjacency matrix

$$
\begin{aligned}
\vec{f}_{g,c_1} &= \vec{h}_{g,v_1} - \vec{h}_{g,v_2} + \vec{\phi}_{g_1} - \vec{\phi}_{g_0},\\
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\end{aligned}
$$

Framework

$$\Box^2|\vec{\phi}\rangle = -|\vec{f}_{body}\rangle$$

Equation defining the auxiliary fields

Given a contact network and a set of body forces, solution is unique

If the solution violates torque balance/static friction condition, network will rearrange

Framework

$$\Box^2|\vec{\phi}\rangle = -|\vec{f}_{body}\rangle$$ Equation defining the auxiliary fields

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Disorder of contact network represented by network Laplacian

Diffusion on a random network: Localization ?

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Diffusion on a random network: Localization ?

Eigenfunction expansion $\Box^2 = \sum_{i=1}^{N} \lambda_i |\lambda_i\rangle\langle\lambda_i|$

One zero mode: $\lambda_1 = 0 \;, |\lambda_1\rangle = (1\ 1\ 1 \ldots 1)$

Localized ? $|\vec{\phi}\rangle = \sum_{i=1}^{N} \frac{1}{\lambda_i} \langle\lambda_i|\vec{f}_{body}\rangle|\lambda_i\rangle$

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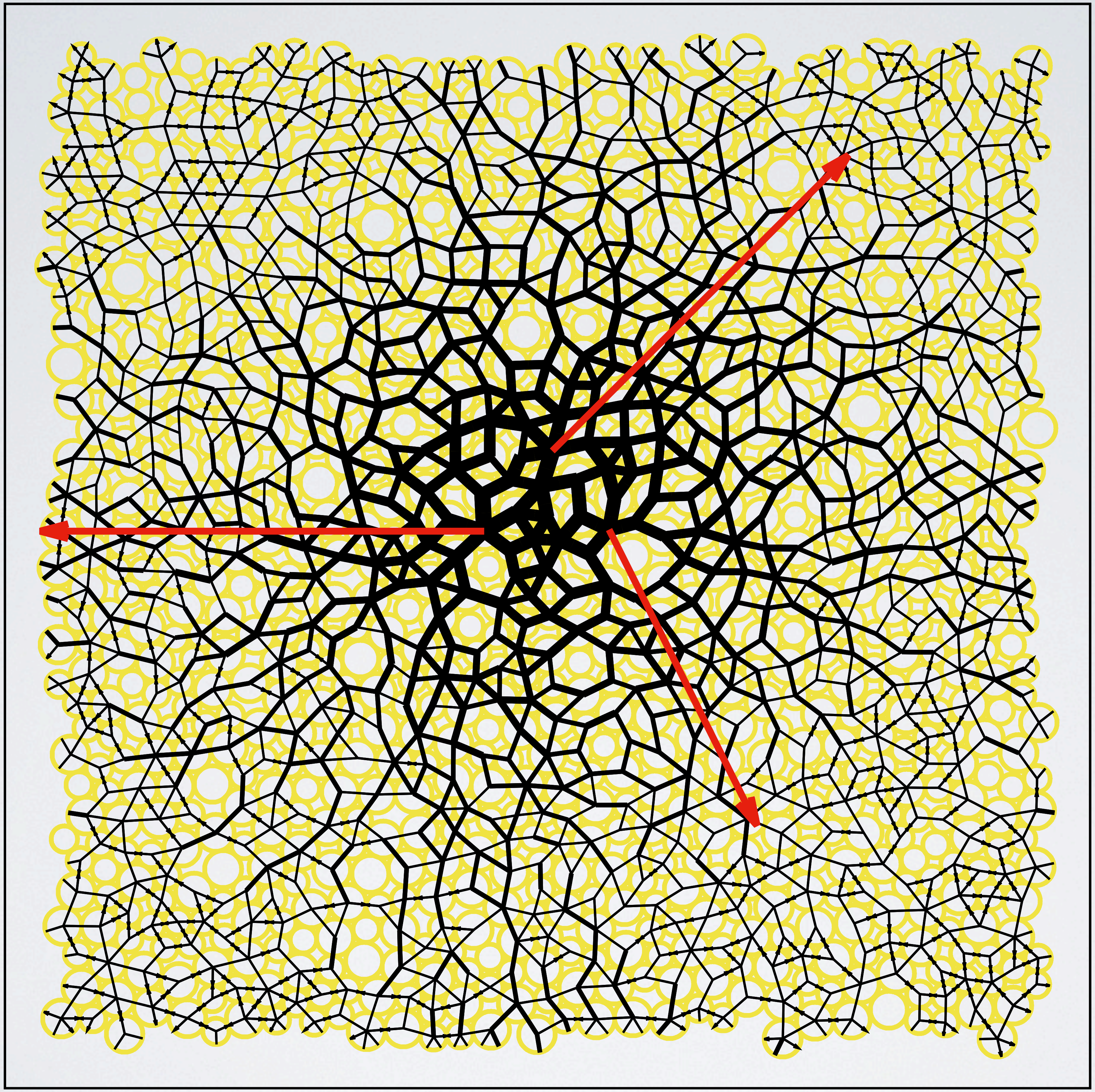
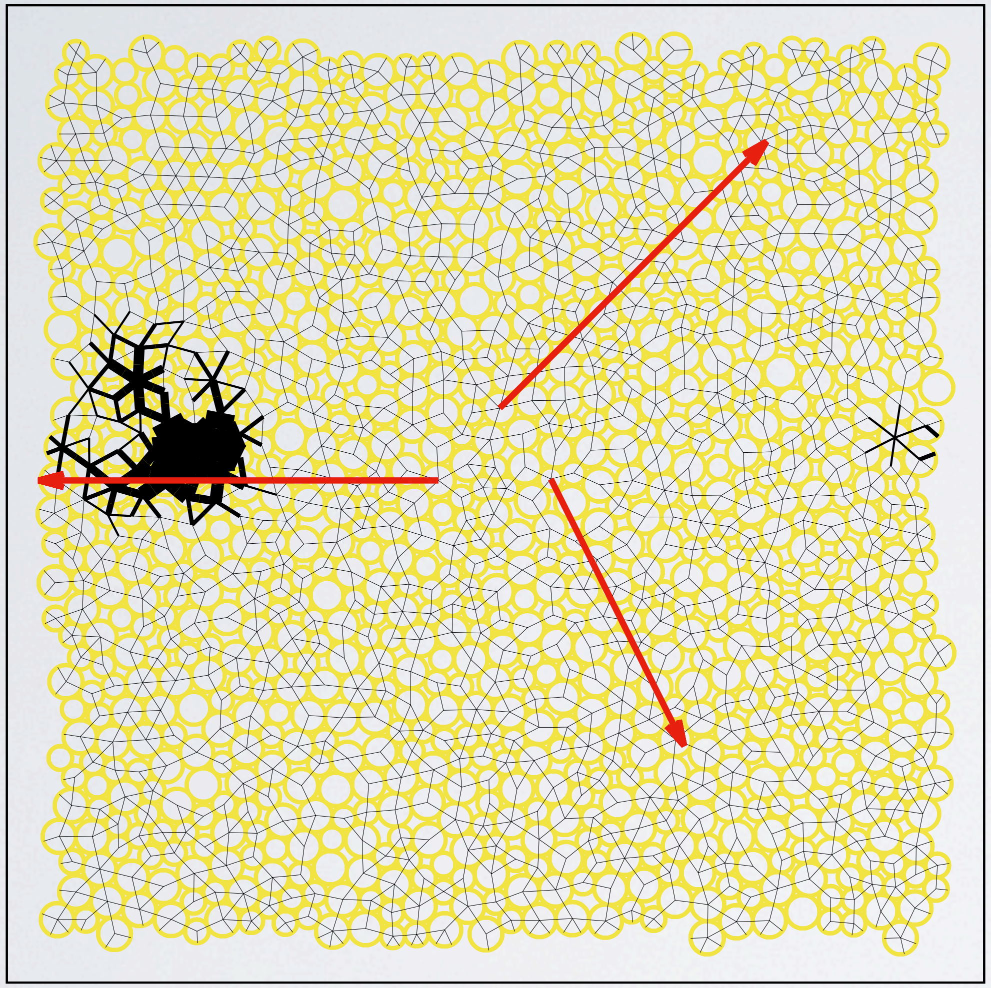
Different from q model: On a given contact network, how the force perturbation gets distributed among contacts is completely determined by the constraints of force balance

Constitutive Law determined by statistical properties of the ensemble of Laplacians

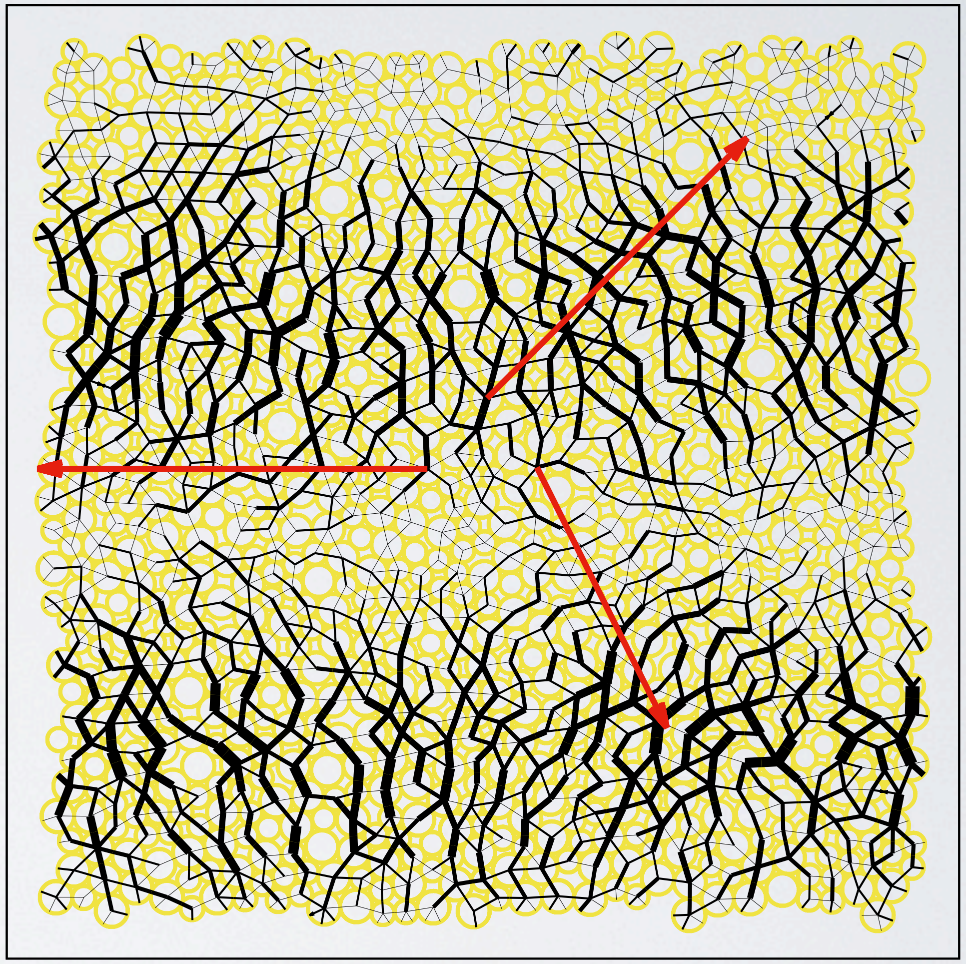
Response of a frictionless granular solid



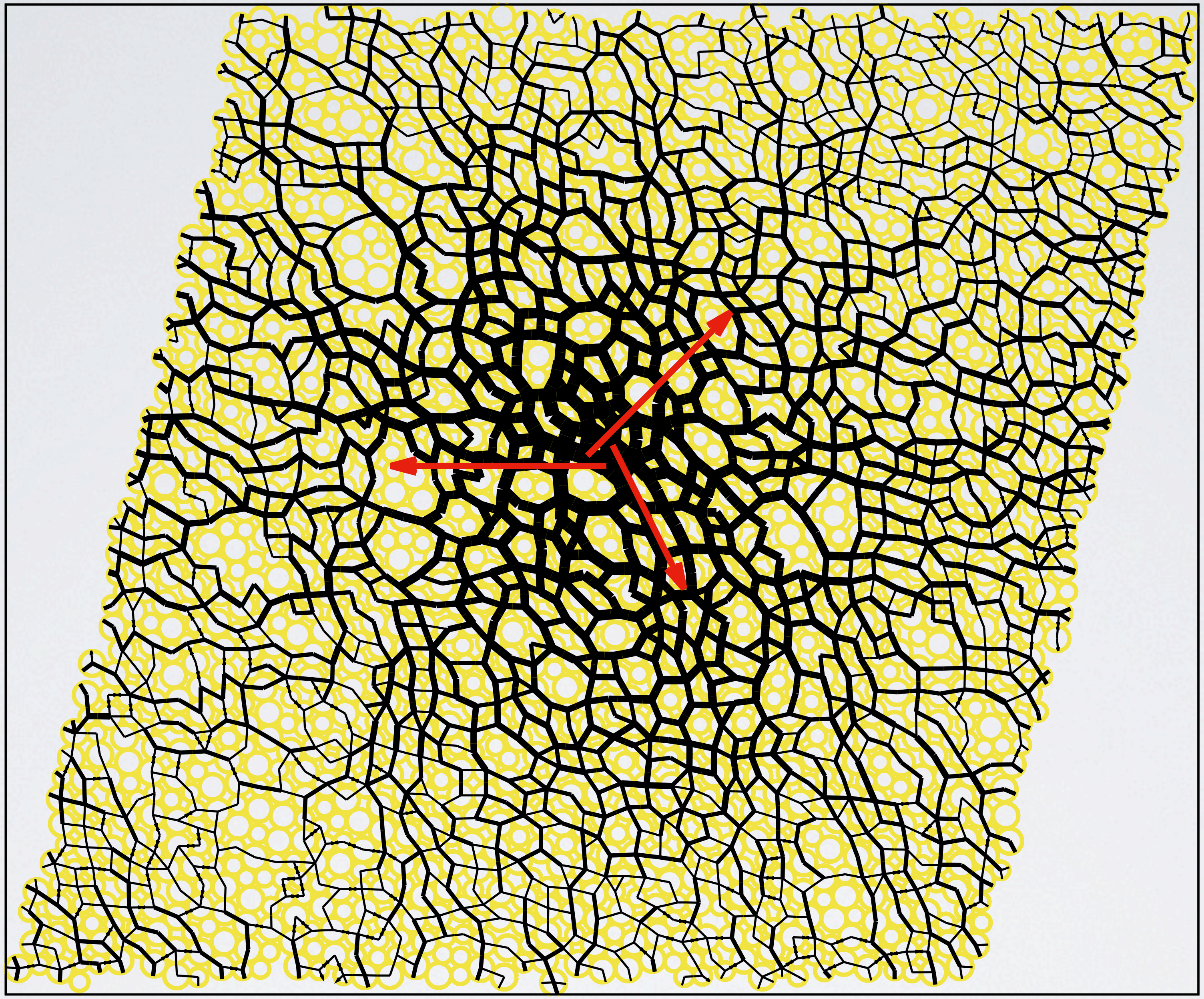
Highest eigenvalue:
strongly localized



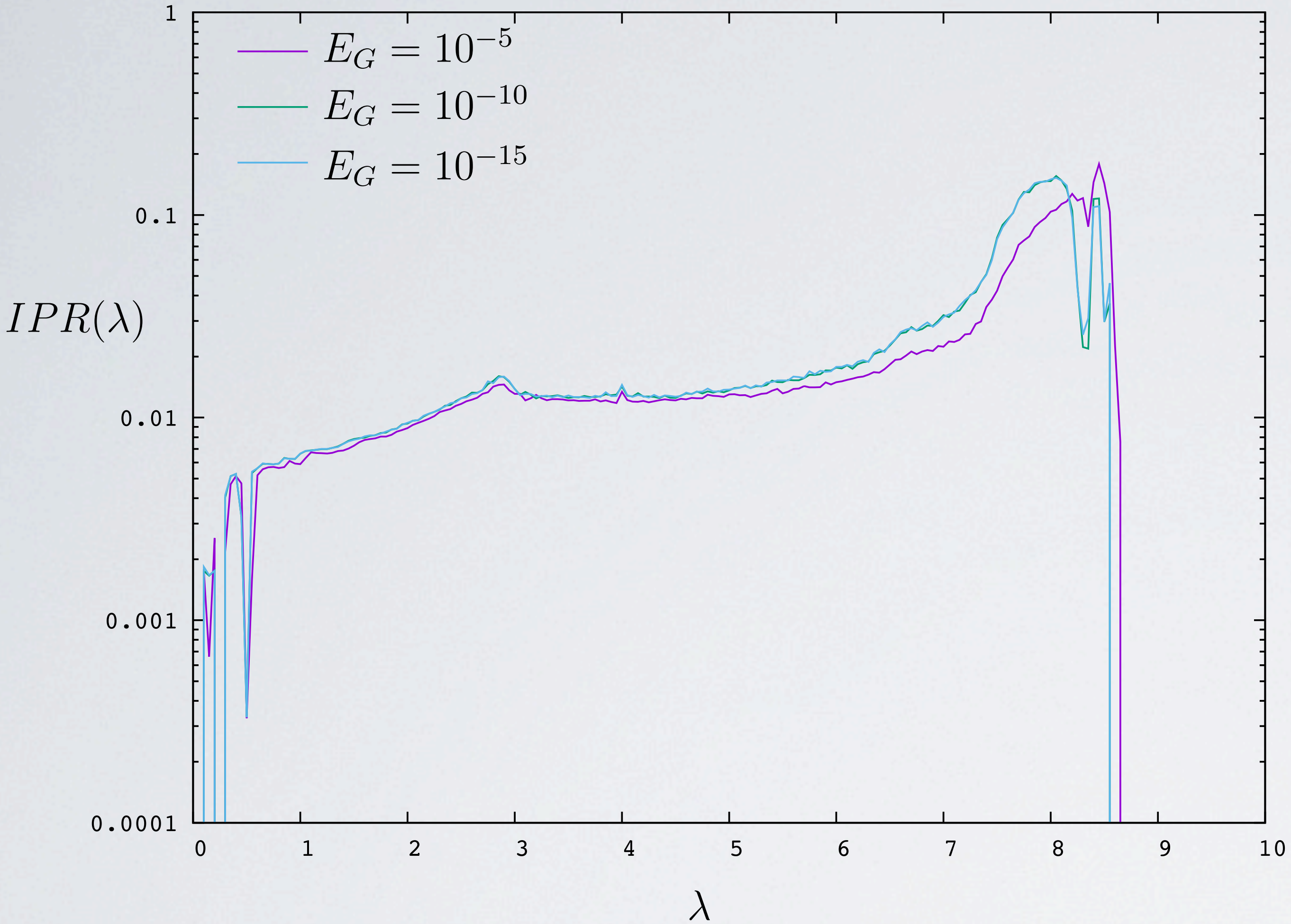
Lowest eigenvalue:
delocalized



Response of a sheared, frictional granular solid

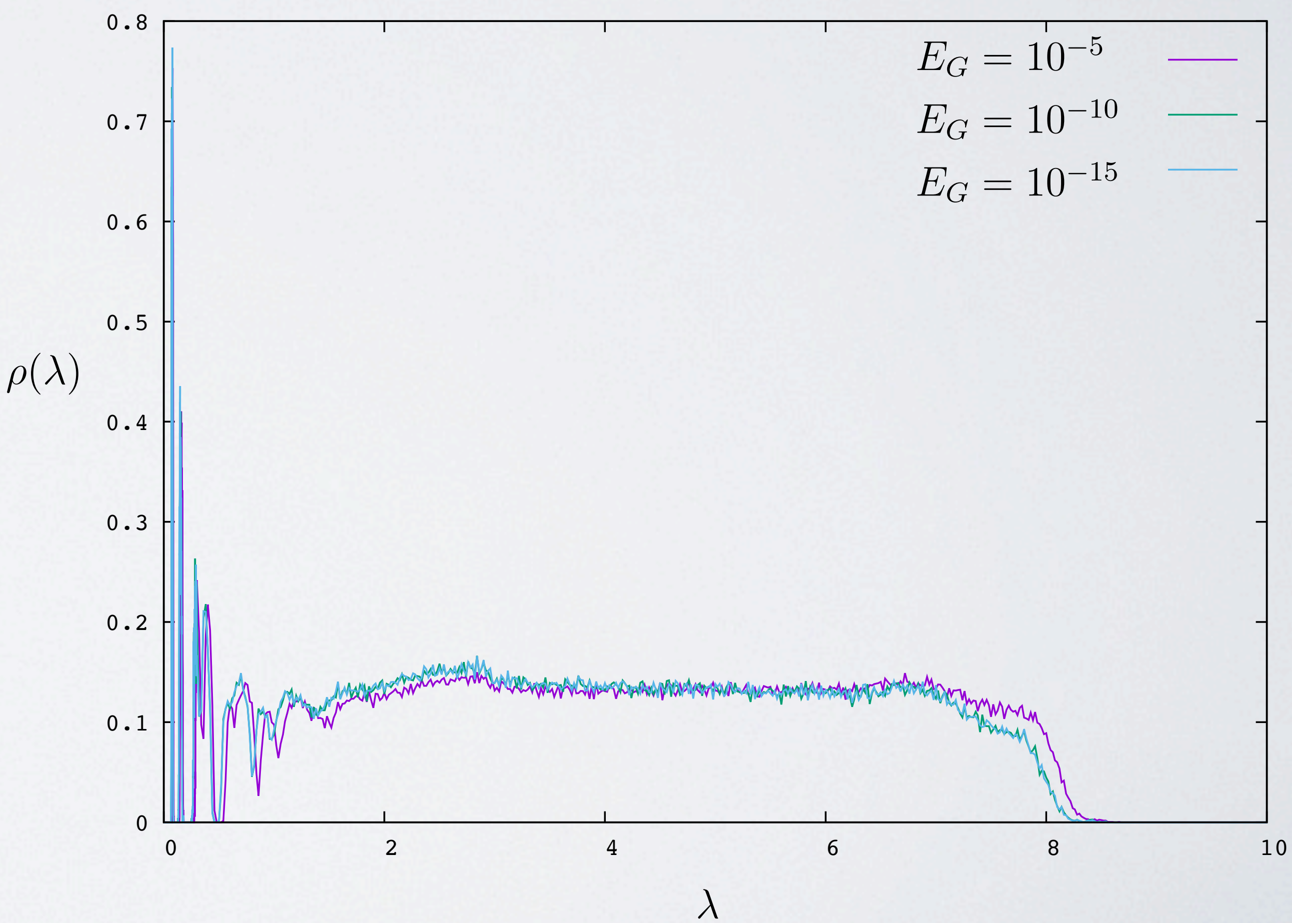


Ensemble average: Spectral properties



Inverse Participation Ratio

Density of States



Statistical Mechanics of Granular Media

- Dual Networks: Contacts and Force Tilings

- History Dependence:

Including forces in defining microstates takes away that indeterminacy

- Contact Networks are random but can characterize ensembles relevant for stress transmission

- Pattern formation in height fields: Distinguishes phases