1. Visintin's model

Assume that λ_1 and λ_2 are real constants such that $\lambda_2 > 0$. Let D be a bounded, possibly multi-connected domain in $\mathbb{R}^3$. Physically, D is the set occupied by a ferromagnetic body. M will denote the magnetization field. We will write $M(t,x)$ for the value of M at time t and space $x \in D$. We assume that there are three types of energy in our system: the anisotropy energy $\mathcal{E}_{\rm an}$, the exchange energy $\mathcal{E}_{\rm ex}$, the energy $\mathcal{E}_{\rm fi}$ due to the magnetic field H and the total magnetic energy $\mathcal{E}_{\rm mag}$. These are defined, for a suitable function $M : D \to \mathbb{R}^3$, as follows.

(i) Suppose that $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a convex Does it has to be convex? Where is this assumption used? function. Physically, ϕ depends on the crystal structure of the material. We put,

$$\mathcal{E}_{\rm an}(M) := \int_D \phi(M(x))\, dx. \tag{1.1}$$

(ii) Suppose that $[a_{lm}]_{l,m=1}^3$ is a symmetric and positive definite matrix. We put,

$$\mathcal{E}_{\rm ex}(M) := \frac{1}{2}\int_D \sum_{l,m} a_{lm} \frac{\partial M(x)}{\partial x_l}\frac{\partial M(x)}{\partial x_m}\, dx. \tag{1.2}$$

(iii) For a magnetic field $H : \mathbb{R}^3 \to \mathbb{R}^3$ we put,

$$\mathcal{E}_{\rm fi}(H) := \int_{\mathbb{R}^3} \frac{1}{2}|H(x)|^2\, dx. \tag{1.3}$$

It will be more convenient to use a related vector field defined by $B = H + \tilde{M}$, where

$$\tilde{M}(x) = \begin{cases} M(x), & \text{if } x \in D,\\ 0, & \text{otherwise}\end{cases}$$

and called the the magnetic induction field. Then the above definition of the energy $\mathcal{E}_{\rm fi}$ can be rewritten as

$$\mathcal{E}_{\rm fi}(M,B) := \int_{\mathbb{R}^3} \frac{1}{2}|B(x)-\tilde{M}(x)|^2\, dx = \int_D \frac{1}{2}|B(x)-M(x)|^2\, dx + \int_{\mathbb{R}^3\setminus D}\frac{1}{2}|B(x)|^2\, dx \tag{1.4}$$

(iii)

$$\begin{aligned}\mathcal{E}_{\rm mag}(M,B) &:= \mathcal{E}_{\rm an}(M) + \mathcal{E}_{\rm ex}(M) + \mathcal{E}_{\rm fi}(M,B) = \int_D \phi(M(x))\, dx\\ &+ \frac{1}{2}\int_D \sum_{l,m} a_{lm}\frac{\partial M(x)}{\partial x_l}\frac{\partial M(x)}{\partial x_m}\, dx + \int_{\mathbb{R}^3}\frac{1}{2}|B(x)-\tilde{M}(x)|^2\, dx.\end{aligned} \tag{1.5}$$

Let us observe that

$$e_{\rm mag}(M)(x) := \begin{cases} \phi(M(x)) + \frac{1}{2}\sum_{l,m} a_{lm}\frac{\partial M(x)}{\partial x_l}\frac{\partial M(x)}{\partial x_m} + \frac{1}{2}|M(x)-B(x)|^2, & \text{if } x \in D,\\ \frac{1}{2}|B(x)|^2, & \text{if } x \in \mathbb{R}^3\setminus D\end{cases}$$

is the density of the magnetic energy, i.e.

$$\mathcal{E}_{\rm mag}(M,B) = \int_{\mathbb{R}^3} e_{\rm mag}(M,B)(x)\,dx.$$

(iv) Define finally the total electro-magnetic energy $\mathcal{E}_{\rm el.mag.}$, for $M \in \mathbb{L}^2(D)$, $B \in \mathbb{L}^2_{\rm sol}(\mathbb{R}^3)$ and $E \in \mathbb{L}^2(\mathbb{R}^3)$, by the following formula

$$\begin{aligned} \mathcal{E}_{\rm el.mag}(M,B,E) &:= \mathcal{E}_{\rm mag}(M,B) + \frac{1}{2}\int_{\mathbb{R}^3} |E(x)|^2\,dx = \frac{1}{2}\int_{\mathbb{R}^3} (|B(x)-\tilde{M}(x)|^2 + |E(x)|^2)\,dx \\ &+ \int_D \big[\phi(M(x)) + \frac{1}{2}\sum_{l,m} a_{lm}\frac{\partial M(x)}{\partial x_l}\frac{\partial M(x)}{\partial x_m}\big]\,dx \end{aligned} \tag{1.6}$$

The Landau-Lifshitz' equation read as follows[1].

$$\begin{aligned} \frac{\partial M(t,x)}{\partial t} &= \lambda_1 M(t,x)\times\rho(M,B)(t,x) \\ &- \lambda_2 M(t,x)\times(M(t,x)\times\rho(M,B)(t,x)),\ t>0,\ x\in D, \end{aligned} \tag{1.7}$$

$$\frac{\partial B(t)}{\partial t} = -\nabla\times E(t), \tag{1.8}$$

$$\frac{\partial E(t)}{\partial t} = \nabla\times(B(t)-\tilde{M}(t)) - 1_D(E+f) \tag{1.9}$$

where

$$\begin{aligned} \rho(M,B) &= -\frac{\partial\phi(M)}{\partial M} - M + \sum_{l,m} a_{lm}\frac{\partial^2 M}{\partial x_l\partial x_m} + B(t,x) \\ &= -\varphi'(M) + (A-1)M + B \end{aligned}$$

Motivated by the recent papers [24] and [39] our aim is to study a stochastic version of the above problem with ρ replaced by

$$\tilde{\rho}(M,H) := \rho(M,B) + \sum_i g_i(M,B)\frac{dw_i}{dt}, \tag{1.10}$$

where $g_i : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$, for $i = 1,\cdots,d$ is a smooth function such that

$$\boxed{\langle g_i(x,y),x\rangle = 0,\ \text{ for } \ \text{all } x,y\in\mathbb{R}^3} \tag{1.11}$$

and $w(t) = (w_i(t))_{i=1}^d$, $t \geq 0$, is a $\mathrm{K} := \mathbb{R}^d$-valued Wiener process defined on some probability space. The condition (1.11) above means that if $x \in S^2$ then $g_i(x,y)$ is tangent to S^2.

Let us introduce now some useful functional spaces. We will try to follow the notation used by Visintin in [63].

[1]What was denoted by $H^{\rm e}$ we will denote by ρ.

Let

$$W := H^{\mathrm{curl}}(\mathbb{R}^3) \tag{1.12}$$

$$V := H^{\nabla}(D,\mathbb{R}^3) \tag{1.13}$$

where for an open set $\mathcal{O}\subset\mathbb{R}^3$ we put

$$H^{\mathrm{curl}}(\mathcal{O}) := \{u\in\mathbb{L}^2(\mathcal{O}) : \mathrm{curl}\, u\in\mathbb{L}^2(\mathcal{O})\}, \tag{1.14}$$

$$H^{\nabla}(\mathcal{O},\mathbb{R}^3) := \{u\in\mathbb{L}^2(\mathcal{O}) : \nabla u\in L^2(\mathcal{O},\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3))\}. \tag{1.15}$$

Here, as elsewhere, we will follow a convention that $\mathbb{L}^2(\mathcal{O})$ stands for $L^2(\mathcal{O},\mathbb{R}^3)$. Let us notice, that both these spaces are Hilbert spaces with naturally defined scalar products and that $H^{\nabla}(\mathcal{O},\mathbb{R}^3)$ is equal to the usual Sobolev space $H^{1,2}(\mathcal{O},\mathbb{R}^3)$. The latter will also be denoted by $\mathbb{H}^{1,2}(\mathcal{O})$. It should be clear that $H^{\mathrm{curl}}(\mathcal{O},\mathbb{R}^3)\subset H^{\nabla}(\mathcal{O},\mathbb{R}^3)$ but the reverse inclusion is not true. For any open subset $\mathcal{O}$ of $\mathbb{R}^3$, by $\mathbb{L}^2_{\mathrm{sol}}(\mathcal{O})$ we will denote the space of all $u\in\mathbb{L}^2(\mathcal{O})$ such that the weak divergence of u is equal to zero, i.e.

$$\boxed{\mathbb{L}^2_{\mathrm{sol}}(\mathcal{O}) := \{u\in\mathbb{L}^2(\mathcal{O}) : \mathrm{div}\, u = 0\}.} \tag{1.16}$$

Let us also put

$$\boxed{\mathrm{X} := \mathbb{L}^2(D)\times\mathbb{L}^2_{\mathrm{sol}}(\mathbb{R}^3)\times\mathbb{L}^2(\mathbb{R}^3).} \tag{1.17}$$

The elements of the space X will be denoted by $u=(M,B,E)$.

It is well known, see e.g. [61], who however uses different notation, that $\mathbb{L}^2_{\mathrm{sol}}(\mathcal{O})$ is a closed subspace of the Hilbert space $\mathbb{L}^2(\mathcal{O})$.

We assume that $T>0$ is fixed and that

$$M_0 \in \mathbb{L}^\infty(D) \tag{1.18}$$

$$B_0 \in \mathbb{L}^2_{\mathrm{sol}}(\mathbb{R}^3) \tag{1.19}$$

$$E_0 \in \mathbb{L}^2(\mathbb{R}^3) \tag{1.20}$$

$$f \in M^2(0,T;\mathbb{L}^2(D)). \tag{1.21}$$

Here, if Y is a separable Banach space and $p\in[1,\infty)$, by by $M^p(0,T;Y)$ we understand the space of all progressively measurable Y-valued processes f, to be more precise the space of equivalence classes etc, such that

$$\mathbb{E}\int_0^T |f(t)|_Y^2\, dt < \infty.$$

Our aim is to find an X-valued processes $u=(M,E,B)$ such that, with $H(t) := B(t) - \tilde{M}(t)$, $t\geq 0$, the following system of equations will be satisfied

$$
\begin{aligned}
dM(t,x) &= \lambda_1 M(t,x)\times[\rho(M,B)(t,x)\,dt+\sum_i g_i(M)\circ dw_i(t)] \\
&- \lambda_2 M(t,x)\times(M(t,x) \\
&\quad \times\big[\rho(M,B)(t,x))\,dt+\sum_i g_i(M)\circ dw_i(t)\big],\ t>0,\ x\in D,
\end{aligned} \tag{1.22}
$$

$$
\frac{\partial B(t,x)}{\partial t} = -\nabla\times E(t,x), \tag{1.23}
$$

$$
\frac{\partial E(t,x)}{\partial t} = \nabla\times H(t,x)-1_D(E+\tilde f)(t,x) \tag{1.24}
$$

$$
(M(0),E(0),B(0))=(M_0,E_0,B_0). \tag{1.25}
$$

In agreement with the usual definition of the Stratonovich integral, see e.g. [13], the equation (1.22) is understood as the Itô equations with a correction term, i.e.

$$
\begin{aligned}
dM(t,x) &= \lambda_1 M(t,x)\times\rho(M,B)(t,x)\,dt+\lambda_1\sum_i M(t,x)\times g_i(M)dw_i(t) \\
&+ \frac{\lambda_1^2}{2}\sum_i\Big[M\times\big(\frac{\partial g_i}{\partial x}(M,B)(M\times g_i(M,B)\big)-|g_i(M,B)|^2M\Big]\,dt \\
&+ \lambda_2|M|^2\rho(M,B)(t,x)\,dt+\lambda_2\sum_i|M|^2g_i(M)dw_i(t), \\
&- \frac{\lambda_2^2}{2}\sum_i\Big[2|M|^2\langle M,g_i(M)\rangle+|M|^4\big(\frac{\partial g_i}{\partial x}(M,B)g_i(M)\big)\Big]\,dt,\ t>0,\ x\in D,
\end{aligned}
$$

where by $\frac{\partial g_i}{\partial x}(x_0,y_0)\in\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$ we mean the Fréchet directional derivative. Note that because g_i satisfies the condition (1.11), in the second line the term $-|g_i(M)|^2M$ appears. Otherwise, it would have to be written as $(M\times g_i(M,M))\times g_i(M,B)$. For the same reason the term $2|M|^2\langle M,g_i(M)\rangle$ in the fourth line is equal to 0, but we keep it as it may be useful in the future.

The definition of the weak solution will be preceded by some formulae, mostly following the Visintin's paper [63].

Let us first define,

$$
AM(x) = \sum_{l,m} a_{lm}\frac{\partial^2 M(x)}{\partial x_l\partial x_m},\ x\in D,\ M\in D(A); \tag{1.26}
$$

$$
D(A) = \{u\in\mathbb{H}^{2,2}(D):\frac{\partial M}{\partial\nu_A}=0\ \text{ on }\ \partial D\}. \tag{1.27}
$$

In the above, we put

$$
\frac{\partial M}{\partial\nu_A}:=\sum_{l,m}a_{l,m}\frac{\partial M}{\partial x_l}\nu_m,
$$

where $\nu=(\nu_1,\nu_2,\nu_3)$ is the unit outward normal vector field on ∂D.

Proposition 1.1. *If $v \in V$ and $M \in D(A)$, then*

$$\int_D \langle M(x) \times AM(x), v(x)\rangle = \sum_{l,m} \int_D a_{lm} \frac{\partial M(x)}{\partial x_l} \frac{\partial (v(x) \times M(x))}{\partial x_m}\, dx, \tag{1.28}$$

Remark 1.2. Let us introduce the following formal notation.

$$\nabla_A M := \sum_{l,m} a_{l,m} \frac{\partial M}{\partial x_l} \nu_m,$$

$$\int_D \langle M(x) \times AM(x), v(x)\rangle = \sum_{l,m} \int_d a_{lm} \frac{\partial M(x)}{\partial x_l} \frac{\partial (v(x) \times M(x))}{\partial x_m}\, dx \tag{1.29}$$

Proof of formula (1.28). Because $M \in H^{2,2}(D, \mathbb{R}^3)$, by the Sobolev-Gagliardo inequalities, $M \in L^\infty(D, \mathbb{R}^3)$ and $AM \in L^2(D, \mathbb{R}^3)$. Hence, $M(x) \times AM$ belongs to $L^2(D, \mathbb{R}^3)$ and hence the LHS of (1.28) makes sense. By a similar argument, since by our assumptions, $v \in L^2(D, \mathbb{R}^3) \cap L^\infty(D, \mathbb{R}^3)$, $D_j v \in L^2(D, \mathbb{R}^3)$, $j = 1, 2, 3$, and $M \in L^\infty(D, \mathbb{R}^3)$ and $D_j M \in L^2(D, \mathbb{R}^3)$, $j = 1, 2, 3$, the RHS of equality (1.28) also makes sense. Notice next that because of formula (4.2) we have $\int_D \langle M(x) \times AM(x), v(x)\rangle = \int_D \langle AM(x), v(x) \times M(x)\rangle = -\sum_{l,m} a_{lm} \int_D \langle \frac{\partial^2 M(x)}{\partial x_l \partial x_m}, v(x) \times M(x)\rangle\, dx$. Hence, by applying the Stokes Theorem to the last integral, see e.g. [61] p.??, we infer that

$$\int_D \langle M(x) \times AM(x), v(x)\rangle = \int_D \sum_{l,m} a_{lm} \langle \frac{\partial M(x)}{\partial x_l}, \frac{\partial (v(x) \times M(x))}{\partial x_m}\rangle\, dx$$

$$- \int_{\partial D} \sum_{l,m} a_{lm} \nu_m \langle \frac{\partial M(x)}{\partial x_l}, v(x) \times M(x)\rangle\, d\sigma_x$$

$$= \int_D \sum_{l,m} a_{lm} \langle \frac{\partial M(x)}{\partial x_l}, \frac{\partial (v(x) \times M(x))}{\partial x_m}\rangle\, dx \quad - \quad \int_{\partial D} \langle \frac{\partial M(x)}{\partial \nu_A}, v(x) \times M(x)\rangle\, d\sigma_x,$$

where $d\sigma_x$ is the surface measure on ∂D. Hence the result follows. ∎

Proof of formula (1.29). As before, but this time using the formula (4.2) we get $\int_D \langle M \times (M \times AM), v\rangle = \int_D \langle AM, (v \times M) \times M\rangle = -\sum_{l,m} a_{lm} \int_D \langle \frac{\partial^2 M(x)}{\partial x_l \partial x_m}, (v(x) \times M(x)) \times M(x)\rangle\, dx$. ∎

Remark 1.3. Using the operator A we can write down the following formula for ρ:

$$\boxed{\rho(M, B) = -\phi(M) + AM + 1_D B - M, \ M \in D(A), \ B \in \mathbb{L}^2_{\mathrm{sol}}(\mathbb{R}^3).} \tag{1.30}$$

Moreover, the deterministic part of the right-hand side of equation (1.22) can be written in a short way as follows. For $(M,B,E) \in D(A) \times \mathbb{L}^2_{\rm sol}(\mathbb{R}^3) \times \mathbb{L}^2(\mathbb{R}^3)$

$$\begin{aligned} F_1(M,B,E) &= \lambda_1 M \times \rho(M,B) - \lambda_2 M \times (M \times \rho(M,B)),\\ F_2(M,B,E) &= -\nabla \times E,\\ F_3(M,B,E) &= \nabla \times (B - \tilde{M}) - 1_D(E + \tilde{f}). \end{aligned}$$

We also define a map $G : \mathrm{X} \to R(\mathrm{K},\mathrm{X})$, where $R(\mathrm{K},\mathrm{X})$ is the space of γ-radonifying operators[2] from (K to X), by $G(M,B,E)(k) = (G_1(M,B)(k),0,0) \in \mathrm{X})$, $u = (M,B,E) = \mathrm{X})$, $k \in \mathrm{K}$ and

$$G_1(M,B)k = \sum_j \lambda_1 M \times g_j(M,B)k_j - \sum_j \lambda_2 M \times (M \times g_j(M,B)k_j),\ k = (k_j)_j \in \mathrm{K}.$$

Lemma 1.4. *The function $\mathcal{E}_{en} : \mathbb{L}^2(D) \to \mathbb{R}$ defined by formula (1.3) is Fréchet differentiable and*

$$d_{M_0}\mathcal{E}_{en}(M) = \int_D \phi(M_0(x))M(x)\,dx,\ M, M_0 \in \mathbb{L}^2(D). \tag{1.31}$$

Moreover, $\mathcal{E}_{el.mag}$ is a C^∞ map from X to $\mathbb{R}$ and for any $(M_0,B_0,E_0) \in \mathrm{X}$, we have

$$\begin{aligned} d_{(M_0,B_0,E_0)}\mathcal{E}_{el.mag}(M,B,E) &= \langle B_0 - \tilde{M}_0, B - \tilde{M}\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \langle E_0, E\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\\ \int_D (d_{M_0(x)}\phi)(M(x))\,dx &+ \int_D \sum_{l,m} a_{lm} \frac{\partial M_0(x)}{\partial x_l}\frac{\partial M(x)}{\partial x_m}\,dx,\ (M,B,E) \in \mathrm{X}, \end{aligned} \tag{1.32}$$

$$\begin{aligned} d^2_{(M_0,B_0,E_0)}\mathcal{E}_{el.mag}\big((M_1,B_1,E_1),(M_2,B_2,E_2)\big) &= \int_D (d^2_{M_0(x)}\phi)(M_1(x),M_2(x))\,dx\\ &+ \int_D \sum_{l,m} a_{lm} \frac{\partial M_1(x)}{\partial x_l}\frac{\partial M_2(x)}{\partial x_m}\,dx,\ (M_1,B_1,E_1),(M_2,B_2,E_2) \in \mathrm{X}. \end{aligned} \tag{1.33}$$

2. A priori bounds

Lemma 2.1. *Under appropriate assumptions we have*

$$\begin{aligned} \langle E, F_3(M,B,E)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} &= \langle E, \nabla \times (B - \tilde{M}) - 1_D(E + f)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\\ &= \langle \nabla \times E, (B - \tilde{M})\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - |1_D E|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \langle 1_D E, 1_D f\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \end{aligned} \tag{2.1}$$

Proof. Since

$$\langle E, \nabla \times (B - \tilde{M})\rangle = \langle \nabla \times E, B - \tilde{M}\rangle + \mathrm{div}\,((B - \tilde{M}) \times E)$$

[2] Since both (K and X) are Hilbert spaces, $R(\mathrm{K},\mathrm{X})$ is equal to the space of all Hilbert-Schmidt operators between (K and X).

and $\int \mathrm{div}\, dx = 0$, we have

$$\begin{aligned}
\langle E, \nabla \times (B - \tilde{M}) - 1_D(E+f)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} &= \langle E, \nabla \times (B - \tilde{M})\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - \langle E, 1_D(E+f)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
= \langle \nabla \times E, (B - \tilde{M})\rangle_{\mathbb{L}^2(\mathbb{R}^3)} &+ \int_{\mathbb{R}^3} \mathrm{div}\,((B - \tilde{M}) \times E)\, dx \\
&\quad - \langle 1_D E, 1_D E + 1_D f\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
= \langle \nabla \times E, (B - \tilde{M})\rangle_{\mathbb{L}^2(\mathbb{R}^3)} &- |1_D E|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \langle 1_D E, 1_D f\rangle_{\mathbb{L}^2(\mathbb{R}^3)}
\end{aligned}$$

This concludes the proof. ∎

We also have

Lemma 2.2. *Under appropriate assumptions we have*

$$\begin{aligned}
\langle (1_D B - M), F_1(M,B,E)\rangle_{\mathbb{L}^2(D)} &= \langle (1_D B - M), \lambda_1 M \times \rho(M,B) \\
&\quad - \lambda_2 M \times (M \times \rho(M,B))\rangle_{\mathbb{L}^2(D)} \\
= \lambda_2 \int_D |M \times \rho|^2\, dx &+ \int_D \phi(M(x))(F_1(M,B,E))(x)\, dx \\
&+ \int_D AM(x)(F_1(M,B,E))(x)\, dx. \qquad (2.2)
\end{aligned}$$

Proof. From the definition (1.30) of the functional ρ we have that

$$1_D B - M = \rho(M,B) + \phi(M) - AM.$$

Hence, with $\rho = \rho(M,B)$, we have

$$\begin{aligned}
\langle 1_D B - M, \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)\rangle_{\mathbb{L}^2(D)} &= \langle \rho, \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)\rangle_{\mathbb{L}^2(D)} \\
+\langle \phi(M), \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)\rangle_{\mathbb{L}^2(D)} &+ \langle AM, \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)\rangle_{\mathbb{L}^2(D)}
\end{aligned}$$

Using the vector formulae (4.4) and (4.5) we infer that the 1st term on the RHS of the last formula satisfies

$$\begin{aligned}
\langle \rho, \lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)\rangle_{\mathbb{L}^2(D)} &= \lambda_1 \langle \rho, M \times \rho\rangle_{\mathbb{L}^2(D)} \\
- \lambda_2 \langle \rho, M \times (M \times \rho)\rangle_{\mathbb{L}^2(D)} &= \lambda_2 |M \times \rho|^2_{\mathbb{L}^2(D)}.
\end{aligned}$$

This concludes the proof. ∎

Theorem 2.3. *Suppose that an* X*-valued process $u = (M,B,E)$ is a strong solution of the stochastic Landau-Lifshitz equations (1.22), (1.23), (1.24). Denote for simplicity the function $\mathcal{E}_{el.mag}$ by ψ. Then,*

$$\begin{aligned}
\psi(u(t)) &= \mathcal{E}_{el.mag}(M(t),B(t),E(t)) = \mathcal{E}_{el.mag}(M(0),B(0),E(0)) \\
&- \lambda_2 \int_0^t \int_D |M(s,x) \times \rho(M(s,x),B(s,x))|^2\, dx\, ds - \int_0^t \int_D |E(s,x)|^2\, dx\, ds \qquad (2.3) \\
&- \int_0^t \int_D \langle E(s,x), f(s,x)\rangle\, dx\, ds + ???
\end{aligned}$$

Proof. By the Itô formula,

$$\begin{aligned}\psi(u(t)) &= \psi'(u(t))(F(u(t)))\,dt + \psi'(u(t))(G(u(t)))\,dw(t) \\ &+ \frac{1}{2}\mathrm{tr}_{\mathrm{K}}[\psi''(u(t))(F(u(t)))\circ(G(u(t),G(u(t))\,dt\end{aligned} \tag{2.4}$$

Since,

$$\psi'(u)(F(u)) = \boxed{\frac{\partial\psi(u)}{\partial M}(F_1(u))} + \frac{\partial\psi(u)}{\partial B}(F_2(u)) + \frac{\partial\psi(u)}{\partial E}(F_3(u)) \tag{2.5}$$

and since by the definition of ψ, formula (1.32) and Lemmata 2.2 we have

$$\begin{aligned}\frac{\partial\psi(u)}{\partial M}(F_1(u)) &= \int_D \phi(M(x))F_1(u(x))\,dx + \int_D \langle AM(x), F_1(u(x))\rangle\,dx - \langle B-\tilde{M}, \tilde{F}_1\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\ &= \int_D \phi(M(x))F_1(M,B,E)\,dx + \int_D \langle AM(x), F_1(M,B,E)\rangle\,dx \\ &\qquad - \int_D \langle 1_D B - M, F_1(M,B,E)\rangle\,dx \\ &= \int_D \phi(M(x))F_1(M,B,E)\,dx + \int_D \langle AM(x), F_1(M,B,E)\rangle\,dx \\ &- \lambda_2\int_D |M\times\rho|^2\,dx - \int_D \phi(M(x))(F_1(M,B,E))(x)\,dx \\ &\qquad - \int_D AM(x)(F_1(M,B,E))(x)\,dx \\ &- \lambda_2\int_D |M\times\rho(M,B,E)|^2\,dx.\end{aligned}$$

Therefore, by Lemmata 1.4 and 2.1 we infer that

$$\begin{aligned}\psi'(u)(F(u)) &= -\lambda_2\int_D |M\times\rho(M,B,E)|^2\,dx + \langle B-\tilde{M}, F_2\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \langle E, F_3\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\ &= -\lambda_2\int_D |M\times\rho(M,B,E)|^2\,dx + \langle B-\tilde{M}, -\nabla\times E\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\ &+ \int_{\mathbb{R}^3}\langle\nabla\times E, B-\tilde{M}\rangle\,dx - \int_D |E(x)|^2\,dx - \int_D \langle E(x), f(x)\rangle\,dx \\ &= -\lambda_2\int_D |M\times\rho(M,B,E)|^2\,dx - \int_D |E(x)|^2\,dx - \int_D \langle E(x), f(x)\rangle\,dx\end{aligned}$$

∎

I use notation from Visinit's paper. Let $M \in L^2(D, R^3)$. Then we denote by $\bar{M}$ the extension of M by 0 in D^c. The map $L^2(D, R^3) \ni M \mapsto \bar{M} \in L^2(R^3, R^3)$ is obviously linear and bounded. Then we put $H = -\pi(\bar{M})$, where π is defiend in definition 10. So we put

$$\psi : L^2(D, R^3) \ni M \mapsto -\pi(\bar{M}) \in L^2(R^3, R^3).$$

Obviously ψ is linear and bounded. We also denote $H = \psi(M)$.

Claims 1. H defined above satisfies (2.1) (with ∇u replaced by H).

1. H defined above satisfies the three conditions listed between (2.1) and (2.2).

3. Gradient flow

Assume that H, $(\cdot,\cdot)$ is a real separable Hilbert space. Assume that

$$E : D(E) \to \mathbb{R}$$

is a "smooth" function, where $D(E)$ is a subset of H. We have the following non-rigorous result

Proposition 3.1. *If* $u : (a,b) \to \mathrm{H}$ *is a solution to*

$$\frac{du}{dt} = -\nabla E(u(t)) \tag{3.1}$$

Then,

$$E(u(t)) - E(u(s)) = -\int_s^t |\nabla E(u(r))|^2\,dr \quad s < t, s, t \in (a,b). \tag{3.2}$$

Proof. We have, on an informal level,

$$\frac{d}{dt}E(u(t)) \;=\; (\nabla E(u(t)), \frac{du}{dt}) = (\nabla E(u(t)), -\nabla E(u(t))) = |\nabla E(u(t))|^2.$$

∎

The above result is also true in a curved space.

Assume that M is a riemannian submanifold of the Hilbert space H, $(\cdot,\cdot)$. In particular, the tangent space T_mM, $m \in M$ is a Hilbert space endowed with the inner product $(\cdot,\cdot)_{T_m}$ being the restriction of $(\cdot,\cdot)$ to T_mM.

Assume that

$$E : D(E) \to \mathbb{R}$$

is a "smooth" function, where $D(E)$ is a subset of M.

Denote, for $m \in M$, by $\nabla_m E$ the unique element of the tangent space T_mM such that

$$(\nabla_m E, y)_{T_m} = (d_m E)(y), \quad y \in T_u M. \tag{3.3}$$

We have the following non-rigorous result

Proposition 3.2. *If* $u : (a,b) \to M$ *is a solution to*

$$\frac{du}{dt} = -\nabla_{u(t)} E \tag{3.4}$$

Then,

$$E(u(t)) - E(u(s)) = -\int_s^t |\nabla E(u(r))|^2_{T_{u(r)}}\,dr = -\int_s^t |\nabla E(u(r))|^2\,dr, \quad s < t, s, t \in (a,b). \tag{3.5}$$

Proof. We have, on an informal level,

$$\frac{d}{dt}E(u(t)) \;=\; (\nabla_{u(t)} E, \frac{du}{dt}) = (\nabla_{u(t)} E, -\nabla_{u(t)} E) = |\nabla_{u(t)} E|^2.$$

∎

Now consider the following finite dimensional example. Let $\mathrm{H} = \mathbb{R}^3$ endowed with the euclidean scalar product and

$$M = \mathbb{S}^2.$$

Then, $a \in M$, the tangent space T_a is equal to

$$T_a M = \{y \in \mathbb{R}^3 : \langle a, y \rangle = 0\}.$$

For $a \in M$, let $\pi_a : \mathbb{R}^3 \to T_a M$ ne the orthogonal projection:

$$\pi_a : \mathbb{R}^3 \ni y \mapsto y - (a, y) \in T_a M.$$

We have

Lemma 3.3. *For $a \in M$, then*

$$\pi_a(y) = -a \times (a \times y), \quad y \in \mathbb{R}^3, \tag{3.6}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$.

Proof. Denote

$$T(y) = -a \times (a \times y), \quad y \in \mathbb{R}^3, \tag{3.7}$$

It is enough to show that

$$|Ty|^2 = (y, Ty), \quad y \in \mathbb{R}.$$

By (4.5) we have

$$(y, Ty) = -(y, a \times (a \times y)) = -(b, a \times c) = (a \times b, c) = (a \times y, a \times y) = |a \times y|^2.$$

On the other hand,

$$|a \times (a \times y)|^2$$

∎

Consider now a the following result.

Proposition 3.4. *Assume that $f : \mathbb{R}^3 \to \mathbb{R}$ is sufficiently regular vector function. If $u : (a, b) \to \mathbb{S}^2$ is a solution to*

$$\frac{du}{dt} = -u \times \big(u \times (-\nabla_u f)\big) = u \times (u \times \nabla_u f), \tag{3.8}$$

where now $\nabla_u f \in \mathbb{R}$. Then,

$$f(u(t)) - f(u(s)) = -\int_s^t |u(r) \times \nabla_{u(r)} f|^2 \quad s < t, s, t \in (a, b). \tag{3.9}$$

Proof. We have, on an informal level,

$$\begin{aligned} \frac{d}{dt} f(u(t)) &= (\nabla_{u(t)} f, \frac{du}{dt}) = (\nabla_{u(t)} f, u(t) \times (u(t) \times \nabla_{u(t)} f)) \\ &= (b, a \times c) = -(a \times b, c) \\ &= -(u(t) \times \nabla_{u(t)} f, u(t) \times \nabla_{u(t)} f) = -|u(t) \times \nabla_{u(t)} f|^2. \end{aligned}$$

∎

If we denote by ϕ the restriction of f to S^2, then since

$$u \times (u \times \nabla_u f) = \pi_u \circ \nabla_u f = \nabla_u \phi,$$

where $\nabla_u \phi \in T_n \mathbb{S}^2$, we infer that the equation (3.8) is just

$$\frac{du}{dt} = u \times (u \times \nabla_u f) = -\nabla_u \phi, \tag{3.10}$$

and since, for $u \in \mathbb{S}^2$

$$|u \times \nabla_u f| = |u \times (u \times \nabla_u f)|,$$

we infer that the equality (3.9) becomes

$$\phi(u(t)) - \phi(u(s)) = -\int_s^t |\nabla_{u(r)} \phi|^2 \quad s < t, s, t \in (a, b). \tag{3.11}$$

Hence, we see that Proposition 3.4 is just a special case of Proposition 3.1.

Consider now the following generalisation of Proposition 3.4.

Proposition 3.5. *Assume that $f : \mathbb{R}^3 \to \mathbb{R}$ is sufficiently regular vector function. If $u : (a, b) \to \mathbb{S}^2$ is a solution to*

$$\frac{du}{dt} = -u \times \big(u \times (-\nabla_u f)\big) + \lambda_1 u \times (-\nabla_u f), \tag{3.12}$$

where, as before, $\nabla_u f \in \mathbb{R}$. Then[3],

$$f(u(t)) - f(u(s)) = -\int_s^t |u(r) \times \nabla_{u(r)} f|^2 \quad s < t, s, t \in (a, b). \tag{3.13}$$

Proof. We have, on an informal level,

$$\begin{aligned} \frac{d}{dt} f(u(t)) &= (\nabla_{u(t)} f, \frac{du}{dt}) = (\nabla_{u(t)} f, u(t) \times (u(t) \times \nabla_{u(t)} f)) - \lambda_1 (\nabla_{u(t)} f, u(t) \times \nabla_{u(t)} f \\ &= (b, a \times c) = -(a \times b, c), \quad (a, b \times a) = 0 \\ &= -(u(t) \times \nabla_{u(t)} f, u(t) \times \nabla_{u(t)} f) = -|u(t) \times \nabla_{u(t)} f|^2. \end{aligned}$$

∎

[3]Note that the identity (3.13) below is the same sa the equality (3.9)

4. Appendix

In this Appendix we will list all algebraic identities used in this paper.
Assume that $a, b, c, d \in \mathbb{R}^3$. Then

$$
\begin{aligned}
&(4.1) & a \times b &= -b \times a \\
&(4.2) & \langle a \times (b \times c), d\rangle &= \langle c, (d \times a) \times b\rangle \\
&(4.3) & \langle a \times b, c\rangle &= \langle b, c \times a\rangle \\
&(4.4) & \langle a \times b, b\rangle &= 0, \\
&(4.5) & -\langle a \times b, c\rangle &= \langle b, a \times c\rangle \\
&(4.6) & a \times (b \times c) &= \langle a, c\rangle b - \langle a, b\rangle c \\
&(4.7) & |a \times b| &\leq |a||b|. \\
&(4.8) & (a \times b) \times b &= -|b|^2 a, \text{ if } \langle a, b\rangle = 0 \\
&(4.9) & a \times (a \times b) &= -|a|^2 b, \text{ if } \langle a, b\rangle = 0
\end{aligned}
$$

Proof of (4.8) *and* (4.9). If $\langle a, b\rangle = 0$, then

$$(a \times b) \times b = b \times (b \times a) = \langle b, a\rangle b - \langle b, b\rangle a = -|b|^2 a$$

and

$$a \times (a \times b) = \langle a, b\rangle a - \langle a, a\rangle b = -|a|^2 b$$

∎

Corollary 4.1.

$$\langle a \times (a \times b), b\rangle = -|a \times b|^2 \tag{4.10}$$

Proof. Apply, (4.3) and then (4.1).

∎

5. Stochastic integration in M-type 2 Banach spaces

We begin with recalling one of the most fundamental notions in our work.

Definition 5.1. *A Banach space X is called M-type 2 iff there exists a constant $C = C_2(X) > 0$ such that for any X-valued finite martingale $\{M_k\}$ the following holds*

$$\sup_k \mathbb{E}|M_k|^2 \leq C \sum_k \mathbb{E}|M_k - M_{k-1}|^2. \tag{5.1}$$

For a definition of a Banach space valued martingale see [49] section 8.3 (p.41).
It is known, see [106], that X is of M-type 2 if and only if one of the following two conditions holds.

• There exists an equivalent norm $\|\cdot\|$ and $k > 0$ on X such that its modulus of smoothness $\rho_X(t)$ satisfies

$$\rho_X(t) \;\leq\; kt^2, \qquad t \in (0,1], \tag{5.2}$$

where

$$\rho_X(t) := \sup_{\|x\|=\|y\|=1} \frac{1}{2}\left(\|x+ty\| + \|x-ty\| - 1\right).$$

•• There exists a constant $A > 0$ such that

$$|x+y|^2 + |x-y|^2 \;\leq\; |x|^2 + A|y|^2, \qquad x, y \in X. \tag{5.3}$$

The following is based on [51].

Definition 5.2. *For separable Hilbert and Banach spaces H and X we put*

$$M(H,X) := \{L : H \to X : \; L \text{ is linear bounded and } \gamma\text{–radonifying } \}. \tag{5.4}$$

Thus, a bounded linear operator $L :\; H \to X$ belongs to $M(H,X)$ iff the image $L(\gamma_H)$ of the canonical finitely additive gaussian function γ_H on H by L is σ-additive on the algebra of cylindrical sets in X. This measure will be denoted by ν_L.
For $L \in M(H,X)$ we put

$$\|L\|_{M(H,X)} := \left\{ \int_X |x|^2 d\nu_L(x) \right\}^{\frac{1}{2}}, \tag{5.5}$$

In view of the Landau-Shepp-Fernique Theorem, $\|L\|$ is a finite number.

We have, see Neidhard [51], the following

Theorem 5.3.1. *$M(H,X)$ is a separable Banach space.*
2. *If $i : H \to E$ is an abstract Wiener space (AWS) and*

$$\tilde{i} \;:\; \mathcal{L}(E,X) \ni A \mapsto A \circ i \in M(H,X),$$

then the map $\tilde{i}$ is continuous, i.e. there is $C > 0$ such that for any $A \in \mathcal{L}(E,X)$

$$\|\tilde{i}(A)\|_{M(H,X)} \;\leq\; C\|A\|_{\mathcal{L}(E,X)},$$

the range of $\tilde{i}$ is a dense subspace of $M(H,X)$, and, for $A \in \mathcal{L}(E,X)$,

$$\nu_{\tilde{i}(A)} = \mu \bar{\circ} A,$$

where $\mu\bar{\circ}A$ is the image by A of the canonical Gaussian measure μ on E.

Remark 5.4. Note the following trivial fact. If Y is another separable Banach space then there exists $C > 0$ such that for any $A \in M(H,X)$ and $B \in \mathcal{L}(E,Y)$, $BA \in M(H,X)$ and $\|BA\|_{M(H,X)} \leq C\|B\|_{\mathcal{L}(E,Y)}\|A\|_{M(H,X)}$. In particular, $A \circ i \in M(H,X)$ if $A \in \mathcal{L}(E,X)$ so that the range of $\tilde{i}$ is indeed contained in $M(H,X)$ (and not only in $\mathcal{L}(H,X)$), as we tacitly assumed in part 2. of the above Theorem.
Baxendale in [3] proved that if $L \in M(H,X)$ and $T \in \mathcal{L}(G,H)$ (where G is another separable Hilbert space) then also $L \circ T \in M(G,X)$.

From now on we shall assume that $i : H \to E$ is an AWS. By μ we will denote the canonical Gaussian (and centred) probability measure on E and by $\{w(t)\}_{t\geq 0}$ an E-valued Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t$, $t \geq 0$ be the standard augmented filtration induced by $w(t)$, $t \geq 0$. Note that, in particular μ is the law of $t^{-\frac{1}{2}} w(t)$, for each $t > 0$. If S is a normed vector space endowed with some σ-algebra ρ, then for $0 \leq a < b \leq \infty$, $\mathcal{N}(a, b; S)$ denotes the set of all progressively measurable S-valued processes $\eta : [a, b) \times \Omega \to S$. If $p \in [1, \infty)$, then we set $\widetilde{\mathcal{M}}$

$$
\begin{aligned}
(5.6) \qquad \tilde{\mathcal{M}}^p(a, b; S) &:= \{\xi \in \mathcal{N}(a, b; S) : \mathbb{E} \int_a^b |\xi(t)|_S^p \, dt < \infty\}, \\
(5.7) \qquad \tilde{\mathcal{N}}^p(a, b; S) &:= \{\xi \in \mathcal{N}(a, b; S) : \int_a^b |\xi(t)|_S^p \, dt < \infty \text{ a.s.}\}, \\
\mathcal{N}_{\text{step}}(a, b; S) &:= \{\xi \in \mathcal{N}(a, b; S) : \exists \pi = \{a = t_0 < t_1 < \ldots < t_n = b\} : \\
(5.8) \qquad & \qquad \xi(t) = \xi(t_k), \; t \in [t_k, t_{k+1}), \; k = 1, \ldots, n-1\}.
\end{aligned}
$$

Then, we define $\mathcal{M}^p(a, b; S)$ to be the space of all equivalence classes of elements of $\tilde{\mathcal{M}}^p(a, b; S)$ with respect to a natural equivalence relation, $\xi \sim \eta$ iff $\mathbb{E} \int_a^b |\xi(t) - \eta(t)|^p \, dt = 0$. Note that $\mathcal{M}^p(a, b; S)$ is complete if S is. Analogously we define $\mathcal{N}^p(a, b; S)$ with the only difference that now $\xi \sim \eta$ iff $\int_a^b |\xi(t) - \eta(t)|^p \, dt = 0$ a.s..

Two typical examples of S are $S = M(H, X)$ or $S = \mathcal{L}(E, X)$. In the former case one considers $\rho = \tau_{M(H,X)}$, the Borel σ-algebra $\mathcal{B}(M(H, X))$ on $M(H, X)$. One can prove, see Lemma 44 in [51], that $\mathcal{B}(M(H, X))$ is generated by sets of the form

$$\{A \in M(H, X) : \; \varphi A h \in U\}, \; h \in H, \; \varphi \in X^*, \; U \subset \mathbb{R} \text{ open}.$$

It follows that if $\tau_{\mathcal{L}(E,X)}$ denotes the σ-algebra on $\mathcal{L}(E, X)$ generated by sets of the form

$$\{A \in \mathcal{L}(E, X) : \; \varphi A y \in U\}, \; y \in E, \; \varphi \in X^*, \; U \subset \mathbb{R} \text{ open}.$$

then

Proposition 5.5. ***(Neidhardt)*** *Assume that $i : H \to E$ is an AWS. Then the natural imbedding map*

$$\tilde{\imath} : \mathcal{L}(E, X) \;\to\; M(H, X)$$

is $\big(\tau_{\mathcal{L}(E,X)}; \mathcal{B}(M(H, X))\big)$–measurable.

For simplicity of notation let $\mathcal{M}^p_{\text{step}}(a, b; S)$ be the space of equivalence classes (in $\mathcal{M}^p(a, b; S)$) of elements of $\tilde{\mathcal{M}}^p(a, b; S) \cap \mathcal{M}_{\text{step}}(a, b; S)$.
For $\xi \in \mathcal{M}^2_{\text{step}}(a, b; \mathcal{L}(E, X))$ (with partition $\pi = \{a = t_0 < t_1 < \ldots < t_n = b\}$) let $I(\xi)$ be a measurable map from Ω into X defined by

$$I(\xi) := \sum_{k=0}^{n-1} \xi(t_k) \left(w(t_{k+1}) - w(t_k) \right). \tag{5.9}$$

We have

Lemma 5.6. *Assume that X is an M-type 2 Banach space and $i : H \to E$ is an AWS. Then for any $\xi \in \mathcal{M}^2_{\text{step}}(a, b; \mathcal{L}(E, X))$*

$$\begin{aligned} I(\xi) &\in L^2(\Omega, X), \\ \mathbb{E}I(\xi) &= 0 \end{aligned}$$

and

$$\mathbb{E}|I(\xi)|^2 \le C_2(X)\mathbb{E}\int_a^b \|\xi(t)\|^2_{M(H,X)}\, dt. \tag{5.10}$$

Remark 5.7. Neidhard proved Lemma 5.6 by using property ••) of X. An alternative approach based on using the M-type 2 property of X (i.e. (5.1)) was proposed independently in [69]. In both approaches one uses the standard properties of the conditional expectation and the following

Lemma 5.8. *If ξ is $\mathcal{L}(E, X)$ and $t > s$, then*

$$\mathbb{E}|\xi(w(t) - w(s))|^2 \;=\; (t - s)\int_X |z|^2\, d\nu_{\tilde{i}(\xi)}(z). \tag{5.11}$$

Lemma 5.8 is a simple consequence of part 2. of Theorem 5.3. See also Lemma III.2 in [36] and the remarks following its proof.

□

The fundamental property of the mapping I is that it extends uniquely to a bounded linear map from $\mathcal{M}^2(a, b; M(H, X)$ into $L^2(\Omega, X)$. This follows easily from (5.10) and the fact (proven in [51]) that $\mathcal{M}^2_{\text{step}}(a, b; \mathcal{L}(E, X))$ is dense in $\mathcal{M}^2(a, b; M(H, X)$. The value of this extension at $\xi \in \mathcal{M}^2(a, b; M(H, X)$ will be denoted by $\int_a^b \xi(s)\, dw(s)$.

If $\tau_1 \le \tau_2$ are two stopping times (with respect to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$) such that $\tau_2 - \tau_1$ is accessible and $\Omega \times [\tau_1, \tau_2) := \{(\omega, t) \in \Omega \times [0, \infty] : \tau_1(\omega) \le t \le \tau_2(\omega)\}$ then by $\mathcal{M}^p(\tau_1, \tau_2; S)$ we denote the space of classes of all progressively measurable processes $\xi : \Omega \times [\tau_1, \tau_2) \to S$ for which

$$\mathbb{E}\int_{\tau_1}^{\tau_2} \|\xi(t)\|^p_{M(H,X)} dt < \infty.$$

We denote by $\mathcal{M}^p_{\text{loc}}(\tau_1, \tau_2; S)$ the space of classes of all S-valued progressively measurable processes $\xi(t)$, $\tau_1 \le t < \tau_2$ for which there exists an increasing sequence $(T_n)_{n=1}^\infty$ of stopping times such that a.s. $\tau_1 \le T_n < \tau_2$ and $\lim_n T_n = \tau_2$, and, for any $n \in \mathbb{N}$,

$$\mathbb{E}\int_{\tau_1}^{T_n} |\xi(t)|^p\, dt < \infty.$$

Analogously, one defines $\mathcal{N}^p_{\text{loc}}(\tau_1, \tau_2; S)$ to be the space of classes of all S-valued progressively measurable processes $\xi(t)$, $\tau_1 \le t < \tau_2$ for which there exists a sequence $(T_n)_{n=1}^\infty$ of stopping times such that $T_n \nearrow \tau_2$ a.s. and, for any $n \in \mathbb{N}$, a.s.

$$\int_{\tau_1}^{T_n} |\xi(t)|^p\, dt < \infty.$$

Remark 5.9. Our notation differs slightly from the one used by Revuz & Yor [57]. The reason for this difference is that while these authors treat a whole family of martingales we are mostly interested in the E-valued Wiener process (or, maybe more precise, H-cylindrical Wiener process) $w(t)$. Thus, for example, what we denote $\mathcal{M}^p(0,\infty;\mathbb{R})$, Yor & Revuz denote by $L^p(w)$ (with $w(t)$, $t \geq 0$ being the standard 1-dimensional Wiener process). Note however, that in [57] only real valued martingales are studied.
For a careful discussion of cylindrical Wiener processes see [65]. One should point out that for purposes of Itô integration the use of H-cylindrical Wiener process is more natural than the use of E-valued canonical Wiener process. Indeed, E plays in this setting only an auxiliary rôle. This approach has been adopted in e.g. [19]. However in the case of Stratonovich integral the space E plays a more specific rôle and we have decided to use this approach throughout the whole paper.

With all this, one can define $\int_{\tau_1}^{\tau_2} \xi(s)\,dw(s) \in L^2(\Omega, X)$ for $\xi \in \mathcal{M}^2(\tau_1,\tau_2;\, M(H,X))$, having the following properties

$$\begin{aligned} \mathbb{E}\int_{\tau_1}^{\tau_2} \xi\,dw(s) &= 0, \\ \mathbb{E}\left|\int_{\tau_1}^{\tau_2} \xi(s)\,dw(s)\right|^2 &\leq C_2(X)\mathbb{E}\int_{\tau_1}^{\tau_2} \|\xi(t)\|^2_{M(H,X)}\,dt, \end{aligned} \tag{5.12}$$

We have, see [25] and Proposition 3.4 in [69], the following Burkholder inequality

Theorem 5.10. *Assume that $i : H \to E$ is an AWS and X is an M-type 2 Banach space. Assume that $\xi \in \mathcal{M}^2_{\mathrm{loc}}(0,\infty; M(H,X))$ and $I(t) := \int_0^t \xi(s)\,dw(s)$.*
Then, $I(t)$ is a continuous X-valued martingale and for any $r \in (1,\infty)$ there exists a constant $C_r > 0$ such that for any finite stopping time[4] $\tau > 0$

$$\mathbb{E}\sup_{0\leq s\leq \tau} |I(s)|^r \leq C_r\mathbb{E}\left[\int_0^\tau \|\xi(s)\|^2_{M(H,X)}\,ds\right]^{\frac{r}{2}}. \tag{5.13}$$

We have also the following localization property of the Itô integral introduced above, see also [36] and [32]

Proposition 5.11. *Assume that $i : H \to E$ is an AWS and X is an M-type 2 Banach space. Assume that $\xi_k \in \mathcal{M}^2_{\mathrm{loc}}(0,\infty; M(H,X))$, $k = 1,2$. Assume that $\Omega^0 \in \mathcal{F}$ with $P(\Omega^0) > 0$ and τ is a stopping time such that $\tau > 0$ on Ω^0. Assume that the process $\xi_1(t) = \xi_2(t)$ a.s. on $\Omega^0_t(\tau) = \{\omega \in \Omega^0 : t < \tau(\omega)\}$. Then, for any stopping time σ s.th. $\sigma \leq \tau$ a.s. on Ω^0 and $\xi_k \in \mathcal{M}^2(0,\sigma; M(H,X)$, $k = 1,2$ one has, a.s. on Ω^0.*

$$\int_0^\sigma \xi_1(t)\,dw(t) = \int_0^\sigma \xi_2(t)\,dw(t). \tag{5.14}$$

Remark 5.12. In view of Remark 5.4 one may replace $M(H,X)$ by $\mathcal{L}(E,X)$. In particular, $\int_0^t \xi(s)\,dw(s)$ exists for any $\xi \in \mathcal{M}^2_{\mathrm{loc}}(0,\infty;\mathcal{L}(E,X))$ and satisfies, for any finite stopping

[4]A stopping time τ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is finite iff $\tau < \infty$ a.s.

time τ,

$$\mathbb{E}\sup_{0\le s\le\tau}|\int_0^s \xi(u)\,dw(u)|^r \le C_r\mathbb{E}\left[\int_0^\tau |\xi(s)|^2_{\mathcal{L}(E,X)}ds\right]^{\frac{r}{2}} \tag{5.15}$$

Returning to the discussion from before the formula (5.10) we consider a measurable semiflow $(\Theta_t)_{t\ge0}$ on Ω, i.e. a semigroup of measurable mappings $\Theta_t:\Omega\to\Omega$. We assume that for all $t,s\ge0$ one has $w(t)\circ\Theta_s = w(t+s)-w(s)$ a.s. Assume that $(\Theta_t)_{t\ge0}$ is jointly measurable, i.e. the map $\Theta:[0,\infty)\times\Omega\ni(t,\omega)\mapsto\Theta_t(\omega)$ is measurable. Before we formulate our next result we need to spend some time on explaining the notation in (5.16) below. Since Θ is jointly measurable, $\Theta_\tau := \Theta\circ(\tau,\mathrm{id}_\Omega):\Omega\to\Omega$ is measurable. As we already know I is a continuous process and so is also progressively measurable. Hence the LHS map $I\circ(\eta,\Theta_\tau):\Omega\to\Omega$ is also measurable. On the other hand, $f\in\mathcal{M}^2(0,\tau;S)$ and thus $f(\cdot-\tau,\Theta_\tau)$ is progressively measurable and in addition belongs to $\mathcal{M}^2(\sigma,\tau+\sigma;S)$. This explains the meaning of the RHS.

We have, see also Remark III.7D from [36].

Proposition 5.13. *Assume that $f\in\mathcal{M}^2_{\mathrm{loc}}(0,\infty;M(H,X))$ and let $I(t)$, $t\ge0$ be the continuous version of the stochastic integral $\int_0^t f(s)\,dw(s)$. Then, for any two finite stopping times σ and τ the following holds a.s.*

$$I(\sigma,\Theta_\tau) \;=\; \int_\tau^{\tau+\sigma} f(s-\tau,\Theta_\tau)\,dw(s). \tag{5.16}$$

PROOF. First we assume that $\sigma=t$ a.s. for some fixed $t\in[0,\infty)$. It is sufficient to consider f of the form

$$f(s,\omega) \;=\; \sum_{i=0}^N f(t_i,\omega)1_{[t_i,t_{i+1})}(s),\; s\ge0,$$

where $0\le t_0<t_1<\ldots<t_N<t_{N+1}<\infty$, and $f(t_i,\cdot)$ is $\big(\mathcal{F}_{t_i},\tau_{\mathcal{L}(E,X)}\big)$ measurable, $i=0,\ldots,N-1$. Then, a.s.

$$I(t) \;=\; \sum f(t_i)[w(t_{i+1}\wedge t)-w(t_i\wedge t)].$$

By approximating τ by finite valued stopping times we see that $w(t)\circ\Theta_\tau = w(\tau+t)-w(\tau)$ a.s. Thus

$$\begin{aligned}
I(t,\Theta_\tau) &= I(t)\circ\Theta_\tau = \sum f(t_i)\circ\theta_\tau[w(t_{i+1}\wedge t+\tau)-w(t_i\wedge t+\tau)]\\
&= \sum f(t_i+\tau-\tau)\circ\theta_\tau\left[w\left((t_{i+1}+\tau)\wedge(t+\tau)\right)-w\left((t_i+\tau)\wedge(t+\tau)\right)\right]\\
&= \int_\tau^{\tau+t} f(s-\tau)\circ\theta_\tau\,dw(s).
\end{aligned}$$

The proof in the general case remains practically the same.

■

A more general class of processes for which one can introduce a reasonably defined Itô integral is $\mathcal{N}^2(\tau_1,\tau_2;M(H,X))$, see also [9]. One can prove, see [51] Lemma 18 and [9]

Lemma 2.6 for the case $\tau_1 = 0$ and τ_2 a constant, that for $\xi \in \mathcal{N}^2(0,T;M(H,X))$ there exists a sequence of (classes of) processes $\{\xi_n\} \in \mathcal{M}^2(\tau_1,\tau_2;M(H,X))$ such that

$$\int_{\tau_1}^{\tau_2} \|\xi_n(s)\|^2_{M(H,X)}\,ds \;\le\; \int_{\tau_1}^{\tau_2} \|\xi(s)\|^2_{M(H,X)}\,ds, \quad \text{a.s.,} \tag{5.17}$$

$$\int_{\tau_1}^{\tau_2} \|\xi(s)-\xi_n(s)\|^2_{M(H,X)}\,ds \;\to\; 0, \quad \text{a.s..} \tag{5.18}$$

With this, see Lemmata 34, 51, 54 and 56 in [51] and Theorem 2.7 in [9] for the particular case mentioned above, we have

Theorem 5.14. *Assume that $\tau_1 \le \tau_2$ are two stopping times such that $\tau_2 - \tau_1$ is accessible and $\xi \in \mathcal{N}^2(\tau_1,\tau_2;M(H,X))$ and a sequence of processes $\{\xi_n\} \in \mathcal{N}^2(\tau_1,\tau_2;M(H,X))$ satisfies the conditions* (5.17)–(5.17) *above. Then the sequence of processes*

$$x_n(t) := \int_{\tau_1}^{t} \xi_n(s)\,dw(s),\ \tau_1 \le t < \tau_2,$$

converges, locally-uniformly w.r.t. $t \in [\tau_1,\tau_2)$, in probability, to some admissible X–valued process $x(t)$, $\tau_1 \le t < \tau_2$. The latter process will denoted by

$$\int_{\tau_1}^{t} \xi(s)\,dw(s) := x(t),\ t \in [\tau_1,\tau_2).$$

The process $x(t)$, $\tau_1 \le t < \tau_2$, is independent of the approximating sequence ξ_n: if ξ_n' is another sequence satisfying the above conditions and if $x'(t)$ is the corresponding limit process, then the processes x and x' are equivalent.
Moreover, the process $x(t)$, $\tau_1 \le t < \tau_2$ is a.s. bounded, i.e.

$$\sup_{\tau_1 \le t < \tau_2} |x(t)| < \infty, \quad a.s.$$

■

Remark 5.15. Theorem 5.14 can be generalized so that the space $\mathcal{N}^2(\tau_1,\tau_2;M(H,X))$ can be replaced by $\mathcal{N}^2_{\rm loc}(\tau_1,\tau_2;M(H,X))$. Take for example $\tau_1 = 0$ and $\tau_2 = \tau$.
If $\xi \in \mathcal{N}^2_{\rm loc}(0,\tau;M(H,X))$ and T_n is an increasing sequence of stopping times such that a.s. $T_n < \tau$ and $T_n \to \tau$ then $1_{[0,T_n0)}\xi \in \mathcal{N}^2(0,\tau;M(H,X))$ and so $x_n(t)$, $t < \tau$ is an admissible X-valued process. There exists an admissible X-valued process $x(t)$, $t < \tau$ such that $x(t \wedge T_n) = x_n(t)$ a.s. for any $n \in \mathbb{N}$. In particular, x_n converges in probability to x, locally-uniformly w.r.t. $t \in [0,\tau)$.
Note however, that now we cannot claim that the process $x(t)$, $0 \le t < \tau$ is a.s. bounded.

Our next aim is to state the Itô Lemma. For this we need to introduce some additional notation.

Definition 5.16. *Assume that $L \in M(H,X)$ is fixed and let ν_L be the Gaussian measure on X related to L, see Definition 5.2. Then for any bounded bilinear bounded map $A : X \times X \to Y$, we write $A \in \mathbb{L}(X,X;Y)$, let us define*

$$\mathrm{tr}_L A := \int_X A(x,x)\,d\nu_L(x). \tag{5.19}$$

In the special case of an AWS $i : H \to E$ we will simply write $\mathrm{tr} A$ *instead of* $\mathrm{tr}_i A$.

Let us note that in view of the Landau-Shepp-Fernique Theorem, $\mathrm{tr}_L : \mathbb{L}(E, E; Y) \to Y$ is a bounded linear map:

$$|\mathrm{tr}_L A| \le \|A\|_{\mathbb{L}(X,X;Y)} \|L\|^2_{M(H,X)}. \tag{5.20}$$

Theorem 5.17. ***(Itô Formula)*** *Assume that $i : H \to E$ is an AWS and X, Y are two M-type 2 Banach spaces. Let $0 \le c < d \le \infty$. Assume that a function $f : [c, d) \times X \to Y$ is of $C^{1,2}$ class, i.e. f is Fréchet differentiable, the Fréchet derivative $f' : [c, d) \times X \to \mathcal{L}(\mathbb{R} \times X, Y)$ is continuous and differentiable in the X-direction with the resulting derivative being continuous*[5]*. Let, for $a \in \mathcal{N}^1_{\mathrm{loc}}(c, d; X)$ and $b \in \mathcal{N}^2_{\mathrm{loc}}(c, d; M(H, X))$,*

$$\xi(t) = \xi(c) + \int_c^t a(s)\, ds + \int_c^t b(s)\, dw(s), \ t \in [c, d]. \tag{5.21}$$

Then for all $t \in [c, d)$, a.s,

$$\begin{aligned} f(t, \xi(t)) - f(c, \xi(c)) &= \int_c^t \frac{\partial f}{\partial s}(s, \xi(s))\, ds + \int_c^t \frac{\partial f}{\partial x}(s, \xi(s)) a(s)\, ds \\ &+ \int_c^t \frac{\partial f}{\partial x}(s, \xi(s)) b(s)\, dw(s) \\ &+ \frac{1}{2} \int_c^t \mathrm{tr}_{b(s)} \frac{\partial^2 f}{\partial x^2}(s, \xi(s))\, ds. \end{aligned} \tag{5.22}$$

Remark 5.18. *(i)* If we assume that $b \in \mathcal{M}^2_{\mathrm{loc}}(c, d; \mathcal{L}(E, X))$ then the last term in 5.22 takes the form

$$\frac{1}{2} \int_c^t \mathrm{tr} \left[\frac{\partial^2 f}{\partial x^2}(s, \xi(s)) \circ (b(s), b(s)) \right] ds,$$

where $\frac{\partial^2 f}{\partial x^2} : [c, d) \times X \to \mathcal{L}(X, \mathcal{L}(X, Y)) \cong \mathcal{L}(X, X; Y) = \mathbb{L}(X, X; Y)$.
(ii) Theorem 5.17 is proved in [51] in the case $a \in \mathcal{M}^1(c, d; X)$ and $b \in \mathcal{M}^2(c, d; N(H, X))$. Our result is twofold more general. First, we allow processes defined up to a stopping time. Secondly, we allow processes belonging to the class $\mathcal{N}^2$ and not $\mathcal{M}^2$ as Neidhard does. These generalizations are essential for constructing a satisfactory theory of stochastic differential equations on (Banach) manifolds.

We will conclude this section with a short discussion of stochastic (ordinary!) differential equations on M-type 2 Banach spaces. We begin by recalling a result from [51] (Theorem 144).

Theorem 5.19. *Assume that $i : H \to E$ is an AWS, $\{w(t)\}_{t \ge 0}$ is an Wiener process on E and X is an M-type 2 Banach space. Let $T \in (0, \infty]$ be fixed and the functions*

$$\begin{aligned} f &: [0, T] \times X \to X \\ g &: [0, T] \times X \to M(H, X) \end{aligned}$$

satisfy[6]

[5]Simply, $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous on $[c, d) \times X$ with values in the appropriate space.
[6]In the case $T = \infty$ we will understand the interval $[0, T]$ to be equal the interval $[0, \infty)$.

1. (LINEAR GROWTH CONDITION) *there exists* $K > 0$ *such that*

$$\max\{|f(t,x)|,\ \|g(t,x)\|_{M(H,X)}\} \le K(1+|x|), \quad t \in [0,T],\ x \in X; \tag{5.23}$$

2. ("LOCAL" LIPSCHITZ CONDITION) *for any* $R > 0$ *there exists* $C_R > 0$ *such that*

$$\max\{|f(t,x)-f(t,y)|,\ \|g(t,x)-g(t,y)\|_{M(H,X)}\} \le C_R|x-y|, \tag{5.24}$$

for all $t \in [0,T]$ *and all* $x, y \in X$ *such that* $|x|, |y| \le R$.

Then, for any $\mathcal{F}_0$ *measurable* $\xi_0 : \Omega \to X$, *there exists a unique solution* $\xi(t)$, $t \in [0,T]$, *to the (stochastic) initial value problem*

$$\begin{aligned} d\xi(t) &= f(t,\xi(t))\,dt + g(t,\xi(t))\,dw(t), \quad t \ge 0, \\ \xi(0) &= \xi_0. \end{aligned} \tag{5.25}$$

Moreover, ξ *is a continuous process and for any* $r \ge 1$ *there exists* $C_r > 0$ *such that for all* $t \in [0,T)$

$$\mathbb{E} \sup_{s\in[0,t]} |\xi(t)|^r \le C_r t^r. \tag{5.26}$$

Remark 5.20. The notion of a (strong) solution referred to in the previous Theorem (as in the Theorem which follows) is standard; for the Hilbert case one can see [36], for the Banach [51].

An admissible process $\xi(t)$, $t \in [0,T]$ is a (strong) solution to the problem (5.26) iff for all $t \in [0,T]$,

$$\xi(t) = \xi_0 + \int_0^t f(t,\xi(t))\,dt + \int_0^t g(t,\xi(t))\,dw(t), \quad \text{a.s.}. \tag{5.27}$$

One can extend the above result to the case when not only the coefficients a and b are random functions but also to the case when their Lipschitz constants K and C_R above are random as well.

Theorem 5.21. *Assume that* $i : H \to E$ *is an AWS and* $\{w(t)\}_{t\ge 0}$ *is an* E*-valued Wiener process. Suppose also that* X *is an* M*-type 2 Banach space. Let* $T \in (0,\infty]$ *be fixed and the adapted random functions*

$$\begin{aligned} f &: [0,T] \times X \times \Omega \to X \\ g &: [0,T] \times X \times \Omega \to M(H,X) \end{aligned}$$

satisfy (see the footnote on 20)

1. (LINEAR GROWTH CONDITION) *there exists a random variable* $K : \Omega \to [0,\infty)$ *such that*

$$\max\{|f(t,x,\omega)|,\ \|g(t,x,\omega)\|_{M(H,X)}\} \le K(\omega)(1+|x|), \quad t \in [0,T],\ x \in X, \omega \in \Omega; \tag{5.28}$$

2. ("LOCAL" LIPSCHITZ CONDITION) *for any* $R > 0$ *there exists a random variable* $C_R : \Omega \to [0,\infty)$ *such that a.s in* $\omega \in \Omega$,

$$\max\{|f(t,x,\omega)-f(t,y,\omega)|,\ \|g(t,x,\omega)-g(t,y,\omega)\|_{M(H,X)}\} \le C_R(\omega)|x-y|, \tag{5.29}$$

for all $t \in [0,T]$ *and all* $x, y \in X$ *such that* $|x|, |y| \le R$.

Then, for any $\mathcal{F}_0$ *measurable* $\xi_0 : \Omega \to X$, *there exists a unique solution* $\xi(t)$, $t \in [0,T]$, *to the (stochastic) initial value problem* (5.25).

In what follows we will pay some attention to the case when the coefficients a, b no longer satisfy the linear growth condition.

Definition 5.22. *Let $\tau \leq \infty$ be an accessible stopping time. An admissible stochastic process $\xi(t)$, $t < \tau$, is a local solution to equation* (5.25) *iff there exists an increasing sequence σ_n of stopping times such that a.s. $\sigma_n < \tau$, $\sigma_n \to \tau$ and for any $t \geq 0$, any $n \in \mathbb{N}$, a.s.*

$$\xi(\sigma_n \wedge t) = \xi_0 + \int_0^{\sigma_n \wedge t} f(s, \xi(s))\, ds + \int_0^{\sigma_n \wedge t} g(s, \xi(s))\, dw(s). \tag{5.30}$$

A local solution $\xi(t)$, $t < \tau$ is called global iff $\tau = T$ a.s. Thus, in particular, if $c = \infty$ a local solution $\xi(t)$, $t < \tau$ is called global iff $\tau = \infty$ a.s.

Remark 5.23. The Definition 5.22 of a local solution is independent of the sequence σ_n. A proof of this fact follows from continuity of the process $\xi(t)$, $t < \tau$ and is based on the following three principles.
(i) If τ is an accessible stopping time then there exist an increasing sequence τ_n of *discrete* stopping times such that a.s. $\tau_n < \tau$ and $\tau_n \to \tau$;
(ii) if τ is an accessible stopping time and $\sigma \leq \tau$ is a stopping time then σ is also accessible.
(ii) if $\xi(t)$, $t < \tau$ is a local solution to then (5.30) holds with t being any discrete stopping time.
It follows, that that the following is an equivalent definition of a local solution.
An admissible stochastic process $\xi(t)$, $t < \tau$, where τ be an accessible stopping time, is a local solution to equation (5.25) iff for every accessible stopping time σ such that $\sigma < \tau$, for every $t \geq 0$, a.s.

$$\xi(\sigma \wedge t) = \xi_0 + \int_0^{\sigma \wedge t} f(s, \xi(s))\, ds + \int_0^{\sigma \wedge t} g(s, \xi(s))\, dw(s). \tag{5.31}$$

Definition 5.24. *Let $\tau \leq \infty$ be an accessible stopping time. A local solution $\xi(t)$, $t < \tau$ to* (5.25)*, is a maximal solution to equation* (5.25) *iff for any other local solution $\hat{\xi}(t)$, $t < \hat{\tau}$, $\hat{\tau} \leq \infty$ being a stopping time such that $\mathbb{P}\{\hat{\tau} > \tau\} > 0$ there exists a measurable set $\hat{\Omega} \subset \{\hat{\tau} > \tau\}$ such that $\mathbb{P}(\hat{\Omega}) > 0$ and $\xi(\tau) \neq \hat{\xi}(\tau)$ on $\hat{\Omega}$. Thus, a local solution $\xi(t)$, $t < \tau$ is not a maximal one iff there exists another local solution $\hat{\xi}(t)$, $t < \hat{\tau}$, such that*

$$P\left(\hat{\tau} > \tau, \xi(\tau) = \xi(\hat{\tau})\right) > 0.$$

If $\xi(t)$, $t < \tau$ to (5.25)*, is a maximal solution to equation* (5.25)*, the stopping time τ is called the explosion time of ξ.*

Definition 5.25. *A local solution $\xi(t)$, $t < \tau$ to problem* (5.25) *is unique iff for any other local solution $\hat{\xi}(t)$, $t < \hat{\tau}$ to (5.25) the restricted processes $\xi_{[0,\tau\wedge\hat{\tau})\times\Omega}$ and $\hat{\xi}_{[0,\tau\wedge\hat{\tau})\times\Omega}$ are equivalent.*

Remark 5.26. In the Definition 5.22 $\xi(t)$, $t < \tau$ is an admissible (so that a continuous) process. Thus, the process $g(s, \xi(s)$, $0 \leq s < \tau$ belongs to $\mathcal{N}^2_{\rm loc}(0, \tau; M(H, X))$ and so, in view of Remark 5.15 the Itô integral

$$\int_0^{t \wedge \sigma} g(s, \xi(s))\, dw(s)$$

makes sense, for any stopping time $\sigma < \tau$ and any $t \geq 0$.

In the following generalization of Theorem 5.19 the assumption of Lipschitz continuity of the coefficients on all bounded balls is replaced by a weaker one: a true local Lipschitz condition.

Theorem 5.27. *Assume that $i : H \hookrightarrow E$ is an AWS and $\{w(t)\}_{t\geq 0}$ is an E-valued Wiener process. Assume that X is an M-type 2 Banach space, $T \in (0, \infty]$ is fixed and the functions*

$$\begin{aligned} f &: [0,T) \times X \to X, \\ g &: [0,T) \times X \to M(H, X) \end{aligned}$$

satisfy the local Lipschitz condition:
for any $x_0 \in X$ there exists $r_0 > 0$ and $L_0 > 0$ such that for any $x_1, x_2 \in \bar{B}(x_0, r_0) := \{x \in X : |x - x_0| \leq r_0\}$ and all $t \in [0, T)$

$$\max\{|f(t,x_2) - f(t,x_1)|, \; \|g(t,x_2) - g(t,x_1)\|_{M(H,X)}\} \leq L_0 |x_2 - x_1|. \tag{5.32}$$

Then, for any $\mathcal{F}_0$ measurable $\xi_0 : \Omega \to X$ there exists a unique process $\{\xi(t)\}$, $t < \tau$ (with an accessible stopping time $\tau > 0$) which is a maximal solution to (5.25).
Moreover, if the Linear Growth Condition from Theorem 5.19 holds, then the solution $\xi(t)$, $t < \tau$ is global, i.e. $\tau = T$ a.s. and

$$\mathbb{E}|\xi(t)|^2 \leq 3e^{3K^2(C_2+T)t} \left(\mathbb{E}|\xi_0|^2 + K^2 T (C_2 + T) \right), \quad t \in [0, T]. \tag{5.33}$$

Remark 5.28. It follows from the Definition of uniqueness, see Definition 5.25, that ξ is a unique maximal solution to (5.25) iff for any local solution $\hat{\xi}(t)$, $t < \hat{\tau}$ to (5.25),

$$\begin{aligned} \hat{\tau} &\leq \tau, \text{ a.s.}, \\ \hat{\xi} &= \xi \text{ a.s. on } [0, \tau \wedge \hat{\tau}) \times \Omega. \end{aligned}$$

Before we embark on prove of Theorem 5.27 we present some discussion needed later on.

One can also replace the initial condition in (5.25), deterministic in the sense that initial time is 0, by a random one in the following sense. Let T be a finite stopping time and let $h : \Omega \to X$ be $\mathcal{F}_T$ measurable. Then one seeks a stopping time $\tau > T$ and an admissible X-valued process $\xi(t)$, $T \leq t < \tau$ which is a solution to

$$\begin{aligned} d\xi(t) &= f(\xi(t))\,dt + g(\xi(t))\,dw(t), \\ \xi(T) &= h, \text{ a.s..} \end{aligned} \tag{5.34}$$

Local, resp. maximal solution to the problem (5.34) we define analogously as the corresponding notions with respect to problem (5.25), see Definitions 5.22 and 5.24. In particular, in the definition of local solution we require the existence of an increasing sequence of $\{\sigma_n\}_n$ of stopping times such that $T \leq \sigma_n < \tau$ a.s., $\sigma_n \to \tau$ and for any $t \geq 0$, any $n \in \mathbb{N}$, a.s. on $\{t \geq T\}$,

$$\xi(\sigma_n \wedge t) = h + \int_T^{\sigma_n \wedge t} f(s, \xi(s))\,ds + \int_T^{\sigma_n \wedge t} g(s, \xi(s))\,dw(s). \tag{5.35}$$

Existence and uniqueness of maximal solutions to problem (5.34) holds as in Theorem 5.27. We have the following

Corollary 5.29. *Assume that $\xi(t)$, $t<\hat{T}$ is a solution to (5.25) and T is an accessible stopping time such that $T<\hat{T}$ a.s.. Let $h(\omega):=\xi(T(\omega),\omega)$, $\omega\in\Omega$ and let $\eta(t)$, $T\le t<\tau$ be a solution to (5.34).*

Then the process $x(t)$, $t<\tau$ defined by

$$x(t,\omega) \;:=\; \begin{cases}\xi(t,\omega), & \text{if } t\le T(\omega),\\ \eta(t,\omega), & \text{if } T(\omega)<t<\tau(\omega),\end{cases}$$

is a solution to (5.34)

Proof. First of all we notice h is $\mathcal{F}_T$-measurable, see e.g. [47] Proposition 2.18, ch. I.

To prove that the process $x(t)$, $0\le t<\tau$, is a solution, we need to prove that, firstly, it is admissible and, secondly, that the integral identities (5.35) hold true for an appropriate choice of stopping times. To prove the first property we observe that since obviously the paths of the process x are continuous, we only need to show that for each $t\ge 0$, the function $x(t)=x(t,\cdot):\{t<\tau\}\to X$ is $\mathcal{F}_t$ measurable. To prove the latter we observe that

$$x(t)=\xi(t)1_A+\eta(t)1_B,$$

where $A=\{T\ge t\}$, $B=\{T<t,\ t<\tau\}=\{T<t\}\cap\{t<\tau\}$. Since τ and T are stopping times, the sets A and B belong to the σ-algebra $\mathcal{F}_t$. Since moreover $\xi(t)$ and $\eta(t)$ are $\mathcal{F}_t$-measurable, we infer $\mathcal{F}_t$-measurability of $x(t)$.

We finish by observing that by using the additivity of the Itô and the Riemann integrals with respect to limits of integration we infer that the process $x(t)$, $0\le t<\tau$ is a (local) solution.

□

Corollary 5.30. *If $\xi(t)$, $0\le t<\tau$ is the maximal solution to* (5.25) *then*

$$\mathbb{P}\left\{w\in\Omega:\tau(\omega)<\infty,\exists\lim_{t\nearrow\tau(\omega)}\xi(t)(\omega)\in X\right\}=0.$$

Proof of Corollary 5.30. We argue by contradiction. Assume that there exists $\Omega_1\subset\Omega$ such that $\mathbb{P}(\Omega_1)>0$ and such that for any $\omega\in\Omega_1$ we have $\tau(\omega)<\infty$ and $\lim_{t\nearrow\tau(\omega)}\xi(t)(\omega)=\bar{\eta}(\omega)\in X$.
Take an increasing sequence $\{\sigma_n\}_{n=1}^{\infty}$ of stopping times such that $\sigma_n<\tau$ a.s. and $\sigma_n\nearrow\tau$ a.s. Since the function $f_n(\omega):=\xi(\sigma_n(\omega))(\omega)$, $\omega\in\Omega$ is $\mathcal{F}_{\sigma_n}$–measurable, see e.g. [47] Proposition 2.18, ch. I, and hence $\mathcal{F}_\tau$–measurable, the set $\Omega_2:=\{\omega\in\Omega:\tau(\omega)<\infty$ and $\lim_n f_n(\omega):=\bar{\eta}(\omega)$exists$\}$ belongs to the σ-algebra $\mathcal{F}_\tau$ and the function $\bar{\eta}$ is $\mathcal{F}_\tau$–measurable. Note that obviously $\Omega_1\subset\Omega_2$ so that $\mathbb{P}(\Omega_2)>0$.
Let $\eta(t)$, $\tau(\omega)\le t<\tau_2(\omega)$, be the solution to (5.25) with initial condition $\eta_{\tau(\omega)}(\omega)=\bar{\eta}(\omega)1_{\Omega_2}(\omega)$. Then, by Corollary 5.29 the process $\tilde{\xi}(t)$ defined by

$$\tilde{\xi}(t)(\omega)=\begin{cases}\xi(t)(\omega), & \text{if } t<\tau(\omega),\\ \eta(t)(\omega), & \text{if } \omega\in\Omega_2 \text{ and } t\in[\tau(\omega),\tau_2(\omega))\end{cases}$$

is a solution to (5.25). Obviously this contradicts the maximality of the solution $\xi(t)$.

□

Proof of Theorem 5.27. *Existence* Let U_0 and U be two open and bounded subsets of X such that $U_0 \subset \bar{U}_0 \subseteq U$ and both f and g satisfy the condition (5.32) in U. For each such a pair of sets there exist functions $f_U : X \to X$ and $g_U : [0,T) \times X \to \mathcal{L}(E,X)$ which satisfy the conditions of Theorem 5.19 and coincide on $[0,T) \times U_0$ with f and g respectively. There exists a global solution $\xi_U(t)$ to the problem

$$\left\{ \begin{array}{rcl} d\xi_U(t) & = & f_U(\xi_U(t))\, dt + g_U(\xi_U(t))\, dw(t), \\ \xi_U(0) & = & \xi_0 1_{\{\xi_0 \in U_0\}}. \end{array} \right. \tag{5.36}$$

Let τ_U be the exit time of the process $\xi_U(t)$ from U_0. Note that $\tau_U = 0$ if $\{\xi_0 \notin U_0\}$. If $P(\xi_0 \in U_0) > 0$, then the process $\xi(t)$, $t < \tau_U$ is a solution to (5.25).

Let Ξ denote the set of all solutions $\xi(t)$, $t < \tau$ to (5.25). We have just seen Ξ to be a nonempty set. Let $\mathcal{D}$ be the set of all stopping times τ, with $\xi(t)$, $t < \tau$ belonging to Ξ. Next we need the following result, see Lemma III.B in [36].

Lemma 5.31. *Let $\mathcal{A}$ be a family of stopping times, each with values in $[0,\infty]$. Assume that $\tau := \sup\{\alpha : \alpha \in \mathcal{A}\}$ is > 0 a.s. Assume that for each $\alpha \in \mathcal{A}$, $I_\alpha : [0,\alpha) \times \Omega \to X$ is an adapted process. Assume that for any $t < \infty$ and any $\alpha, \beta \in \mathcal{A}$*

$$\mathbb{P}\left(\{\omega \in \Omega : t < \alpha(\omega) \wedge \beta(\omega) \ \text{ and } \ I_\alpha(t,\omega) \neq I_\beta(t,\omega)\}\right) = 0.$$

Then there exists an adapted process $I : [0,\tau) \times \Omega \to X$ such that for all $t < \infty$, $\alpha \in \mathcal{A}$

$$\mathbb{P}\left(\{\omega \in \Omega : t < \alpha(\omega),\ I_\alpha(t,\omega) \neq I(t,\omega)\}\right) = 0. \tag{5.37}$$

Moreover,

(i) *If $\tilde{I} : [0,\tau) \times \Omega \to X$ is any process satisfying* (5.37) *then the process $\tilde{I}$ is a version of the process I, i.e. for any $t \in [0,\infty)$*

$$\mathbb{P}\left(\left\{\omega \in \Omega : t < \tau(\omega),\ I(t,\omega) \neq \tilde{I}(t,\omega)\right\}\right) = 0.$$

(ii) *If each element of $\mathcal{A}$ is an accessible stopping time then also τ is an accessible stopping time.*

(iii) *If also each I_α is admissible, then I can be chosen to be admissible as well.*

Proof of Lemma 5.31. We simply repeat the arguments of the corresponding proof from [36].

□

Using the above result we find a stopping time $\hat{\tau} : \Omega \to [0,T]$ and an admissible X-valued process $\hat{\xi}(t)$, $t < \hat{\tau}$ such that for each $\tau \in \mathcal{D}$ a.s. on $\Omega_t(\tau)$,

$$\hat{\xi}(t) \ = \ \xi(t).$$

We need to prove that $\hat{\xi}(t)$, $t < \hat{\tau}$ is a solution to (5.25) with $\hat{\tau} > 0$ an accessible stopping time. We begin with the latter. For this let $\{B_n\}$ be a sequence of balls in X such that, if $B_n = B(x_n, r_n)$, $\bigcup_n (x_n, \frac{r_n}{2}) = X$ and the condition (5.32) is satisfied in each ball B_n. Let $\Omega_n := \{\xi_0 \in B(x_n, \frac{r_n}{2})\}$. Then $P(\bigcup_n \Omega_n) = 1$. If $\xi_n(t)$, $t < \tau_n$ is a local solution to (5.25) constructed as above (with $U = B_n$, $U_0 = B(x_n, \frac{r_n}{2})$ then $\hat{\tau} \geq \tau_n$. Since $\tau_n > 0$ a.s. on Ω_n if $P(\Omega_n) > 0$ it follows (as $P(\bigcup_n \Omega_n) = 1$) that $\hat{\tau} > 0$ a.s.

Now we shall prove that $\hat{\xi}(t)$, $t < \hat{\tau}$ is a solution to (5.25).

Denote

$$\eta(t) = \hat{\xi}(0) + \int_0^t f(s,\hat{\xi}(s))\,ds + \int_0^t g(s,\hat{\xi}(s))\,dw(s),\ \ 0 \le t < \hat{\tau}. \tag{5.38}$$

If a process $\xi(t)$, $t < \tau$ belongs to Ξ, then a process η defined by (5.38) by replacing $\hat{\xi}$ by ξ, is a version of $\xi(t)$, $0 \le t < \tau$. Indeed, $\xi(t)$, $0 \le t < \tau$ is a solution to (5.25). Therefore by the uniqueness part (i) of Lemma 5.31 we infer that $\hat{\eta}$, $0 \le t < \hat{\tau}$ is a version of $\hat{\xi}$, $0 \le t < \hat{\tau}$. This proves that $\xi(t)$, $0 \le t < \hat{\tau}$ is a solution to (5.25) and thus the proof of the first part of Theorem 5.27 is finished.

Our aim is to show that in the presence of the Linear Growth Condition $\tau = T$ a.s. By localization we may assume that ξ_0 is bounded. In particular, $\mathbb{E}|\xi_0|^p < \infty$ for all $p \ge 1$. For simplicity we assume that $f = 0$.

Lemma 5.32. *Suppose that a.s. on a measurable set* Ω_1,

$$\tau < T, \tag{5.39}$$

$$\limsup_{t \nearrow \tau} |\xi(t)| < \infty. \tag{5.40}$$

Then $P(\Omega_1) = 0$.

Proof of Lemma 5.32. By localization we may suppose that T is finite, a.s. on Ω_1, $\tau \le T$ and $|\xi(t)| \le R$, $0 \le t < \tau$ for some $R \in (0,\infty)$. We shall prove that this implies that $\xi(t)$ has a limit as $t \nearrow \tau$, what in view of Corollary 5.30 implies that $P(\Omega_1) = 0$. In what follows we suppose that $\Omega_1 = \Omega$. Denote

$$x(t) = 1_{\{t<\tau\}} g(t,\xi(t)),\ \ t \in [0,T].$$

Then, since $\|g(t,\xi(t))\|^2 \le K(1+R^2)$ (by the Linear Growth Condition) and $T < \infty$, $x \in \mathcal{M}^p(0,T;M(H,X))$ for any $p \ge 2$. Therefore the process $y(t)$, $0 \le t \le T$ defined by

$$y(t) = \int_0^t x(s)\,dw(s),\ t \in [0,T],$$

is a.s. continuous on $[0,T]$. On the other hand, $\xi(t) = y(t)$ a.s. on Ω_t. Recall that $\Omega_t = a_t = \{\omega \in \Omega : t < \tau(\omega)\}$. Hence, as ξ is an admissible process, a.s. on Ω, $\xi(t) = x(t)$ when $t < \tau$. The proof of the Lemma is complete.

It follows from Lemma 5.32 that for each $n \in \mathbb{N}$

$$\sigma_n := \inf\{t < \tau : |\xi(t)| > n\} \tag{5.41}$$

is a well defined finite stopping time. First we shall show that

$$\lim_{n\to\infty} \sigma_n = T. \tag{5.42}$$

Since $\xi(t)$ is a solution to (5.25),

$$\xi(t\wedge\sigma_n) = \xi_0 + \int_0^{t\wedge\sigma_n} g(s,\xi(s))\,dw(s) + \int_0^{t\wedge\sigma_n} f(s,\xi(s))\,ds,\ \ t \in [0,T). \tag{5.43}$$

From formula (5.15) in our paper and the Linear growth condition above we get

$$\begin{aligned}
\mathbb{E}|\xi(t\wedge\sigma_n)|^2 &\le 3\mathbb{E}|\xi_0|^2 + 3C_2\mathbb{E}\int_0^{t\wedge\sigma_n} |g(s,\xi(s))|^2\,ds + 3T\mathbb{E}\int_0^{t\wedge\sigma_n} |f(s,\xi(s))|^2\,ds \\
&\le 3\mathbb{E}|\xi_0|^2 + 3(C_2+T)K^2\mathbb{E}\int_0^{t\wedge\sigma_n} \left(1+|\xi(s)|^2\right)\,ds \\
&\le 3\mathbb{E}|\xi_0|^2 + 3(C_2+T)K^2\mathbb{E}\int_0^{t\wedge\sigma_n} \left(1+|\xi(s\wedge\sigma_n)|^2\right)\,ds \\
&\le 3\mathbb{E}|\xi_0|^2 + 3(C_2+T)K^2\int_0^{t} \left(1+\mathbb{E}|\xi(s\wedge\sigma_n)|^2\right)\,ds \\
&= 3\mathbb{E}|\xi_0|^2 + 3K^2T(C_2+T) + 3K^2(C_2+T)\mathbb{E}\int_0^t |\xi(s\wedge\sigma_n)|^2\,ds
\end{aligned}$$

Hence by the Gronwall Lemma, applied to function $\varphi(t) = \mathbb{E}|\xi(t\wedge\sigma_n)|^2$, $t\in[0,T)$, we infer that

$$\mathbb{E}|\xi(t\wedge\sigma_n)|^2 \le 3\left(\mathbb{E}|\xi_0|^2 + K^2T(C_2+T)\right)e^{3K^2(C_2+T)t},\ \ t\in[0,T). \tag{5.44}$$

Let us fix $t\in[0,T)$. Since

$$\mathbb{E}|\xi(t\wedge\sigma_n)|^2 = \int_{\{t>\sigma_n\}} |\xi(\sigma_n)|^2 + \int_{\{t\le\sigma_n\}} |\xi(t)|^2 \tag{5.45}$$

and $|\xi(\sigma_n)| = n$ it follows that

$$n^2\mathbb{P}\{\sigma_n < t\} \le C$$

for all $n\in\mathbb{N}$ and some constant $C>0$. Therefore, $\mathbb{P}\{\sigma_n<t\}\to 0$ as $n\to\infty$ and hence $\sigma_n \nearrow T$ as $n\to\infty$.

Since $\sigma_n\le\tau$ we infer that $\tau = T$ as required.

Next we will use (5.44) again. For a fixed t we take $n\to\infty$. Then from (5.44) we see that (5.33) holds.

Uniqueness. We return to the consideration from the beginning of the *Existence* part of the proof. Thus, U_0 and U are be two open and bounded subsets of X such that $U_0\subset\bar{U_0}\subseteq U$ and both f and g satisfy the condition (5.32) in U. Moreover, both $f_U : X\to X$ and $g_U : [0,T)\times X\to\mathcal{L}(E,X)$ satisfy the conditions of Theorem 5.19 and both coincide on $[0,T)\times U_0$ with f and g respectively. Let $\xi(t)$, $t<\tau$ be a local solution to problem (5.25). Let τ_U be the exit time of the process $\xi(t)$ from U_0. Then $\xi(t)$, $t<\tau\wedge\tau_U$ is also a local solution to problem (5.25). By the properties of f_U and g_U we infer that $\xi(t)$, $t<\tau\wedge\tau_U$ is also a local solution to problem (5.36). Since the latter problem has a unique global solution, we infer that the processes $\xi(t)$, $t<\tau\wedge\tau_U$ and $\xi_U(t)$, $t<\tau\wedge\tau_U$ are equivalent. This proves the following

Lemma 5.33. *Let $\xi_1(t)$, $t<\tau_1$ and $\xi_2(t)$, $t<\tau_2$ be two local solutions to problem* (5.25)*. Set $\tau=\tau_1\wedge\tau_2$. Then there exists a positive accessible stopping time σ such that the processes $\xi_1(t)$, $t<\tau\wedge\sigma$ and $\xi_2(t)$, $t<\tau\wedge\sigma$ are equivalent.*

Let now ξ_1, ξ_2 and τ be as in Lemma 5.33. We need to show that the processes $\xi_1(t)$, $t<\tau$ and $\xi_2(t)$, $t<\tau$ are equivalent. Define $\sigma := \inf\{t\in[0,\tau) : \xi_1(t)\ne\xi_2(t)\}$. Since the diagonal is a closed subset of X^2, σ is an accessible stopping time. We need to show that $\sigma = \tau$ a.s. Suppose the set $\hat{\Omega} = \{\sigma < \tau\}$ has positive measure. Since obviously $\hat{\Omega} \in \mathcal{F}_\sigma$ and $\xi_1(\sigma) = \xi_2(\sigma)$ a.s. on $\hat{\Omega}$ are $\mathcal{F}_\sigma$ measurable, by employing a random initial value modification of Lemma 5.33, we find out that there exist a stopping time $\hat{\sigma}$ such that $\hat{\sigma} > \sigma$ a.s. on $\hat{\Omega}$ and the processes $\xi_1(t)$, $t < \tau \wedge \hat{\sigma}$ and $\xi_2(t)$, $t < \tau \wedge \hat{\sigma}$ are equivalent. This contradicts the definition of σ and thus the uniqueness part of the proof of Theorem 5.27 is also completed.

■

Corollary 5.34. *In addition to the assumptions of Theorem 5.27 let us assume that the coefficients a and b do not depend on time, i.e.*

$$a : X \to X, \quad b : X \to M(H, X)$$

are such that for any $R >$ there exists $C = C_R > 0$ such that

$$\max\{|a(x_2) - a(x_1)|_X, |b(x_2) - b(x_1)|_{M(H,X)}\} \leq C_R|x_2 - x_1|, \quad \textit{for } |x_1|, |x_2| \leq R. \tag{5.46}$$

Then the maximal solution $\xi(t)$, $t < \tau$ to the following homogeneous in time version of the problem (5.25)

$$\begin{cases} d\xi(t) &= f(\xi(t))\,dt + g(\xi(t))\,dw(t), \quad t \geq 0 \\ \xi(0) &= \xi_0 \end{cases} \tag{5.47}$$

satisfies a.s. on $\{\tau < \infty\}$,

$$|\xi(t)| \to \infty \textit{ as } t \nearrow \tau. \tag{5.48}$$

Proof of Corollary 5.34. Since both f and g are uniformly Lipschitz continuous on every ball in X, the proof is completely analogous to the finite dimensional case, see e.g. the proof of Theorem 3.4.5 in [48].

■

Remark 5.35. It follows from the proof of Theorem 5.27 that if $c < \infty$ and the linear growth condition holds, the maximal solution $\xi(t)$, $t \in [0, T)$ has a.s. a limit as $t \nearrow T$.

The following result has already been used in [12], see Proposition 4.6 therein. We use notation from Theorem 5.19

Proposition 5.36. *Suppose that $\alpha(t)$ and $\beta(t)$ are two progressively measurable, locally bounded $\mathcal{L}(X; M(H; X))$ and respectively $M(H; X)$–valued stochastic processes. Assume that v_0 is an $\mathcal{F}_0$ measurable, X–valued random variable. Assume finally that for some fixed $q \geq 2$ and $T > 0$*

$$\mathbb{E}|v_0|^q < \infty, \tag{5.49}$$

$$\mathbb{E}\int_0^T \|\beta(t)\|^q\,dt < \infty. \tag{5.50}$$

Suppose that an admissible X–valued process $v(t)$, $t \in [0, T]$, is a solution to the following problem

$$\begin{aligned} dv(t) &= \alpha(t)v(t)\,dw(t) + \beta(t)\,dw(t), \ t \geq 0 \\ v(0) &= v_0. \end{aligned} \tag{5.51}$$

If $n > 0$ and $\tau_n := \inf\{t \in (0,T] : |\alpha(t)| > n\}$, then

$$\mathbb{E} \sup_{0\le t\le T} |v(t\wedge\tau_n)|^q \le 3^{q-1}\left(\mathbb{E}|v_0|^q + C_q\mathbb{E}(\int_0^{T\wedge\tau_n} \|\beta(t)\|^2\,dt)^{q/2}\right) e^{3^{q-1}C_q n^q T^{\frac{q}{2}-1}}. \tag{5.52}$$

Remark 5.37. An admissible X–valued process $v(t)$, $t \in [0,T]$, is a solution to the problem (5.51-5.52) iff for every $n \in \mathbb{N}$ and all $t \in [0,T]$,

$$v(t\wedge\tau_n) = v(0) + \int_0^{t\wedge\tau_n} (\alpha(s)v(s) + \beta(s))\,dw(s), \ \text{ a.s..}$$

Under the conditions of Proposition 5.36 one can show existence of a unique solution to the problem (5.51-5.52).

Proof. As in the proof of Lemma 5.32 we apply the Gronwall Lemma to the L^q-norm of the stopped process $v(t\wedge\tau_n)$, $t\in[0,T]$. We make an essential use of the Burkholder inequality (5.13).

□

Remark 5.38. The main differences between our treatment and that given in [5] are first that Belopolskaja and Daletskii considered Banach spaces with strong differentiability conditions on their norms. Whereas we use the more general class of M-type 2 spaces (we need to do this since the $W^{\vartheta,p}$ spaces we need to use in section **??** below are M-type 2 but do not have an equivalent norm satisfying the conditions of [5]). In fact a Banach space with norm twice differential with both derivatives bounded is a Hilbert Space, see Fact 1.0 p. 184 in [27], so caution is needed in such differential assumptions. Secondly we use Stratonovich calculus and our noise for Stratonovich equations has to be a genuine Wiener process rather than a cylindrical process as used in [5], or for Itô equations in, for example, Theorem 5.21 above. In other words our noise coefficients b takes values in $\mathcal{L}(E,X)$ rather than $M(H,X)$. This is discussed in more detail in the Appendix below.

6. Stochastic Integration in M-type 2 Banach Spaces

The following definition is fundamental for our work.

Definition 6.1. *A Banach space X is called M-type 2 if and only if there exists a constant $C(X) > 0$ such that for any X-valued martingale $\{M_k\}$ the following inequality holds*

$$\sup_k \mathbb{E}\big[|M_k|^2\big] \le C(X)\sum_k \mathbb{E}\big[|M_k - M_{k-1}|^2\big]. \tag{6.1}$$

Any Hilbert space is an M-type 2 Banach space. In such a case we then have equality in (6.1) with $C(X) = 1$. The Lebesgue Function spaces L^p, $p > 2$, are examples of M-type 2 Banach spaces which are not Hilbert spaces.

The theory of stochastic integration in infinite dimensional Hilbert spaces has been developed and is well understood. However, for general separable Banach spaces there are difficulties in defining a meaningful Itô integral. In an unpublished thesis by Neidhardt, [51], a theory of stochastic integration was developed for a certain class of Banach spaces known as 2-uniformly smooth Banach spaces. A Banach space X is said to be 2-uniformly smooth if and only if for each x, $y \in X$

$$\frac{1}{2}\big(|x+y|_X^2 + |x-y|_X^2\big) \le |x|_X^2 + A|y|_X^2, \tag{6.2}$$

for some constant $A > 0$. If X is a Hilbert space then equality holds in (6.2) with $A = 1$, i.e., the norm $|\,.\,|_X$ satisfies the parallelogram law. Independently of Neidhardt, similar work on stochastic integrals was carried out by Dettweiler, see [83] and references therein. It is known, see [106], that a Banach space is 2-uniformly smooth if and only if it is M-type 2. Either of the above two inequalities make it possible to define a meaningful Itô integral for this class of Banach spaces. However, the M-type 2 inequality (6.1) will prove to be the most useful for our needs. We briefly outline the construction of the Itô integral in M-type 2 Banach spaces and refer the reader to [73] and [77] for a more detailed summary and further references.

Definition 6.2. *For separable Hilbert and Banach spaces H and X we set*

$$R(H,X) := \{T : H \to X : T \in L(H,X) \text{ and } T \text{ is } \gamma\text{-radonifying}\}, \tag{6.3}$$

where $L(H,X)$ denotes the Banach space of bounded linear operators between H and X. By T being γ-radonifying we mean that the image $T(\gamma_H) := \gamma_H \circ T^{-1}$ of the canonical finitely additive Gaussian measure γ_H on H is σ-additive on the algebra of cylindrical sets in X.

Remark 6.3. The algebra of cylindrical sets in X generates the Borel σ-algebra, $\mathcal{B}(X)$ on X, see [42]. Thus $T(\gamma_H)$ extends to a Borel measure on $\mathcal{B}(X)$ which we denote by ν_T. In particular, ν_T is a Gaussian measure on $\mathcal{B}(X)$, i.e., for each $\lambda \in X^*$ (the dual of X), the image measure $\lambda(\nu_T)$ is a Gaussian measure on $\mathcal{B}(\mathbb{R})$. The covariance operator of ν_T equals $TT^* : E^* \to E$.

For $T \in R(H,X)$ we put

$$\|T\|_{R(H,X)}^2 := \int_X |x|^2 \,\mathrm{d}\nu_T(x). \tag{6.4}$$

As ν_T is Gaussian, then by the Fernique–Landau–Shepp Theorem, see [42], $\|T\|_{R(H,X)}$ is finite. Furthermore, see [51], $R(H,X)$ is a separable Banach space endowed with the norm (6.4).

Definition 6.4. *Let E be a separable Banach space. We say that $i : H \hookrightarrow E$ is an Abstract Wiener Space, AWS, if and only if i is a linear, one-to-one map and $i \in R(H,E)$. If $i : H \hookrightarrow E$ is an AWS, then the Gaussian measure ν_i on E will be denoted by μ and called the canonical Gaussian measure on E.*

Remark 6.5. Many authors require $i(H)$ to be dense in E in the definition of an AWS. This is an unnecessary restriction for us. In fact, Sato, [107], proved that given a separable Banach space with Gaussian measure μ, then there always exists a Hilbert subspace $H \subset E$ such that $i : H \hookrightarrow E$ is an AWS, with $\mu = \nu_i$, where i is the inclusion mapping. The imbedding i is not dense in general.

Remark 6.6. The Hilbert space H appearing in the above definition is often referred to as the reproducing kernel Hilbert space, RKHS, of (E,μ).

Suppose that a triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and let $i : H \hookrightarrow E$ be an AWS. Let $w(t)$, $t \geq 0$, denote the corresponding E-valued Wiener process, i.e., a continuous process on E such that:

(i) $w(0) = 0$ a.s.;
(ii) the law of the random function $t^{-1/2}w(t) : \Omega \to E$ equals μ, for any $t > 0$;
(iii) if $\mathcal{F}_s$ is the σ-algebra generated by $w(r)$, $r \in [0, s]$, then $w(t) - w(s)$ is independent of $\mathcal{F}_s$ for any $t \geq s \geq 0$.

Remark 6.7. In view of (ii) it is not difficult to show that for $p \geq 0$,

$$m_p := \mathbb{E}\left[\left|\frac{w(t) - w(s)}{(t - s)^{1/2}}\right|_E^p\right] = \int_E |z|_E^p \, \mathrm{d}\mu(z). \tag{6.5}$$

Furthermore by the Fernique–Landau–Shepp Theorem, see [42], $m_p < \infty$ for each $p \geq 0$.

Let X be an M-type 2 Banach space and $T \in (0, \infty)$. For $p \geq 1$, let $M^p(0, T; L(E, X))$ be the space of (equivalence classes of) progressively measurable functions $\xi : [0, T] \times \Omega \to L(E, X)$ which satisfy

$$\mathbb{E}\left[\int_0^T |\xi(t)|_{L(E,X)}^p \, \mathrm{d}t\right] < \infty$$

(with an analogous definition for the space $M^p(0, T; R(H, X))$).

Let $M^p_{\text{step}}(0, T; L(E, X))$ be the subspace of those $\xi \in M^p(0, T; L(E, X))$ for which there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ such that $\xi(t) = \xi(t_k)$ for $t \in [t_k, t_{k+1})$, $0 \leq k \leq n - 1$, $k \in \mathbb{N}$.

For $\xi \in M^2_{\text{step}}(0, T; L(E, X))$ define a measurable map $I(\xi) : \Omega \to X$ by

$$I(\xi) := \sum_{j=1}^{n-1} \xi(t_k)\big(w(t_{k+1}) - w(t_k)\big). \tag{6.6}$$

The following lemma is crucial for the successful construction of the Itô integral.

Lemma 6.8. *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, X is an M-type 2 Banach space and $T \in (0, \infty)$. Then for $\xi \in M^p_{\text{step}}(0, T; L(E, X))$, $I(\xi) \in L^2(\Omega; X)$, $\mathbb{E}[I(\xi)] = 0$ and*

$$\mathbb{E}\big[|I(\xi)|_X^2\big] \leq C \int_0^T \mathbb{E}\big[\|\xi(t) \circ i\|_{R(H,X)}^2\big] \, \mathrm{d}t. \tag{6.7}$$

Remark 6.9. Lemma 6.8 may be proved using either the inequality (6.1) or the inequality (6.2), along with the fact that $L(E, X)$ is contained in $R(H, X)$ via the continuous map

$$L(E, X) \ni \xi \longmapsto \xi \circ i \in R(H, X).$$

Remark 6.10. In the case when X is a Hilbert space (6.7) reads

$$\mathbb{E}\big[|I(\xi)|_X^2\big] = \mathbb{E}\left[\int_0^T \|\xi(t) \circ i\|_{R(H,X)} \, \mathrm{d}t\right],$$

which, of course, is the well-known Itô Isometry. The existence of the Itô Isometry is due to the 'nice' geometrical properties of the Hilbert space, i.e., the existence of an inner product. In general Banach spaces we lose the notion of 'geometry' and this is where the difficulty lies when one wishes to construct an Itô Integral. Although we do not have the Itô Isometry, the inequality (6.7) is enough to ensure that we can control the 'size' of the random variable $I(\xi)$ given by (6.6).

The fundamental property of the map I is that it extends uniquely to a bounded linear map from $M^2(0,T;R(H,X))$ into $L^2(\Omega;X)$. This is a consequence of (6.7) and the fact, proven in [51], that $M^2_{\rm step}(0,T;L(E,X))$ is dense in $M^2(0,T;R(H,X))$. For $\xi \in M^2(0,T;R(H,X))$, the value of this extension will be denoted by $\int_0^T \xi(t)\,\mathrm{d}w(t)$. Furthermore, we have

Theorem 6.11. *Suppose $i : H \to E$ is an AWS with corresponding E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Assume that for $T > 0$, $\xi \in M^2(0,T;R(H,X))$ and let $I(r) := \int_0^r \xi(t)\,\mathrm{d}w(t)$ for $r > 0$. Then, $I(r)$ is a continuous X-valued martingale and for any $p \in (1,\infty)$ there exists a constant $C_p > 0$, independent of T and ξ, such that*

$$\mathbb{E}\Big[\sup_{0\le r\le T} |I(r)|_X^p\Big] \le C_p\Big(\int_0^T \mathbb{E}\big[\|\xi(t)\|^2_{R(H,X)}\big]\,\mathrm{d}t\Big)^{p/2}. \tag{6.8}$$

The inequality (6.8) is the Burkholder inequality. The case $p = 2$ was proved in [51] and later, using the M-type 2 inequality, was proved in [83] for any $p \in (1,\infty)$.

Remark 6.12. In the above we may replace $R(H,X)$ by $L(E,X)$, see Remark 6.9. In particular, $\int_0^T \xi(t)\,\mathrm{d}w(t)$ exists for any $\xi \in M^2(0,T;L(E,X))$ and satisfies

$$\mathbb{E}\Big[\sup_{0\le r\le T}\Big|\int_0^r \xi(t)\,\mathrm{d}w(t)\Big|_X^p\Big] \le C_p\Big(\int_0^T \mathbb{E}|\xi(t)|^2_{L(E,X)}\,\mathrm{d}t\Big)^{p/2}. \tag{6.9}$$

For suitable maps $f : X \to X$ and $g : X \to R(H,X)$ we consider the following problem

$$\begin{cases} \mathrm{d}\xi(t) &= f\big(\xi(t)\big)\,\mathrm{d}t, + g\big(\xi(t)\big)\,\mathrm{d}w(t) \\ \xi(0) &= \xi_0, \end{cases} \tag{6.10}$$

where $\xi_0 : \Omega \to X$ is $\mathcal{F}_0$-measurable. A continuous and adapted process $\xi : [0,T]\times\Omega \to X$ is said to be a solution to the Itô equation (6.10) if and only if for all $t \in [0,T]$

$$\xi(t) = \xi(0) + \int_0^t f\big(\xi(r)\big)\,\mathrm{d}r + \int_0^t g\big(\xi(r)\big)\,\mathrm{d}w(r) \qquad \text{a.s.} \tag{6.11}$$

We have the following existence and uniqueness theorem (see Theorem 2.26 in [73], where only the case $p = 2$ was studied; however, the proof carries over to any $p \in [1,\infty)$ without any substantial difference).

Theorem 6.13. *Assume that $i : H \hookrightarrow E$ is an AWS, $\{w(t)\}_{t\geq 0}$ the corresponding Wiener process on E and X is an M-type 2 Banach space. Let $T > 0$ be fixed. Suppose the maps $f : X \to X$ and $g : X \to R(H,X)$ satisfy the following linear growth and Lipschitz conditions:*

(i) *(Linear Growth Condition) there exists $K > 0$ such that for each $x \in X$*

$$\max\big\{|f(x)|_X, \|g(x)\|_{R(H,X)}\big\} \le K(1 + |x|_X);$$

(ii) *(local Lipschitz continuity) for any $x_0 \in X$ there exists an $r_0 > 0$ and $L_0 > 0$ such that for any $x, y \in \bar{B}(x_0, r_0) := \{x \in X : |x - x_0| \le r_0\}$*

$$\max\big\{|f(x) - f(y)|_X, \|g(x) - g(y)\|_{R(H,X)}\big\} \le L_0|x - y|_X.$$

Let $p \ge 1$ and $\xi_0 : \Omega \to X$ be $\mathcal{F}_0$-measurable such that $\mathbb{E}[|\xi_0|_X^p] < \infty$.

Then there exists a unique $\xi \in M^p(0, T; X)$ which is the solution to the problem (6.10)*. Moreover, the following estimate holds:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\xi(t)|_X^p\right] \le C_p\big(\mathbb{E}\big[|\xi_0|_X^p\big] + T^p\big). \tag{6.12}$$

Remark 6.14. One should point out that the local Lipschitz condition (ii) above is weaker than the usual one:

(ii′) (Lipschitz continuity on balls) for any $R > 0$ there exists $C_R > 0$ such that

$$\max\big\{|f(x) - f(y)|_X, \|g(x) - g(y)\|_{R(H,X)}\big\} \le C_R|x - y|_X$$

for all $x, y \in X$ with $|x|_X, |y|_X \le R$.

The condition (ii) is more suitable for studying equations on Banach manifolds. Both conditions are equivalent if $\dim X < \infty$.

So far we have introduced the Itô integral and defined what we mean by a solution to an Itô equation. We now turn to Stratonovitch integrals and Stratonovitch equations. We first need to introduce some notation. By $L_2(E; X)$ we denote the space of bounded bilinear maps, $\Lambda : E \times E \to X$. Let $i : H \to E$ be an AWS. We define the map $\mathrm{tr} : L_2(E; X) \to X$ by

$$\mathrm{tr}\,\Lambda := \int_E \Lambda(e, e)\,\mathrm{d}\mu(e), \tag{6.13}$$

where μ is the canonical Gaussian measure on E. In view of the Fernique–Landau–Shepp Theorem, tr is a bounded linear map. Note that the tr map depends on the choice of AWS.

The following two definitions are taken from [73].

Definition 6.15. *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \ge 0$, and X is an M-type 2 Banach space. Let $T \in (0, \infty)$ and $\xi(t)$, $t \in [0, T]$ be a stochastic process such that for any $t \ge 0$*

$$\xi(t) = \xi(0) + \int_0^t a(r)\,\mathrm{d}r + \int_0^t b(r)\,\mathrm{d}w(r) \qquad a.s.,$$

where $a \in M^1(0, T; X)$ and $b \in M^2(0, T; L(E, X))$. For a C^1 map $g : X \to L(E, X)$ we define the Stratonovitch Integral of $g(\xi(t))$ as

$$\int_0^t g\big(\xi(r)\big) \circ \mathrm{d}w(r) := \int_0^t g\big(\xi(r)\big)\,\mathrm{d}w(r) + \frac{1}{2}\int_0^t \mathrm{tr}\big[g'\big(\xi(r)\big)\,b(r)\big]\,\mathrm{d}r. \tag{6.14}$$

Remark 6.16. By a C^1 map we mean that $g : X \to L(E, X)$ is Fréchet differentiable with continuous Fréchet derivative $g' : X \to L(X, L(E, X))$. Furthermore, note that

$$g'\big(\xi(r)\big)\, b(r) \in L\big(E, L(E, X)\big) \simeq L_2(E; X)$$

so that $\mathrm{tr}[g'(\xi(r))\, b(r)]$ appearing in (6.14) is well defined.

Remark 6.17. In the definition of the Stratonovitch Integral, it is not accidental that we have chosen

$$b \in M^2\big(0, T; L(E, X)\big) \qquad \text{rather than} \qquad b \in M^2\big(0, T; R(H, X)\big).$$

For a discussion why one needs to consider processes in $M^2(0, T; L(E, X))$ and not in the larger space $M^2(0, T; R(H, X))$, see [73], Appendix A.

Definition 6.18. *Suppose $i : H \to E$ is an AWS with canonical E-valued Wiener process $w(t)$, $t \geq 0$, and X is an M-type 2 Banach space. Let $T \in (0, \infty)$. Let g be as above and let $f : X \to X$ be a continuous function. We say that an adapted and continuous X-valued process $\xi(t)$, $t \in [0, T]$, is a solution to the Stratonovitch equation*

$$\mathrm{d}\xi(t) = f\big(\xi(t)\big)\,\mathrm{d}t + g\big(\xi(t)\big) \circ \mathrm{d}w(t) \tag{6.15}$$

if and only if it is a solution to the Itô equation

$$\mathrm{d}\xi(t) = \Big(f\big(\xi(t)\big) + \frac{1}{2}\,\mathrm{tr}\big[g'\big(\xi(t)\big)\, g\big(\xi(t)\big)\big]\Big)\,\mathrm{d}t + g\big(\xi(t)\big)\,\mathrm{d}w(t). \tag{6.16}$$

Thus $\xi(t)$ is a solution to (6.15) *if and only if it satisfies for each $t \geq 0$*

$$\begin{aligned}\xi(t) = \xi(0) + \int_0^t f\big(\xi(r)\big)\,\mathrm{d}r \\ + \frac{1}{2}\int_0^t \mathrm{tr}\big[g'\big(\xi(r)\big)\, g\big(\xi(r)\big)\big]\,\mathrm{d}r + \int_0^t g\big(\xi(r)\big)\,\mathrm{d}w(r) \qquad a.s.\end{aligned} \tag{6.17}$$

6.1. **From Brz+Elw paper.** It follows from the discussion in Remark 6.16 that $\mathrm{tr}(g'(\xi(t))g(\xi(t)))$ is an element of the Banach space X.

Remark 6.19. If $g : X \to \mathcal{L}(E, Y)$, $g'(x)g(x)$ is meaningless unless $Y \subset X$.

We have introduced two different notions: Stratonovich integral and a solution to a stochastic Stratonovich equation. They have been defined independently although one can suspect they are intimately related. The following result shows that it is the case indeed.

Theorem 6.20. *Assume that the functions $f : X \to X$ and $g : X \to \mathcal{L}(E, X)$ are, respectively, continuous and continuously differentiable. Let a function $\varphi : X \to Y$ be of $\mathcal{C}^2$ class. Assume that an admissible X-valued process $\xi(t)$, $0 \leq t < \tau$ is a local solution to* (6.14)*. Then the process $\eta(t) = \varphi(\xi(t))$, $t < \tau$ is representable via a Stratonovich equation as below*

$$\eta(t) - \eta_0 \;=\; \int_0^t \varphi'(\xi(s))f(\xi(s))\,ds + \int_0^t \varphi'(\xi(s))g(\xi(s)) \circ dw(s). \tag{6.18}$$

Here, using the notation (6.20) below, the last term in (6.18) *is the Stratonovich integral, i.e.*

$$\begin{aligned}\int_0^t \varphi'(\xi(s))g(\xi(s)) \circ dw(s) &= \int_0^t g(\xi(s)) \circ dw(s) \\ &:= \int_0^t G(\xi(s))\, dw(s) + \frac{1}{2}\int_0^t \mathrm{tr}[G'(\xi(s))g(\xi(s))]\, ds.\end{aligned}$$

Taking $X = Y$ and $\varphi = \mathrm{id}$ we get

Corollary 6.21. *If* $\xi(t)$*,* $t < \tau$ *is a local solution to the Stratonovich equation* (6.15) *then*

$$\xi(t \wedge \sigma) - \xi(0) = \int_0^{t\wedge\sigma} f(\xi(s))\, ds + \int_0^{t\wedge\sigma} g(\xi(s)) \circ dw(s)$$

for any $t \in [0, \infty)$ *and for any stopping time* $\sigma < \tau$*, where the last term is the Stratonovich integral*

$$\int_0^{t\wedge\sigma} g(\xi(s)) \circ dw(s) := \int_0^{t\wedge\sigma} G(\xi(s))\, dw(s) + \frac{1}{2}\int_0^{t\wedge\sigma} \mathrm{tr}\,[g'(\xi(s))g(\xi(s))]\, ds.$$

Proof of Theorem 6.20. Since $\xi(t)$, $t < \tau$ is a (local) solution to the Stratonovich equation (6.16) by using the Itô formula (Theorem 5.17) we see that $\eta(t)$, $t < \tau$ is a (local) solution to the Itô equation

$$\begin{aligned} d\eta(t) &= \varphi'(\xi(t)\Big[f(\xi(t)) + \frac{1}{2}\mathrm{tr}\big[g'(\xi(t))g(\xi(t))\big]\Big]\, dt + \varphi'(\xi(t))g(\xi(t))\, dw(t) \\ &+ \frac{1}{2}\mathrm{tr}[\varphi''(\xi(t))(g(\xi(t)), g(\xi(t)))]\, dt \end{aligned} \tag{6.19}$$

Having in mind that $\varphi'(x) \in \mathcal{L}(X, Y)$ for $x \in X$, we set

$$\begin{aligned} G &: X \ni x \mapsto \varphi'(x)g(x) \in \mathcal{L}(E, Y), \\ F &: X \ni x \mapsto \varphi'(x)f(x) \in Y. \end{aligned} \tag{6.20}$$

Then,

$$G'(x) = \varphi'(x) \circ (g'(x)\cdot) + (\varphi''(x)\cdot) \circ g(x) \in \mathcal{L}(X, \mathcal{L}(E, Y)), \qquad x \in X$$

and so identifying $\mathcal{L}(E, \mathcal{L}(E, Y))$ with $\mathbb{L}(E, E; Y)$,

$$G'(x)g(x) \in \mathbb{L}(E, E; Y), \qquad x \in X.$$

Since also

$$\begin{aligned} G'(x)g(x) &= \varphi'(x) \circ (g'(x)g(x)) + (\varphi''(x)g(x)) \circ g(x) \\ &= \varphi'(x) \circ (g'(x)g(x)) + \varphi''(x)(g(x), g(x)). \end{aligned}$$

we find out that

$$\mathrm{tr}\,\{\varphi'(x)[g'(x)] + \varphi''(x)(g(x), g(x))\} = \mathrm{tr}[G'(x)g(x)], \qquad x \in X. \tag{6.21}$$

From (6.19), (6.20) and (6.21), by taking into account Definition 6.15 we infer that $\eta(t)$ satisfies

$$\begin{aligned} d\eta(t) &= F(\xi(t))\,dt + G(\xi(t))\,dw(t) + \frac{1}{2}\mathrm{tr}[G'(\xi(t))g(\xi(t))]\,dt \\ &= F(\xi(t))\,dt + G(\xi(t)) \circ dw(t). \end{aligned} \tag{6.22}$$

∎

We have the following immediate and important

Corollary 6.22. *Assume in addition that $\varphi : X \to Y$ is a diffeomorphism of class $\mathcal{C}^2$. Then an admissible X-valued process $\xi(t)$, $t < \tau$ is a local solution to* (6.15) *iff the process $\eta(t) := \varphi(\xi(t))$, $t < \tau$ is a local solution to*

$$d\eta(t) = \hat{f}(\eta(t))\,dt + \hat{g}(\eta(t)) \circ dw(t), \tag{6.23}$$

where

$$\begin{aligned} \hat{f} :& \quad Y \ni y \mapsto \varphi'(\varphi^{-1}y)f(\varphi^{-1}y) \in Y, \\ \hat{g} :& \quad Y \ni y \mapsto \varphi'(\varphi^{-1}y)g(\varphi^{-1}y) \in \mathcal{L}(E, Y). \end{aligned} \tag{6.24}$$

Proof of Corollary 6.22. It's enough to consider only the "only if" part of the Corollary. With G, F defined in (6.20) we have

$$\hat{f} = H \circ \varphi^{-1}, \qquad \hat{g} = F \circ \varphi^{-1}. \tag{6.25}$$

Therefore (6.23) follows from (6.18). ∎

We will find useful the following

Lemma 6.23. *In the framework from Corollary 6.22*

$$\mathrm{tr}[G'(x)g(x)] = \mathrm{tr}[\hat{g}'(y)\hat{g}(y)], \; y = \varphi(x) \in Y. \tag{6.26}$$

Proof of Lemma 6.23. Since for $x \in X$ $\hat{g}'(y)G'(\varphi^{-1}y) \circ d_y\varphi^{-1}$ and $d_y\varphi^{-1} \circ \varphi'(\varphi^{-1}y) = id_X$, we have, for $y = \varphi(x)$,

$$\hat{g}'(y)\hat{g}(y) = G'(\varphi^{-1}y) \circ d_y\varphi^{-1} \circ \varphi'(\varphi^{-1}y)G'(\varphi^{-1}y)g(\varphi^{-1}y).$$

This proves the Lemma. ∎

We have the following variation on Theorem 6.20.

Theorem 6.24. *Assume that a, b satisfy* (**??**) *and ξ satisfies* (**??**)*. Let Z be an M-type* 2 *Banach space and $\varphi : X \to Z$ and $g : Z \to \mathcal{L}(E, Y)$ be respectively functions of $\mathcal{C}^2$ and $\mathcal{C}^1$ class. Put $\eta(t) = \varphi(\xi(t))$. Then the Stratonovich integrals $\int_0^t g(\eta(s)) \circ dw(s)$, $t < \tau$ and $\int_0^t (g \circ \varphi)(\xi(s)) \circ dw(s)$, $t < \tau$ are equivalent processes.*

Proof of Theorem 6.24. As in the proof of Theorem 6.20 $\eta(t)$ satisfies

$$d\eta(t) = \left[\varphi'(\xi(t)) + \frac{1}{2}\mathrm{tr}[\varphi''(\xi(t))(b(t), b(t))]\right] dt + \varphi'(\xi(t))b(t)\, dw(t).$$

Therefore, by Definition 6.15

$$\int_0^t g(\eta(s)) \circ dw(s) = \int_0^t g(\eta(s))\, dw(s) + \frac{1}{2}\int_0^t \mathrm{tr}[g'(\eta(s))\varphi'(\xi(s))b(s)]\, ds,$$

in the sense that the processes on the LHS is a version of RHS. On the other hand, $g \circ \varphi : X \to \mathcal{L}(E, Y)$ is of $\mathcal{C}^1$ class and so by Definition 6.15

$$\int_0^t (g \circ \varphi)(\xi(s)) \circ dw(s) = \int_0^t (g \circ \varphi)(\xi(s))\, dw(s) + \frac{1}{2}\int_0^t \mathrm{tr}[(g \circ \varphi)'(\xi(s))b(s)]\, ds,$$

in the sense, as before, that the processes on the LHS and RHS are equivalent. Since $(g \circ \varphi)(\xi(s)) = g(\eta(s))$ and $(g \circ \varphi)'(\xi(s)) = g'(\eta(s))\varphi'(\xi(s))$ the proof is complete. ∎

It is obvious how to reformulate Definition 6.15, Theorem 6.24 and Corollary 6.22 in terms of processes defined up to a stopping times and local diffeomorphisms.

7. Approximations of SDEs with Lipschitz and bounded coefficients

Let X be an M-type 2 Banach space and $i : H \hookrightarrow E$ an AWS with corresponding E-valued Wiener process $w(t)$, $t \geq 0$. We impose the following conditions on the coefficients f and g.

- (**A1**) $f : X \to X$ is a C^1-map which is Lipschitz and bounded.
- (**B1**) $g : X \to L(E, X)$ is a C^1 map such that the maps g and g' are Lipschitz and bounded.

We should point out that as a consequence of (**B1**), the map $\mathrm{tr}(g'g) : X \to X$ is Lipschitz and bounded, where $\mathrm{tr}(g'g)(x) := \mathrm{tr}[g'(x)\, g(x)]$, $x \in X$, see (6.13). Let $x_0 \in L^p(\Omega, X)$, $p \geq 2$ and $T > 0$, be fixed but arbitrary. In view of Theorem 6.13 there exists a unique continuous progressively measurable process $x : [0, T] \times \Omega \to X$ such that for each $t \in [0, T]$,

$$x(t) = x(0) + \int_0^t f\big(x(r)\big)\, \mathrm{d}r + \frac{1}{2}\int_0^t \mathrm{tr}\big[g'\big(x(r)\big)\, g\big(x(r)\big)\big]\, \mathrm{d}r + \int_0^t g\big(x(r)\big)\, \mathrm{d}w(r), \qquad \text{a.s.} \tag{7.1}$$

Moreover, we have the estimate

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |x(t)|_X^p\right] \leq C_p\big(\mathbb{E}\big[|x_0|^p\big] + T^p\big). \tag{7.2}$$

Note x is a solution to the Stratonovitch equation

$$\mathrm{d}x(t) = f\big(x(t)\big) + g\big(x(t)\big) \circ \mathrm{d}w(t) \tag{7.3}$$

and x may be written as

$$x(t) = x(0) + \int_0^t f\big(x(r)\big)\,\mathrm{d}r + \int_0^t g\big(x(r)\big) \circ \mathrm{d}w(r), \tag{7.4}$$

where the last integral on the RHS of (7.4) is the Stratonovitch integral.

For each $n \in \mathbb{N}$, let π_n be a partition of $[0,T]$, i.e.,

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N(n)} = T.$$

We assume that each partition satisfies

$$\operatorname{mesh} \pi_n := \max_{0 \le k \le N(n)-1} |t_{k+1} - t_k| \le \frac{C_1}{n}, \tag{7.5}$$

$$N(n) \le C_2 n, \tag{7.6}$$

where C_1 and C_2 are constants independent of n. For each partition π_n, $n \in \mathbb{N}$, we consider the following piece-wise linear approximation of the E-valued Wiener process $w(t)$:

$$w_{\pi_n}(t) = w(t_i) + \frac{t - t_i}{t_{i+1} - t_i}\big(w(t_{i+1}) - w(t_i)\big), \quad t \in [t_i, t_{i+1}],\ 0 \le i < N(n).$$

Let $x_{\pi_n} : [0,T] \times \Omega \to X$ be the solutions to the family of ODEs (indexed by $\omega \in \Omega$)

$$\begin{cases} \dfrac{\mathrm{d}x_{\pi_n}(t)}{\mathrm{d}t} &= f\big(x_{\pi_n}(t)\big) + g\big(x_{\pi_n}(t)\big)\dfrac{\mathrm{d}w_{\pi_n}(t)}{\mathrm{d}t}, \\ x_{\pi_n}(0) &= x_0. \end{cases} \tag{7.7}$$

The family of equations (7.7) may sometimes be written

$$\begin{cases} \mathrm{d}x_{\pi_n}(t) &= f\big(x_{\pi_n}(t)\big)\,\mathrm{d}t + g\big(x_{\pi_n}(t)\big)\,\mathrm{d}w_{\pi_n}(t) \\ x_{\pi_n}(0) &= x_0 \end{cases}$$

In particular, for $t \in (t_i, t_{i+1})$, $i = 0, \dots, N(n)-1$, x_{π_n} takes the form

$$x_{\pi_n}(t) = x_{\pi_n}(t_i) + \int_{t_i}^t f\big(x_{\pi_n}(s)\big)\,\mathrm{d}s + \int_{t_i}^t g\big(x_{\pi_n}(s)\big)\left(\frac{w(t_{i+1}) - w(t_i)}{t_{i+1} - t_i}\right)\mathrm{d}s.$$

Using the above notation, we now state our first result.

Theorem 7.1. *For $p > 2$ and $n \in \mathbb{N}$*

$$\mathbb{E}\left[\sup_{t \in [0,T]} |x(t) - x_{\pi_n}(t)|_X^p\right] \le C n^{1-p/2} = \frac{C}{n^{p/2-1}}, \tag{7.8}$$

where C is a constant independent of n and depending only on the space X, p, T, m_p (see (6.5)), C_1, C_2 and the bounds and Lipschitz constants of f, g, g' and $\operatorname{tr}(g'g)$.

Corollary 7.2. *For each $T > 0$ $x_{\pi_n}(\,.\,) \to x(\,.\,)$ in $C(0,T;X)$ in probability, i.e., for each $\varepsilon > 0$*

$$\mathbb{P}\big\{\omega : |x(\,.\,,\omega) - x_{\pi_n}(\,.\,,\omega)|_{C(0,T;X)} > \varepsilon\big\} \longrightarrow 0 \tag{7.9}$$

as $\operatorname{mesh} \pi_n \to 0$. Here $C(0,T;X)$ is the space of X valued continuous functions on the interval $[0,T]$.

Corollary 7.3. *For each $T > 0$,*

$$x_{\pi_n}(\,.\,) \longrightarrow x(\,.\,) \qquad \text{in } C(0,T;X) \text{ almost surely as } n \to \infty. \tag{7.10}$$

Remark 7.4. Theorem 7.1 is an extension of a result proved in the PhD thesis by Dowell, [86]. There, the case $p = 2$ with X being a Hilbert space was treated. In particular, Dowell proved the following two results (more or less independently of one another), see Theorems 5.2 and 5.7 in [86]:

- For each $T > 0$

$$\sup_{t\in[0,T]} \mathbb{E}\big[|x(t) - x_{\pi_n}(t)|_X^2\big] \longrightarrow 0 \qquad \text{as mesh}\,\pi_n \to 0. \tag{7.11}$$

- For each $T > 0$ and $\varepsilon > 0$

$$\mathbb{P}\Big\{\omega : \sup_{t\in[0,T]} |x(t,\omega) - x_{\pi_n}(t,\omega)|_X > \varepsilon\Big\} \longrightarrow 0 \qquad \text{as mesh}\,\pi_n \to 0. \tag{7.12}$$

Our result is a much stronger and more general result than Dowell's for several reasons. Firstly, Theorem 7.1 holds in the case when X is an M-type 2 Banach space. Secondly, we have convergence in $L^p(\Omega; C(0,T;X))$, $p \geq 2$, whereas Dowell only proved a weaker form of convergence, i.e., uniform convergence in $L^2(\Omega;X)$, see (7.11). With this stronger form of convergence, convergence in $C(0,T;X)$ in probability is then a simple consequence of the Chebyshev inequality and this gives us Corollary 7.2. Finally, for $p > 2$ we prove estimates which give a rate of convergence, see (7.8). Using these estimates it is straightforward to prove almost sure convergence in $C(0,T;X)$ (see Corollary 7.3). Indeed, the estimates (7.8) imply that for $p > 4$,

$$\mathbb{E}\Big[\sum_{n=1}^{\infty} |x - x_{\pi_n}|^p_{C(0,T;X)}\Big] \leq C \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty.$$

Thus, almost surely

$$\sum_{n=1}^{\infty} |x - x_{\pi_n}|^p_{C(0,T;X)} < \infty,$$

which implies that almost surely

$$|x - x_{\pi_n}|_{C(0,T;X)} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

The method Dowell uses to prove (7.11), which itself is a generalization of a similar result in [105], carries over to some extent to the case $p > 2$ and X is an M-type 2 Banach space. The Burkholder inequality (6.9) is the main tool we use here. However, although Dowell was familiar with stochastic integration in 2-uniformly smooth Banach spaces and the Burkholder inequality (via the thesis of Neidhardt), he was not able to deal with the Banach space case because of the term involving the tr map. There is a considerable level of difficulty in dealing with the tr map in Banach spaces as opposed to Hilbert spaces. We deal with this problem by making use of the M-type 2 property of our space X, in particular, the inequality (6.1).

Proof of Theorem 7.1. Fix a partition $\pi = \pi_n = \{0 \le t_0 \le t_1 \cdots \le t_{N(n)} = T\}$ and denote x_π by y. Set $x_j = x(t_j)$, $y_j = y(t_j) = x_\pi(t_j)$, $\Delta_j t = t_{j+1} - t_j$ and $\Delta_j w = w(t_{j+1}) - w(t_j)$. To simplify the notation we put f identically zero. This will not affect the result owing to the conditions put on f. Moreover, C will denote a generic constant depending only on the space X, p, T, m_p, C_1, C_2 the bounds and Lipschitz constants of g, g' and tr.

For $t \in [0,T]$, let k be the largest integer such that $t_k \le t$. Moreover, for $r \in [0,T]$, set $R(n) = \max\{m : t_m \le r\}$. Then, using the triangle inequality, we have

$$\mathbb{E}\Big[\sup_{0\le t\le r} |x(t) - y(t)|_X^p\Big] \le C\,\mathbb{E}\Big[\sup_{0\le t\le r}\Big(|x(t) - x(t_k)|_X^p + |y(t_k) - y(t)|_X^p\Big)\Big] + C\,\mathbb{E}\Big[\sup_{0\le k\le R(n)} |x(t_k) - y(t_k)|_X^p\Big]. \tag{7.13}$$

Suppose, for the time being, we have the following estimates

$$\mathbb{E}\Big[\sup_{t\in[0,r]}\Big(|x(t) - x(t_k)|_X^p + |y(t_k) - y(t)|_X^p\Big)\Big] \le C\eta(\pi), \tag{7.14}$$

$$\mathbb{E}\Big[\sup_{0\le k\le R(n)} |x(t_k) - y(t_k)|_X^p\Big] \le \eta(\pi) + C\int_0^r \mathbb{E}[\gamma(s)]\,\mathrm{d}s, \tag{7.15}$$

where

$$\gamma(s) = \sup_{t\in[0,s]} |x(t) - y(t)|_X^p \tag{7.16}$$

and $\eta(\pi)$ satisfies

$$\eta(\pi) \le C\,n^{1-p/2}.$$

(Note, for example, that $N(n)(\operatorname{mesh}\pi)^{p/2}$ is a term of the form $\eta(\pi)$.) From (7.13), (7.14), (7.15) and (7.16) we may deduce that for all $r \in [0,T]$:

$$\mathbb{E}[\gamma(r)] = \mathbb{E}\Big[\sup_{0\le t\le r} |x(t) - y(t)|_X^p\Big] \le C\,\eta(\pi) + C\int_0^r \mathbb{E}[\gamma(s)]\,\mathrm{d}s.$$

An application of Gronwall's Lemma implies that

$$\mathbb{E}[\gamma(T)] \le C\,\eta(\pi)\exp(CT),$$

i.e.,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |x(t) - y(t)|_X^p\Big] \le C\,n^{1-p/2}.$$

To complete the proof of Theorem 7.1 we need to prove the estimates (7.14) and (7.15). We begin with (7.14). ∎

Lemma 7.5. *With the above notation,*

$$\mathbb{E}\Big[\sup_{t\in[0,r]}\Big(|x(t) - x(t_k)|_X^p + |y(t_k) - y(t)|_X^p\Big)\Big] \le C\,N(n)(\operatorname{mesh}\pi)^{p/2}. \tag{7.17}$$

Proof. Note first that from (7.1) and the boundedness of the maps g and $\mathrm{tr}(g'g)$ we have

$$\mathbb{E}\left[\sup_{0\le t\le r} |x(t)-x(t_k)|_X^p\right] \le C\,(\mathrm{mesh}\,\pi)^p + C\,\mathbb{E}\left[\sup_{0\le t\le r}\left|\int_{t_k}^t g(x(s))\,\mathrm{d}w(s)\right|_X^p\right].$$

It then follows using the Burkholder inequality and the boundedness of g that

$$\mathbb{E}\left[\sup_{0\le t\le r} |x(t)-x(t_k)|_X^p\right] \le CT^{p/2}(\mathrm{mesh}\,\pi)^{p/2}.$$

Recall Taylor's formula in integral form, see [78]:

$$y(a)-y(b) = \int_0^1 y'\big(b+r(a-b)\big)(a-b)\,\mathrm{d}r. \tag{7.18}$$

For some $0\le s\le 1$, we have, using (7.18), (7.7) and the boundedness of g,

$$\begin{aligned}
|y(t)-y(t_k)|_X^p &= |y(t_k+s\,\Delta_k t)-y(t_k)|_X^p \\
&= \left|\int_0^1 y'(t_k+r(s\,\Delta_k t))(s\,\Delta_k t)\,\mathrm{d}r\right|_X^p \\
&= \left|\int_0^s y'(t_k+r\,\Delta_k t)(\Delta_k t)\,\mathrm{d}r\right|_X^p \\
&= \left|\int_0^s g(y(t_k+r\,\Delta_k t))(\Delta_k w)\,\mathrm{d}r\right|_X^p \\
&\le C|\Delta_k w|_E^p. &(7.19)
\end{aligned}$$

Using (6.5) we infer that

$$\mathbb{E}\left[\sup_{t\in[0,r]} |y(t)-y(t_k)|_X^p\right] \le C\,(\mathrm{mesh}\,\pi)^{p/2}.$$

This completes the proof of Lemma 7.5. ∎

Fix an interval $[t_i,t_{i+1}]$ in the partition π. We quote another form of Taylor's formula, see [78]:

$$y(a)-y(b) = y'(b)(a-b) + \int_0^1 (1-s)y''\big(b+s(a-b)\big)(a-b,a-b)\,\mathrm{d}s. \tag{7.20}$$

Using (7.20), the chain rule and (7.7) we obtain

$$\begin{aligned}
y(t_{i+1})-y(t_i) &= y'(t_j)\,\Delta_j t + \int_0^1 (1-s)\,y''(t_j+s\,\Delta_j t)(\Delta_j t,\Delta_j t)\,\mathrm{d}s \\
&= g(y_j)\,\Delta_j w \\
&+ \int_0^1 (1-s)\Big(g'\big(y(t_j+s\,\Delta_j t)\big)\,g\big(y(t_j+s\,\Delta_j t)\big)(\Delta_j w,\Delta_j w)\Big)\,\mathrm{d}s.
\end{aligned}$$

It then follows, denoting $s_j := t_j + s\,\Delta_j t$, that

$$\begin{aligned} y(t_k) - y(0) &= \sum_{j=0}^{k-1} (y_{j+1} - y_j) \\ &= \sum_{j=0}^{k-1} \Big(g(y_j)\,\Delta_j w + \frac{1}{2}\, g'(y_j)\, g(y_j)(\Delta_j w, \Delta_j w) \Big) \\ &\quad + \sum_{j=0}^{k-1} \int_0^1 (1-s)\, g'\big(y(s_j)\big)\, g\big(y(s_j)\big)(\Delta_j w, \Delta_j w)\, \mathrm{d}s \\ &\quad - \sum_{j=0}^{k-1} \int_0^1 (1-s)\, g'(y_j)\, g(y_j)(\Delta_j w, \Delta_j w)\, \mathrm{d}s. \end{aligned}$$

Recalling that

$$x(t_k) = x(0) + \int_0^{t_k} g\big(x(s)\big)\, \mathrm{d}w(s) + \frac{1}{2} \int_0^{t_k} \mathrm{tr}\big[g'\big(x(s)\big)\, g\big(x(s)\big)\big]\, \mathrm{d}s,$$

we may write

$$y(t_k) - x(t_k) = A_k + B_k + \frac{1}{2}\, \overline{C}_k + D_k + \frac{1}{2}\, E_k + \frac{1}{2}\, F_k,$$

where

$$\begin{aligned} A_k &= \sum_{j=0}^{k-1} \int_0^1 (1-s) \Big(g'\big(y(s_j)\big)\, g\big(y(s_j)\big) - g'(y_j)\, g(y_j) \Big) (\Delta_j w, \Delta_j w)\, \mathrm{d}s \\ B_k &= \sum_{j=0}^{k-1} \big(g(y_j) - g(x_j) \big)\, \Delta_j w \\ \overline{C}_k &= \sum_{j=0}^{k-1} \Big(g'(y_j)\, g(y_j) - g'(x_j)\, g(x_j) \Big) (\Delta_j w, \Delta_j w) \end{aligned}$$

$$\begin{aligned} D_k &= \sum_{j=0}^{k-1} g(x_j)\, \Delta_j w - \int_0^{t_k} g\big(x(s)\big)\, \mathrm{d}w(s) \\ E_k &= \sum_{j=0}^{k-1} \big(g'(x_j)\, g(x_j)(\Delta_j w, \Delta_j w) - \mathrm{tr}[g'(x_j)\, g(x_j)]\, \Delta_j t \big) \\ F_k &= \sum_{j=0}^{k-1} \mathrm{tr}[g'(x_j)\, g(x_j)]\, \Delta_j t - \int_0^{t_k} \mathrm{tr}\big[g'\big(x(t)\big)\, g\big(x(t)\big)\big]\, \mathrm{d}t. \end{aligned}$$

We begin with proving:

Lemma 7.6. *Using the above notation we have*

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)} |A_k + D_k + E_k + F_k|_X^p\Big] \le C\,(\operatorname{mesh}\pi)^{p/2}.$$

Proof. Consider first the term $A_k = \sum_{j=0}^{k-1} \Gamma_j$, where

$$\Gamma_j := \int_0^1 (1-s)\Big(g'\big(y(s_j)\big)\, g\big(y(s_j)\big)(\Delta_j w, \Delta_j w) - g'(y_j)\, g(y_j)(\Delta_j w, \Delta_j w)\Big)\,\mathrm{d}s.$$

The boundedness and Lipschitz properties of g' and g, along with (7.19), imply that

$$\begin{aligned} |\Gamma_j|_X \;&\le\; \int_0^1 \Big|\Big(g'\big(y(s_j)\big) - g'(y_j)\Big)\, g\big(y(s_j)\big)(\Delta_j w, \Delta_j w)\Big|_X\,\mathrm{d}s \\ &\quad + \int_0^1 \Big|g'(y_j)\Big(g\big(y(s_j)\big) - g(y_j)\Big)(\Delta_j w, \Delta_j w)\Big|_X\,\mathrm{d}s \\ &\le\; C|\Delta_j w|_E^2\, |y(s_j) - y_j|_X \\ &\le\; C|\Delta_j w|_E^3. \end{aligned} \tag{7.21}$$

Using (7.21) and Hölder's inequality for sums we have

$$\mathbb{E}\Big[\sup_{1\le k\le N(n)} |A_k|_X^p\Big] \le CN(n)^{p-1}\,\mathbb{E}\Bigg[\sum_{j=0}^{N(n)-1} |\Delta_j w|_E^{3p}\Bigg].$$

Applying (6.5) (with p replaced by $3p$) gives us

$$\mathbb{E}\Big[\sup_{1\le k\le N(n)} |A_k|_X^p\Big] \le CN(n)^{p-1} \sum_{j=0}^{N(n)-1} |\Delta_j t|^{3p/2}.$$

It then follows, using (7.5) and (7.6), that

$$\mathbb{E}\Big[\sup_{1\le k\le N(n)} |A_k|_X^p\Big] \le C\, n^{p-1}(\operatorname{mesh}\pi)^{3p/2} n \le C(\operatorname{mesh}\pi)^{p/2}. \tag{7.22}$$

Consider then the term $D_k = \sum_{j=0}^{k-1} g(x_j)\,\Delta_j w - \int_0^{t_k} g(x(s))\,\mathrm{d}w(s)$. Define

$$\tilde g(s) = \begin{cases} g(x_j) & \text{for } t_j \le s < t_{j+1}, \\ 0 & \text{if } s > t_k. \end{cases}$$

$\tilde g(s)$ is well-defined, adapted to the filtration $\{\mathcal{F}_s\}_{s\ge 0}$ and moreover, the integral $\int_0^t \tilde g(s)\,\mathrm{d}w(s)$ makes sense for all $t \in [0,T]$. We may write

$$D_k = \int_0^{t_k} \Big(\tilde g(s) - g\big(x(s)\big)\Big)\,\mathrm{d}w(s).$$

Using the Burkholder inequality, the Lipschitz property of g and the properties (7.5) and (7.6), it follows that

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{1\le k\le R(n)}|D_k|_X^p\Big] &\le \mathbb{E}\Big[\sup_{0\le t\le r}\Big|\int_0^t\Big(\tilde g(s)-g\big(x(s)\big)\Big)\,\mathrm{d}w(s)\Big|_X^p\Big]\\
&\le C\,\mathbb{E}\Big[\Big(\int_0^r\big|\tilde g(s)-g\big(x(s)\big)\big|^2_{L(E,X)}\,\mathrm{d}s\Big)^{p/2}\Big]\\
&= C\,\mathbb{E}\Big[\Big(\sum_{j=0}^{R(n)-1}\int_{t_j}^{t_{j+1}}\big|g(x_j)-g\big(x(s)\big)\big|^2_{L(E,X)}\,\mathrm{d}s\Big)^{p/2}\Big]\\
&\le C\,\mathbb{E}\Big[\Big(\sum_{j=0}^{R(n)-1}\int_{t_j}^{t_{j+1}}|x_j-x(s)|_X^2\,\mathrm{d}s\Big)^{p/2}\Big]\\
&\le C\,\mathbb{E}\Big[\sup_{0\le t\le r}|x(t_l)-x(t)|_X^p\Big]
\end{aligned}
$$

where l is such that $t\in[t_l,t_{l+1})$. Using Lemma 7.5 we deduce that

$$
\mathbb{E}\Big[\sup_{1\le k\le R(n)}|D_k|_X^p\Big]\le C(\operatorname{mesh}\pi)^{p/2}. \tag{7.23}
$$

Consider next the term F_k. We have

$$
\begin{aligned}
|F_k|_X &= \Big|\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\Big(\mathrm{tr}[g'(x_j)\,g(x_j)]-\mathrm{tr}\big[g'\big(x(t)\big)\,g\big(x(t)\big)\big]\Big)\,\mathrm{d}t\Big|_X\\
&\le \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\Big|\mathrm{tr}\big[g'(x_j)\,g(x_j)-g'\big(x(t)\big)\,g\big(x(t)\big)\big]\Big|_X\,\mathrm{d}t\\
&\le C\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\big|g'(x_j)\,g(x_j)-g'\big(x(t)\big)\,g\big(x(t)\big)\big|_{L_2(E,X)}\,\mathrm{d}t\\
&\le C\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\Big(\big|g'(x_j)\,g(x_j)-g'(x_j)\,g\big(x(t)\big)\big|_{L_2(E,X)}\\
&\qquad +\big|g'(x_j)\,g\big(x(t)\big)-g'\big(x(t)\big)\,g\big(x(t)\big)\big|_{L_2(E,X)}\Big)\,\mathrm{d}t.
\end{aligned}
$$

Using the boundedness and the Lipschitz properties of functions g and g', we deduce that

$$
\begin{aligned}
|F_k|_X &\le C\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}|x_j-x(t)|_X\,\mathrm{d}t\\
&\le CT\sup_{0\le t\le r}|x(t_l)-x(t)|_X,
\end{aligned}
$$

where l is such that $t \in [t_l, t_{l+1})$. Again, using Lemma 7.5, we conclude that

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)} |F_k|_X^p\Big] \le C(\operatorname{mesh}\pi)^{p/2}. \tag{7.24}$$

Finally, we deal with the term E_k and we will prove

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)} |E_k|_X^p\Big] \le C(\operatorname{mesh}\pi)^{p/2}. \tag{7.25}$$

This part of the proof differs considerably from [86]. Dowell proves (7.25) using the properties of the inner product on a Hilbert space and the proof is quite straightforward. We do not have an inner product to work with and instead we make use of the M-type 2 property of our space X. Let $E_k = \sum_{j=0}^{k-1} \Lambda_j$, where

$$\Lambda_j = g'(x_j)\, g(x_j)(\mathit{\Delta}_j w, \mathit{\Delta}_j w) - \operatorname{tr}\big(g'(x_j)\, g(x_j)\big)\, \mathit{\Delta}_j t \tag{7.26}$$

We first show that E_k is an X-valued martingale with respect to the discrete filtration $\{\mathcal{F}_{t_k}\}_{1\le k\le R(n)}$. For $0 \le j \le k-1$, $x_j : \Omega \to X$ is $\mathcal{F}_{t_j}$-measurable and $w(t_{j+1}) - w(t_j) : \Omega \to E$ is $\mathcal{F}_{t_{j+1}}$-measurable. Using the continuity of the maps g, g' and $\operatorname{tr}(g'g)$ it follows that each Λ_j is $\mathcal{F}_{t_{j+1}}$-measurable. We deduce that E_k is $\mathcal{F}_{t_k}$-measurable. To prove E_k is a martingale we are left with showing that $\mathbb{E}[E_k \,|\, \mathcal{F}_{t_{k-1}}] = E_{k-1}$. For this it suffices to prove that $\mathbb{E}[\Lambda_{k-1} \,|\, \mathcal{F}_{t_{k-1}}] = 0$.

Denote

$$\Psi_{k-1} := g'(x_{k-1})\, g(x_{k-1})(\mathit{\Delta}_{k-1} w, \mathit{\Delta}_{k-1} w).$$

Then

$$\begin{aligned}
\mathbb{E}[\Psi_{k-1} \,|\, \mathcal{F}_{k-1}] &= \mathbb{E}[g'(x_{k-1})\, g(x_{k-1})(\mathit{\Delta}_{k-1} w, \mathit{\Delta}_{k-1} w) \,|\, \mathcal{F}_{t_{k-1}}] \\
&= g'(x_{k-1})\, g(x_{k-1}) \mathbb{E}[(\mathit{\Delta}_{k-1} w, \mathit{\Delta}_{k-1} w)] \\
&= (t_k - t_{k-1}) \int_E g'(x_{k-1})\, g(x_{k-1})(e,e)\, \mathrm{d}\mu(e) \\
&= (\mathit{\Delta}_{k-1} t) \operatorname{tr}\big(g'(x_{k-1})\, g(x_{k-1})\big) \\
&= \mathbb{E}\big[(\mathit{\Delta}_{k-1} t) \operatorname{tr}\big(g'(x_{k-1})\, g(x_{k-1})\big) \,\big|\, \mathcal{F}_{t_{k-1}}\big]. \qquad (7.27)
\end{aligned}$$

As x_{k-1} is $\mathcal{F}_{t_{k-1}}$-measurable, then so is $\operatorname{tr}(g'(x_{k-1})\, g(x_{k-1}))$, which explains the final step. Thus (7.26) and (7.27) imply that $\mathbb{E}[\Lambda_{k-1} \,|\, \mathcal{F}_{t_{k-1}}] = 0$. We conclude that $\{E_k\}_{k=1}^{R(n)}$ is an X-valued martingale with respect to the discrete filtration $\{\mathcal{F}_{t_k}\}_{1\le k\le R(n)}$. Since X is an M-type 2 Banach space it follows that, see (6.1),

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)} |E_k|_X^p\Big] \le C\, \mathbb{E}\Bigg[\bigg(\sum_{j=1}^{R(n)-1} |E_j - E_{j-1}|_X^2\bigg)^{p/2}\Bigg].$$

Thus

$$\mathbb{E}\Bigg[\sup_{1\le k\le R(n)} \bigg|\sum_{j=0}^{k-1} \Lambda_j\bigg|_X^p\Bigg] \le C\, \mathbb{E}\Bigg[\bigg(\sum_{j=1}^{R(n)} |\Lambda_{j-1}|_X^2\bigg)^{p/2}\Bigg].$$

Applying the Hölder inequality for sums gives

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)}\Big|\sum_{j=0}^{k-1}\Lambda_j\Big|_X^p\Big] \le CR(n)^{p/2-1}\,\mathbb{E}\Big[\sum_{j=1}^{R(n)}|\Lambda_{j-1}|_X^p\Big]$$

$$\le CN(n)^{p/2-1}\sum_{j=1}^{N(n)}\mathbb{E}\big[|\Lambda_{j-1}|_X^p\big]. \tag{7.28}$$

Note that

$$\mathbb{E}\big[|\Lambda_j|_X^p\big] \le \mathbb{E}\Big[\Big(\big|g'(x_j)\,g(x_j)(\mathit{\Delta}_j w,\mathit{\Delta}_j w)\big|_X + \big|\mathrm{tr}\big(g'(x_j)\,g(x_j)\big)(\mathit{\Delta}_j t)\big|_X\Big)^p\Big]$$

$$\le C\,\mathbb{E}\big[|\mathit{\Delta}_j w|_E^{2p} + |\mathit{\Delta}_j t|^p\big]$$

$$\le C(\mathit{\Delta}_j t)^p. \tag{7.29}$$

It follows from (7.28) and (7.29) that

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)}|E_k|_X^p\Big] \le CN(n)^{p/2-1}\sum_{j=1}^{N(n)}(\mathit{\Delta}_j t)^p$$

$$\le CN(n)^{p/2-1}\sum_{j=1}^{N(n)}(\mathrm{mesh}\,\pi)^p$$

$$\le CN(n)^{p/2}(\mathrm{mesh}\,\pi)^p \le C(\mathrm{mesh}\,\pi)^{p/2}. \tag{7.30}$$

Lemma 7.6 now follows from (7.22), (7.23), (7.24) and (7.30). ∎

Lemma 7.7. *For a constant C independent of k and r,*

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)}\big|B_k+\overline{C}_k\big|_X^p\Big] \le C\int_0^r \mathbb{E}[\gamma(s)]\,\mathrm{d}s \tag{7.31}$$

Proof. As in the proof of Lemma 7.6, define

$$Y(s) = \begin{cases} g(y_j)-g(x_j) & \text{if } t_j\le s<t_{j+1}, \text{ where } 0\le j\le k-1,\\ 0 & \text{if } s>t_k.\end{cases}$$

$Y(s)$ is well-defined, adapted to the filtration $\{\mathcal{F}_s\}_{s\ge 0}$ and $\int_0^t Y(s)\,\mathrm{d}w(s)$ makes sense for all $t\in[0,T]$. Moreover,

$$|B_k|_X^p = \Big|\int_0^{t_k} Y(s)\,\mathrm{d}w(s)\Big|_X^p.$$

Using the Burkholder inequality and the Lipschitz properties of g, it follows that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{1\le k\le R(n)}|B_k|_X^p\Big] &= \mathbb{E}\Big[\sup_{0\le t\le r}\Big|\int_0^t Y(s)\,\mathrm{d}w(s)\Big|_X^p\Big] \\
&\le C\,\mathbb{E}\Big[\Big(\int_0^r |Y(s)|^2_{L(E,X)}\,\mathrm{d}s\Big)^{p/2}\Big] \\
&= C\,\mathbb{E}\Big[\Big(\sum_{j=0}^{R(n)-1}\int_{t_j}^{t_{j+1}} |g(y_j)-g(x_j)|^2_{L(E,X)}\,\mathrm{d}s\Big)^{p/2}\Big] \\
&\le C\,\mathbb{E}\Big[\Big(\sum_{j=0}^{R(n)-1} |y_j-x_j|_X^2\,\Delta_j t\Big)^{p/2}\Big] \\
&\le C\,\mathbb{E}\Big[\Big(\sum_{j=0}^{R(n)-1} \gamma(t_j)^{2/p}\,\Delta_j t\Big)^{p/2}\Big].
\end{aligned}$$

Applying the Hölder inequality for sums gives

$$\begin{aligned}
\mathbb{E}\Big[\sup_{1\le k\le R(n)}|B_k|_X^p\Big] &\le CR(n)^{p/2-1}\,\mathbb{E}\Big[\sum_{j=0}^{R(n)-1}\gamma(t_j)(\Delta_j t)^{p/2}\Big] \\
&\le CN(n)^{p/2-1}(\mathrm{mesh}\,\pi)^{p/2-1}\sum_{j=0}^{R(n)-1}\mathbb{E}[\gamma(t_j)\,\Delta_j t] \\
&\le C\int_0^r \mathbb{E}[\gamma(s)]\,\mathrm{d}s,
\end{aligned}$$

which constitutes the first in proving Lemma 7.7. Consider the final term $\overline{C}_k$. Then

$$\begin{aligned}
\big|\overline{C}_k\big|_X &= \Big|\sum_{j=0}^{k-1}\big(g'(y_j)g(y_j)-g'(x_j)g(x_j)\big)(\Delta_j w,\Delta_j w)\Big|_X \\
&\le \sum_{j=0}^{k-1}\Big(\big|\big(g'(y_j)-g'(x_j)\big)\,g(x_j)(\Delta_j w,\Delta_j w)\big|_X \\
&\qquad +\big|g'(y_j)\big(g(y_j)-g(x_j)\big)(\Delta_j w,\Delta_j w)\big|_X\Big) \\
&\le C\sum_{j=0}^{k-1}|x_j-y_j|_X\,|\Delta_j w|_E^2.
\end{aligned}$$

Applying the Hölder inequality gives

$$\big|\overline{C}_k\big|^p \le CN(n)^{p-1}\sum_{j=0}^{k-1}|x_j-y_j|_X^p\,|\Delta_j w|_E^{2p}.$$

On taking supremum over k and then expectations we get

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)}\big|\overline{C}_k\big|_X^p\Big] \le CN(n)^{p-1}\sum_{j=0}^{R(n)-1}\mathbb{E}\big[|x_j-y_j|_X^p\,|\Delta_j w|_E^{2p}\big].$$

Since both x_j and y_j are $\mathcal{F}_{t_j}$-measurable and $\Delta_j w$ is independent of $\mathcal{F}_{t_j}$ then using the properties of conditional expectation and (6.5) we have

$$\begin{aligned}\mathbb{E}\big[|x_j-y_j|_X^p|\Delta_j w|_{E_0}^{2p}\big] &= \mathbb{E}\Big[\mathbb{E}\big[|x_j-y_j|_X^p\,|\Delta_j w|_E^{2p}\mid\mathcal{F}_{t_j}\big]\Big]\\ &= \mathbb{E}\Big[|x_j-y_j|_X^p\,\mathbb{E}\big[|\Delta_j w|_E^{2p}\mid\mathcal{F}_{t_j}\big]\Big]\\ &= \mathbb{E}\Big[|x_j-y_j|_X^p\,\mathbb{E}\big[|\Delta_j w|_E^{2p}\big]\Big]\\ &\le C|\Delta_j t|^p\,\mathbb{E}\big[|x_j-y_j|_X^p\big].\end{aligned}\tag{7.32}$$

It then follows using (7.32), (7.5) and (7.6) that

$$\begin{aligned}\mathbb{E}\Big[\sup_{1\le k\le R(n)}\big|\overline{C}_k\big|_X^p\Big] &\le CN(n)^{p-1}\sum_{j=0}^{R(n)-1}\mathbb{E}\big[|x_j-y_j|_X^p\big]|\Delta_j t|^p\\ &\le CN(n)^{p-1}(\mathrm{mesh}\,\pi)^{p-1}\sum_{j=0}^{R(n)-1}(\Delta_j t)\,\mathbb{E}\Big[\sup_{0\le r\le t_j}|x(r)-y(r)|_X^p\Big]\\ &\le C\sum_{j=0}^{R(n)-1}\mathbb{E}[(\gamma(t_j)\,\Delta_j t].\end{aligned}$$

Since $\gamma(s)$ is non-decreasing we can conclude that

$$\mathbb{E}\Big[\sup_{1\le k\le R(n)}\big|\overline{C}_k\big|_X^p\Big] \le C\int_0^r \mathbb{E}[\gamma(s)]\,\mathrm{d}s,$$

which concludes the proof of Lemma 7.7. The proof of Theorem 7.1 is now complete. ∎

Remark 7.8. In a very recent preprint [94] by M. Ledoux, T. Lyons and Z.Qian, the authors extend the main results of [95] to a wide class of Banach spaces. The finite dimensional case of the rough path theory, see [95], gives deep understanding of what approximation procedure leads to Stratonovitch stochastic differential equations. The infinite dimensional case discussed in the above cited preprint should give greater understanding of Corollary 7.3. On the other hand, our results could be used to show that the rough path theory agrees with classical theory of stochastic differential equations in M-type 2 Banach spaces. One can point out a difference concerning regularity assumptions between our paper and [95], [94]. While we assume that the coefficient g is of C^2-class (i.e., g' is Lipschitz), the assumption in the above two papers is that g is of $C^{2+\varepsilon}$-class for some $\varepsilon > 0$ depending on the roughness of the driving rough path.

In another recent work [82] the author employs the Euler method to prove local existence of solutions to differential equations in finite dimensional spaces driven by a finite dimensional rough path. It would be interesting to extend his result to an infinite dimensional case and to also consider the global existence of solutions when the input is a p-rough path with $p > 2$. Such results would help to give a better understanding of the relationship between our paper and the T. Lyons theory, in particular with the above mentioned preprint [94]. The authors would like to thank the anonymous referee for informing them about the interesting paper by A.M. Davie [82].

8. Stochastic differential equations on Banach manifolds

Let us give a list of basic assumptions valid throughout the whole section. Assume that $i : H \to E$ is an AWS, $w(t)$, $t \geq 0$ is an E-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X is an M-type 2 Banach space.

Assume that M is a metrizable differentiable manifold modelled on the Banach space X. Let $\mathbb{F}$ denote the vector bundle $\mathcal{L}(E, TM)$ over M with fibres $\mathbb{F}_x = \mathcal{L}(E, T_xM)$, $x \in M$. Assume that

$$f : M \to TM,$$

where TM denotes the tangent vector bundle, is of $\mathcal{C}^{1,0}$ class, i.e. f is (Frèchet) differentiable and its derivative $f' : TM \to T^2M$ is locally Lipschitz continuous. Assume also that

$$g : M \to \mathbb{F}$$

is of $\mathcal{C}^{2,0}$ class, i.e. the second Frèchet derivative $g'' : T^2M \to T^2\mathbb{F}$ is locally Lipschitz continuous.

Our aim is to study the following stochastic differential equation on M

$$d\xi(t) = f(\xi(t))\,dt + g(\xi(t)) \circ dw(t) \tag{8.1}$$

Definition 8.1. *Let $\tau : \Omega \to (0, \infty]$ be an accessible stopping time. An admissible M-valued process $\{\xi(t)\}$, $t < \tau$, is a local solution to equation (8.1) iff for any M-type 2 Banach space Y and for any $\mathcal{C}^2$ function $\psi : U \to Y$, where U is an open subset of M, the process $\zeta(t) := \psi(\xi(t))$, $0 \leq t < \tau$, where $\tau := \inf\{t \in [0, \tau) : \xi(t) \notin U\}$ is the first exit time of $\xi(t)$ from U, satisfies the following*

$$d\zeta(t) = \psi'(\xi(t))f(\xi(t))\,dt + \psi'(\xi(t))g(\xi(t)) \circ dw(t), \tag{8.2}$$

where $\circ$ denotes the Stratonovich differential.

The main result of this section is the existence of maximal local solutions to SDE (8.2). The proof of this result will be preceded by a discussion concerning the meaning of a solution.

Theorem 8.2. *In the framework described at the beginning of the section 8, if ξ_0 is an $\mathcal{F}_0$ measurable M–valued random variable, then there exists a unique maximal solution $\xi(t)$, $t < \tau$ to the problem* (8.1).

Remark 8.3. Taking into account Definition 6.15 we see that $\zeta(t)$ satisfies (8.2) iff it satisfies

$$d\zeta(t) = \psi'(\xi(t))g(\xi(t))\,dw(t) + \left\{\psi'(\xi(t))f(\xi(t)) + \frac{1}{2}\mathrm{tr}[g'_\psi(\xi(t))g(\xi(t))]\right\}dt \tag{8.3}$$

where

$$g_\psi : M \ni x \mapsto \psi'(x)g(x) \in \mathcal{L}(E,Y) \tag{8.4}$$

so that

$$g'(x)g(x) \in \mathcal{L}(T_xM, \mathcal{L}(E,Y)) \circ \mathcal{L}(E, T_xM) \hookrightarrow \mathcal{L}(E, \mathcal{L}(E,Y)) \cong \mathbb{L}(E,E;Y).$$

We have first

Proposition 8.4. *Assume that τ is an accessible stopping time and $\xi(t)$, $t < \hat{\tau}$ is an M-valued admissible process. Then the following three conditions are equivalent.*

(i) *The process $\xi(t)$, $t < \tau$ is a local solution to (8.1);*

(ii) *for any open subset U of M and any diffeomorphism $\varphi : U \to V$, where V is an open subset of X, the process $\zeta(t) := \varphi(\xi(t))$, $t < \tau$ is a solution to*

$$d\zeta(t) = \hat{f}(\eta(t))\,dt + \hat{g}(\eta(t)) \circ dw(t), \tag{8.5}$$

where

$$\begin{aligned} \hat{f} : V \ni y &\mapsto \varphi'(\varphi^{-1}y)f(\varphi^{-1}y) \in X, \\ \hat{g} : V \ni y &\mapsto \varphi'(\varphi^{-1}y)g(\varphi^{-1}y) \in \mathcal{L}(E,X). \end{aligned} \tag{8.6}$$

(iii) *There exists an atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M consisting of $\mathcal{C}^2$ diffeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$ onto some open subset V_α of X, such that for each α, the process $\eta_\alpha(t) = \varphi_\alpha(\xi(t))$, $t < \tau_{U_\alpha}$ is a solution to (8.5).*

Corollary 8.5. *An admissible M-valued process $\xi(t)$, $t < T$, where T is an accessible stopping time, is a local solution to (8.1) iff for any coordinate system $\varphi : U \to V \subset X$ and for any stopping time $S < T$, there exists a stopping time τ such that*

$$\begin{cases} \tau > S, & \text{if } \xi(S) \in U, \\ \tau = S, & \text{otherwise} \end{cases}$$

and such that the process $\eta(t) := \varphi(\xi(t))$, $S \le T \le \tau$, is a solution to (8.5), with (random) initial condition at time S:

$$\eta(S) = \varphi(\xi(S))1_{\{\xi(S)\in U\}}.$$

Remark 8.6. The characterization of a solution to (8.1) as given in Corollary 8.5 is the definition of a solution suggested first by Clark in [21].

Proof of Proposition 8.4. *(i)* $\Rightarrow$ *(ii)*. We observe first that for $x \in M$

$$\hat{f}(\varphi(x)) = \varphi'(x)g(x),\ \hat{g}(\varphi(x)) = \varphi'(x)g(x)$$

and hence (with $\tilde{g}(x) = \varphi'(x)g(x) \in \mathcal{L}(E,X)$, $x \in M$) by (6.26)

$$\mathrm{tr}[\hat{g}'\hat{g}](\varphi(x)) \;=\; \mathrm{tr}[\tilde{g}'(x)g(x)].$$

Thus by using Definition 8.1 and (the second part of) Definition 6.15 we infer that *(ii)* holds.

(ii) $\Rightarrow$ *(iii)*. Obvious.

(iii) $\Rightarrow$ *(i)*. Let us fix φ as in (iii) and let $\psi :\ M \to Y$ be of $\mathcal{C}^2$ class where Y is an M-type 2 Banach space. We want to show that $\zeta(t) := \psi(\xi(t))$, $t < \tau$ satisfies (8.3) (with $\tilde{g}$ as before). Since $\eta(t) = \varphi(\xi(t))$, $\eta(t) = (\psi \circ \varphi^{-1})(\eta(t))$, $t < \tau$.

Since $\eta(t)$ is a solution to (8.5), by applying Theorem 6.20 to $\eta(t)$ and $\psi \circ \varphi^{-1}$ we see that

$$d\eta(t) = (\psi \circ \varphi^{-1})'(\eta(t))\hat{f}(\eta(t))\, dt + (\psi \circ \varphi^{-1})'(\eta(t))\hat{g}(\eta(t)) \circ dw(t).$$

From (8.6) we have ($x := \varphi^{-1}y$)

$$\begin{aligned} \hat{f}(y) &= \varphi'(x)f(x), \qquad \hat{g}(y) = \varphi'(x)g(x) \\ (\psi \circ \varphi^{-1})'(y)\hat{f}(y) &= \psi'(x)(\varphi'(x))^{-1}\varphi'(x)f(x) \end{aligned}$$

and similarly

$$(\psi \circ \varphi')(y)\hat{g}(y) = \psi'(x)g(x).$$

Therefore, as $\varphi^{-1}(\eta(t)) = \xi(t)$,

$$d\eta(t) = \psi'(\xi(t))f(\xi(t))\, dt + \psi'(\xi(t))g(\xi(t)) \circ dw(t),$$

what concludes the proof.

■

PROOF OF THEOREM 8.2. We practically repeat the proof of Theorem 5.27.

■

The following is a manifold version of Corollary 5.30 and is of some importance.

Corollary 8.7. *If $\xi(t), t < \tau$ is the maximal solution to the SDE* (8.2) *on the manifold M then*

$$\mathbb{P}\left\{ w \in \Omega : \tau(\omega) < \infty, \exists \lim_{t \nearrow \tau(\omega)} \xi(t)(\omega) \in M \right\} = 0.$$

The next result deals with problems related to submanifolds. Suppose that $N \subset M$ is a submanifold and the vector fields f and g defined on M are tangent (in an appropriate sense) to N. One may ask then if a solution to (8.1) with $\xi_0 \in N$ a.s. is also a solution to

$$d\xi(t) = f_N(\xi(t))\, dt + g_N(\xi(t)) \circ dw(t), \tag{8.7}$$

with f_N and g_N being restrictions of f and g to N? One may also ask if $\xi(t)$ is a solution to (8.1) when it is a solution to (8.7). For the Hilbert manifold case see section VII.3 in [36]. In our case we have

Theorem 8.8. *Assume that $i: H \hookrightarrow E$ is an AWS and $\{w(t)\}_{t \geq 0}$ is an E-valued Wiener process. Assume that Y, Y_2 and X are Banach spaces such that $Y \oplus Y_2 = X$ (topological direct sum). Suppose that all three spaces are of M-type 2. Let M and N be Banach manifolds modelled respectively on X and Y. Suppose that N is a split closed submanifold of M. Let $f : M \to TM$ be a $\mathcal{C}^1$ vector field such that $h(x) \in T_xN$ for any $x \in N$ and denote*

$$f_N : N \ni x \mapsto f(x) \in T_xN.$$

Let $\mathbb{F}$ be the vector bundle $\mathcal{L}(E, TM)$ over M with a fibres $\mathcal{L}(E, T_xM)$, $x \in M$ and let $\mathbb{F}_N$ be a vector bundle $\mathcal{L}(E, TN)$ over N with fibres $\mathcal{L}(E, T_xN)$, $x \in N$.
Assume that a $\mathcal{C}^2$ section g of the vector bundle $\mathbb{F}$, $g : M \to \mathbb{F}$, is such that $g(x) \in \mathcal{L}(E, T_xN)$, $x \in N$, thus giving rise to a $\mathcal{C}^2$ section g_N of the vector bundle $\mathbb{F}_N$:

$$g_N : N \ni x \mapsto g(x) \in \mathbb{F}_N(x).$$

Denote by $j : N \hookrightarrow M$ the natural imbedding. Let $\xi(t)$, $t < \tau$ be an admissible N-valued process.
Then
(i) *if $\xi(t)$, $t < \tau$ is a local solution to* (8.7)*, the M-valued process $j\xi(t)$, $t < \tau$ is a solution to* (8.1)*;*
(ii) *if the M-valued process $j\xi(t)$, $t < \tau$ is a solution to* (8.1)*, then the process $\xi(t)$, $t < \tau$ is equivalent to a solution $\hat{\xi}(t)$, $t < \tau$ of* (8.7).

PROOF. First we shall prove (i). Let Z is an M-type 2 Banach space, U open subset of M and let $\psi : U \to Z$ be of $\mathcal{C}^2$ class. Then the process $\eta(t) := (\psi \circ j)(\xi(t))$, $t < \tau$ satisfies

$$d\eta(t) = (\psi \circ i)'(\xi(t)) f_N(\xi(t)) + (\psi \circ i)'(\xi(t)) g_N(\xi(t)) \circ dw(t).$$

However, for $x \in N$

$$\begin{aligned} (\psi \circ i)'(x) h_N(x) &= \psi'(x) f(x), \\ (\psi \circ i)'(x) g_N(x) &= \psi'(x) g(x).(g_N)_{\psi \circ i}(x) = (\psi \circ i)'(x) g_N(x) = g_\psi(x). \end{aligned}$$

Hence, for $x \in N$,

$$(g_N)'_{\psi \circ i}(x) g_N(x) = G'_\psi(ix) \circ i \circ g(x) = g'_\psi(x) g(x).$$

In particular

$$\mathrm{tr}((g_N)'_{\psi \circ i}(x) g_N(x)) = \mathrm{tr}(g'_\psi(x) g(x)).$$

Hence, $j\xi(t)$ is a solution to (8.1).
Now we are ready to prove (ii). Let $\hat{\xi}(t)$, $t < T$ be the maximal solution to (8.7). Then by (i) the process $j\hat{\xi}(t)$, $t < T$ is a solution to (8.1). Moreover, since the embedding $j : N \to M$ is closed, employing the same argument as in the proof of Theorem VII.3 in [36], the latter is in fact a maximal solution. Hence, by uniqueness $\tau \le T$ a.s. and $j\xi$ is equivalent to $j\hat{\xi}_{|[0,\tau)\times\Omega}$. This proves (ii) as j is one-to-one.

∎

As usual solutions can be started at time $r > 0$. Also to deal with explosion times it is convenient to add a 'coffin' state Δ to M and topologise $M^+ := M \cup \{\Delta\}$ as a disjoint union. For $\mathcal{F}_r$-measurable $u : \Omega \to M+$ we let $F^r_s(u) : \Omega \to M^+$, $0 \le s < \infty$ be the maximal solution with $F^r_r(u) = u$; Δ being treated as trap. When u is constant say $u(\omega) = x_0$ for all ω we will write $F^r_s(u)$ as $F^r_s(x_0)$ to get $F^r_s : M^+ \times \Omega \to M^+$. For $0 \le r \le t < \infty$ let G^r_t be the σ-algebra generated by the increments $\{w(s) - w(r) : r \le s \le t\}$, with $G^r = \bigvee_{t \ge r} G^r_t$. Then $\mathcal{F}_r$ is independent of G^r. The following is sometimes called the "flow Markov property". From it follows easily the semigroup property for the associated diffusion semigroup and the Markov property of the solutions to our s.d.e.. Its proof is just as that of the corresponding result for Hilbert manifolds, Theorem 3B of Chapter IX, in [36]. We have

Proposition 8.9. *For each $r \ge 0$ there is a version of $F^r_t : M^+ \times \Omega \to \Omega$, $r \le t < \infty$, which is Borel$(M^+)\bar{\times}G^r$-measurable and adapted to $\{G^r_t : r \le t < \infty\}$. For such versions, for each $s \ge r \ge 0$ and $x \in M$*

$$F^s_t\left(F^r_s(x,\omega),\omega\right) = F^r_t(x,\omega) \text{ for all } t \ge s \text{ a.s.}$$

9. Stochastic Geometric heat flow on $\mathbb{S}^1 \times \mathbb{R}_+$

In this section we assume that M is a compact riemannian manifold that is isometrically embedded into an Euclidean space $\mathbb{R}^d$. We consider the following one-dimensional stochastic geometric heat flow equation

$$\partial_t u = \mathbf{D}_x \partial_x u + Y_u \circ \dot{W}, \tag{9.1}$$

with initial data

$$u(0,\cdot) = \xi(\cdot), \tag{9.2}$$

where $\mathbb{S}^1$ is the unit circle (usually identified with the interval $[0,2\pi)$, Y is a C^1-class section of a certain vector bundle $\mathbb{M}$ over M, see Theorem 9.6, and $\xi : \mathbb{S}^1 \to M$ is a continuous map. We assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration such that $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets. Let us denote by $\mathfrak{S} = (e^{-tA})_{t\geq 0}$ the C_0 analytic semigroup of bounded linear operators on the space $L^2(\mathbb{S}^1, \mathbb{R}^d)$, generated by an operator $A := -\Delta$ whose domain $\mathrm{Dom}(A)$ is equal to the Sobolev space $H^{2,2}(\mathbb{S}^1, \mathbb{R}^d)$.

Let S denote the second fundamental tensor (form) of the manifold M with respect to the above mentioned isometric embedding $M \subset \mathbb{R}^d$. In particular, for each $p \in M$, $S_p : T_pM \times T_pM \to N_pM$, where $N_p := \mathbb{R}^d \ominus T_pM$ is the normal space to p with respect to the standard scalar product in $\mathbb{R}^d$. The operator $\mathbf{D}_x \partial_x$ that appears in equation (9.1) acts on smooth curves $\gamma : \mathbb{S}^1 \to M$ and is defined by the formula (see for instance [18, section 2] and references therein)

$$\mathbf{D}_x \partial_x \gamma(x) = \partial_{xx}\gamma(x) - S_{\gamma(x)}(\partial_x \gamma(x), \partial_x \gamma(x)), \;\; x \in \mathbb{S}^1. \tag{9.3}$$

We note that the following fundamental property of the operator $\mathbf{D}_x \partial_x$, see [18, formula (2.6)],

$$\langle \partial_{xx}\gamma(x) - S_{\gamma(x)}(\partial_x \gamma(x), \partial_x \gamma(x)), \partial_{xx}\gamma(x) \rangle = |\partial_{xx}\gamma(x) - S_{\gamma(x)}(\partial_x \gamma(x)|^2, \; x \in I. \tag{9.4}$$

As far as the noise is concerned, we make the following standing assumption.

Assumption 9.1. *$W = (W(t))_{t\geq 0}$ is an E-valued $\mathbb{F}$-Wiener process, where E is a separable Banach space such that for some fixed natural number n, $E \subset H^1(\mathbb{S}^1, \mathbb{R}^n)$ continuously.*

Remark 9.1. It follows from Assumption 9.1 that the Reproducing Kernel Hilbert Space K of the law of $W(1)$ is contained in E (and so in $H^1(\mathbb{S}^1, \mathbb{R}^n)$) and the natural embedding $i : K \hookrightarrow E$ is γ-radonifying.
Let us recall that if $\Lambda : E \to H$ is a bounded linear map, where H is a separable Hilbert space, then $\Lambda \circ i : \mathrm{K} \to H$ is γ-radonifying, i.e. Hilbert-Schmidt.
If $\Lambda : E \times E \to X$ is a bounded bilinear map where X is a separable Banach space then

$$\mathrm{tr}_{\mathrm{K}}(\Lambda) := \sum_j \Lambda(e_j, e_j) \in X, \tag{9.5}$$

where $(e_j)_j$ is an ONB basis of K is well defined. In other words the series on the RHS of equality (9.5) is absolutely convergent, its sum is independent of choice of the ONB $(e_j)_j$ and the map $\mathbb{L}(E,E;X) \ni \Lambda \mapsto \mathrm{tr}_{\mathrm{K}}(\Lambda)$ is linear and bounded. In particular, if $G : X \to \mathcal{L}(E, X)$ is of C^1-class, then for every $a \in X$, $G'(a)G(a) \in \mathcal{L}(E, \mathcal{L}(E,H)) \cong \mathbb{L}(E,E;X)$ and so $\mathrm{tr}_{\mathrm{K}}[G'(a)G(a)] \in X$ is well defined.

For the deterministic version of our problem one can consult the fundamental works by Eells-Sampson [35] and Hamilton [34].

Contrary to the case of a wave equation the solutions to the stochastic heat flow equation can only be defined using an external formulation. However, we hope to be able to find an appropriate definition of an *intrinsic solution*.

What concerns the initial data ξ we make the following assumption.

Assumption 9.2. *The initial data ξ is an $\mathcal{F}_0$-measurable random variable with values in $H^1(\mathbb{S}^1, M)$.*

Remark 9.2. As is [16], see the end of the proof of Theorem 1.1 on page 133. it is sufficient to assume that ξ is such that for some $p > 2$,

$$\mathbb{E}|\xi|^p_{H^1(\mathbb{S}^1,\mathbb{R}^d)} < \infty. \tag{9.6}$$

Definition 9.3. *A process $u : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to M$ is called an* **extrinsic solution** *to equation* (9.1) *if and only if the following five conditions are satisfied*

(i) *$u(t, x, \cdot)$ is $(\mathcal{F}_t)$-progressively measurable for every $x \in \mathbb{S}^1$,*

(ii) *$u(\cdot, \cdot, \omega)$ belongs to $C(\mathbb{R}_+ \times \mathbb{S}^1; M)$ for every $\omega \in \Omega$,*

(iii) *$\mathbb{R}^+ \ni t \mapsto u(t, \cdot, \omega) \in H^1(\mathbb{S}^1, M)$ is continuous for every $\omega \in \Omega$,*

(vii) *for all $t \geq 0$ the following equality holds in $H^{-1}(\mathbb{S}^1, \mathbb{R}^d)$, $\mathbb{P}$ almost surely,*

$$\begin{aligned} u(t) &= u(0) + \int_0^t \left[\partial_{xx} u(s) - S_{u(s)}(\partial_x u(s), \partial_x u(s))\right] ds \\ &+ \int_0^t Y_{u(s)} \circ dW(s). \end{aligned} \tag{9.7}$$

Moreover, if Assumption 9.2 is satisfied, then process $u : \mathbb{R}_+ \times \mathbb{S}^1 \times \Omega \to M$ is called an **extrinsic solution** *of problem* (9.1)*-(9.2) if and only if u is an extrinsic solution to equation* (9.1) *and*

(v) *$u(0, x, \omega) = \xi(x, \omega)$ for every $x \in \mathbb{S}^1$, $\mathbb{P}$-a.s.*

Finally, u is called a **regular extrinsic solution** *if in addition the following two conditions are satisfied*

(viii) *$\mathbb{E} \int_0^T |u(t)|^2_{H^2(\mathbb{S}^1,\mathbb{R}^d)}\, dt < \infty$ for each $T > 0$,*

(ix) *and for all $t \geq 0$ the equality* (9.7) *holds in $L^2(\mathbb{S}^1, \mathbb{R}^d)$, $\mathbb{P}$ almost surely.*

Remark 9.4. Since a function $\xi : \Omega \to C(\mathbb{S}^1; M)$ is $\mathcal{F}$-measurable if and only if for every $x \in \mathbb{S}^1$ the function $i_x \circ \xi \to M$ is $\mathcal{F}$-measurable, in view of the Kuratowski Theorem, Assumption 9.2 is equivalent to the following one.

The initial data ξ is a function taking values in $H^1(\mathbb{S}^1, M)$ such that for every $x \in \mathbb{S}^1$ the function $i_x \circ \xi \to M$ is a $\mathcal{F}_0$-measurable.

In a similar vein, condition (i) in Definition 9.3 can be replaced by the following one

(i′) $\Omega \ni \omega \mapsto u(t, \cdot, \omega) \in H^1(\mathbb{S}^1, M)$ is $\mathcal{F}_t$-measurable for every every $t \geq 0$.

Remark 9.5. Let us observe that the following is an informal version of equation (9.7)

$$\partial_t u = \partial_{xx} u - S_u(u_x, u_x) + Y_u \circ \dot{W}. \tag{9.8}$$

Both equations can also be formulated in the following mild form.

$$u(t) = e^{-tA}\xi - \int_0^t e^{-(t-s)A} S_u(u_x, u_x)\, ds + \int_0^t e^{-(t-s)A} Y_u \circ dW(s), \quad t \geq 0. \tag{9.9}$$

Next we formulate the main result of this part of the paper.

Theorem 9.6. *Let us denote by $\mathbb{M}$ a vector bundle over M whose fiber at $m \in M$ is equal to $\mathcal{L}(\mathbb{R}^n; T_m M)$, where m is a fixed natural number. Assume that Y is a C^1 class section of the vector bundle $\mathbb{M}$. Then there exists an $\mathbb{F}$-adapted process $u = \big(u(t)\big)_{t\geq 0}$ such that u is a* **regular extrinsic solution** *to problem (9.1-9.2).*
Moreover, suppose that $u = \big(u(t)\big)_{t\geq 0}$ and $\bar{u} = \big(\bar{u}(t)\big)_{t\geq 0}$ are two $\mathbb{F}$-adapted processes such that for some $T > 0$, they are **extrinsic solutions** *to problem (9.1-9.2). Then $\bar{u}(t, x, \omega) = u(t, x, \omega)$ for all $x \in \mathbb{S}^1$ and $t \in [0, T)$, $\mathbb{P}$-almost surely.*

In the following generalization of the Itô Lemma, see [17, Lemma 6.5] we denote by $\mathcal{T}_2(\mathrm{K}, H)$ the Hilbert space of Hilbert-Schmidt operators acting between separable Hilbert spaces K and H.

Lemma 9.7. *Let K and H be separable Hilbert spaces, and let f and g be progressively measurable processes with values in H and $\mathcal{T}_2(\mathrm{K}, H)$ respectively, such that*

$$\int_0^T \left\{ |f(s)|_H + \|g(s)\|^2_{\mathcal{T}_2(\mathrm{K},H)} \right\} ds < \infty \quad \textit{almost surely.}$$

For some H-valued $\mathcal{F}_0$-measurable random variable ξ define a process u by

$$u(t) = e^{-tA}\xi + \int_0^t e^{-(t-s)A} f(s)\, ds + \int_0^t e^{-(t-s)A} g(s)\, dW(s), \qquad t \in [0, T],$$

where W is a cylindrical Wiener process on K, and $(e^{-tA})_{t\geq 0}$ is a C_0-semigroup on H with an infinitesimal generator $-A$. Let V be another separable Hilbert space and let $(e^{-tB})_{t\geq 0}$ be a C_0-semigroup on V with an infinitesimal generator $-B$. Suppose that $Q : H \to V$ is a C^2-smooth function such that $Q[D(A)] \subseteq D(B)$ and there exists a continuous function $F : H \to V$ such that

$$-Q'(u)Au = -BQ(u) + F(u), \qquad u \in D(A). \tag{9.10}$$

Then, for all $t \geq 0$,

$$\begin{aligned} Q(u(t)) \;&=\; e^{-tB}Q(\xi) + \int_0^t e^{-(t-s)B} Q'(u(s)) g(s)\, dW(s) \\ &+\; \int_0^t e^{-(t-s)B} \Big[Q'(u(s)) f(s) + F(u(s)) \,+\, \frac{1}{2} \mathrm{tr}_K Q''(u(s)) \circ (g(s), g(s)) \Big] ds. \end{aligned}$$

10. Proof of Theorem 9.6

The basic idea of the proof of the main result comes from [34] and [10]. The nonlinearities S and Y in equation (9.8) are extended from their domains (products of tangent bundles) to the ambient space, and thus we obtain a classical SPDE in Euclidean space for which the existence of global solutions is known. However our proof of the existence of the manifold valued solutions requires, that from the many extensions that can be constructed, we choose those which satisfy certain "symmetry" properties.

10.1. **Differential Geometry preliminaries.** Let us denote by TM and NM the tangent and the normal bundle respectively, and denote by $\mathcal{E}$ the exponential function $T\mathbb{R}^d \ni (p,\xi) \mapsto p+\xi \in \mathbb{R}^d$ relative to the Riemannian manifold $\mathbb{R}^d$ equipped with the standard Euclidean metric. The following result about tubular neighbourhood of M can be found in [55], see Proposition 7.26, p. 200.

Proposition 10.1. *There exists an $\mathbb{R}^d$-open neighbourhood O of M and an NM-open neighbourhood V around the set $\{(p,0) \in NM : p \in M\}$ such that the restriction of the exponential map $\mathcal{E}|_V : V \to O$ is a diffeomorphism. Moreover, V can be chosen in such a way that $(p,t\xi) \in V$ whenever $t \in [-1,1]$ and $(p,\xi) \in V$.*

Remark 10.2. In what follows, we will denote the diffeomorphism $\mathcal{E}|_V : V \to O$ by $\mathcal{E}$, unless there is a danger of ambiguity.
Denote by $i : NM \to NM$ the diffeomorphism $(p,\xi) \mapsto (p,-\xi)$ and define

$$h = \mathcal{E} \circ i \circ \mathcal{E}^{-1} : O \to O. \tag{10.1}$$

The function h defined above is an involution on the normal neighbourhood O of M and corresponds to multiplication by -1 in the fibers, having precisely M for its fixed point set. The identification of the manifold M as a fixed point set of a smooth function enables to prove that solutions of heat equations with initial values on the manifold remain thereon, see [34] for deterministic heat equations in manifolds and [10] for stochastic heat equations in manifolds. Employing a partition of unity argument we may assume that $h : \mathbb{R}^d \to \mathbb{R}^d$ is such that properties (1)-(5) of Corollary 10.3 are valid on O. Therefore, without loss of generality we may assume that the function h is defined on the whole $\mathbb{R}^d$.

Corollary 10.3. *The function h has the following properties: (i) $h : O \to O$ is a diffeomorphism, (ii) $h(h(q)) = q$ for every $q \in O$, (iii) if $q \in O$, then $h(q) = q$ if and only if $q \in M$, (iv) if $p \in M$, then $h'(p)\xi = \xi$, provided $\xi \in T_pM$ and $h'(p)\xi = -\xi$, provided $\xi \in N_pM$.*

Next we define, for $q \in \mathbb{R}^d$ and $a,b \in \mathbb{R}^d$,

$$B_q(a,b) = d_q^2 h(a,b), \quad \mathcal{S}_q(a,b) = \frac{1}{2} B_{h(q)}(h'(q)a, h'(q)b). \tag{10.2}$$

Let us recall that the second fundamental tensor S was introduced before the formula (9.4). We will be studying problem (9.7) with S replaced by $\mathcal{S}$. The following result which is essential for our paper is taken from [18, Proposition 4.2].

Proposition 10.4. *If $p \in M$ and $q \in O$, then*

$$\mathcal{S}_p(\xi,\eta) = \frac{1}{2} B_p(a,b) = S_p(\xi,\eta), \ \xi,\eta \in T_pM, \tag{10.3}$$

$$\mathcal{S}_{h(q)}(h'(q)a, h'(q)b) = h'(q)\mathcal{S}_q(a,b) + B_q(a,b), \ a,b \in \mathbb{R}^d. \tag{10.4}$$

Let us formulate and prove the following result which shows importance of Proposition (10.4).

Corollary 10.5. *Let us put*

$$\underline{\mathbf{\Delta}}(u) = u_{xx} - \mathcal{S}_u(u_x, u_x), \;\; u \in H^2(\mathbb{S}^1, \mathbb{R}^d). \tag{10.5}$$

Then,

$$\underline{\mathbf{\Delta}}(h \circ u) = h'(u)\underline{\mathbf{\Delta}}(u), \;\; u \in H^2(\mathbb{S}^1, \mathbb{R}^d). \tag{10.6}$$

Proof. Assume that $u \in C^2(\mathbb{S}^1, \mathbb{R}^d)$ and put $v = h \circ u$. Then,

$$\begin{aligned} \underline{\mathbf{\Delta}}(v) &= [h \circ u]_{xx} - \mathcal{S}_{h \circ u}((h \circ u)_x, h \circ u)_x) \\ &= h'(u)u_{xx} + h''(u)(u_x, u_x) - \mathcal{S}_{h(u)}(h'(u)u_x, h'(u)u_x) \\ &= h'(u)u_{xx} - h'(u)\mathcal{S}_u(u_x, u_x) = h'(u)\underline{\mathbf{\Delta}}(u), \end{aligned}$$

where the second line above follows from (10.4). ∎

To this end, let π_p, $p \in M$ be the orthogonal projection of $\mathbb{R}^d$ to T_pM and let us define $v_{ij}(p) = S_p(\pi_p e_i, \pi_p e_j)$ for $i, j \in \{1, \ldots, n\}$ and extend the functions $v_{ij} = v_{ji}$ smoothly to the whole $\mathbb{R}^d$.

Now we will shortly recall the construction of extensions of vector fields on M to vector fields on O from [34], cf. [20]. To this end, let us define a new Riemannian metric g on O by

$$g_q(a, b) = \langle a, b\rangle_{\mathbb{R}^d} + \langle h'(q)a, h'(q)b\rangle_{\mathbb{R}^d}, \qquad q \in O, \quad a, b \in \mathbb{R}^d. \tag{10.7}$$

Remark 10.6. $h : (O, g) \to (O, g)$ is an isometric diffeomorphism.

If $q \in O$ then, by Proposition 10.1, there exists a unique $(p, \xi) \in V$ such that $q = \mathcal{E}(p, \xi)$. We will write $p(q) = p$ for this dependence. Moreover, also by Proposition 10.1, $\mathcal{E}(p, t\xi) \in O$ for $t \in [0, 1]$. Hence we can define the curve $\gamma_q : [0, 1] \ni t \mapsto \mathcal{E}(p, t\xi) \in O$. If $a \in \mathbb{R}^d$ and $(X(t))_{t\in[0,1]}$, $X(0) = a$ is the parallel translation of a along γ_q in (O, g) then we denote by $P_q a$ the endpoint vector $X(1)$.

Proposition 10.7. [17, Proposition 3.9] $P : O \to \mathcal{L}^{\rm isom}(\mathbb{R}^d, \mathbb{R}^d)$ *is a smooth function. Moreover,* $P_q = I$ *for* $q \in M$ *and*

$$h'(q)P_q = P_{h(q)}h'(p(q)), \qquad q \in O.$$

Due to this setting, it is possible to extend conveniently various mappings defined on the manifold M to its neighbourhood O, c.f. [20]. For example, if X is a vector field on M, i.e. a section of the tangent bundle, then we can define a map $\mathbf{X} : O \to \mathbb{R}^d$ by the following formula

$$\mathbf{X}_q = P_q X_{p(q)}, \;\; q \in O. \tag{10.8}$$

10.2. **Existence of solutions to approximating equations.** Note that the tangent bundle $T\mathbb{R}^d$ is isomorphic to $\mathbb{R}^d \times \mathbb{R}^d$. Using formula (10.8) and Proposition 10.7 we can find a C^1-class map $\mathbf{Y} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ that such

$$\mathbf{Y}_m = Y_m : \mathbb{R}^n \to T_m M, \quad m \in M, \tag{10.9}$$

where we identify $T_m M$ with the corresponding subspace of $\mathbb{R}^d$, and

$$\mathbf{Y}_{h(q)} = h'(q) \circ \mathbf{Y}_q, \; q \in O. \tag{10.10}$$

Note that both sides of (10.10) belong to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$.

Let us fix $T > 0$. In what follows we put $H^1 = H^1(\mathbb{S}^1, \mathbb{R}^d)$, $L^1 = L^1(\mathbb{S}^1, \mathbb{R}^d)$ etc. We will also denote by $H^1(\mathbb{S}^1, M)$ the Hilbert manifold consisting of those $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^d)$ which satisfy $\gamma(x) \in M$ for all $x \in \mathbb{S}^1$.

Let us recall that E is a Banach space from Assumption 9.1.

We define maps $\mathbf{F} : H^1 \to L^1$, $\mathbf{G} : H^1 \to \mathcal{L}(E, H^1)$ and $\mathbf{Q} : H^1 \to H^1$ by the following formulae,

$$\begin{aligned} [\mathbf{F}(u)](x) &= -\mathcal{S}_{u(x)}(u_x(x), u_x(x)), \; u \in H^1, \; x \in \mathbb{S}^1, \\ \big(\mathbf{G}(u)\xi\big)(x) &= \mathbf{Y}_{u(x)}(\xi(x)), \; u \in H^1, \; \xi \in E, \; x \in \mathbb{S}^1, \\ \mathbf{Q}(u) &= h \circ u, \; u \in H^1. \end{aligned}$$

We begin with the following result which follows directly from Corollary 10.3 parts (3) and (4) and the definitions of the function $\mathbf{Q}$.

Lemma 10.8. *If $u \in H^1(\mathbb{S}^1, M)$, then $\mathbf{Q}(u) = u$. Conversely, if $u \in H^1(\mathbb{S}^1, \mathbb{R}^d)$ is such that for all $x \in \mathbb{S}^1$ $u(x) \in O$ and $\mathbf{Q}(u) = u$, then $u \in H^1(\mathbb{S}^1, M)$.*

Let π_n be the projection from H^1 onto the ball $B(0, n) \subset H^1$ defined by

$$\pi_n(v) = \begin{cases} v, & \text{if } |v|_{H^1} \le n, \\ \frac{n}{|v|_{H^1}} v, & \text{if } |v|_{H^1} > n. \end{cases} \tag{10.11}$$

It is well known, see for instance [11, Lemma 2.3], that the map $\pi_n : H^1(0,1) \to H^1(0,1)$ is globally Lipschitz with Lipschitz constant 1. Moreover,

$$\begin{aligned} |(D_x(\pi_n u))^2 - (D_x(\pi_n v))^2|_{L^1} &\le 2n|u - v|_{H^1}, \text{ for all } u, v \in H^1, \\ |(\pi_n u)^2|_{L^1} &\le \big[n \wedge |u|_{H^1}\big]|u|_{H^1}, \text{ for all } u \in H^1. \end{aligned} \tag{10.12}$$

Next we define maps $\mathbf{F}_n : H^1 \to L^2$ and $\mathbf{G}_n : H^1 \to \mathcal{L}(E, H^1)$ by analogous formulae

$$\begin{aligned} \mathbf{F}_n(u) &= -\mathcal{S}_u((\pi_n \circ u)_x, (\pi_n \circ u)_x), \; u \in H^1, \\ \mathbf{G}_n(u) &= G(\pi_n u), \; u \in H^1. \end{aligned}$$

Note that $\mathbf{F}_n = \mathbf{F}$ and $\mathbf{G}_n = \mathbf{G}$ on on the ball $B(0, n)$ in H^1. Moreover, since the function G is Lipschitz continuous on the ball $B(0, n)$ in H^1, it follows, see for instance the proof of [10, Corollary 3] that $\mathbf{G}_n : H^1 \to H^1$ is globally Lipschitz. Finally, the Lipschitz continuity and boundedness of the map $\mathbf{F}_k : H^1 \to L^1$ can be derived as in [11]. Thus have the following result.

Lemma 10.9. *The functions* $\mathbf{F} : H^1 \to L^1$ *and* $\mathbf{G} : H^1 \to \mathcal{L}(E, H^1)$ *are Lipschitz continuous on balls, the function* $\mathbf{G}$ *is of* C^1*-class and the the function*

$$\mathrm{tr}_K(G' \circledast G) : H^1 \ni u \mapsto \mathrm{tr}_K(G'(u)G(u)) \in H^1$$

is Lipschitz continuous on balls. For each $k \in \mathbb{N}$*, the functions* $\mathbf{F}_k : H^1 \to L^1$*,* $\mathbf{G}_k : H^1 \to \mathcal{L}(E, H^1)$ *and* $\mathrm{tr}_K(\mathbf{G}_k' \circledast \mathbf{G}_k) : H^1 \to H^1$ *are globally Lipschitz continuous, i.e. there exists a constant* C_k *such that for all* $u, v \in H^1$

$$\begin{aligned} |\mathbf{F}_k(u) - \mathbf{F}_k(v)|_{L^1} \;&+\; |\mathbf{G}_k(u) - \mathbf{G}_k(v)|_{\mathcal{L}(E,H^1)} \\ &+\; |\mathrm{tr}_K(\mathbf{G}_k'(u)\mathbf{G}_k(u)) - \mathrm{tr}_K(\mathbf{G}_k'(v)\mathbf{G}_k(v))|_{H^1} \le C_k|u - v|_{H^1}. \end{aligned} \tag{10.13}$$

We will also need the following result which is related to the proofs of Lemmae 2.11 and 2.12 in [11].

Lemma 10.10. *Let* $\left(e^{-tA}\right)_{t\ge 0}$ *be the heat semigroup on the scale of Banach spaces* $L^p(\mathbb{S}^1, \mathbb{R}^d)$*,* $p \in [1, \infty)$*. Then for each* $\alpha \ge 0$*, there exists a constant* $C = C_\alpha > 0$ *such that*

$$\|e^{-tA}\|_{\mathcal{L}(L^1, H^{\alpha,2})} \le C t^{-\frac{1}{4} - \frac{\alpha}{2}}, \; t > 0. \tag{10.14}$$

In particular, for each for each $T > 0$ *and* $\alpha \in [0, \frac{3}{2})$ *and for any bounded and strongly-measurable function* $v : (0, t) \to L^1(0, 1)$ *the following inequality holds*

$$\sup_{t\in[0,T]} |\int_0^t |e^{-(t-s)A} v(s)\, ds|_{H^{\alpha,2}} \le C_\alpha T^{\frac{3}{4} - \frac{\alpha}{2}} \sup_{t\in[0,T]} |v(t)|_{L^1}. \tag{10.15}$$

The following result can be proved by using Lemma 10.9 and employing similar methods as used in the proof of Theorem 2.14 in [11]. One should point out here that this result is different from more standard existence results as those for instance in Theorem 4.3 in [9].

Proposition 10.11. *Let us fix* $p > 2$*. Let the initial data* ξ *from Assumptions 9.2 satisfy in addition condition* (9.6)*.*

Then there exists a unique H^1*-valued continuous process* u_k *satisfying*

$$\mathbb{E} \sup_{t\in[0,T]} |u_k(s)|_{H^1}^p < \infty \tag{10.16}$$

and such that for all $t \in [0, T]$*,* $\mathbb{P}$*-a.s.*

$$\begin{aligned} u_k(t) \;&=\; e^{-tA}\xi + \int_0^t e^{-(t-s)A} \mathbf{F}_k(u_k(s))\, d\, s + \int_0^t e^{-(t-s)A} \mathbf{G}_k(u_k(s))\, dW(s) \\ &+\; \int_0^t e^{-(t-s)A} \mathrm{tr}_K[\mathbf{G}_k' \circledast \mathbf{G}_k](u_k(s))\, d\, s. \end{aligned} \tag{10.17}$$

Moreover,

$$\mathbb{E} \int_0^T |u_k(s)|_{H^2}^2\, ds < \infty. \tag{10.18}$$

Proof of Proposition 10.11. As mentioned above the proof of the first part follows the ideas from the proof of Theorem 2.14 in [11]. The proof of the second part uses ideas from [15]. Since $\xi \in L^p(\Omega, H^1) \subset L^2(\Omega, H^1)$ and $H^1 = \mathrm{Dom}(A^{1/2}) = (D(A), L^2)_{1/2,2}$, by invoking [46] we infer that the 1st term on the RHS of (10.17) satisfies the condition (10.18). Because u_k satisfies the condition (10.16) in view of the Lipschitz condition (10.13) satisfied by $\mathbf{G}_k$ and $\mathrm{tr}_{\mathrm{K}}(\mathbf{G}'_k \circledast \mathbf{G}_k)$, both the 3rd and the 4th terms on the RHS of (10.17) satisfy the condition (10.18). The only difficulty lies with the 2nd term because so far we only know that for each $t > 0$, $\mathbf{F}_k(u_k(t))$ belongs to L^1. By [46] again, it is enough to show that $\mathbb{E}\int_0^T |\mathbf{F}_k(u_k(s))|_{L^2}^2\, ds < \infty$. In view of the definition of $\mathbf{F}_k$, it is enough to show that

$$\mathbb{E}\int_0^T |(u_k)_x(s)|_{L^4}^4\, ds < \infty. \tag{10.19}$$

Obviously, it is enough to show that each term on the RHS of (10.17) satisfies the above condition (10.19). To this end it is sufficient to verify the following two claims.
Claim 1. If $v \in L^2(0,T;H^1) \cap L^\infty(0,T;L^2)$ then $v \in L^4(0,T;L^4)$ and there exists a constant $C > 0$ such that

$$\int_0^T |v(s)|_{L^4}^4\, ds \le CT^{1/2} \sup_{t\in[0,T]} |v(t)|_{L^2}^2 \Big(\int_0^T |v(s)|_{L^2}^2 \Big)^{1/2}.$$

Claim 2. If $\alpha > \frac{5}{4}$, then there exists a constant $C > 0$ such that for all v

$$\int_0^T |v_x(s)|_{L^4}^4\, ds \le C \sup_{t\in[0,T]} |v(t)|_{H^\alpha}^4.$$

Claim 1 follows from a special case of the Gagliardo-Nirenberg inequality $|v|_{L^4}^4 \le C|v|_{L^2}^3 |v_x|_{L^2}$. Claim 2 follows from a special case of the Sobolev embedding (valid for $\beta > \frac{1}{4}$) that

$$|v|_{L^4} \le C|v|_{H^\beta}.$$

Therefore, the proof of Proposition 10.11 is completed by applying Claim 2 to the 2nd term on the RHS of (10.17) and Claim 1 to all three remaining terms. ∎

We will apply Lemma 9.7 to the process u_k and the function $\mathbf{Q}$. We have the following result.

Lemma 10.12. *The map $\mathbf{Q} : H^1 \to H^1$ is of C^2-class and, with $u, v, z \in H^1$, it satisfies*

$$\mathbf{Q}'(u)v = h'(u)v, \quad \mathbf{Q}''(u)(v,z) = h''(u)(v,z),$$

$$\mathbf{Q}'(u)[-Au] = -A\mathbf{Q}(u) + \mathbf{L}(u), \quad u \in H^2, \tag{10.20}$$

$$\mathbf{Q}'(u)[\mathbf{G}(u)] = [\mathbf{G} \circ \mathbf{Q}](u), \quad u \in H^1(\mathbb{S}^1, O), \tag{10.21}$$

where, with the map B being defined in (10.2)*, the function $\mathbf{L} : H^2 \to L^2$ is defined by*

$$\mathbf{L}(u) = B_u(u_x, u_x).$$

Proof. Identity (10.20) can be proved in the same way as identity (10.6). Identity (10.21) is a consequence of the invariance property (10.10). Indeed, if $\xi \in E$, then

$$\begin{aligned} \mathbf{Q}'(u)[\mathbf{G}(u)\xi] &= h'(u)Y_u\xi = Y_{h(u)}\xi \\ &= \mathbf{G}(h(u))(\xi) = (\mathbf{G}(\mathbf{Q}(u))(\xi). \end{aligned}$$

∎

Define now the following two auxiliary functions

$$\widetilde{\mathbf{F}}_k : H^1 \ni u \mapsto \mathbf{Q}'(u)\big(\mathbf{F}_k(u)\big) - \mathbf{L}(u) = h'(u)(\mathbf{F}_k(u)) - B_u(u_x,u_x) \in L^1, \tag{10.22}$$

$$\widetilde{\mathbf{G}}_k : H^1 \ni u \mapsto \mathbf{Q}'(u) \circ \mathbf{G}_k(u) = h'(u)(\mathbf{G}_k(u)) \in \mathcal{L}(E,H^1). \tag{10.23}$$

This leads to the following result that rigorously expresses the fact that $\widetilde{\mathbf{F}}_k$, resp. $\widetilde{\mathbf{G}}_k$ is the push forward by $\mathbf{Q}$ of $\mathbf{F}_k$, resp. $\mathbf{G}_k$.

Proposition 10.13. *The following identities hold.*

$$\widetilde{\mathbf{F}}_k(u) = \mathbf{F}_k(\mathbf{Q}(u)), \quad \widetilde{\mathbf{G}}_k(u) = \mathbf{G}_k(\mathbf{Q}(u)), \ u \in H^1, \tag{10.24}$$

$$\widetilde{\mathbf{G}}_k'(u)\widetilde{\mathbf{G}}_k(u) = \mathbf{G}_k'(\mathbf{Q}(u))\mathbf{G}_k(\mathbf{Q}(u)), \ \ u \in H^1(\mathbb{S}^1,O). \tag{10.25}$$

Proof of Proposition 10.13. It is enough to prove the identities for the original operators, i.e. without the subscript k.

We begin with the second identity in (10.24). By the invariance property (10.10) of the function Y and the identities (10.21) and (10.23) we have,

$$\begin{aligned}[\mathbf{G}(\mathbf{Q}(u))](\xi) &= Y_{\mathbf{Q}(u)(\cdot)}\xi(\cdot) = Y_{h(u(\cdot))}\xi(\cdot) \\ &= h'(u(\cdot))Y_{u(\cdot)}\xi(\cdot) = [h'(u)\mathbf{G}(u)](\xi) = [\widetilde{\mathbf{G}}(u)](\xi).\end{aligned}$$

To prove the first part of (10.24) we can argue as in the proof of identity (10.6).

Identity (10.25) is a consequence of identity (10.21). ∎

Let us also observe that it follows from Lemma 10.12 that the assumptions of Lemma 9.7 are satisfied with the linear operator B being equal to A. Thus we have the following fundamental result.

Corollary 10.14. *Let u_k be the solution to* (10.17) *as in Proposition 10.11 and a process $\widetilde{u}_k$ be defined by the following formula*

$$\widetilde{u}_k = \mathbf{Q} \circ u_k. \tag{10.26}$$

Then for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\widetilde{u}_k(t) &= e^{-tA}\mathbf{Q}(\xi) + \int_0^t e^{-(t-s)A}\widetilde{\mathbf{F}}_k(u_k(s))\,ds + \int_0^t e^{-(t-s)A}\widetilde{\mathbf{G}}_k(u_k(s))\,dW(s) \\ &+ \int_0^t e^{-(t-s)A}\mathrm{tr}_K[\widetilde{\mathbf{G}}_k'(u_k(s))\widetilde{\mathbf{G}}_k(u_k(s))]\,d\,s.\end{aligned} \tag{10.27}$$

10.3. **Construction of a maximal local solution.** In the first part of this subsection we will show that the approximate solutions stay on the manifold M. This will follow from Corollary 10.14. As usual, we begin with some notation.

Let for each $k \in \mathbb{N}$, et u_k be the solution to problem (10.17). Let us define the following four $[0,\infty]$-valued functions on Ω.

$$\begin{aligned}\tau_k^1 &= \inf\{t \in [0,T] : |u_k(t)|_{H^1} > k\},\\ \tau_k^2 &= \inf\{t \in [0,T] : |\widetilde{u}_k(t)|_{H^1} > k\},\\ \tau_k^3 &= \inf\{t \in [0,T] : \exists x \in \mathbb{S}^1 :\ u_k(t,x) \notin O\},\\ \tau_k &= \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3.\end{aligned}$$

The following result is borrowed from [18], see Lemma 5.4.

Lemma 10.15. *Each function τ_k^j, $j = 1, 2, 3$, $k \in \mathbb{N}$, is a stopping time.*

Proposition 10.16. *The process u_k and $\widetilde{u}_k$ coincide on $[0, \tau_k)$ almost surely. In particular, $u_k(t, x) \in M$ for $x \in \mathbb{S}^1$ and $t \le \tau_k$ almost surely. Consequently,*

$$\tau_k = \tau_k^1 = \tau_k^2 \le \tau_k^3.$$

Proof. By Proposition 10.13 we infer that for all $s \in [0, T]$, $x \in \mathbb{S}^1$, $\mathbb{P}$-a.s.

$$\begin{aligned}
1_{[0,\tau_k)}(s)[\widetilde{\mathbf{F}}_k(u_k(s))](x) &= 1_{[0,\tau_k)}(s)[\mathbf{F}_k(\widetilde{u}_k(s))](x), \\
1_{[0,\tau_k)}(s)[\widetilde{\mathbf{G}}_k(u_k(s))e](x) &= 1_{[0,\tau_k)}(s)[\mathbf{G}_k(\widetilde{u}_k(s))e](x), \quad e \in \mathrm{K}, \\
1_{[0,\tau_k)}(s)\mathrm{tr}_{\mathrm{K}}[\widetilde{\mathbf{G}}_k' \circledast \widetilde{\mathbf{G}}_k](u_k(s)) &= 1_{[0,\tau_k)}(s)\mathrm{tr}_{\mathrm{K}}[\mathbf{G}_k' \circledast \mathbf{G}_k](\widetilde{u}_k(s)).
\end{aligned}$$

Let us denote

$$p(t) = |u_k(t) - \widetilde{u}_k(t)|_{L^2}^2, \quad t \in [0, T]. \tag{10.28}$$

Then the process p stopped at τ_k is continuous and uniformly bounded. Note that since $\xi = \mathbf{Q}(\xi)$, we infer that $p(0) = 0$. Moreover, by the Itô Lemma from [56] and Lemma 10.9, we can find a continuous martingale I with $I(0) = 0$ such that for all $k \in \mathbb{N}$,

$$\begin{aligned}
p(t \wedge \tau_k) &\le \int_0^t 1_{[0,\tau_k)(s)}|\mathbf{F}_k(u_k(s)) - \mathbf{F}_k(\widetilde{u}_k(s))|_{H^{-1}}^2 \, ds \\
&+ \int_0^t 1_{[0,\tau_k)(s)}|\mathbf{G}_k'(u_k(s))\mathbf{G}_k(u_k(s)) - \mathbf{G}_k'(u_k(s))\mathbf{G}_k(\widetilde{u}_k(s))|_{L^2}^2 \, ds \\
&+ \int_0^t 1_{[0,\tau_k)(s)}|\mathbf{G}_k(u_k(s)) - \mathbf{G}_k(\widetilde{u}_k(s))|_{\mathcal{T}_2(\mathrm{K},L^2)}^2 \, ds + I(t \wedge \tau_k) \\
&\le 3C \int_0^t 1_{[0,\tau_k)}(s)|u_k(s) - \widetilde{u}_k(s)|_{H^1}^2 \, ds + I(t \wedge \tau_k) \\
&\le 3C \int_0^t p(s \wedge \tau_k) \, ds + I(t \wedge \tau_k), \ t \in [0, T].
\end{aligned}$$

Therefore, by taking the expectation and then applying the the Gronwall lemma, we infer that $p = 0$ on $[0, \tau_k]$ almost surely. In other words, $\mathbb{P}$ almost surely, $u_k = \widetilde{u}_k$ on $[0, \tau_k]$. Consequently, $\mathbb{P}$-a.s. $u_k(t, x) \in O$ and $u_k(t, x) = h(u_k(t, x))$ for $x \in \mathbb{S}^1$ and $t \le \tau_k$. Hence, by Corollary 10.3 (or Lemma 10.8), $\mathbb{P}$-a.s. $u_k(t, x) \in M$ for $x \in \mathbb{S}^1$ and $t \in [0, \tau_k]$. Therefore, $\tau_k \le \tau_k^3$ and so $\tau_k = \tau_k^1 \wedge \tau_k^2$. Finally, since $p = 0$ on $[0, \tau_k]$ we infer that $\tau_k^1 = \tau_k^2$. ∎

Remark 10.17. Although the process $u_k - \widetilde{u}_k$ is H^1-valued, there is no error in considering the L^2 norm of it (and not the H^1 norm) in order to prove that this process is equal to a 0 process. Moreover, we had to use the framework of Pardoux for the Gelfand triple $H^1 \subset L^2 \subset H^{-1}$ (and not the $H^2 \subset H^1 \subset L^2$ one) because we had to use the Lipschitz property of $\mathbf{F}_k$. We have implicitly used an embedding $L^1 \subset H^{-1}$.

The same comments apply to the proof of Proposition 10.18 below.

In the second part of this subsection we will show that the approximate solutions extend each other. To be precise we will prove the following result.

Proposition 10.18. *Let $k \in \mathbb{N}$. Then $u_{k+1}(t,x,\omega) = u_k(t,x,\omega)$ on $x \in \mathbb{S}^1$, $t \leq \tau_k(\omega)$, and $\tau_k(\omega) \leq \tau_{k+1}(\omega)$ almost surely.*

Proof. Define a process p as before by formula (10.28). As in the proof of Proposition 10.16, we apply the Itô Lemma from [56]. Since $p(0) = 0$ we can find continuous martingale I satisfying $I(0) = 0$ such that for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$
\begin{aligned}
p(t \wedge \sigma_k) &\leq \int_0^t 1_{[0,\sigma_k)}(s) |\mathbf{F}_{k+1}(u_{k+1}(s)) - \mathbf{F}_k(u_k(s))|_{L^2}^2 \, ds \\
&+ \int_0^t 1_{[0,\sigma_k)}(s) |1_{[0,\tau_{k+1})} \mathrm{tr}_{\mathrm{K}}[\mathbf{G}'_{k+1} \circledast \mathbf{G}_{k+1}(u_{k+1}(s))] - \mathrm{tr}_{\mathrm{K}}[\mathbf{G}'_k \circledast \mathbf{G}_k(u_k(s))]|_{L^2}^2 \, ds \\
&+ \int_0^t 1_{[0,\sigma_k)}(s) |\mathbf{G}_{k+1}(u_{k+1}(s)) - \mathbf{G}_k(u_k(s))|_{\mathcal{T}_2(\mathrm{K},L^2)}^2 \, ds + I(t \wedge \sigma_k),
\end{aligned}
$$

where $\sigma_k := \tau_k \wedge \tau_{k+1}$. Since for $s \in [0,\sigma_k)$, $\mathbf{F}_k(u_k(s)) = \mathbf{F}(u_k(s)) = \mathbf{F}_{k+1}(u_{k+1}(s))$ and similarly, $\mathbf{G}_k(u_k(s)) = \mathbf{G}(u_k(s)) = \mathbf{G}_{k+1}(u_{k+1}(s))$, by the Lipschitz continuity of the functions $\mathbf{F}_{k+1}$, $\mathbf{F}_{k+1}$ and $\mathbf{G}'_{k+1} \circledast \mathbf{G}_{k+1}$ we infer that for some constant $C > 0$,

$$
p(t \wedge \sigma_k) \leq C \int_0^t 1_{[0,\sigma_k)}(s) p(s) \, ds + I(t \wedge \sigma_k) = C \int_0^{t \wedge \sigma_k} p(s \wedge \sigma_k) \, ds.
$$

Hence by the Gronwall Lemma we infer that $p = 0$ on $[0,\sigma_k]$. This implies that τ_k, $\tau_k \leq \tau_{k+1}$. Indeed, if $|\xi|_{H^1} > k+1$ then $\tau_{k+1} = \tau_k = 0$ and if if $k < |\xi|_{H^1} \leq k+1$ then $\tau_{k+1} > 0$ and $\tau_k = 0$. Thus, one can assume that $|\xi|_{H^1} \leq k$. If τ_{k+1} were smaller than τ_k then by the just proved property we would have $u_k(t) = u_{k+1}(t)$ for $t \in [0,\tau_{k+1}]$. Hence $|u_k(0)|_{H^1} \leq k$ and $|u_k(\tau_{k+1})|_{H^1} \geq k+1$ and therefore we can find $\bar{t} \in [0,\tau_{k+1})$ such that $|u_k(\bar{t})|_{H^1} = k + \frac{1}{2}$. This implies that $\tau_k \leq \bar{t}$ and this contradicts the assumption that $\tau_{k+1} < \tau_k$. The proof is complete. ∎

By Proposition 10.18 the sequence $(\tau_k)_{k=1}^{\infty}$ of stopping times is non-decreasing and so the limit of (τ_k) exists. We denote it by τ, i.e. $\tau = \lim_{k\to\infty} \tau_k$. Moreover, we can define a process $\tilde{u}(t,x)$, $t \in [0,\tau)$, $x \in \mathbb{S}^1$ by $\tilde{u}(t,x,\omega) = u_k(t,x,\omega)$ provided k is so large that $t \in [0,\tau_k(\omega))$. Note that $\tilde{u}(t,\cdot) \in H^1$.

In the following subsection we will show that $\tau = T$ $\mathbb{P}$ almost surely.

10.4. **No explosion for approximate solutions.** In this final subsection we will show that the maximal local solution constructed in the previous subsection is a global solution. We begin with proving that the local maximal solution is a global one.

Proposition 10.19. *$\tau = T$ almost surely.*

Proof. We first notice that we have, for $t \in [0,T]$,

$$\begin{aligned} u_k(t) &= \xi - \int_0^t Au_k(s)\,ds + \int_0^t \mathbf{F}_k(u_k(s))\,ds + \int_0^t [\mathbf{G}_k' \circledast \mathbf{G}_k(u_k(s))]\,ds \\ &+ \int_0^t \mathbf{G}_k(u_k(s))\,dW(s). \end{aligned} \tag{10.29}$$

By applying the Itô Lemma from [56] and Lemma 10.9 we can find a continuous local martingale J_0 such that for $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \frac{1}{2}|\nabla u_k(t)|^2 &+ \int_0^t 1_{[0,\tau_k)}(s)\langle Au_k(s), Au_k(s)\rangle\,ds = \frac{1}{2}|\nabla\xi|^2 + J_0(t) \\ &+ \int_0^t 1_{[0,\tau_k)}(s)\langle Au_k(s), \mathbf{F}_k u_k(s)\rangle\,ds \\ &+ \int_0^t 1_{[0,\tau_k)}(s)\langle \nabla u_k(s), \nabla\mathbf{G}_k'(u_k(s))\mathbf{G}_k(u_k(s))u_k(s)\rangle\,ds, \end{aligned} \tag{10.30}$$

where the norms and the scalar product are those from the $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^d)$ space. Note the following fundamental property. If $u \in D(A)$ then, see (9.4),

$$\langle -\Delta u(x) + F(u(x)), F(u(x))\rangle = 0, \quad \text{for a.a. } x \in \mathbb{S}^1. \tag{10.31}$$

Since for $s \in [0,\tau_k)$, $\mathbf{F}_k(u_k(s)) = \mathbf{F}(u_k(s))$, in view of identity (10.31), equality (10.30) can be rewritten as

$$\begin{aligned} \frac{1}{2}|\nabla u_k(t\wedge\tau_k)|^2 &+ \int_0^t 1_{[0,\tau_k)}(s)|Au_k(s) - \mathbf{F}_k(u_k(s))|^2\,ds - \frac{1}{2}|\nabla\xi|^2 - J_0(t) \\ &= \int_0^t 1_{[0,\tau_k)}(s)\langle \nabla u_k(s), \mathrm{tr}_{\mathrm{K}}[\nabla\mathbf{G}_k'(u_k(s))\mathbf{G}_k(u_k(s))u_k(s)]\rangle\,ds, \\ &\le C\int_0^t 1_{[0,\tau_k)}(s)\big[1 + |\nabla u_k(s)|^2\big]\,ds = C\int_0^{t\wedge\tau_k} \big[1 + |\nabla u_k(s\wedge\tau_k)|^2\big]\,ds. \end{aligned} \tag{10.32}$$

Hence, for each $j \in \mathbb{N}$ there exists a constant K_j such that with $B_j = \big\{\omega \in \Omega : |\nabla\xi(\omega)|_{L^2}^2 \le j\big\}$, one has, by the Gronwall Lemma,

$$\mathbb{E}\,1_{B_j}[1 + |\nabla u_k(t\wedge\tau_k)|] \le K_j, \qquad t \in [0,T], \quad j \in \mathbb{N}. \tag{10.33}$$

Let us now fix $t \in [0,T)$. Then, since $1_{\{\tau_k \le t\}}|u_k(\tau_k)|_{\mathcal{H}_{r-\tau_k}} \ge k 1_{\{\tau_k \le t\}}$, we infer that

$$\log(1+k^2)\mathbb{P}\left(\{\tau_k \le t\} \cap B_j\right) \le \mathbb{E}\,1_{B_j} q(t\wedge\tau_k) \le C_{r,j}. \tag{10.34}$$

Since $\tau_k \nearrow \tau$ as $k \to \infty$, from (10.34) we infer that for all $t \in [0,T)$, $j \in \mathbb{N}$, $\mathbb{P}\left(\{\tau \le t\} \cap B_j\right) = 0$ what in turn implies that $\tau = T$ almost surely. This completes the proof. ∎

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