Exact Counting of Black Hole Microstates

Dileep Jatkar

Harish-Chandra Research Institute, Allahabad

Outline

1. **Introduction**
2. **Quarter BPS black Holes**
3. **N=2 Quiver Quantum Mechanics**

References and References therein!

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- I. Mandal and A. Sen, arXiv:1008.3801,
- J. M. Maldecena, A. Strominger and E. Witten, arXiv:hep-th/9711053,
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- J. Manschot, B. Pioline and A. Sen, arXiv:1207.2230 .

- Understanding Black Holes quantum mechanics is a crucial test for quantum theory of gravity.
- Bekenstein and Hawking showed that a black hole behaves like a thermodynamic system with temperature and entropy

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Puzzles

1. Can we give microscopic description of black hole thermodynamics?
2. How do we reconcile purely thermal evaporation of a black hole with the unitary evolution in quantum theory?

- Here we will deal with the first puzzle, namely we will give microscopic description of entropy of a special class of black holes.
- While understanding black hole evaporation (a.k.a information paradox) requires studying time evolution of the system, exact microscopic description of entropy is best done for black holes which do not radiate, *i.e.*, $T_{BH} = 0$ and hence have trivial time evolution.

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- We will consider quarter BPS (Dyonic) black holes in four dimensional $N = 4$ supersymmetric string theory.
- For illustration we will consider toroidally compactified heterotic string but similar analysis holds for CHL models as well as type II compactifications.
- 28 dimensional electric and magnetic charge vectors $(\vec{Q}, \vec{P})$, specify the black hole, in the heterotic frame.

- Using the Bekenstein-Hawking formula we compute the entropy of this black hole,

$$S_{BH}(\vec{Q}, \vec{P}) = \pi \sqrt{Q^2 P^2 - (\vec{Q} \cdot \vec{P})^2}, \qquad (2)$$

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- Our goal is not to reproduce this result but to obtain exact microstate counting formula whose large charge asymptotics reproduces black hole degeneracy, $\exp(S_{BH})$.
- Subleading terms correspond to two types of corrections to the Bekenstein-Hawking formula:
  1. Corrections due to higher derivative terms in the effective actions (Power suppressed corrections),
  2. Corrections due to subleading saddle points (Exponentially suppressed corrections).

Microstate Counting

- Microstate counting is convenient in type IIB string frame.
- We describe quarter BPS states in terms of D1-D5-P system at the core of Taub-NUT space(KK Monopole) in type IIB string theory compactified on $K3 \times S^1$.

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- We will count microstates in the weak coupling limit and extrapolate them to strong coupling.
- We need to compute a quantity which is protected under such an extrapolation.
- Since quarter BPS states break 12 out of 16 SUSYs, the helicity supertrace B_6 is an appropriate index to be computed.

$$B_6 = \frac{1}{6!} Tr\{(-1)^{2h}(2h)^6\}. \qquad (3)$$

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A Question

How do we relate the index B_6 to BH degeneracy $\exp(S_{BH})$?

Quarter BPS States

- The charge vectors $\vec{Q}$ and $\vec{P}$ specifying a quarter BPS state are not parallel to each other.
- Such a state with generic values of T-duality invariants Q^2, P^2 and $\vec{Q} \cdot \vec{P}$ can be specified by turning on a few charges.
- We will choose
  1. One KK monopole along $\tilde{S}^1$,
  2. One D5 brane wrapping $K3 \times S^1$,
  3. $\tilde{Q}_1 + 1$ D1 branes winding S^1,
  4. $-n$ units of momentum along S^1,
  5. J units of momentum along $\tilde{S}^1$.

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  5. J units of momentum along $\tilde{S}^1$.

- T duality invariants for this charge configuration are

$$Q^2 = 2n, \quad P^2 = 2\tilde{Q}_1, \quad \vec{Q} \cdot \vec{P} = J \,. \tag{4}$$

Weak coupling limit

- In the weak coupling limit D1-D5-P-KK system splits into three non-interacting parts.
  1. Excitations of KK monopole,
  2. Centre of mass motion of D1-D5 system in KK background,
  3. Relative motion of D1-D5 system along $K3$.
- Product of partition funtion of these subsystems gives

$$\begin{aligned} Z(p,q,r) &= Z_{KK}(q)\, Z_{CM}(q,r)\, Z_{D1D5}(p,q,r) \\ &= \sum_{\tilde{Q}_1,n,J} (-1)^J B_6(\tilde{Q}_1,n,J) p^{\tilde{Q}_1} q^n r^J \,. \end{aligned} \tag{5}$$

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\end{aligned}
$$

- We will study each system separately and combine it to get Z(p,q,r).
- BPS states are in the ground state of the right moving sector.

Excitations of KK monopole

Closed string excitations around KK monopole background can be computed by studying world sheet action of $K3$ compactified KK monopole. The world-sheet fields are determined by transverse oscillations as well as cohomology of $K3 \times TN$.

1. Oscillations in 3 $\perp$-direction $\Rightarrow$ 3 non-chiral bosons,
2. Reduction of complex 2-form of type IIB on TN gives 2 non-chiral bosons,
3. Reduction of self-dual 4-form on 4-forms of $K3 \times TN$ gives 19 left chiral and 3 right chiral bosons,
4. KK monopole breaks 8 out of 16 SUSYs of $K3$ compactified type IIB theory. Broken SUSY generators give 8 right-chiral fermionic modes on the world sheet.

Thus we have 24 left-chiral bosons and 8 right-chiral boson-fermion pairs.

BPS Excitations of KK monopole

BPS Excitations corresponds to keeping right-chiral sector in the ground state. Therefore BPS spectrum is captured by the spectrum of 24 left-chiral bosons.

$$Z_{KK}(q) = \eta^{-24}(q) \ . \tag{6}$$

D1D5 CM motion in TN

- TN metric has $U(1)_L \times SU(2)_R$ isometry and $SU(2)_{TL} \times SU(2)_{TR}$ tangent space symmetry.
- TN coordinates transform as $(\mathbf{2},\mathbf{2})$ under $SU(2)_{TL} \times SU(2)_{TR}$ and fermions transform as $(\mathbf{1},\mathbf{2})$+$(\mathbf{2},\mathbf{1})$.
- Left-chiral fermions are neutral under $U(1)_L$ whereas right-chiral fermions are superpartners of bosons and have ± 1 charge with respect to $U(1)_L$.

$$Z_{left-chiral\ Fermions} = \prod_{n=1}^{\infty} (1 - q^n)^4 \ . \tag{7}$$

- Left-chiral boson contribute

$$Z_{left-chiral\ Bosons} = \prod_{n=1}^{\infty} (1 - r\, q^n)^{-2} (1 - r^{-1}\, q^n)^{-2} \ . \tag{8}$$

D1D5 CM in TN

- Zero mode contribution of CM motion is captured by superparticle dynamics in *TN* space.

$$Z_{zero-modes} = \frac{r}{(1-r)^2} \,. \tag{9}$$

- Total partition function of CM motion is

$$Z_{CM} = \prod_{n=1}^{\infty} \frac{(1-q^n)^4}{(1-r\,q^n)^2(1-r^{-1}\,q^n)^2} \frac{r}{(1-r)^2} \,. \tag{10}$$

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- Notice that the zero-mode partition function can also be written as

$$Z_{zero-modes} = \frac{r^{-1}}{(1-r^{-1})^2}\,. \tag{11}$$

- Choice of expansion depends on angle between S^1 and $\tilde{S}^1$.

Relative Motion of D1-D5 system

- This corresponds to transverse fluctuations of D1 brane along D5 world-volume ($K3$).
- For a single D-string wrapped on S^1 and mometa along S^1 and $\tilde{S}^1$, the spectrum is captured by $K3$ elliptic genus.

$$\begin{aligned} F(\sigma, v) &= 8\left[\frac{\vartheta_2(\sigma, v)^2}{\vartheta_2(\sigma, 0)^2} + \frac{\vartheta_3(\sigma, v)^2}{\vartheta_3(\sigma, 0)^2} + \frac{\vartheta_4(\sigma, v)^2}{\vartheta_4(\sigma, 0)^2}\right], \\ &= \sum_{j,n} c(4n - j^2) q^n r^j \,. \end{aligned} \tag{12}$$

- Multi-string partition function is given by partition function of the symmetric product conformal field theory,

$$Z_{D1D5} = \frac{1}{p} \prod_{l,k,j,l\geq 0,k>0} (1 - q^l p^k r^j)^{-c(4lk - j^2)} \,. \tag{13}$$

Full Partition Function

1. Combining all these pieces we have

$$Z(p,q,r) = \frac{1}{pqr} \prod_{\{l,k,j\}} (1 - q^l p^k r^j)^{-c(4lk-j^2)} = \frac{1}{\Phi_{10}} , \quad (14)$$

2. Φ_{10} is the weight 10 Igusa cusp form of $SP(2, Z)$.
3. The helicity supertrace is given by

$$B_6(\vec{Q}, \vec{P}) = (-1)^{\vec{Q}\cdot\vec{P}} \int_C d\rho d\sigma dv \, \frac{p^{-P^2/2} q^{Q^2/2} r^{\vec{Q}\cdot\vec{P}}}{\Phi_{10}} , \quad (15)$$

$$p = \exp(2i\pi\rho), \quad q = \exp(2i\pi\sigma), \quad r = \exp(2i\pi v) \, .$$

4. Large charge asymptotics are controlled by zeros of Φ_{10}. Leading contribution comes from $\rho\sigma - v^2 + v = 0$.

Leading Asymptotics

1. Leading contribution is

$$\ln B_6 = S_{BH} - \phi\left(\frac{\vec{Q}\cdot\vec{P}}{P^2}, \frac{S_{BH}}{\pi P^2}\right) + \cdots\,, \tag{16}$$

$$S_{BH} = \pi\sqrt{Q^2P^2 - (\vec{Q}\cdot\vec{P})^2}\,, \quad \tau = \tau_1 + i\tau_2\,,$$

$$\phi(\tau_1,\tau_2) = 12\ln\tau_2 + 24(\ln\eta(\tau) + \ln\eta(-\bar{\tau}))\,.$$

2. $\phi(\tau_1,\tau_2)$ is a contribution from Gauss-Bonnet term.
3. $\cdots$ contain terms suppressed in $1/Q^2$ as well as exponentially suppressed terms.

Walls of Marginal Stability

- The integrand and the integration measure of the helicity supertrace formula

$$B_6(\vec{Q},\vec{P}) = (-1)^{\vec{Q}\cdot\vec{P}} \int_C d\rho d\sigma dv \; \frac{p^{-P^2/2} q^{Q^2/2} r^{\vec{Q}\cdot\vec{P}}}{\Phi_{10}}\,, \qquad (17)$$

  are duality invariant but the integration contour is not.
- For example, if angle between S^1 and $\tilde{S}^1$ is acute then $Z_{zero-modes}$ is expanded in powers of $1/r$ whereas if the angle is obtuse then the expansion is in powers of r.
- This gives rise to different degeneracies across $v = 0$ wall.
- States lost across this wall correspond to the decay of a quarter BPS state into two half BPS states, $(\vec{Q},\vec{P}) \to (\vec{Q},0) + (0,\vec{P})$.

- Other walls of marginal stability are obtained from this wall using S-duality transformation. For example,

$$(\vec{Q},\vec{P}) \to (a\vec{Q}+b\vec{P}, c\vec{Q}+d\vec{P})+((1-a)\vec{Q}-b\vec{P}, -c\vec{Q}, (1-d)\vec{P})$$

corresponds to the wall at $c\rho - b\sigma + (a-d)v = 0$.

- Structure of walls of marginal stability is well understood and is governed by rank 3 Borcherds-Kac-Moody(BKM) algebra with Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

- This analysis can be repeated for CHL as well as type II models with $N = 4$ SUSY. Again walls are well understood and are related to different rank 3 BKM algebras.

Dyonic BH vs BMPV BH

- Near horizon geometry of 4D dyonic black hole and 5D BMPV black hole is identical but their microstate counting differs.
- This observation led to the understanding that not all microstates are black hole degrees of freedom.
- In dyonic black hole case, all KK monopole excitations and all fermi zero modes live outside the horizon whereas in the BMPV case all fermi zero modes live outside the horizon. Let us call them Hair.

Index vs Degeneracy

- Using the distinction between black hole and hair degrees of freedom we can write

$$B_6 = \frac{1}{6!} Tr\{(-1)^{2h}(2h)^6\} = \frac{1}{6!} Tr\{(-1)^{2h_h+2h_{bh}}(2h_h+2h_{bh})^6\}.$$

- Since hair modes soak up all fermi zero modes, we get

$$B_6 = (-1)^{2h_{bh}} d_{bh}(\vec{Q}_{bh}, \vec{P}_{bh}, h_{bh})\; B_{6,hair}(\vec{Q}_{hair}, \vec{P}_{hair}), \quad (18)$$

  where d_{bh} is black hole degeneracy and h_{bh} is angular momentum carried by the black hole.

- Since SUSY black holes in 4D have zero angular momentum, the Witten index piece drops out.
- Thus starting from helicity index we can determine black hole degeneracy after removing the contribution of hair degrees of freedom.

N=2 BPS black Holes in 4D

- Let us now switch gears and talk about supersymmetric black holes in 4D $N = 2$ string theory.
- Unlike in the $N = 4$ theory, we do not have exact microstate formula in this case.
- Here I will concentrate on the modest goal of reviewing recent progress in Quiver quantum mechanics.

Why Quiver Quantum Mechanics?

- Black holes in $N = 2$ string theory are parametrized a charge vector $\Gamma = \{p^0, p^A, q_A, q_0\}$ which are {D6,D4,D2,D0} charges respectively.

$$\langle \Gamma, \tilde{\Gamma} \rangle = -p^0 \tilde{q}_0 + p^A \tilde{q}_A - q_A \tilde{p}^A + q_0 \tilde{p}^0 \qquad (19)$$

- A generic SUSY black hole corresponds to several of these branes wrapping cycles of Calabi-Yau manifold.
- Any given charge configuration corresponds to quantum mechanics of a quiver gauge theory.
- A microstate counting formula would presumably correspond to summing over all possible quiver gauge theories corresponding to all possible charge vectors.

Summary of Results

- BPS states in $N = 2$ theory are either multi-centered solutions in supergravity(strong coupling description) or they are states in the Coulomb/Higgs branch of quiver gauge theory.
- Coulomb branch states can be shown to be in 1-1 correspondence with states in supergravity but there is no such representation for "Pure Higgs" states.
- "Pure Higgs" states are zero angular momentum states and could potentially contribute to single-centered black hole configurations.
- For quiver quantum mechanics with more than 2 nodes growth of "Pure Higgs" states is exponential.
- "Pure Higgs" states are also known to be stable under wall crossing giving further credence to their relation with single centered black holes.

Quantum Mechanics of Quiver Gauge Theory

- Weak coupling description of D-branes wrapped on cycles of Calabi-Yau, can be given in terms of $N = 4$ supersymmetric $0 + 1$-D quiver quantum mechanics.
- Let us for concreteness consider 3 node quiver. Quantum mechanics contains 3 vector multiplets $\{\vec{x}_p, \lambda_p, D_p\}$ and chiral multiplets corresponding to string stretched between different nodes $\{\phi^{\alpha}_{pq}, \psi^{\alpha}_{pq}, F^{\alpha}_{pq}\}$.
- No. of chiral multiplets stretched between pair of nodes is $\Gamma_{pq} = \langle\Gamma_p, \Gamma_q\rangle$.
- BPS states exist on both Coulomb and Higgs branch. We will concentrate mainly on the Higgs branch.
- We will consider closed 3 node quiver which allows us to write superpotential, which is trilinear in chiral multiplets.

Quiver Gauge Theory Lagrangian

$$\mathcal{L} = \sum_p \frac{m_p}{2}\left(\dot{x}_p^2 + D_p^2 + 2i\bar{\lambda}\dot{\lambda}\right) - \theta_p D_p + \sum_{q\to p}\left(|\dot{\phi}_{pq}|^2 + F_{pq}^2 + i\bar{\psi}_{pq}\dot{\psi}_{pq}\right)$$
$$- \sum_{q\to p}\Big[(x_{pq}^2 + D_{pq})\phi_{pq}^2 + \bar{\psi}_{pq}\sigma^i x_{pq}^i \psi_{pq} - i\sqrt{2}(\bar{\phi}_{pq}\lambda_{pq}\epsilon\psi_{pq} - h.c.\Big]$$
$$+ \sum_{q\to p}\left(\frac{\partial W(\phi)}{\partial \phi_{pq}^a} F_{pq}^a + h.c.\right) + \left(\frac{\partial^2 W(\phi)}{\partial\phi_{pq}^\alpha \partial\phi_{pq}^\beta}\psi^\alpha\epsilon\psi^\beta + h.c.\right),$$

where,

$$W(\phi) = \omega_{\alpha\beta\gamma}\phi_{12}^\alpha\phi_{23}^\beta\phi_{31}^\gamma\,, \tag{20}$$

and θ_p are Fayet-Iliopoulos terms.
Bosonic potential is obtained by integrating out auxilliary fields D_p and F_{pq}.

Higgs Branch

- Supersymmetric minimum corresponds to $\vec{x}_{pq} = 0$ which implies we are in the Higgs branch.
- F and D term constraints are

$$\omega_{\alpha\beta\gamma}\phi_{23}^{\beta}\phi_{31}^{\gamma} = 0, \;\; \omega_{\alpha\beta\gamma}\phi_{12}^{\alpha}\phi_{31}^{\gamma} = 0, \;\; \omega_{\alpha\beta\gamma}\phi_{12}^{\alpha}\phi_{23}^{\beta} = 0,$$

$$|\phi_{12}|^2 - |\phi_{31}|^2 = -\theta_1, |\phi_{23}|^2 - |\phi_{12}|^2 = -\theta_2, |\phi_{31}|^2 - |\phi_{23}|^2 = -\theta_3.$$

- With the condition $\sum_p \theta_p = 0$, choose $\theta_1, \theta_2 < 0$ and $\phi_{31} = 0$.
- D-terms correspond to $CP^{\Gamma_{12}-1} \times CP^{\Gamma_{23}-1} \equiv X$.
- F-term constraint gives Γ_{31} quadratic equations satisfied by coordinates of X.
- Thus Higgs branch is a complete intersection manifold, Y.

Cohomology of Complete Intersection Manifolds

- BPS states on Higgs branch are given by cohomology elements of the Higgs branch.
- Using Lefschetz hyperplane theorem one can show that cohomology of Y is related to that of X except for the middle cohomology.

$$
\begin{aligned}
& = b^i(X), \quad i < \Gamma_{12} + \Gamma_{23} - \Gamma_{31} - 2 \\
b^i(Y) & = b^i(X) + \beta(\Gamma_{12}, \Gamma_{23}, \Gamma_{31}), \ i = \Gamma_{12} + \Gamma_{23} - \Gamma_{31} - 2 \\
& = b^{2\Gamma_{31}+i}(X), \quad i > \Gamma_{12} + \Gamma_{23} - \Gamma_{31} - 2
\end{aligned}
$$

- Thus upto determining $\beta(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$ cohomology of the Higgs branch can be determined entirely from the cohomology of the embedding space X.

Cohomology of CP^N

- CP^N cohomology consists of Kähler form ω and its powers. All odd cohomology groups vanish.

$$b^{2i+1}(CP^N) = 0, \quad b^{2i}(CP^N) = 1 \;\; \forall i \leq N. \tag{21}$$

- Define Lefschetz $SU(2)$ operators

$$J_+ = \omega\wedge\ , \;\; J_- = i(\omega, \cdot)\ , \;\; J_z = \frac{deg - N}{2}\ . \tag{22}$$

- $N+1$ cohomology elements of CP^N forms spin $N/2$ representation of Lefschetz $SU(2)$ algebra.
- Cohomology of $CP^N \times CP^M$ corresponds to tensor product of spin $N/2$ and $M/2$ representation, which decomposes into irreps with spin ranging from $|N-M|/2$ to $(N+M)/2$.

Cohomology of Higgs Branch Y

- Using the cohomology of X we can determine cohomology of Y upto determining $\beta(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$. Let us denote the degeneracy of BPS states derived from the induced cohomology as $N(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$.
- Note that $\beta(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$ does not transform under Lefschetz $SU(2)$.
- In fact, the Lefschetz $SU(2)$ corresponds to spatial rotation symmetry.
- Cohomology elements which transform under Lefschetz $SU(2)$, namely $N(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$, give rise to supergravity states and in the weak coupling belong to states of the Coulomb branch.
- Singlets of Lefschetz $SU(2)$ are the “Pure Higgs” states.

Degeneracy of "Pure Higgs" states

- Since we know cohomology of Y except $\beta(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$, if we can derive the Euler character of Y then we can determine $\beta(\Gamma_{12}, \Gamma_{23}, \Gamma_{31})$.
- Euler character can be determined using the contour integral formula

$$\chi(Y) = \oint dJ_1 \oint dJ_2 \left(\frac{J_1}{1+J_1}\right)^{-\Gamma_{12}} \left(\frac{J_2}{1+J_2}\right)^{-\Gamma_{23}} \left(\frac{J_1+J_2}{1+J_1+J_2}\right)^{-\Gamma_{31}}$$

- One can write the generating function such that Euler character is obtained by an appropriate contour integral.
- Define $x = J_1/(1+J_1)$, $y = J_2/(1+J_2)$ and use the formula $\sum q^a z^a = 1/(1-qz)$.

Degeneracy of “Pure Higgs” states

- The generating function for Euler character and for Coulomb degeneracies then become

$$Z_\chi = \frac{xy(1-xy)}{(1-x)^2(1-y)^2(1-xz-yz-xy+2xyz)}$$

$$Z_N = \frac{xy(1-xy+(2-x-y)xyz)}{(1-x)^2(1-y)^2(1-xy)(1-xz)(1-yz)}$$

- Generating function for $\beta(a,b,c)$ is

$$Z_\beta(x,y,z) = Z_\chi(-x,-y,-z) - Z_N(-x,-y,-z) \qquad (23)$$

Thus

$$Z_\beta(x,y,z) = \frac{x^2y^2z^2}{(1-xy)(1-xz)(1-yz)(1-xy-xz-yz-2xyz)}$$

Degeneracy of “Pure Higgs” states

- Note Z_β is symmetric in x, y and z, implying it is symmetric in Γ_{12}, Γ_{23} and Γ_{31}.
- For large a, b and c,

$$\beta(a,b,c) \sim \frac{2}{\pi}\sqrt{\frac{abc(ABC)^3}{(aA+bB+cC)^7}}\,\frac{a^a b^b c^c}{A^A B^B C^C}\,2^{a+b+c}, \qquad (24)$$

where $A=-a+b+c$, $B=a-b+c$ and $C=a+b-c$.

- Thus we see that degeneracy of “Pure Higgs” states grows exponentially.
- Since these states do not belong to the Coulomb branch they do not change under wall crossing.
- Exponentially large degeneracy, zero angular momentum and stability under wall crossing make “Pure Higgs” states candidates for single centered black hole microstates.

Thank You!