On Late Time Tails in an Extreme Reissner-Nordström Black Hole: Frequency Domain Analysis

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based on

arXiv:1805.10655[gr-qc] with Bhattacharjee, Chow, Paul and Virmani

Annual Review Seminar 2018

Plan of the talk

Important Questions

- Penrose diagrams of extremal black holes are very different from the non-extremal ones.
- The question that has intrigued relativists over many years is whether extremal black holes are classically stable under linearized perturbations.

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- The question that has intrigued relativists over many years is whether extremal black holes are classically stable under linearized perturbations.
- How does a small initial perturbation evolve at late times in an extremal black hole background?
- How to reconcile this with firewall/fuzzball proposals?

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- However all of them have a subtle instability.
- A hint of this instability comes from the fact that test particles encounter null singularity just as they cross the event horizon of extremal black holes. Marolf 2010

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- On the horizon some of the derivatives of the scalar perturbation blow up, leading to an instability known as Aretakis instability. Aretakis '07; Lucietti et. al. '12
- This does not happen for non-extremal black holes.

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- The study of late time tails are important in understanding the no-hair theorems and internal structure of black holes.
- In this talk I will focus on Aretakis instability and Price's law in case of extremal Reissner-Nordström black hole in four space-time dimensions.

Outline of our work

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- We find that initial perturbations with generic regular behaviour across the horizon decays at late times as t^{-2l-2} near timelike infinity ($t \gg r_*$) and as u^{-l-1} near future null infinity ($u \ll r_*$).
- Extreme Reissner-Nordström black hole enjoys a special symmetry. We make use of this symmetry in our analysis.

Motivation

Outline

Aretakis Instability

Couch-Torrence Symmetry

Late time tails

First order correction to late time tails

Summary and Open problems

Aretakis Instability

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- A similar analysis for extremal black hole leads to a conservation law on the horizon.
- This conservation law lies at the heart of Aretakis Instability.

Aretakis Instability

- Consider extreme RN in ingoing Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dvdr + r^2 d\Omega^2. \qquad (1)$$

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- Mode decomposition

$$\Phi = \frac{1}{r}\sum_{l,m}\psi_l(r)Y_{lm}(\theta,\phi)e^{-i\omega t} = \sum_{lm}\phi_l(v,r)Y_{lm}(\theta,\varphi). \qquad (3)$$

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- This is a Aretakis constant.
- Since the combination is conserved both Φ and $\partial_r\Phi$ cannot decay on $\mathcal{H}^+$.
- Aretakis showed that Φ decays, hence $\partial_r\Phi$ does not decay.

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- Act with one more radial derivative on the wave equation

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- At late times as $v \to \infty$, $\partial_r^2 \Phi \Big|_{\mathcal{H}^+} \sim -\frac{I_0}{M^2} v \to \infty$.

Aretakis Constants

- For mode l apply ∂_r^l on the wave equation, the conserved Aretakis constants on the horizon are

$$A_l[\phi_l] = \frac{M^l}{(l+1)!}\partial_r^l[r\partial_r(r\phi_l)]\Big|_{r=M}. \tag{6}$$

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Aretakis constants are $A_l = c_{l+1} + \frac{l}{l+1}c_l$. Ori '13; Sela '15

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- All these properties of CT symmetry are extremely useful for our analysis.

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- The answer is Newman-Penrose constants.

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- These are examples of Newman-Penrose constants.

Relation between Aretakis and Newman-Penrose constants

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$$\begin{aligned}\phi_I &= \frac{1}{r}\left(c_0 + c_1\frac{M}{r} + (c_1 + c_2)\left(\frac{M}{r}\right)^2 \right.\\ &\quad \left. + (c_1 + 2c_2 + c_3)\left(\frac{M}{r}\right)^3 + \ldots\right). \end{aligned} \tag{13}$$

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- Comparing this solution (13) with (11) we find the relation between d and c coefficients.
- Plugging in the expression for the Newman-Penrose constants we have

$$N_l = c_{l+1} + \frac{l}{l+1} c_l, \qquad (14)$$

that is nothing but the Aretakis constant A_l.

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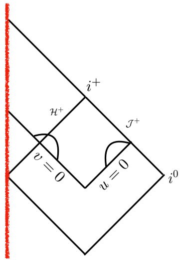
- The effective potential V_l is given by

$$V_l(r) = \left(1 - \frac{M}{r}\right)^2 \left[\frac{2M}{r^3}\left(1 - \frac{M}{r}\right) + \frac{l(l+1)}{r^2}\right].$$

Late time tails



- The initial data is composed of two functions $\psi_l^v(v)$ and $\psi_l^u(u)$. We take these functions of the form (and restrict to this case only)

$$\psi_l^v(v) = \psi_l(u=0,v) = \hat{d}_l \frac{R^l}{r^l} + \text{compactly supported data},$$

$$\psi_l^u(u) = \psi_l(u, v=0) = c_0 + c_1\left(\frac{r}{M}-1\right) + c_2\left(\frac{r}{M}-1\right)^2 + \dots .$$

Late time tails

- Due to the linearity, split the problem in two parts:
  (i) $\psi_l^v(v) = \psi_l(u = 0, v) \neq 0$, $\psi_l^u(u) = 0$,
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- Finally add their contributions to obtain the late time tails.
- Note that the initial data is horizon penetrating and extending to null infinity.
- Hence in general it has non-zero Aretakis and Newman-Penrose constants.

Late time tails

- We use the CT symmetry on $\psi^u(u)$ to obtain

$$\psi_l^v(v) = \hat{c}_0 + \hat{c}_1 \frac{R}{r} + \hat{c}_2 \frac{R^2}{r^2} + \ldots + \hat{c}_l \frac{R^l}{r^l} + \hat{c}_{l+1} \frac{R^{l+1}}{r^{l+1}} + \ldots . \quad (16)$$

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- Using linearity, the effective problem that we need to analyse becomes,

$$\begin{aligned}\psi_l^v(v) =& \hat{c}_0 + \hat{c}_1 \frac{R}{r} + \hat{c}_2 \frac{R^2}{r^2} + \ldots + (\hat{c}_l + \hat{d}_l) \frac{R^l}{r^l} + \hat{c}_{l+1} \frac{R^{l+1}}{r^{l+1}} + \ldots \\ & + \text{compactly supported data,}\end{aligned} \tag{17}$$

with $\psi_l^u(u) = 0$.

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- It is believed that the late time tail arises due to backscattering from the weakly curved asymptotic region. Price '71, Klauder '72, Gundlach '93
- For an initial data eq (17), there is a contribution to the late time tail in an ERN background that is not due to the curvature of the spacetime.
- The term $\hat{c}_{l+1}\frac{R^{l+1}}{r^{l+1}}$ results in a leading order tail even in flat space (Sela).

Late time tails

- The wave equation in flat space is

$$\left(\partial_r^2 - \partial_t^2 - \frac{l(l+1)}{r^2}\right)\psi_l(t,r) = 0. \qquad (18)$$

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- The Fourier transform of the field $\psi_l(t,r)$,

$$\psi_l(\omega,r) = \int_{-\infty}^{\infty} e^{i\omega t}\psi_l(t,r)dt, \qquad (19)$$

satisfies the equation

$$\left(-\omega^2 - \partial_r^2 + \frac{l(l+1)}{r^2}\right)\psi_l(\omega,r) = 0. \qquad (20)$$

Late time tails

- The wave equation in flat space is

$$\left(\partial_r^2 - \partial_t^2 - \frac{l(l+1)}{r^2}\right)\psi_l(t,r) = 0. \qquad (18)$$

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- The general solution to this equation is

$$\psi_l(\omega,r) = A(\omega)\sqrt{r}J_{l+1/2}(\omega r) + B(\omega)\sqrt{r}Y_{l+1/2}(\omega r).$$

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- To obtain regular solutions at $r = 0$ we must set $B(\omega) = 0$. Thus, we get

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- Make the ansatz $A(\omega) = 2\pi A_0\, \omega^p$. Solve for $\psi_l(t, r)$ and match the resulting answer at $u = 0$.
- This gives $p = k - 1/2$, and fixes the constant A_0.

Late time tails

- Finally we obtain

$$\psi_l(t,r) = -\,\frac{\hat{c}_k R^k 2^{k+1}\Gamma(k+1)}{\pi(2l+1)!!}\,\sin(k\pi)\;\Gamma(l-k+1)\; r^{l+1}\; t^{-(k+l+1)}\; F\left(\frac{l+k+2}{2},\frac{l+k+1}{2};l+\frac{3}{2};\frac{r^2}{t^2}\right). \tag{23}$$

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- The leading contribution to the late time tail comes from $k = l+1$.

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- Since flat space is conformal to $AdS_2 \times S^2$, the above results can be related to an AdS_2 analysis. Lucietti, Murata, Reall and Tanahashi 2012

Contributions due to asymptotic curvature of spacetime

- The propagation of linearized scalar waves on black hole backgrounds is described by the Klein-Gordon (KG) equation with an effective potential

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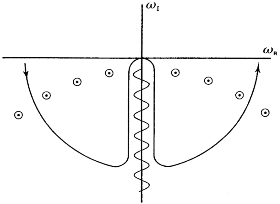
Contributions due to asymptotic curvature of spacetime



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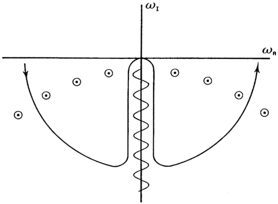

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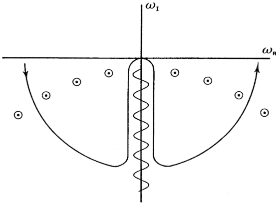

- **Prompt contribution:** Contribution to G coming from the large semicircle corresponds to short time response.
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- **Late time tails**: Contribution comes from the branch cut of $\tilde{G}(r_*, r'_*; \omega)$ along the negative imaginary axis in the complex ω plane.

Contributions due to asymptotic curvature of spacetime

- In order to construct the Green's function we look at solutions of the wave equation satisfying the following boundary conditions
  $\tilde{\psi}_l(r_*, \omega) \to e^{i\omega r_*}$ as $r_* \to \infty$,
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  $\tilde{\psi}_l(r_*, \omega) \to e^{-i\omega r_*}$ as $r_* \to -\infty$.
- For a second order ODE with homogeneous boundary conditions, the Green's function can be uniquely constructed using two auxiliary functions $f(r_*, \omega)$ and $g(r_*, \omega)$, where $f(r_*, \omega)$ satisfies the left boundary condition and $g(r_*, \omega)$ satisfies the right boundary condition.

Contributions due to asymptotic curvature of spacetime

- The Green's function is given by

$$\tilde{G}(r_*, r'_*; \omega) = \begin{cases} \frac{f(r_*,\omega)g(r'_*,\omega)}{W(\omega)}, & \text{if} \quad r_* < r'_* \\ \frac{f(r'_*,\omega)g(r_*,\omega)}{W(\omega)}, & \text{if} \quad r_* > r'_* \end{cases} \tag{28}$$

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- The late time tails come from the branch cut of $\tilde{G}(r_*, r'_*; \omega)$.
- In the low-frequency asymptotic expansion Andersson

$$G^C(r_*, r'_*, t) = -2\pi i M \sqrt{r_* r'_*} \int_0^{-i\infty} \omega \; J_{l+1/2}(\omega r_*) \; J_{l+1/2}(\omega r'_*) \; e^{-i\omega t} d\omega$$

$$\psi_l^C(r_*, t) = -\int_0^{\infty} \partial_t G^C(r_*, r'_*, t) \; \psi_l(0, r'_*) \; dr'_*. \tag{29}$$

Contributions due to asymptotic curvature of spacetime

- At timelike infinity, $\omega r_* \ll 1$ and we get

$$\psi_l(t, r_* | t \gg r_* \gg M) \sim \mu_l M t^{-2l-2}. \qquad (30)$$

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- Near null infinity, $\omega r_* \gg 1$, hence we get

$$\psi_l(t, r_*) \sim \mu_l M u^{-l-1}. \qquad (31)$$

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- The inversion map maps the decay behaviour near future null infinity to the decay behaviour v^{-l-1} near the horizon.
- Using the CT conformal isometry, we relate higher multipole Aretakis and Newman-Penrose constants for a massless scalar in an ERN black hole background. The relations involve Pascal matrices. We find new identities for these matrices.

Open problems

- Our analysis is valid in the asymptotic regions, either near infinity or near the horizon. It will be interesting to compute the correct radial dependence of the tail in full generality.

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  Camps, Hadar, Manton
- Eperon, Reall, Santos have shown that waves on a supersymmetric fuzzball decay differently than on an extremal black hole. It will be interesting to understand Price's law from a microscopic CFT analysis.

Thank You!