Ribbon 2–tubes and automorphisms of the reduced free group

Benjamin Audoux

Aix–Marseille Université

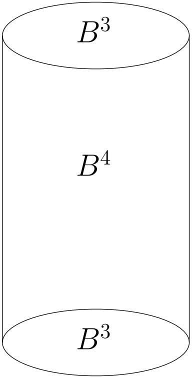
December 18th, 2013

joint work with P. Bellingeri, J-B. Meilhan, E. Wagner

Ribbon 2–tubes

Ribbon 2–tubes

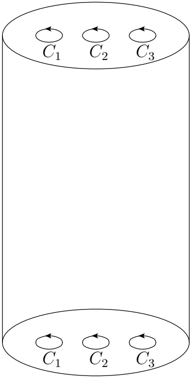
We consider B^4 seen as $B^3 \times I$.



Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.



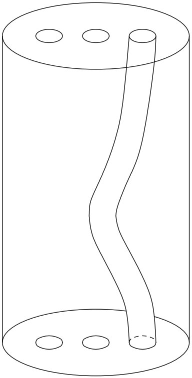
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0,1\}$;



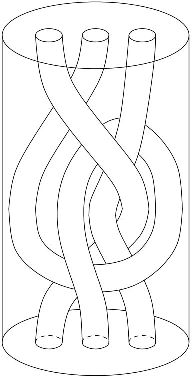
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;



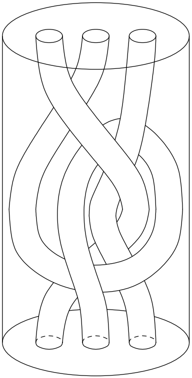
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a ribbon filling



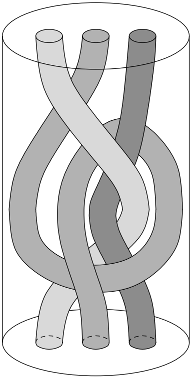
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a ribbon filling, that is immersed 3–balls $B_1, \cdots, B_n$ s.t.
  - $\partial B_i = T_i$;



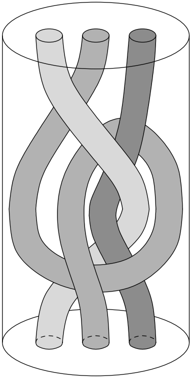
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a ribbon filling, that is immersed 3–balls $B_1, \cdots, B_n$ s.t.
    - $\partial B_i = T_i$;
    - the singular set is a finite union of ribbon disks.



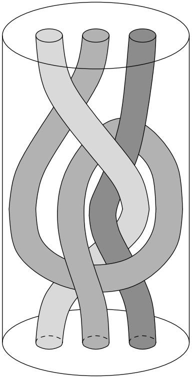
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0,1\}$;
- they admit a ribbon filling, that is immersed 3–balls $B_1, \cdots, B_n$ s.t.
  - $\partial B_i = T_i$;
  - the singular set is a finite union of ribbon disks.



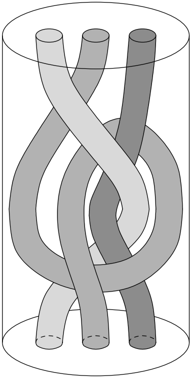
Ribbon 2–tubes

We consider B^4 seen as $B^3 \times I$.

We fix n disjoint and unlinked oriented circles $C_1, \cdots, C_n$ in B^3.

We consider embedded annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a ribbon filling, that is immersed 3–balls $B_1, \cdots, B_n$ s.t.
  - $\partial B_i = T_i$;
  - the singular set is a finite union of ribbon disks.



Definition

We define rT_n as the monoid of ribbon 2–tubes up to isotopy.

Ribbon disk singularities

Definition

A *ribbon disk* is a disk $D \in B_i \cap B_j$ of double points s.t.

Ribbon disk singularities

Definition

A *ribbon disk* is a disk $D \in B_i \cap B_j$ of double points s.t.

- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;

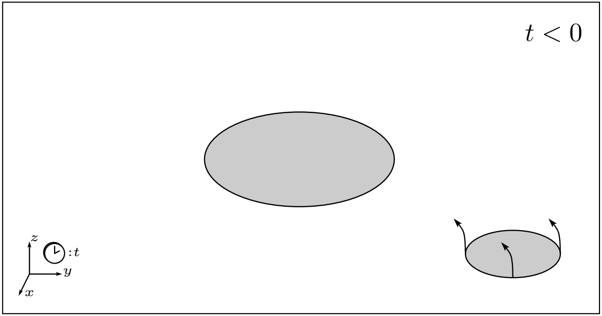
Ribbon disk singularities

Definition

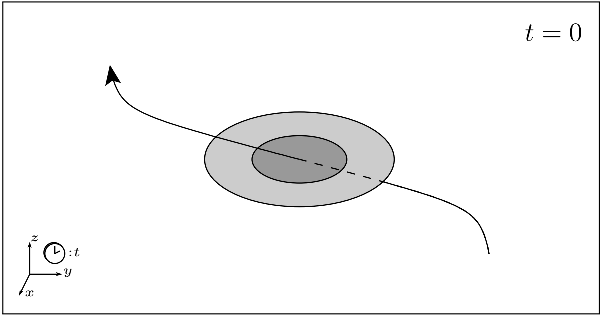
A *ribbon disk* is a disk $D \in B_i \cap B_j$ of double points s.t.

- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;
- $D \subset \mathring{B}_j$.

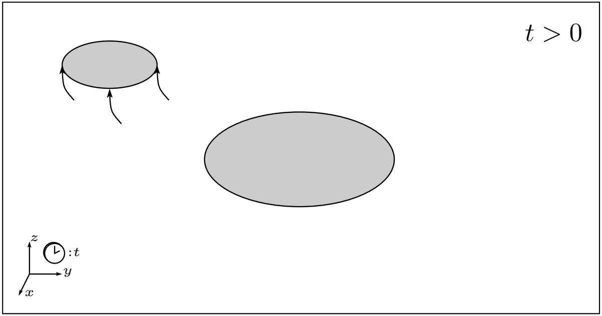
Ribbon disk singularities



Ribbon disk singularities



Ribbon disk singularities



Projection in B^3

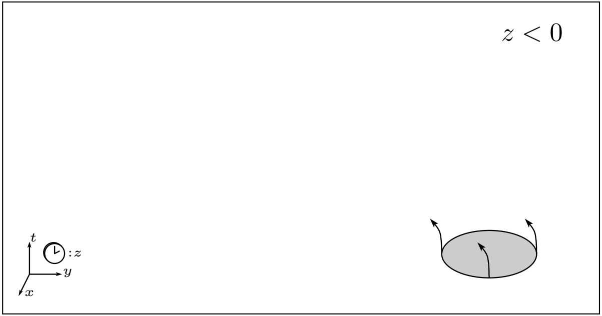
Can it be nicely projected in B^3 ?

Projection in B^3

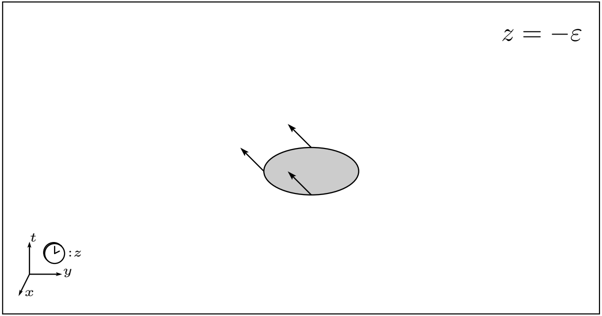
Can it be nicely projected in B^3 ?

↳ In order to project along the height z, let's represent it as the time parameter.

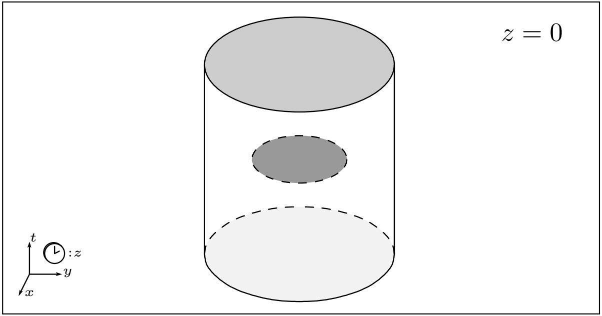
Projection in B^3



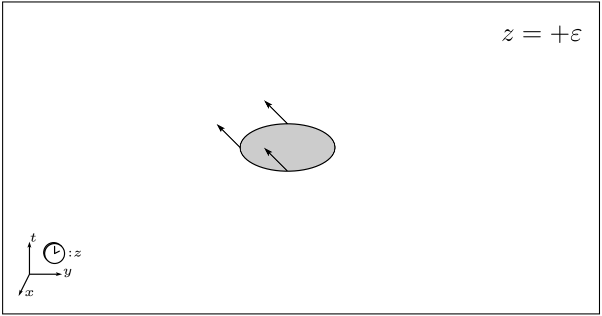
Projection in B^3



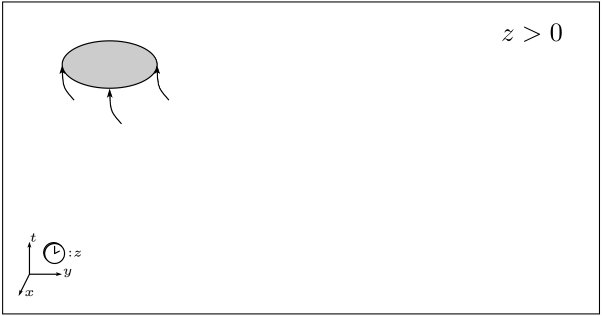
Projection in B^3



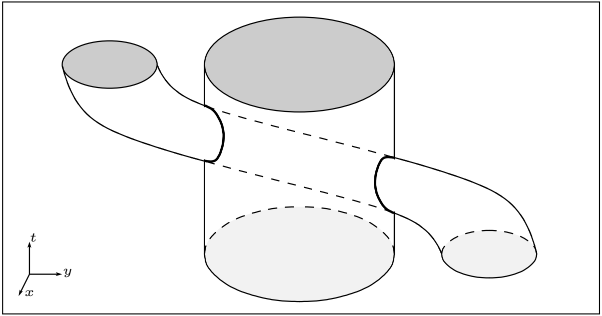
Projection in B^3



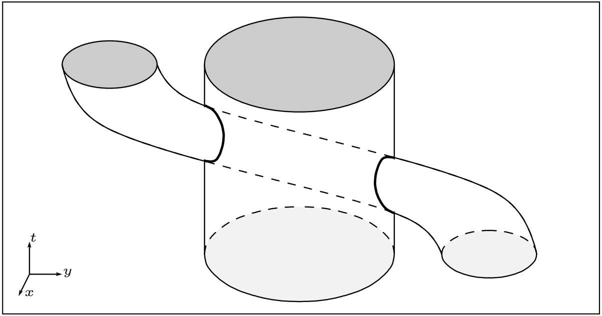
Projection in B^3



Projection in B^3

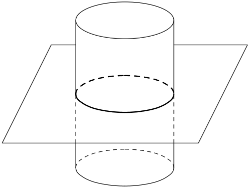


Projection in B^3

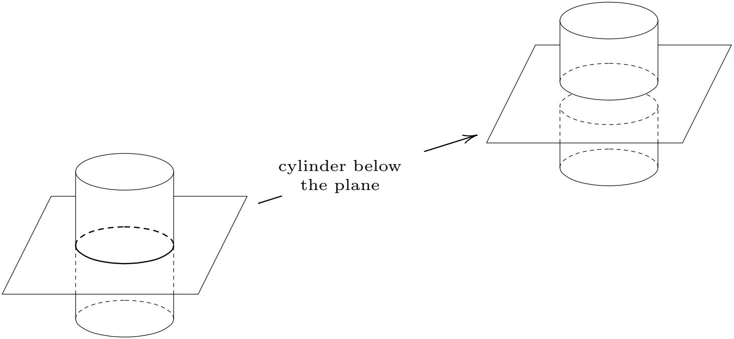


↳ We loose the information that whether the flying disk was moving upward or downward.

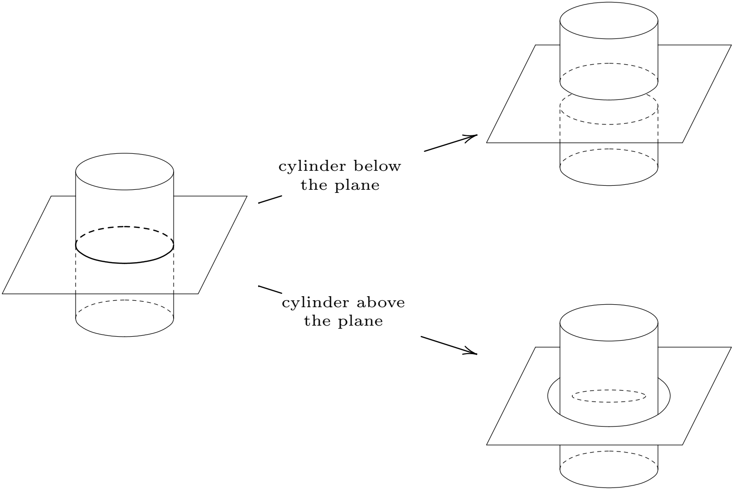
Broken surfaces



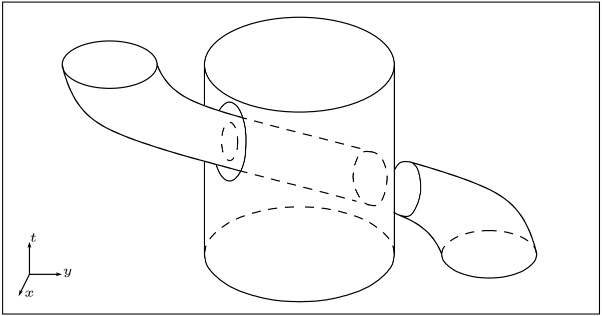
Broken surfaces



Broken surfaces

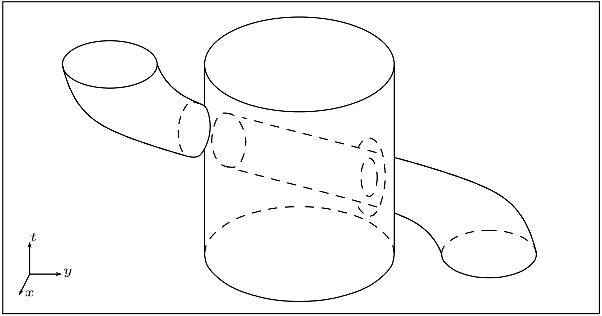


Broken surfaces



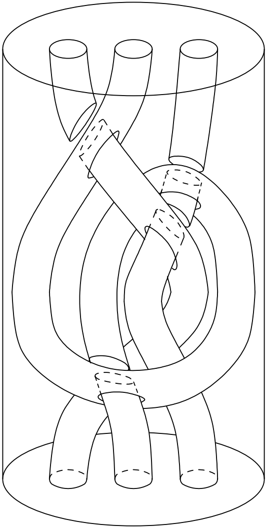
The 3–dimensional projection can be enhanced with over/underpassing decorations.

Broken surfaces

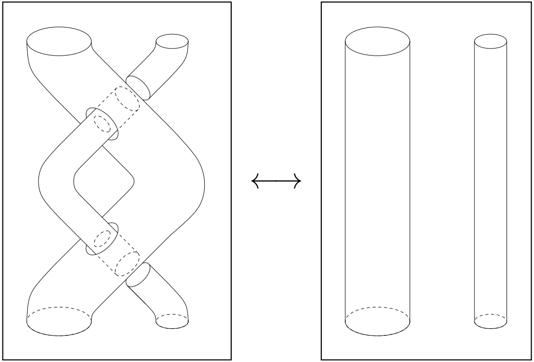


If the flying disk is moving downward, then the decorations are swapped.

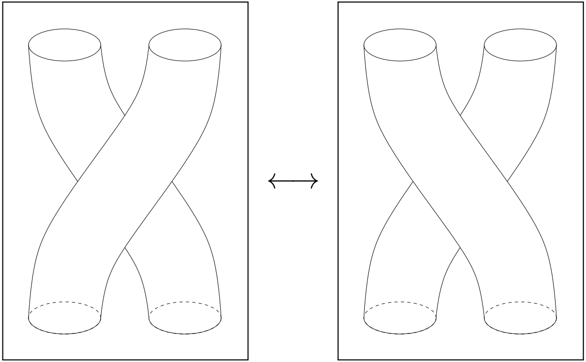
Broken surfaces



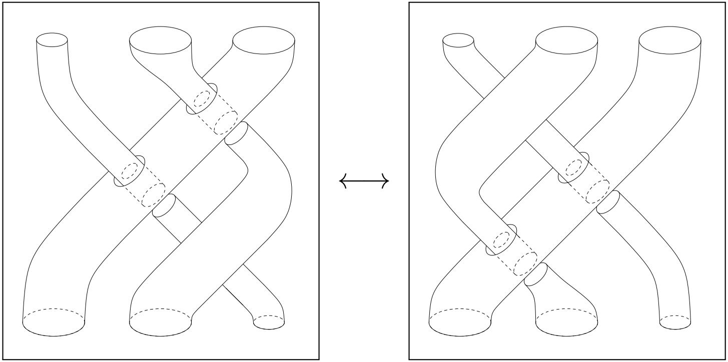
Moves on broken surfaces



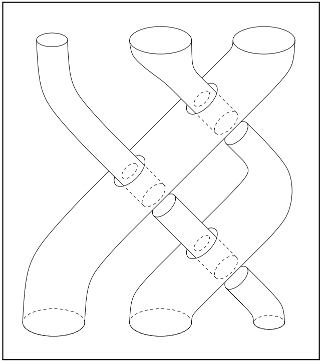
Moves on broken surfaces

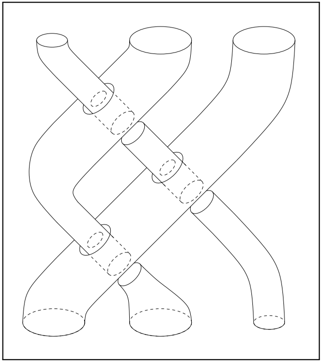


Moves on broken surfaces

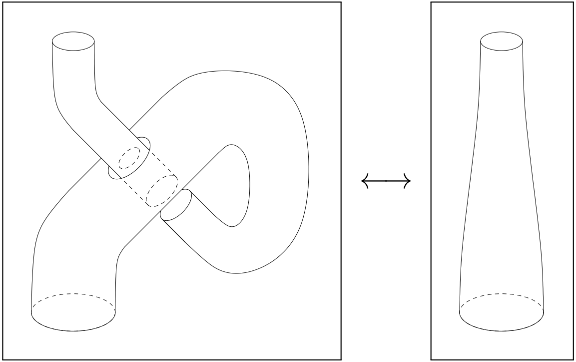


Moves on broken surfaces

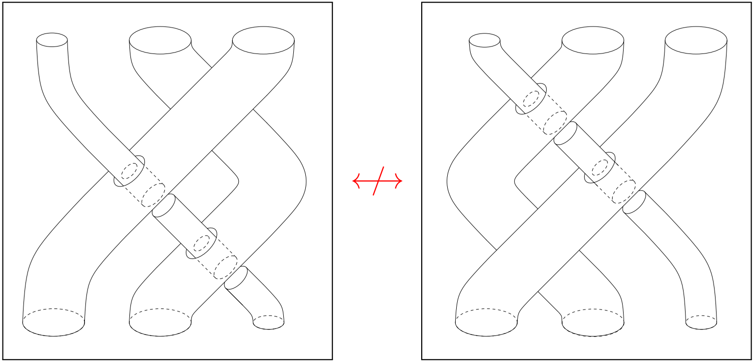




Moves on broken surfaces



Moves on broken surfaces



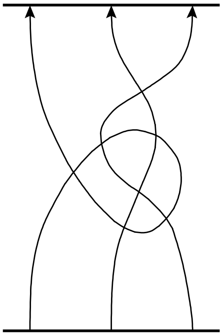
Welded diagrams

Welded diagrams

Definition

A (virtual) *diagram* is an immersion of n oriented arcs $A_1, \cdots, A_n$ in $\mathbb{R} \times I$ s.t.

- $\partial A_i = \{i\} \times \{0, 1\}$;

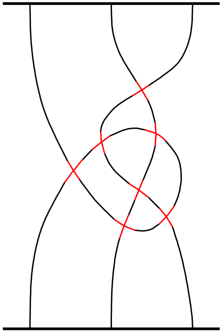


Welded diagrams

Definition

A (virtual) *diagram* is an immersion of n oriented arcs $A_1, \cdots, A_n$ in $\mathbb{R} \times I$ s.t.

- $\partial A_i = \{i\} \times \{0, 1\}$;
- the singular set is a finite number of transverse double points

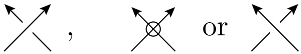


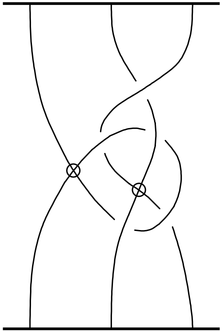
Welded diagrams

Definition

A (virtual) *diagram* is an immersion of n oriented arcs $A_1, \cdots, A_n$ in $\mathbb{R} \times I$ s.t.

- $\partial A_i = \{i\} \times \{0,1\}$;
- the singular set is a finite number of transverse double points decorated as



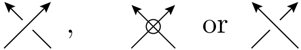


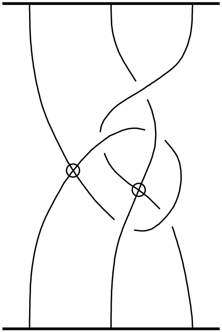
Welded diagrams

Definition

A (virtual) *diagram* is an immersion of n oriented arcs $A_1, \cdots, A_n$ in $\mathbb{R} \times I$ s.t.

- $\partial A_i = \{i\} \times \{0, 1\}$;
- the singular set is a finite number of transverse double points decorated as

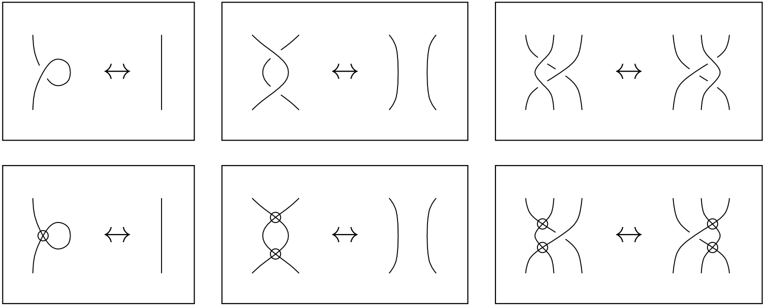




Definition

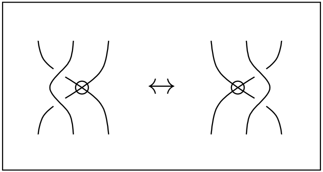
We define wD_n as the monoid of diagrams quotiented by the following relations:

Welded diagrams



Usual and virtual Reidemeister moves

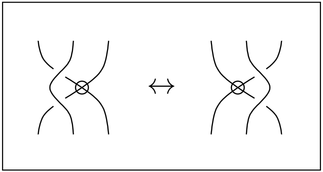
Welded diagrams

authorized: 

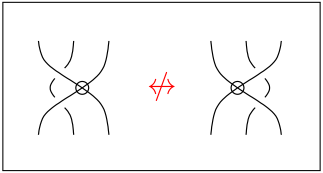
Welded Reidemeister moves

Welded diagrams

authorized:



still forbidden:



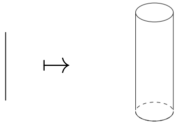
Welded Reidemeister moves

The tube application

For every diagram, one can associate a broken surface, and hence a ribbon 2–tube, by blowing up strings as follows:

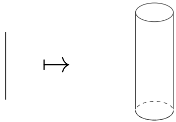
The tube application

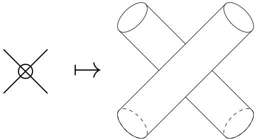
For every diagram, one can associate a broken surface, and hence a ribbon 2–tube, by blowing up strings as follows:



The tube application

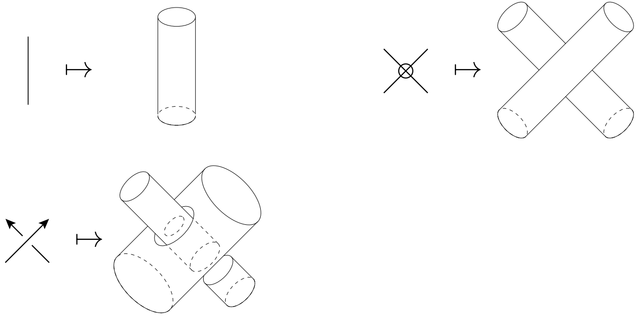
For every diagram, one can associate a broken surface, and hence a ribbon 2–tube, by blowing up strings as follows:





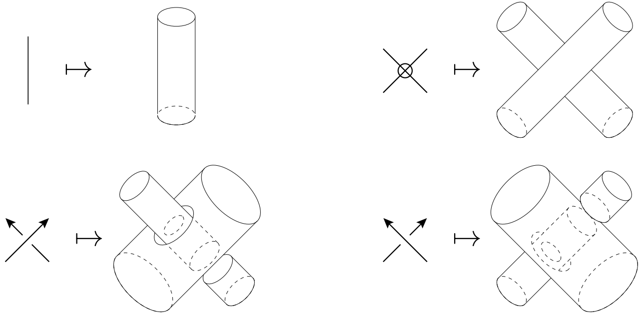
The tube application

For every diagram, one can associate a broken surface, and hence a ribbon 2–tube, by blowing up strings as follows:



The tube application

For every diagram, one can associate a broken surface, and hence a ribbon 2–tube, by blowing up strings as follows:



The tube application

Proposition

This assignment defines a map $Tube : wD_n \to rT_n$.

The tube application

Proposition

This assignment defines a map $Tube : wD_n \to rT_n$.

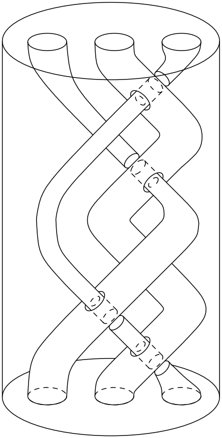
Proposition (Yanagawa,Satoh)

The map $Tube$ *is surjective.*

Monotone ribbon 2–tubes

Definition

A ribbon 2–tubes $T \subset B^3 \times I$ is said *monotone* iff $T \cap \left(B^3 \times \{t\}\right)$ is always a union of n circles.



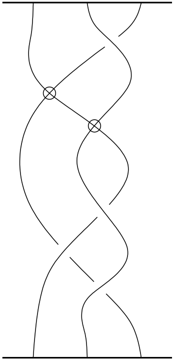
Monotone ribbon 2–tubes

Definition

A ribbon 2–tubes $T \subset B^3 \times I$ is said *monotone* iff $T \cap \left(B^3 \times \{t\}\right)$ is always a union of n circles.

Definition

A diagram $D \subset \mathbb{R} \times I$ is said to be a *pure braid* iff $D \cap \left(\mathbb{R} \times \{t\}\right)$ is always a union of n points.



Monotone ribbon 2–tubes

Definition

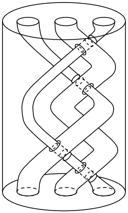
A ribbon 2–tubes $T \subset B^3 \times I$ is said *monotone* iff $T \cap \left(B^3 \times \{t\}\right)$ is always a union of n circles.

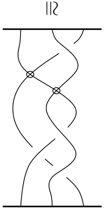
Definition

A diagram $D \subset \mathbb{R} \times I$ is said to be a *pure braid* iff $D \cap \left(\mathbb{R} \times \{t\}\right)$ is always a union of n points.

Theorem (Brendle–Hatcher)

The map Tube yields an isomorphism between welded pure braids and monotone ribbon 2–tubes.





Ribbon 2–tubes with only one tube are already hard to understand.

Ribbon 2–tubes with only one tube are already hard to understand.

Interactions of a tube with other tubes appear to be easier than with itself.

Ribbon 2–tubes with only one tube are already hard to understand.

Interactions of a tube with other tubes appear to be easier than with itself.

↳ we want to get rid of self-knottedness !

Singular ribbon 2–tubes

Singular ribbon 2–tubes

Definition

Singular ribbon 2–tube are immersions of annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a singular ribbon filling.

Singular ribbon 2–tubes

Definition

Singular ribbon 2–tube are immersions of annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a singular ribbon filling.

Singular ribbon 2–tubes

Definition

Singular ribbon 2–tube are immersions of annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a singular ribbon filling.

Singular ribbon 2–tubes

Definition

Singular ribbon 2–tube are immersions of annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a singular ribbon filling, that is immersed 3–balls $B_1, \cdots, B_n$ s.t.
  - $\partial B_i = T_i$;
  - the singular set is the union of a finite number of ribbon disks with a singular ribbon disks.

Singular ribbon 2–tubes

Definition

Singular ribbon 2–tube are immersions of annuli $T_1, \cdots, T_n$ s.t.

- $\partial T_i = C_i \times \{0, 1\}$;
- they admit a singular ribbon filling, that is immersed 3–balls $B_1, \cdots, B_n$ s.t.
  - $\partial B_i = T_i$;
  - the singular set is the union of a finite number of ribbon disks with a singular ribbon disks.

Singular ribbon disks

Singular ribbon disks

Definition (regular ribbon disk)

It is a disk $D \in B_i \cap B_j$ of double points s.t.

- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;
- $D \subset \mathring{B}_j$.

Singular ribbon disks

Definition (regular ribbon disk)

It is a disk $D \in B_i \cap B_j$ of double points s.t.

- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;
- $D \subset \mathring{B}_j$.

Definition (singular ribbon disk)

It is a disk $D \in B_i \cap B_j$ of double points s.t.

- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;
- $D \subset \partial B_j = T_j$.

Singular ribbon disks

Definition (regular ribbon disk)

It is a disk $D \in B_i \cap B_j$ of double points s.t.

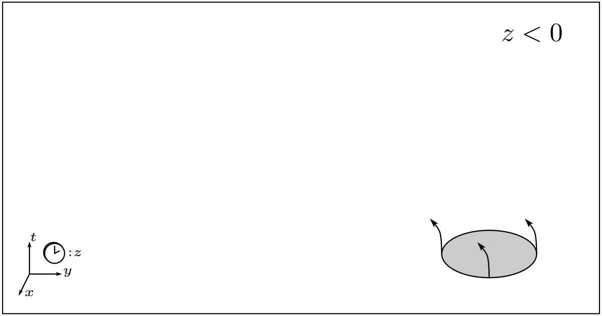
- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;
- $D \subset \mathring{B}_j$.

Definition (singular ribbon disk)

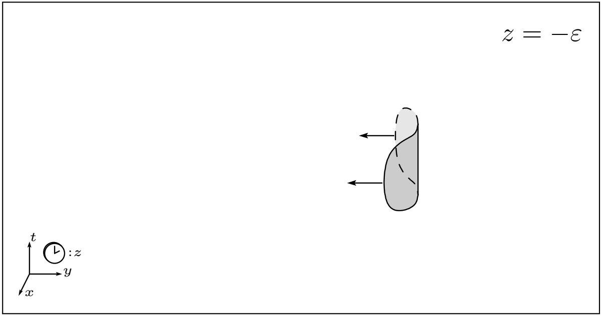
It is a disk $D \in B_i \cap B_j$ of double points s.t.

- $\mathring{D} \subset \mathring{B}_i$ and ∂D is essential in $\partial B_i = T_i$;
- $D \subset \partial B_j = T_j$.

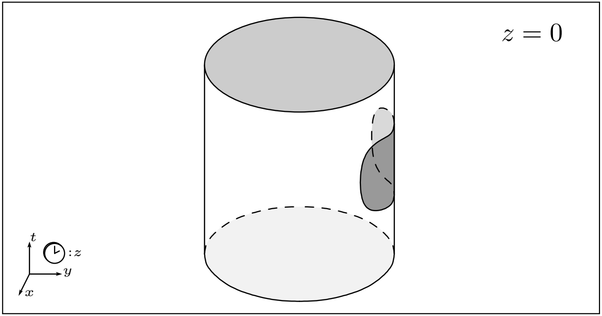
Singular ribbon disks



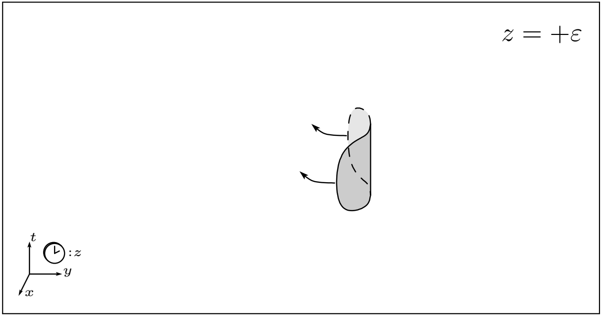
Singular ribbon disks



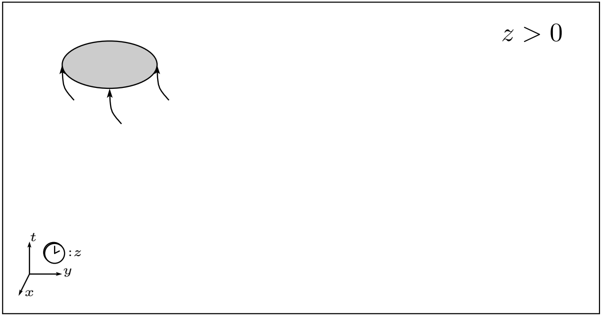
Singular ribbon disks



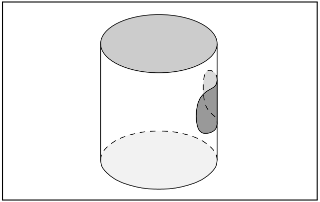
Singular ribbon disks



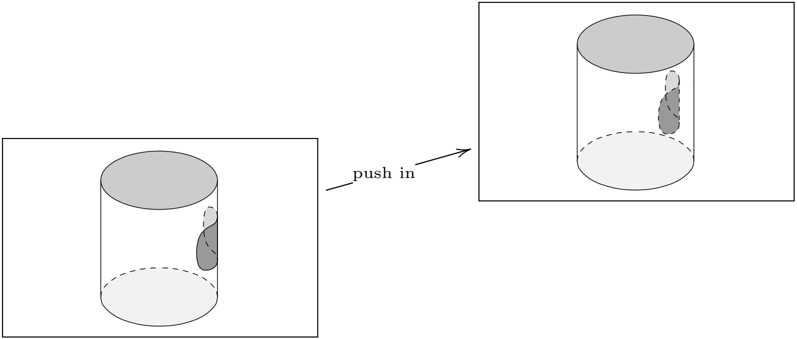
Singular ribbon disks



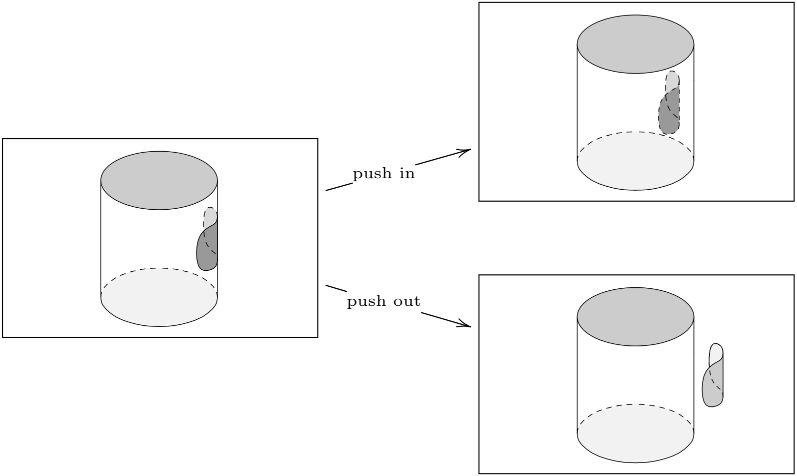
Desingularization



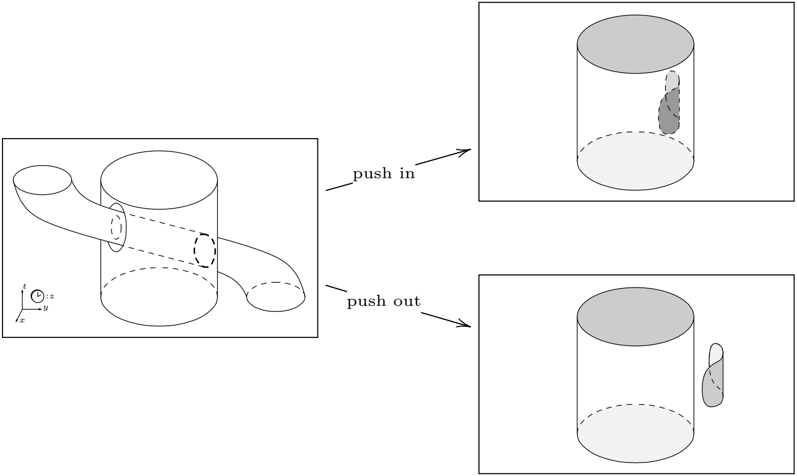
Desingularization



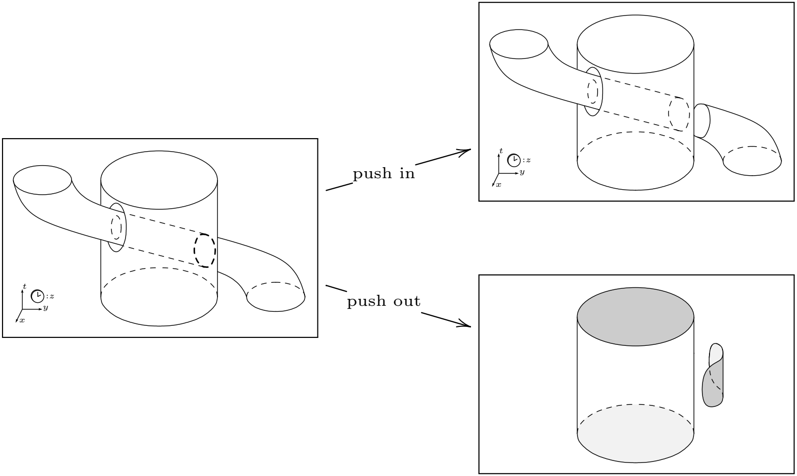
Desingularization



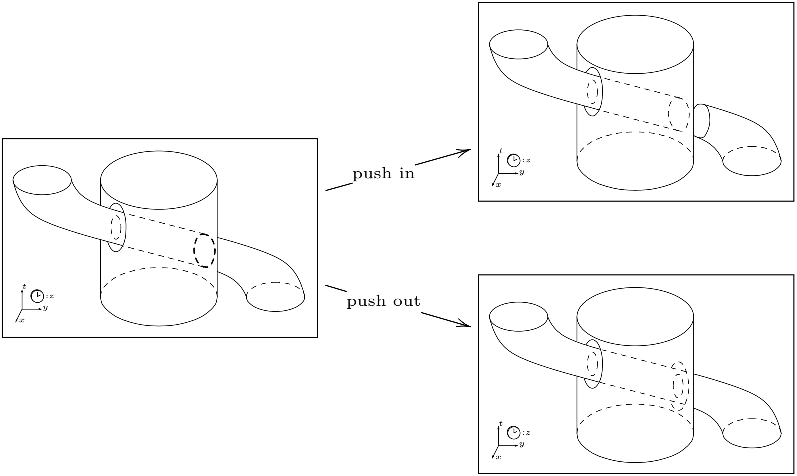
Desingularization



Desingularization



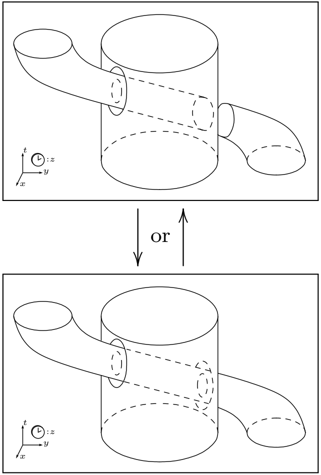
Desingularization



homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.

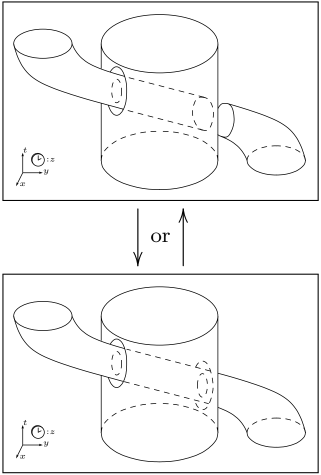


self-homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.

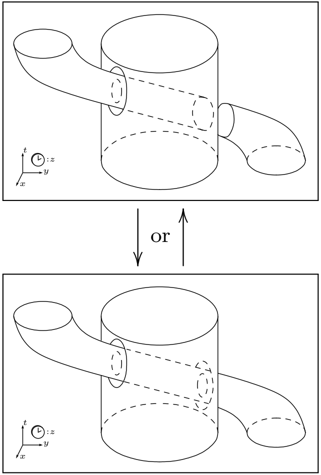
It is a *self*-homotopy iff each singular ribbon disk involves twice the same tube.



self-homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.
It is a *self*-homotopy iff each singular ribbon disk involves twice the same tube.

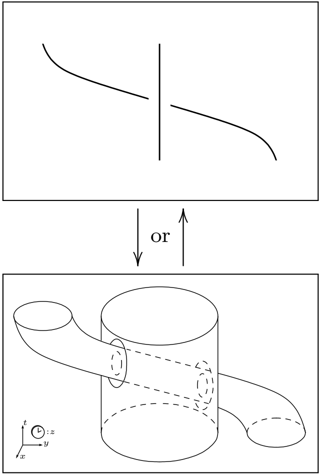


At the level of welded diagrams

self-homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.
It is a *self*-homotopy iff each singular ribbon disk involves twice the same tube.



At the level of welded diagrams

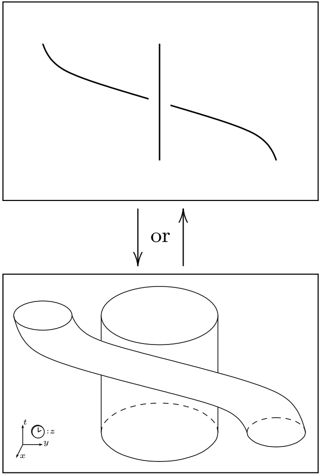
self-homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.

It is a *self*-homotopy iff each singular ribbon disk involves twice the same tube.

At the level of welded diagrams

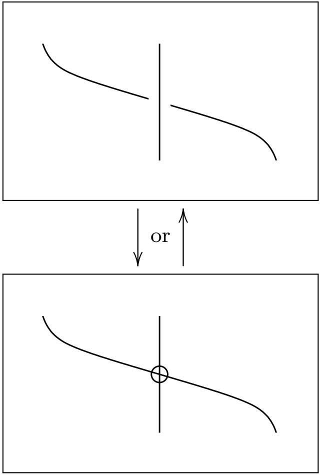


self-homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.
It is a *self*-homotopy iff each singular ribbon disk involves twice the same tube.

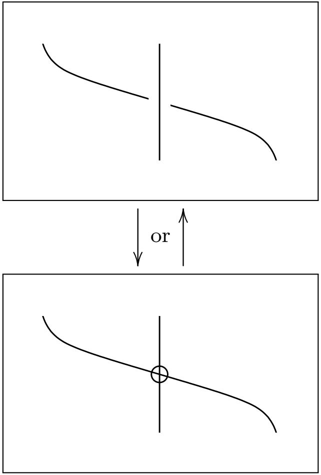
At the level of welded diagrams



self-homotopies

Definition

An *homotopy* is a path of ribbon 2–tubes which may pass a finite number of times through singular ribbon 2–tubes.
It is a *self*-homotopy iff each singular ribbon disk involves twice the same tube.



At the level of welded diagrams, it corresponds to *(de)virtualization* of (self-)crossings.

Ribbon 2–tubes up to self-homotopy

Theorem (ABMW)

Every welded diagram is self-homotopic to a welded pure braid.

Ribbon 2–tubes up to self-homotopy

Theorem (ABMW)

Every welded diagram is self-homotopic to a welded pure braid.

Corollary

Every ribbon 2–tube is self-homotopic to a monotone one.

Ribbon 2–tubes up to self-homotopy

Theorem (ABMW)

Every welded diagram is self-homotopic to a welded pure braid.

Corollary

Every ribbon 2–tube is self-homotopic to a monotone one.

Corollary

$$rT_n^h := {}^{rT_n}\big/_{\{self\text{-}homotopy\}} \cong {}^{wD_n}\big/_{\{self\text{-}homotopy\}}.$$

Reduced free group

Let $\mathrm{F}_n := \langle x_1, \cdots, x_n \rangle$ be the free group on n generators.

Reduced free group

Let $\mathrm{F}_n := \langle x_1, \cdots, x_n \rangle$ be the free group on n generators.

Definition

We define RF_n as $\mathrm{F}_n / [x_i, y^{-1} x_i y]$.

Reduced free group

Let $\mathrm{F}_n := \langle x_1, \cdots, x_n \rangle$ be the free group on n generators.

Definition

We define RF_n as $\mathrm{F}_n / [x_i, y^{-1} x_i y]$.

Definition

We define $\mathrm{Aut}_C(\mathrm{F}_n)$ as the group of automorphisms of RF_n which send every generator to a conjugate of itself.

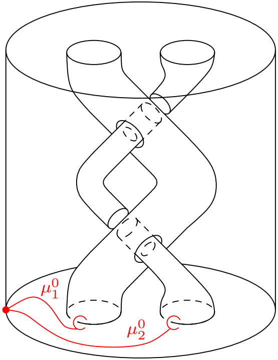
Meridians of ribbon 2–tubes

Let T be a ribbon 2–tubes and fix a base point in $B^4 \setminus T$.

Meridians of ribbon 2–tubes

Let T be a ribbon 2–tubes and fix a base point in $B^4 \setminus T$.

We define μ_i^0 as the based loop which enlaces $C_i \times \{0\}$.



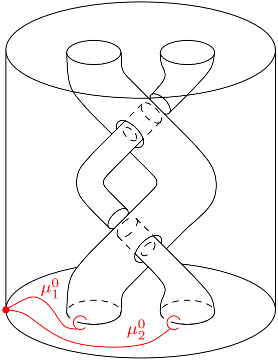
Meridians of ribbon 2–tubes

Let T be a ribbon 2–tubes and fix a base point in $B^4 \setminus T$.

We define μ_i^0 as the based loop which enlaces $C_i \times \{0\}$.

Proposition

$\pi_1(B^4 \setminus T)/[\mu_i^0, \lambda^{-1}\mu_i^0\lambda] \cong RF_n.$



Meridians of ribbon 2–tubes

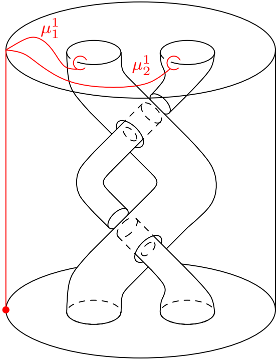
Let T be a ribbon 2–tubes and fix a base point in $B^4 \setminus T$.

We define μ_i^0 as the based loop which enlaces $C_i \times \{0\}$.

Proposition

$\pi_1(B^4 \setminus T)/[\mu_i^0, \lambda^{-1}\mu_i^0\lambda] \cong RF_n$.

We define μ_i^1 as the based loop which enlaces $C_i \times \{1\}$.



Meridians of ribbon 2–tubes

Let T be a ribbon 2–tubes and fix a base point in $B^4 \setminus T$.

We define μ_i^0 as the based loop which enlaces $C_i \times \{0\}$.

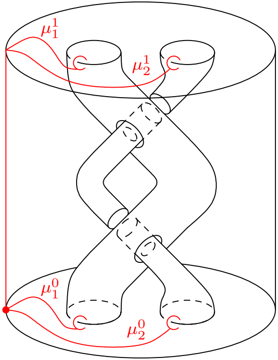
Proposition

$\pi_1(B^4 \setminus T)/[\mu_i^0, \lambda^{-1}\mu_i^0\lambda] \cong RF_n$.

We define μ_i^1 as the based loop which enlaces $C_i \times \{1\}$.

Proposition (ABMW)

Expressing μ_i^1 as a product of μ_j^0 defines a map $\varphi_T \in Aut_C(F_n)$ which depends only on the self-homotopy class of T.



Classification

Theorem (ABMW)

$$\varphi : \begin{array}{ccc} rT_n^h & \to & Aut_C(F_n) \\ T & \mapsto & \varphi_T \end{array}$$ *is a group isomorphism.*

Classification

Theorem (ABMW)

$$\varphi : \begin{array}{ccc} rT_n^h & \to & Aut_C(F_n) \\ T & \mapsto & \varphi_T \end{array}$$ *is a group isomorphism.*

This can be compared with

Theorem (Habegger–Lin)

$$\varphi : \begin{array}{ccc} SL_n^h & \to & Aut_C^0(F_n) \\ T & \mapsto & \varphi_T \end{array}$$ *is a group isomorphism.*

Merci de votre attention.