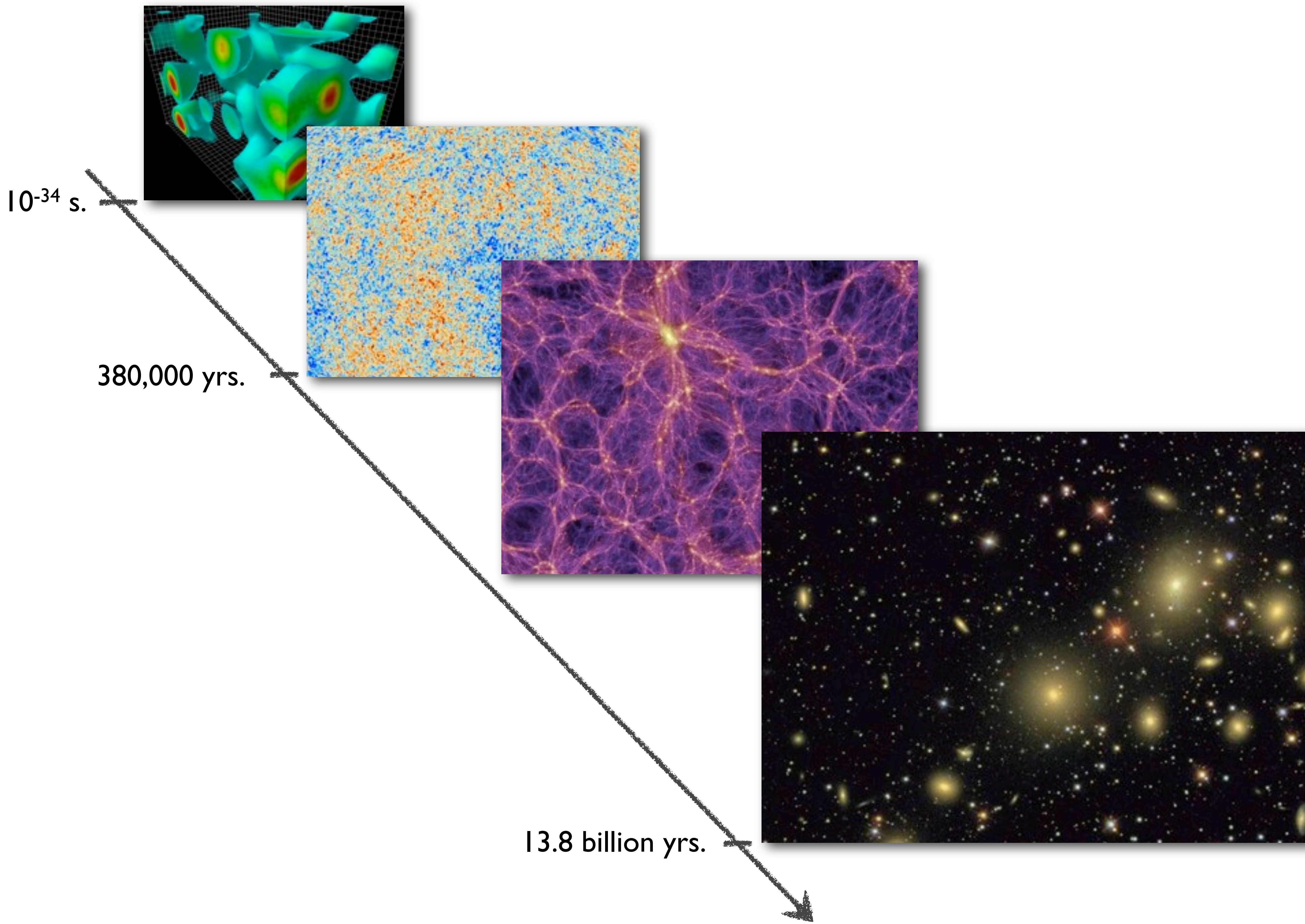
COSMOLOGY for STRING THEORISTS

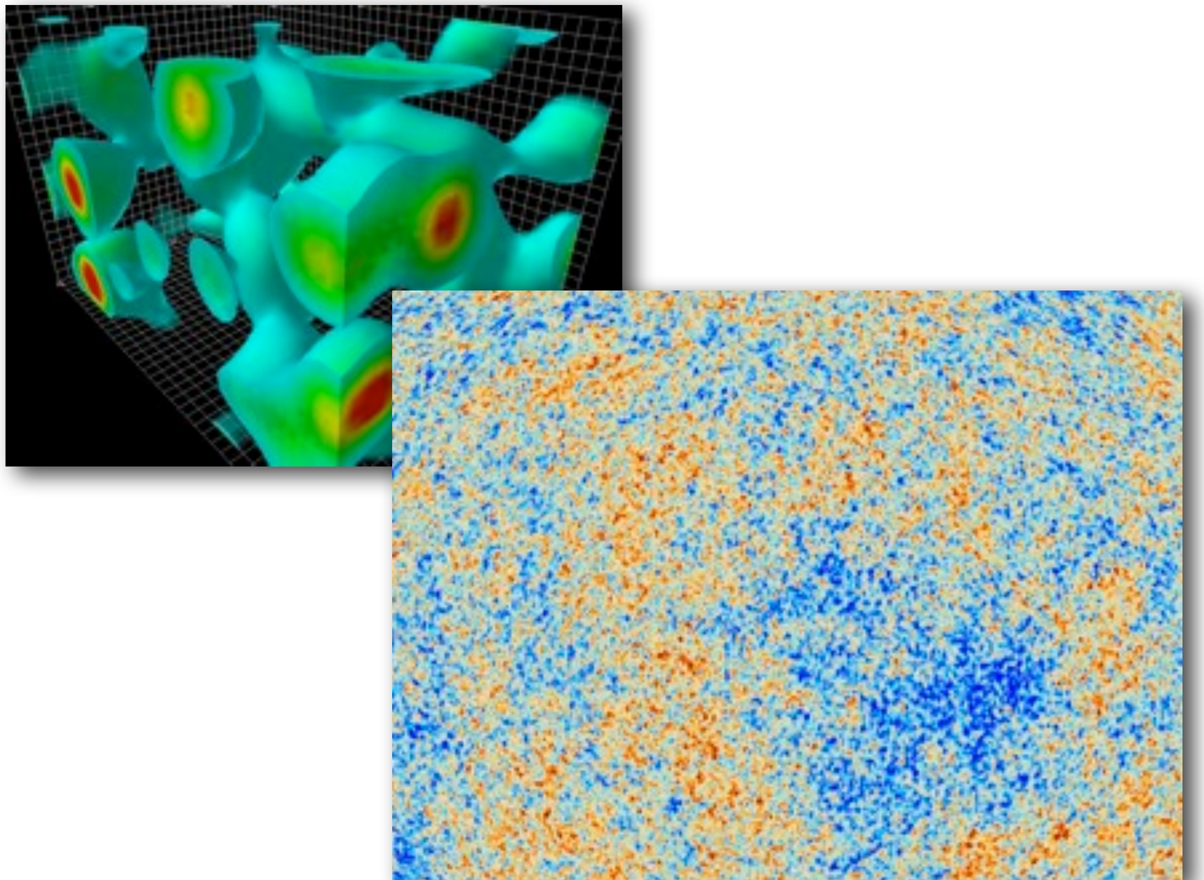
Daniel Baumann
Cambridge University

Asian Winter School on Strings, Particles and Cosmology
Puri, January 2014



10-34 s.
380,000 yrs.
13.8 billion yrs.

Course Outline

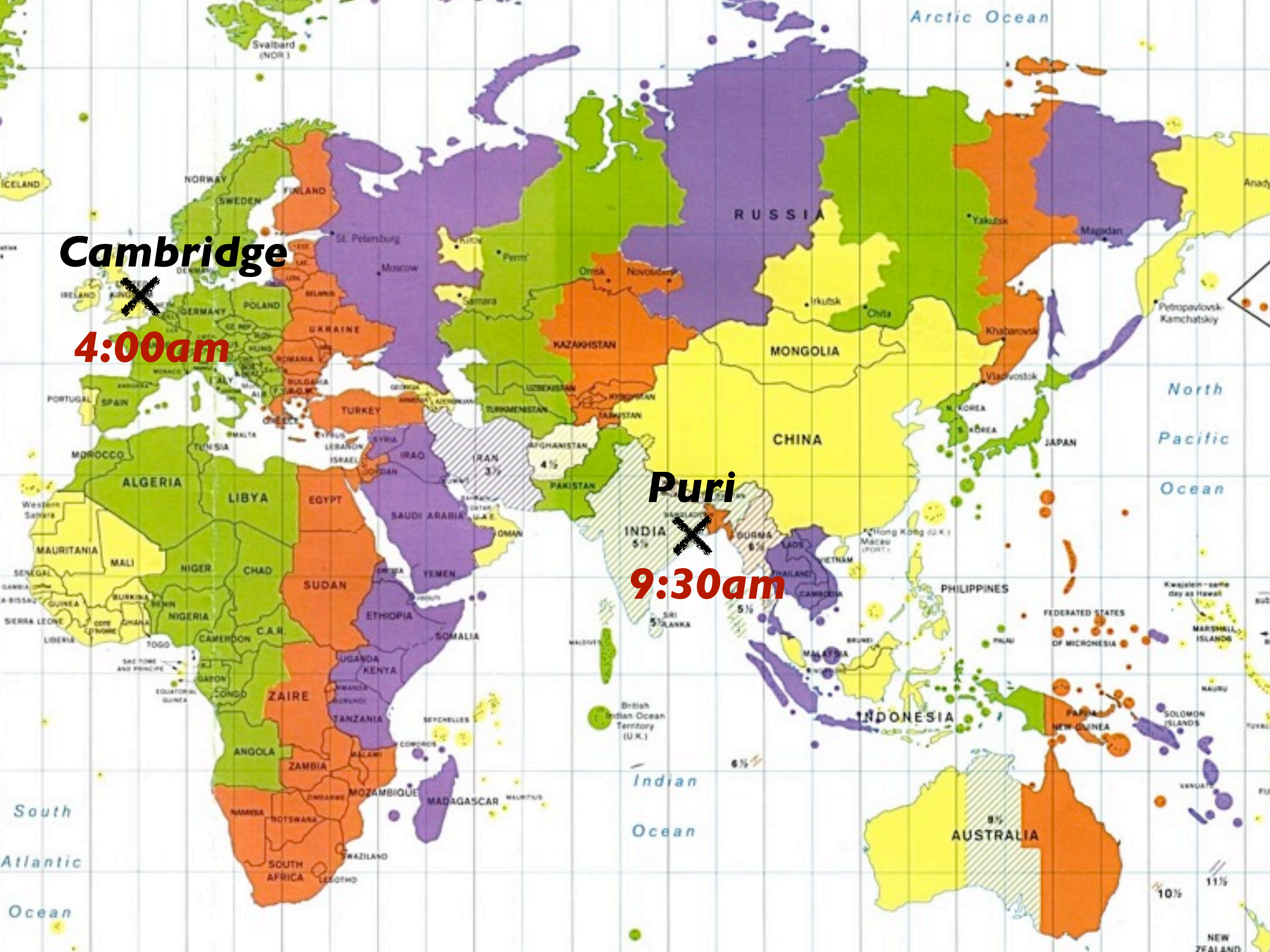


- The Physics of CMB Anisotropies
    - Quantum Initial Conditions
    - Acoustic Dynamics
    - Results from Planck
- Inflation in String Theory
    - Inflation in Effective Field Theory
    - Moduli Stabilization
    - Examples of String Inflation

Course website: www.damtp.cam.ac.uk/user/db275/Puri.pdf

Reference: DB and Liam McAllister, *Inflation and String Theory*

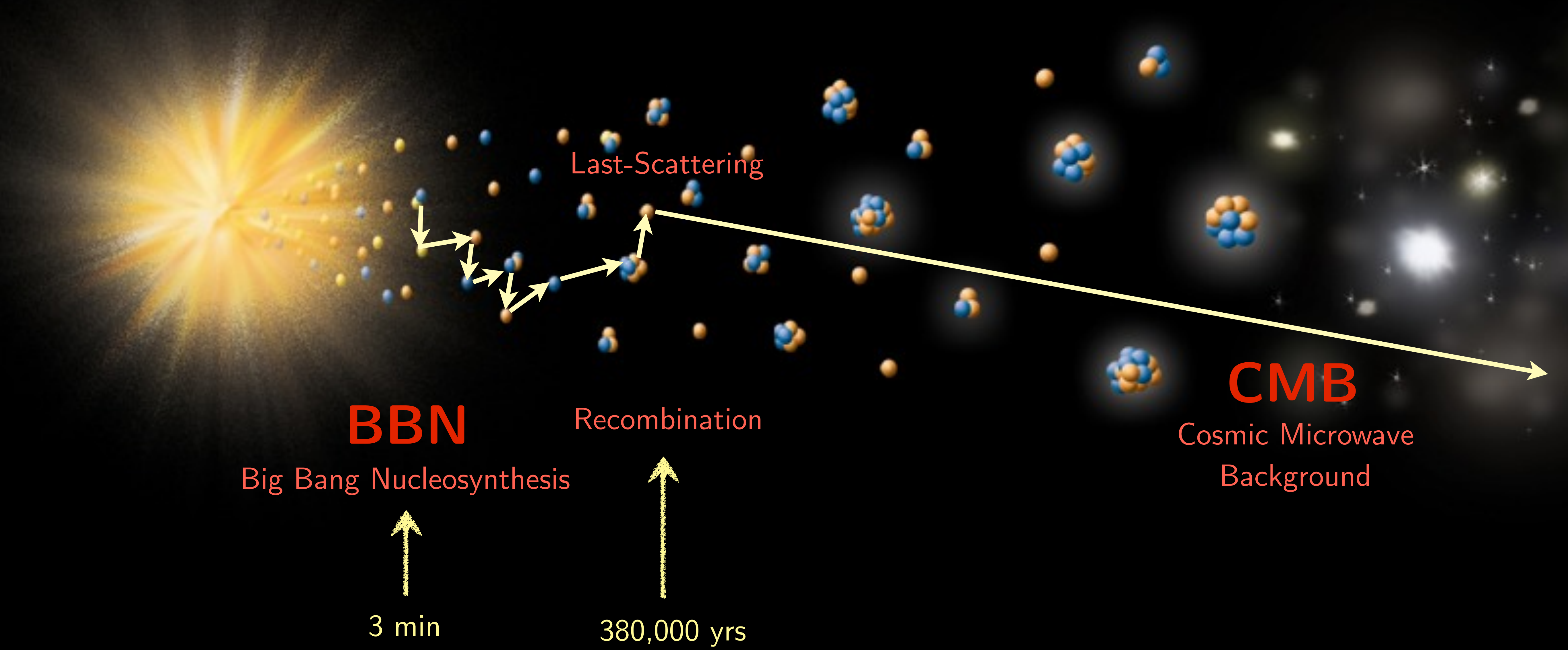
Please ask questions



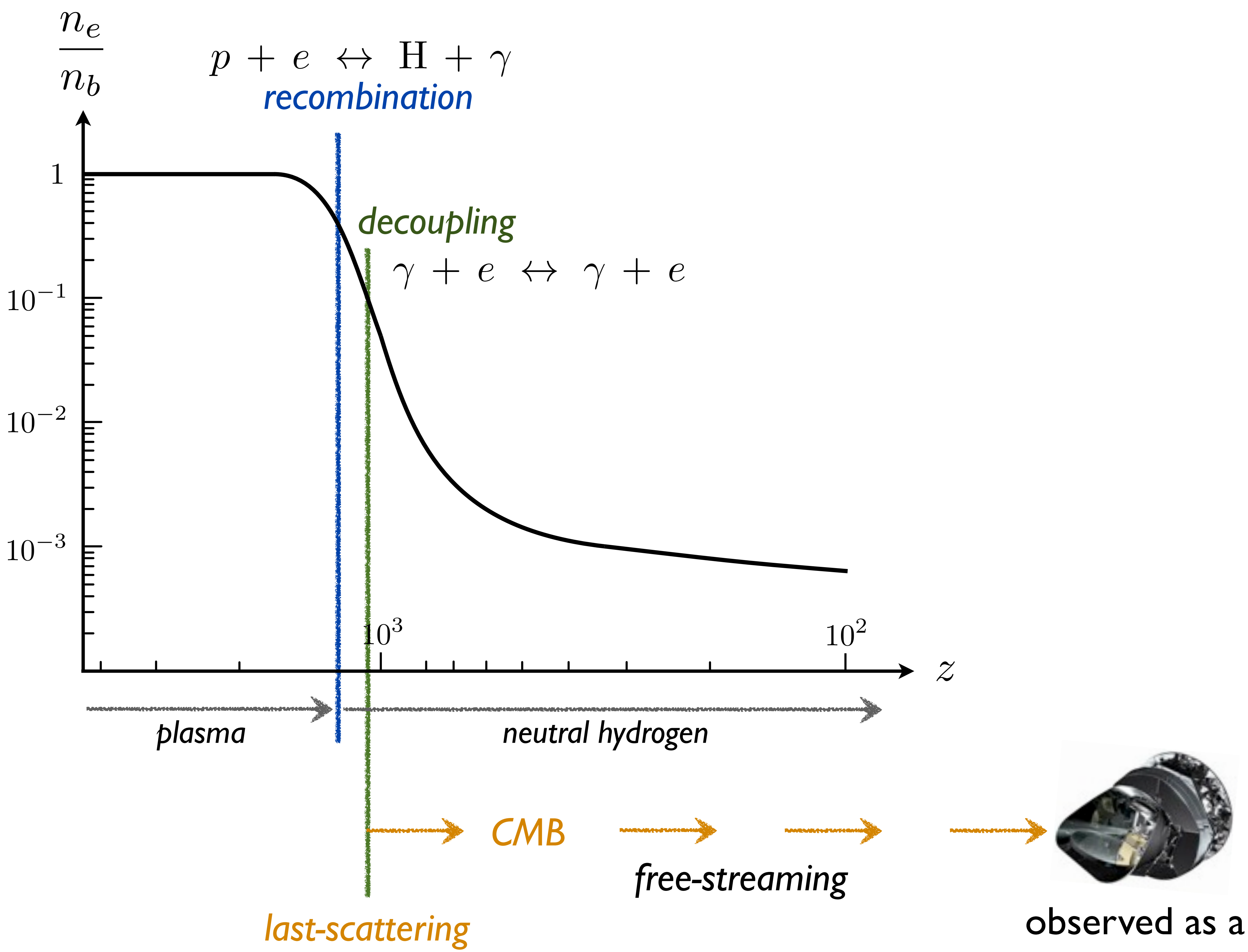
Cambridge
4:00am
Puri
9:30am
Arctic Ocean
North Pacific Ocean
Indian Ocean
South Atlantic Ocean
RUSSIA
CHINA
MONGOLIA
KAZAKHSTAN
INDIA
AUSTRALIA
INDONESIA
PHILIPPINES
JAPAN
TURKEY
IRAN
SAUDI ARABIA
EGYPT
LIBYA
ALGERIA
SUDAN
ETHIOPIA
ZAIRE
ANGOLA
ZAMBIA
NIGERIA
NIGER
CHAD
MALI
MAURITANIA
MOROCCO
SOMALIA
KENYA
TANZANIA
MADAGASCAR
SOUTH AFRICA
UKRAINE
POLAND
GERMANY
FINLAND
SWEDEN
NORWAY
ICELAND
SPAIN
PORTUGAL
PAKISTAN
AFGHANISTAN
MOZAMBIQUE
YEMEN

Lecture I

The Physics of CMB Anisotropies



Cosmic Microwave Background

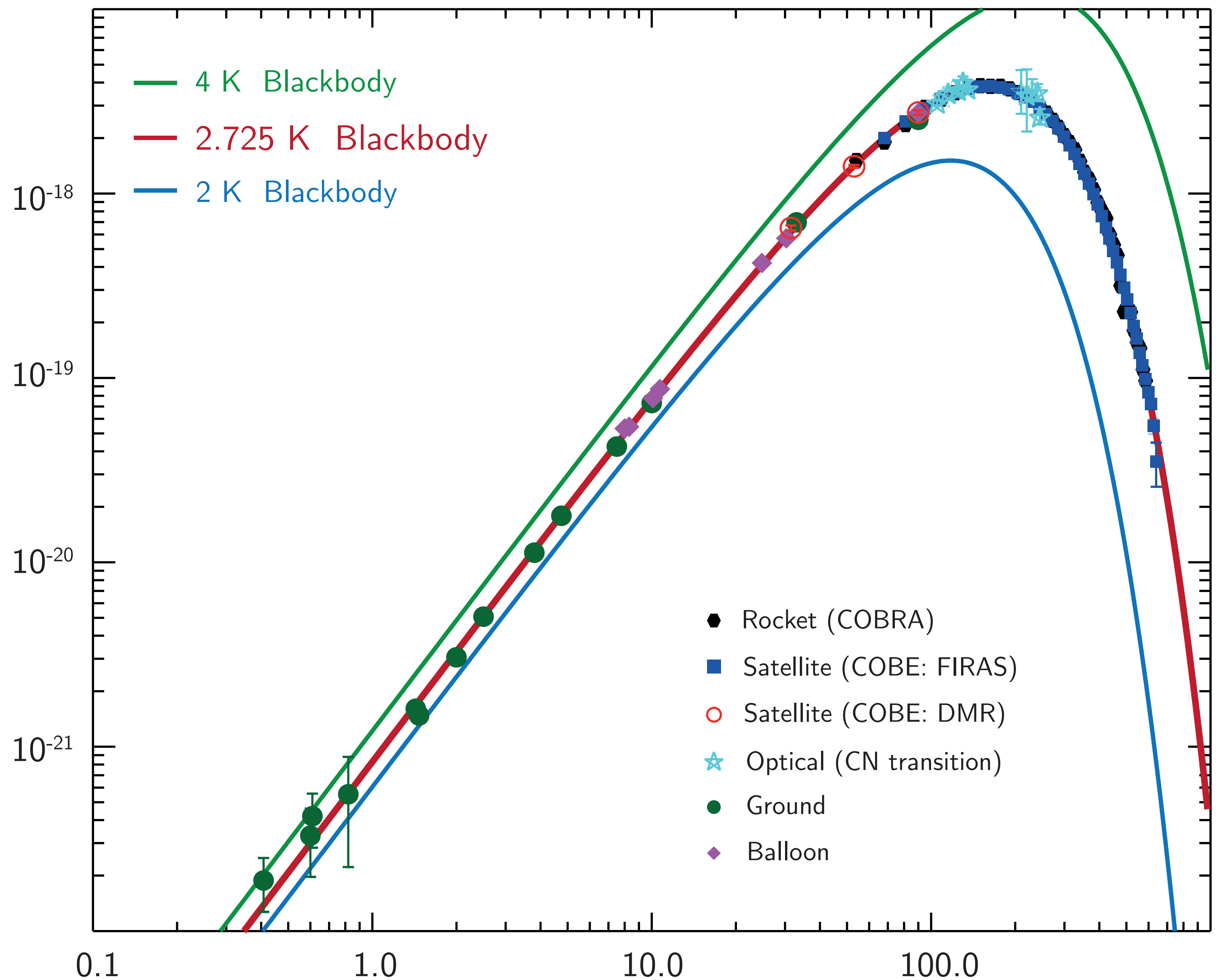


observed as a blackbody spectrum with $\bar{T} = 2.7\,\mathrm{K}$

Cosmic Microwave Background

$$I_\nu = \frac{2h\nu^3}{c^2} f(\nu)$$

intensity

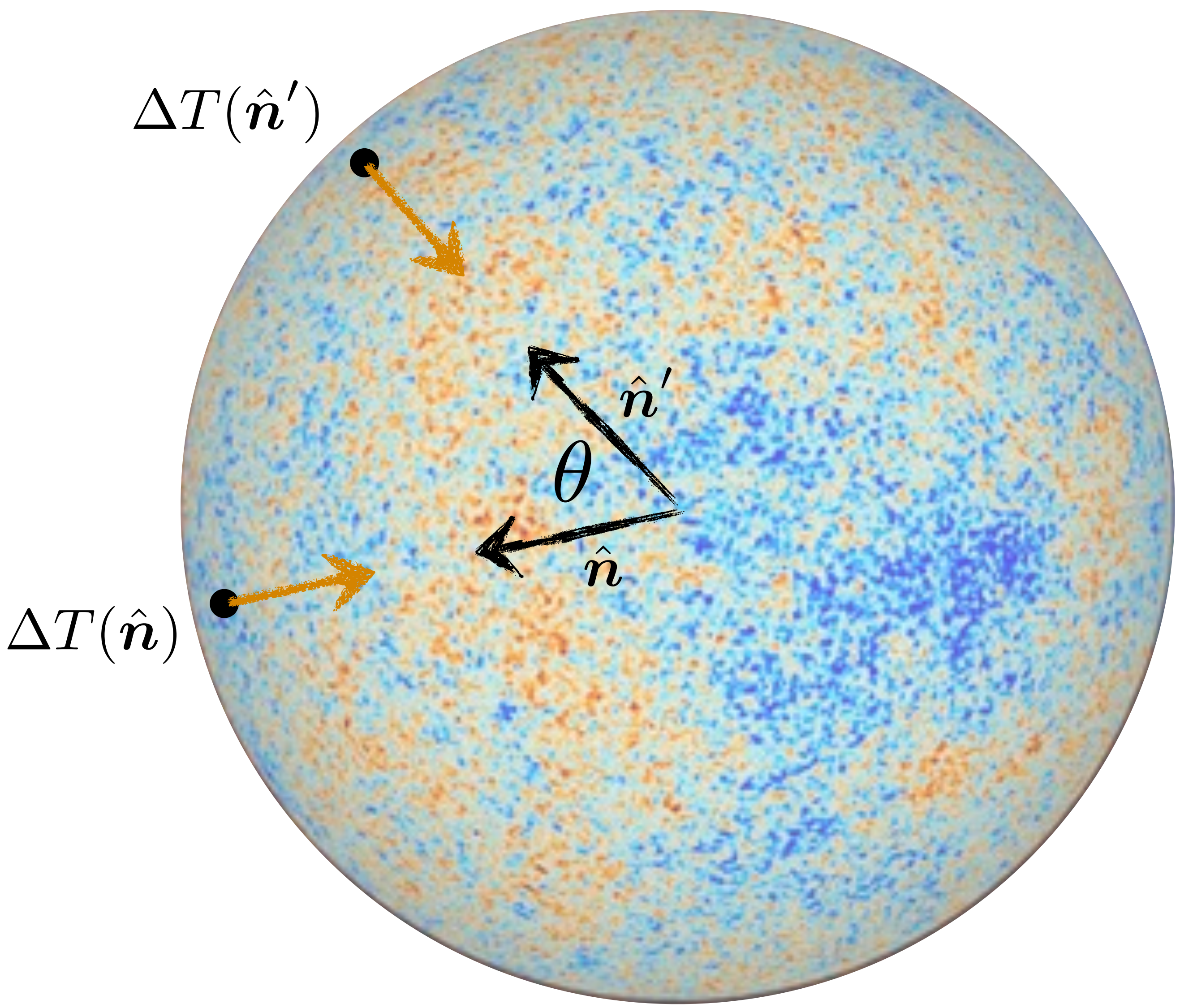


ν [GHz]

frequency

CMB Anisotropies

$$f(\nu, \hat{\boldsymbol{n}}) = [\exp(2\pi\nu/T(\hat{\boldsymbol{n}})) - 1]^{-1}$$



For Gaussian fluctuations, the statistics is determined by the 2-pt function:

$$C(\theta) \equiv \langle \Delta T(\hat{\boldsymbol{n}}) \Delta T(\hat{\boldsymbol{n}}') \rangle$$

ensemble average

CMB Power Spectrum

The same information can be represented by a spherical harmonic expansion of the temperature field

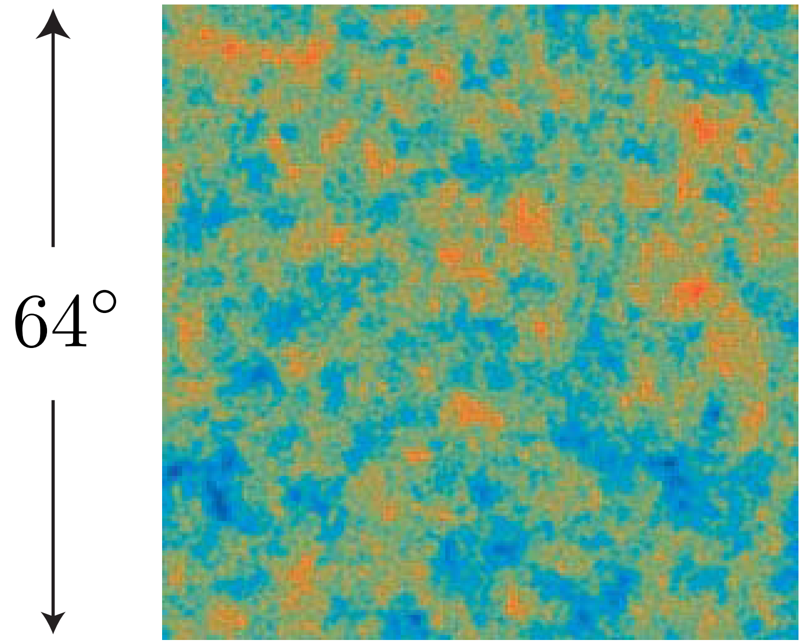
$$\Theta(\hat{\boldsymbol{n}}) \equiv \frac{\Delta T(\hat{\boldsymbol{n}})}{\bar{T}} = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}})$$

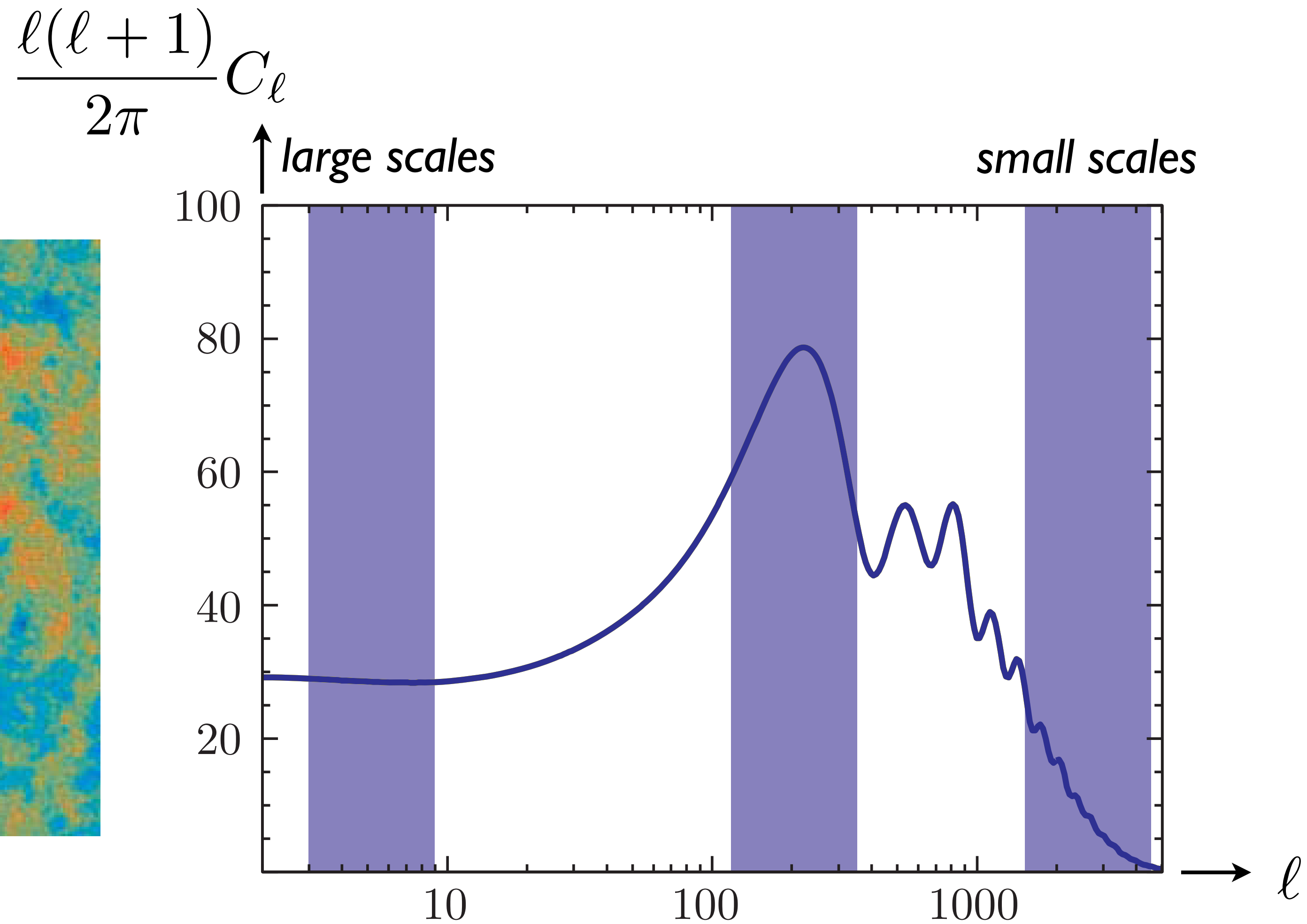
The (angular) power spectrum is

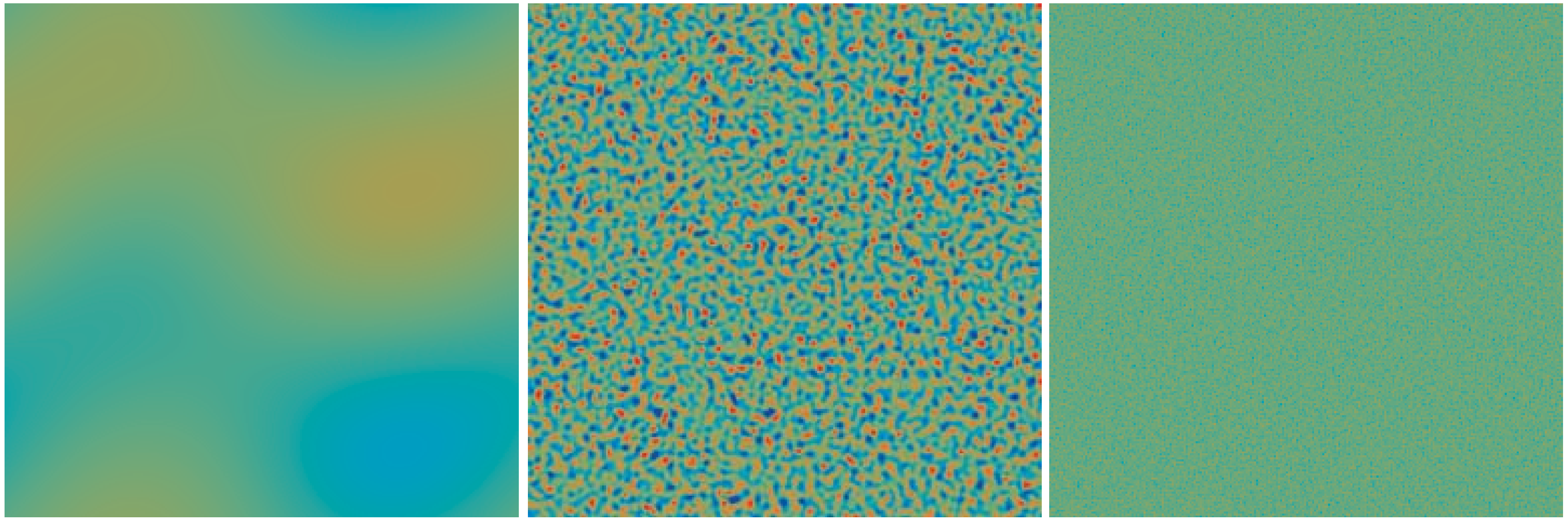
$$\boxed{C_\ell \equiv \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |\Theta_{\ell m}|^2}$$

This compresses the 10^7 pixels of the CMB map into 10^3 multipole moments.

CMB Power Spectrum



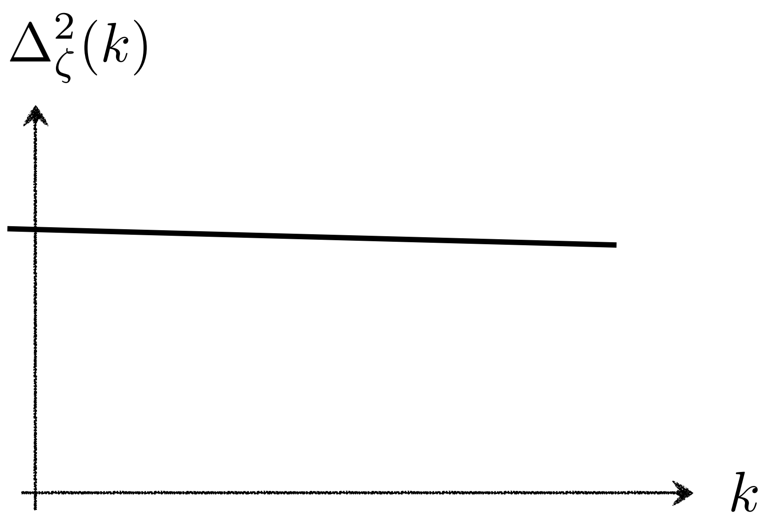




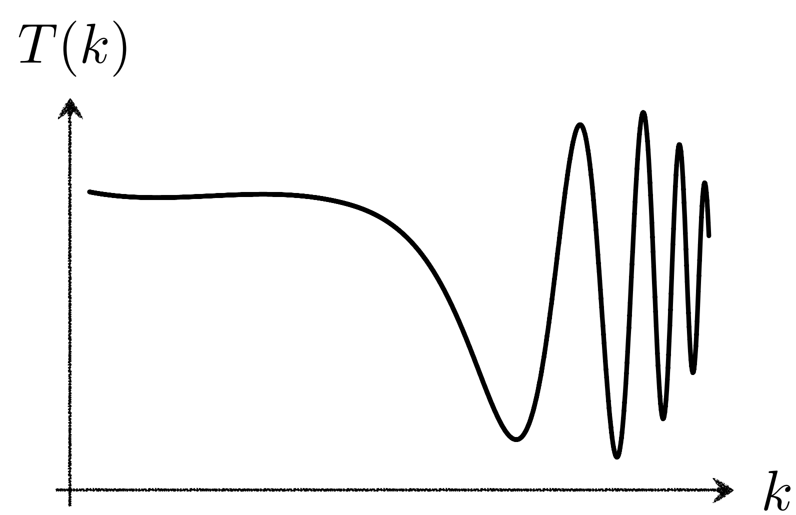
The Physics of CMB Anisotropies

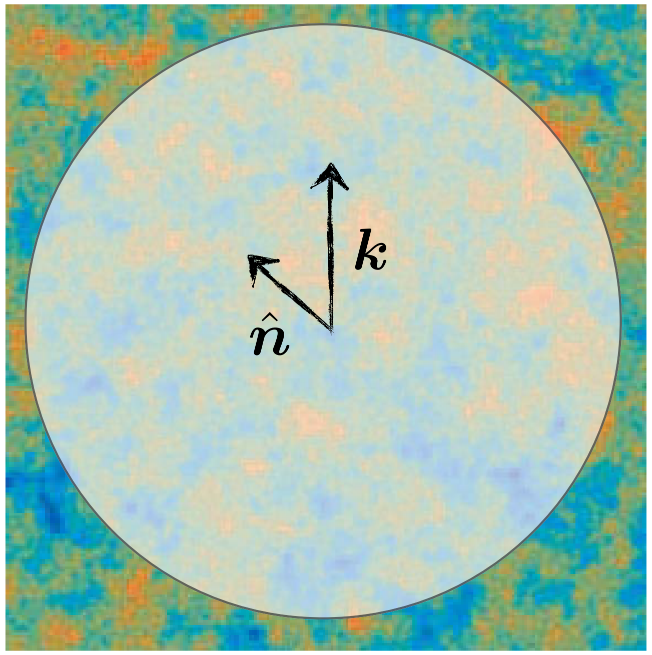
The goal of this lecture is to derive the
CMB power spectrum from first principles.

Initial Conditions + Evolution + Projection



see Leonardo's lectures





Initial Conditions

The CMB measures distortions in space:

ζ

scalar mode

expansion

$$d\ell^2 = a^2(t)\Big[1 + 2\zeta(t, \boldsymbol{x})\Big]\delta_{ij}x^i dx^j$$

curvature perturbation

isotropic stretching

h_{ij}

tensor mode

$$d\ell^2 = a^2(t)\Big[\delta_{ij} + h_{ij}(t, \boldsymbol{x})\Big]x^i dx^j$$

gravitational waves

anisotropic stretching

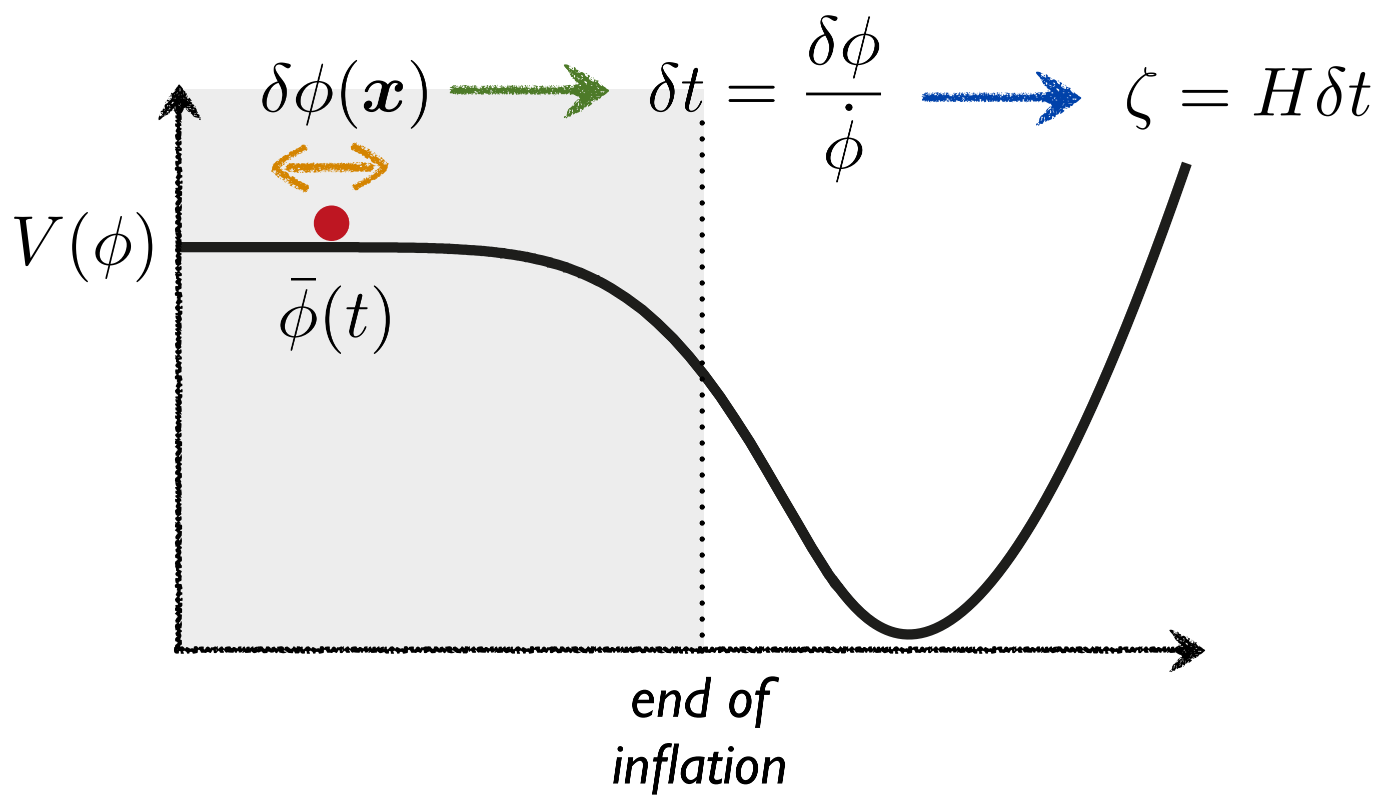
Both are produced by ***quantum fluctuations during inflation***.

$$a(t) = e^{Ht}$$

An inflationary model requires a ***clock*** which determines the amount of inflation still to occur.

see Leonardo's lectures

The clock can be a ***fundamental scalar field*** (the inflaton):



By the ***uncertainty principle***, arbitrarily precise timing is not possible in quantum mechanics.

This leads to ***fluctuations in the end of inflation*** ...

... and to ***curvature perturbations*** after inflation.

Quantum Clocks During Inflation

The ***inflaton fluctuations*** can be computed on the back of an envelope:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

$\phi = \bar{\phi} + \delta\phi$ ↓

$$S \approx \int \mathrm{d}^4x \; a^3 \left[\dot{\delta\phi}^2 - (\nabla \delta\phi)^2 \right]$$

↓

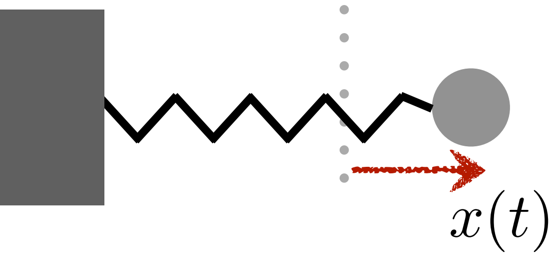
$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \frac{k^2}{a^2(t)}\delta\phi = 0$$

friction

$\omega^2(t)$

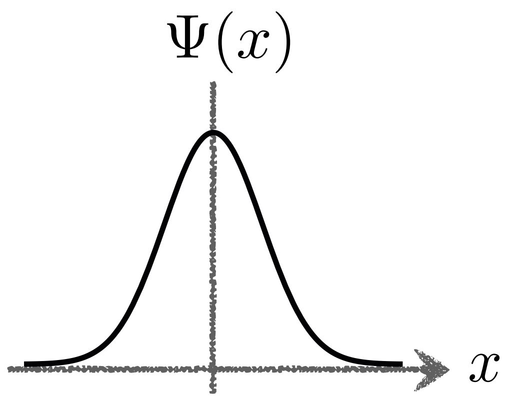
Modes start with $\omega(t) \gg H$ (subhorizon), where they experience zero-pt fluctuations of a harmonic oscillator.

Quantum Clocks During Inflation



$$\ddot{x} + \omega^2 x = 0$$

$$\langle x^2 \rangle = \frac{\hbar}{2\omega}$$



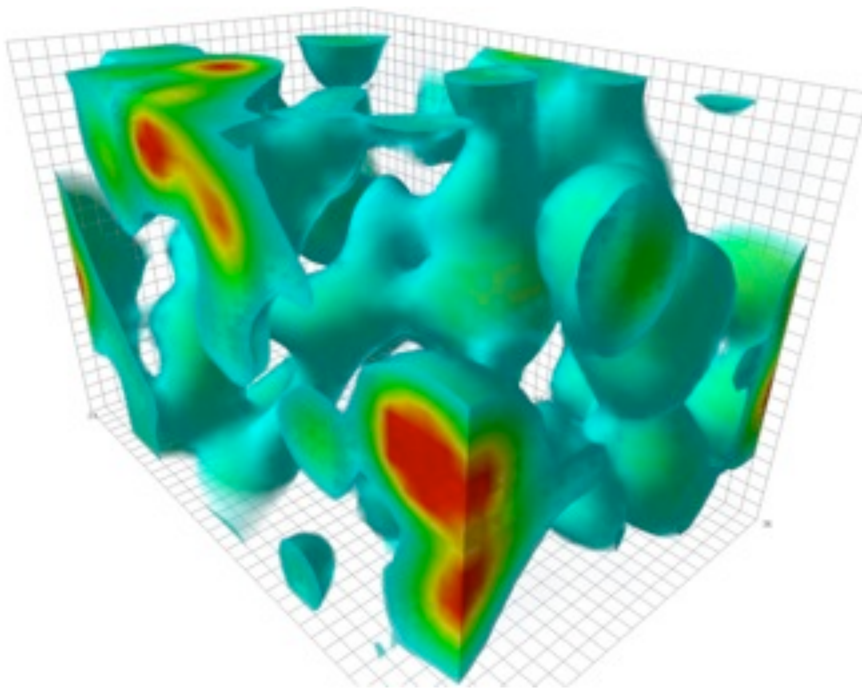
$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \frac{k^2}{a^2(t)}\delta\phi = 0$$

$$\delta\phi_c \equiv a^{-3/2}\delta\phi$$

$$\langle (\delta\phi_c)^2 \rangle = \frac{1}{2\omega}$$

$$\langle (\delta\phi)^2 \rangle = \frac{1}{2}\frac{1}{a^3(t)}\frac{1}{k/a(t)}$$

Quantum Clocks During Inflation



$$\langle (\delta\phi)^2 \rangle = \frac{1}{2}\frac{1}{a^3(t)}\frac{1}{k/a(t)}$$

This holds as long as the mode evolves adiabatically (inside the horizon).

These fluctuations freeze in at horizon crossing ($k/a_\star = H_\star$)

$$\Delta^2_{\delta\phi}(k) \equiv \frac{k^3}{2\pi^2}\langle (\delta\phi)^2 \rangle_\star = \left(\frac{H_\star}{2\pi}\right)^2$$

and become classical curvature perturbations:

$$\Delta^2_\zeta(k) \equiv \left(\frac{H}{\dot\phi}\right)^2 \Delta^2_{\delta\phi}(k)$$

(model-dependent) conversion

Primordial Perturbations from Inflation

We have arrived at a famous result:

$$\Delta_\zeta^2(k) = \frac{1}{8\pi^2}\frac{H^4}{\dot{\phi}^2} \equiv A_s\left(\frac{k}{k_\star}\right)^{n_s-1}$$

Mukhanov and Chibisov
Bardeen, Steinhardt and Turner
Starobinsky
Hawking

evaluated at $k = aH$

Quantum fluctuations also create gravitational waves:

$$\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2}$$

Starobinsky

Evolution

Wayne Hu, arXiv:0802.3688

Two Phases of Evolution

recombination — *today* — *time*

$\zeta(0, \boldsymbol{x})$ — *sound waves* — $\delta T_{\mu\nu}(\tau_{\rm rec}, \boldsymbol{x})$ — *free-streaming* — Θ_0

- free-streaming = photon geodesics in the perturbed spacetime

$$ds^2 = (1 + 2\Psi)\mathrm{d}t^2 - a^2(t)(1 - 2\Phi)\mathrm{d}\boldsymbol{x}^2$$

Newtonian gauge

intrinsic ΔT

gravitational redshift

$$\Theta_0 = (\Theta + \Psi)_{\rm rec} + \text{Doppler} + \text{ISW}$$

observed temperature

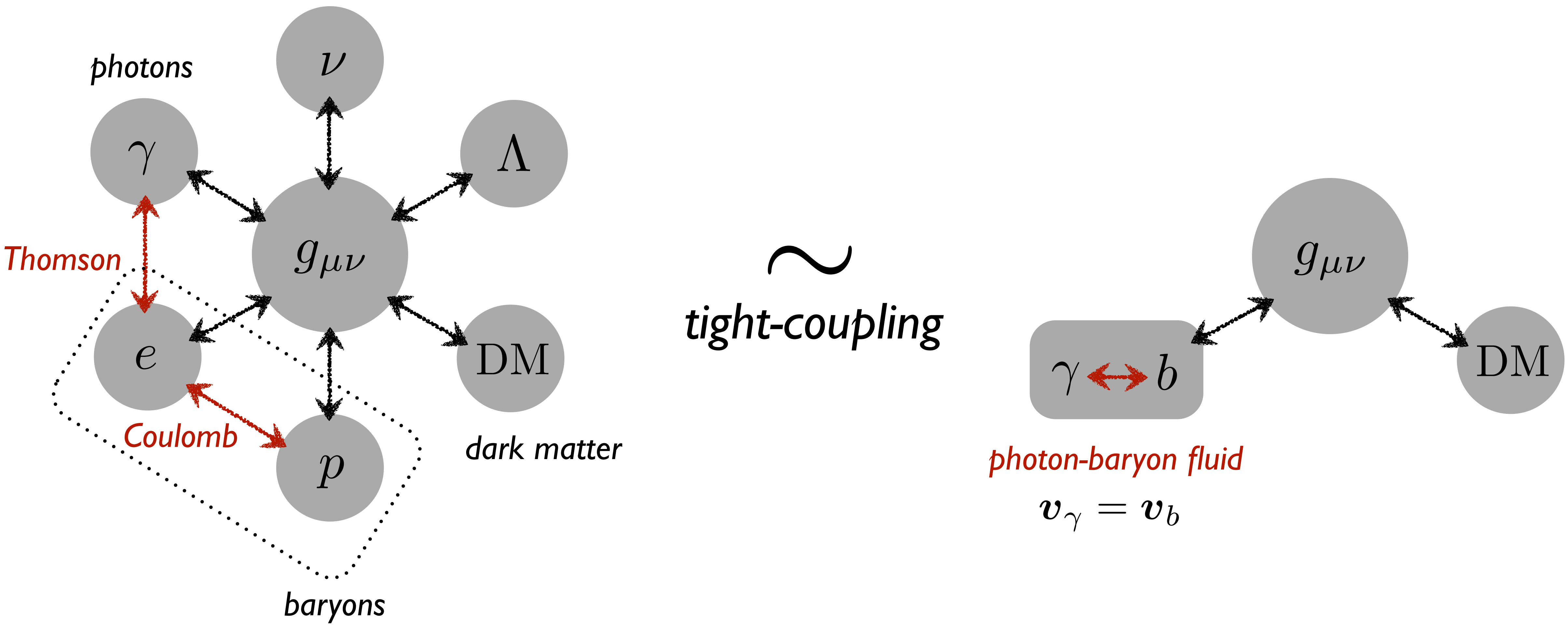
effective temperature

Sachs-Wolfe

- sound waves = oscillations supported by radiation pressure

Tight-Coupling Approximation

- Cast of characters:



- Tight-coupling approximation:

Near recombination the mean free path of photons is 2.5 Mpc (= 0.01 x horizon).

On larger scales, we can treat photons and baryons as a single tightly coupled fluid.

A (Over)Simplified Treatment

We will start with a simplified system and then fix things one by one:

- Neglect the momentum density of the baryons:

$$R \equiv \frac{(\rho_b + P_b)v_b}{(\rho_\gamma + P_\gamma)v_\gamma} = \frac{3}{4}\frac{\rho_b}{\rho_\gamma} \approx 0.6 \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{a_{\rm rec}}\right) < 1$$

- Neglect anisotropic stress: $\Psi \approx \Phi$

- Neglect radiation in the expansion:

$$\frac{\rho_r}{\rho_m} = 0.3 \left(\frac{\Omega_m h^2}{0.15}\right)^{-1}\left(\frac{a}{a_{\rm rec}}\right)^{-1} < 1 \longrightarrow \Psi \approx const.$$

- Ignore gravity!

Fluid Equations

- ***Continuity equation*** = conservation of photon number

$$\boxed{n_\gamma' + 3\,\frac{a'}{a}\,n_\gamma = -\boldsymbol{\nabla}\cdot(n_\gamma \boldsymbol{v}_\gamma)} \qquad {}' \equiv \frac{d}{d\tau}$$

number density — expansion $\bar{n}_\gamma \propto a^{-3}$ — flux

Let $n_\gamma = \bar{n}_\gamma(\tau)\big[1 + 3\Theta(\tau, \boldsymbol{x})\big]$ (recall: $n_\gamma \propto T^3$)

Ex: *Show that*

$$\boxed{\Theta' = -\frac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}_\gamma} \;\rightarrow\; -\frac{i}{3}\boldsymbol{k}\cdot\boldsymbol{v}_\gamma \qquad (1)$$

Fluid Equations

- ***Euler equation*** = momentum conservation

$$[(\rho_\gamma + P_\gamma)\boldsymbol{v}_\gamma]' + 4\,\frac{a'}{a}(\rho_\gamma + P_\gamma)\boldsymbol{v}_\gamma = -\boldsymbol{\nabla} P_\gamma$$

momentum density — expansion — force

Ex: *Using* $P_\gamma = \frac{1}{3}\rho_\gamma$ *and* $\rho_\gamma = \bar{\rho}_\gamma(\tau)\big[1 + 4\Theta(\tau, \boldsymbol{x})\big]$, (recall: $\rho_\gamma \propto T^4$)
show that

$$\boldsymbol{v}'_\gamma = -\boldsymbol{\nabla}\Theta \quad \to \quad -i\boldsymbol{k}\,\Theta \qquad (2)$$

Sound Waves

Combining (1) and (2), we get

$$\Theta'' + c_s^2 k^2 \Theta = 0$$

Simple Harmonic Oscillator

$$c_s^2 \equiv \frac{1}{3}$$

speed of sound

Solution: $\Theta_{\boldsymbol{k}}(\tau) = \Theta_{\boldsymbol{k}}(0)\cos(ks) + \frac{\Theta'_{\boldsymbol{k}}(0)}{kc_s}\sin(ks)$

$$s \equiv \int c_s \, \mathrm{d}\tau$$

"sound horizon"

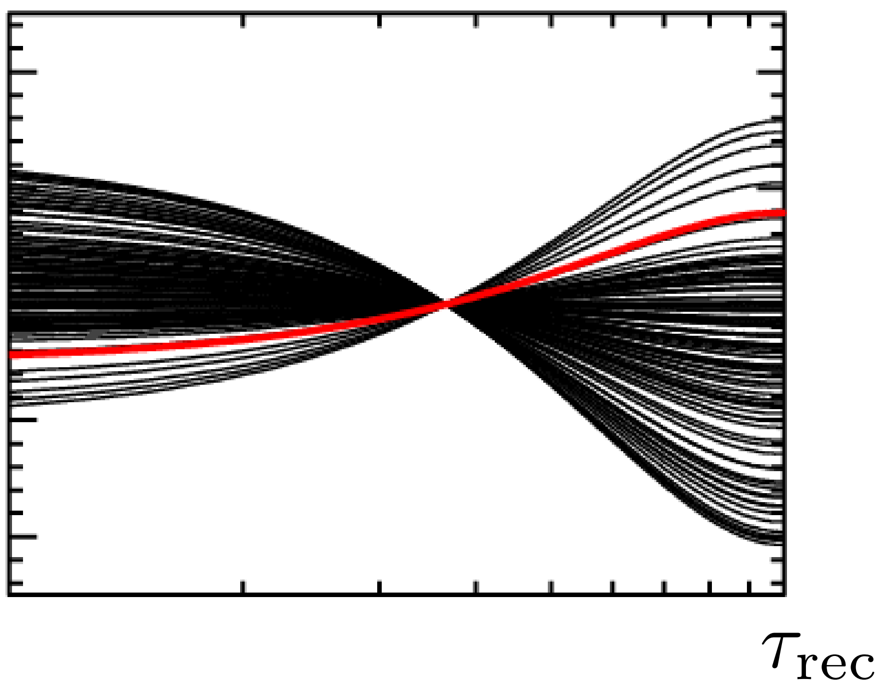
Inflation predicts $\Theta_{\boldsymbol{k}}(0) = \frac{2}{5}\zeta_{\boldsymbol{k}}(0)$ and $\Theta'_{\boldsymbol{k}}(0) = 0$

$$\Theta_{\boldsymbol{k}}(\tau) = \frac{2}{5}\zeta_{\boldsymbol{k}}(0)\cos(ks)$$

Coherent Phases

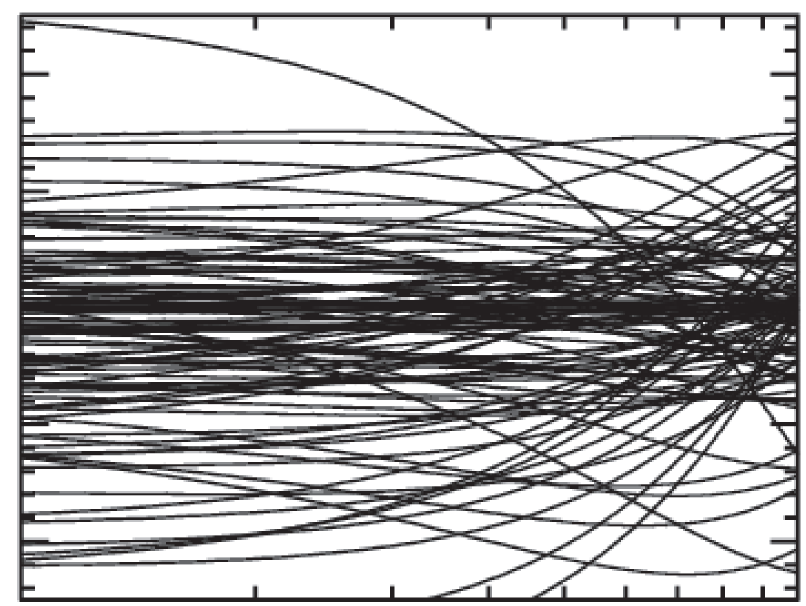
There is a key prediction here:

$\Theta'_{\boldsymbol{k}}(0) = 0$ implies that all $\boldsymbol{k}$-modes start ***in phase!***



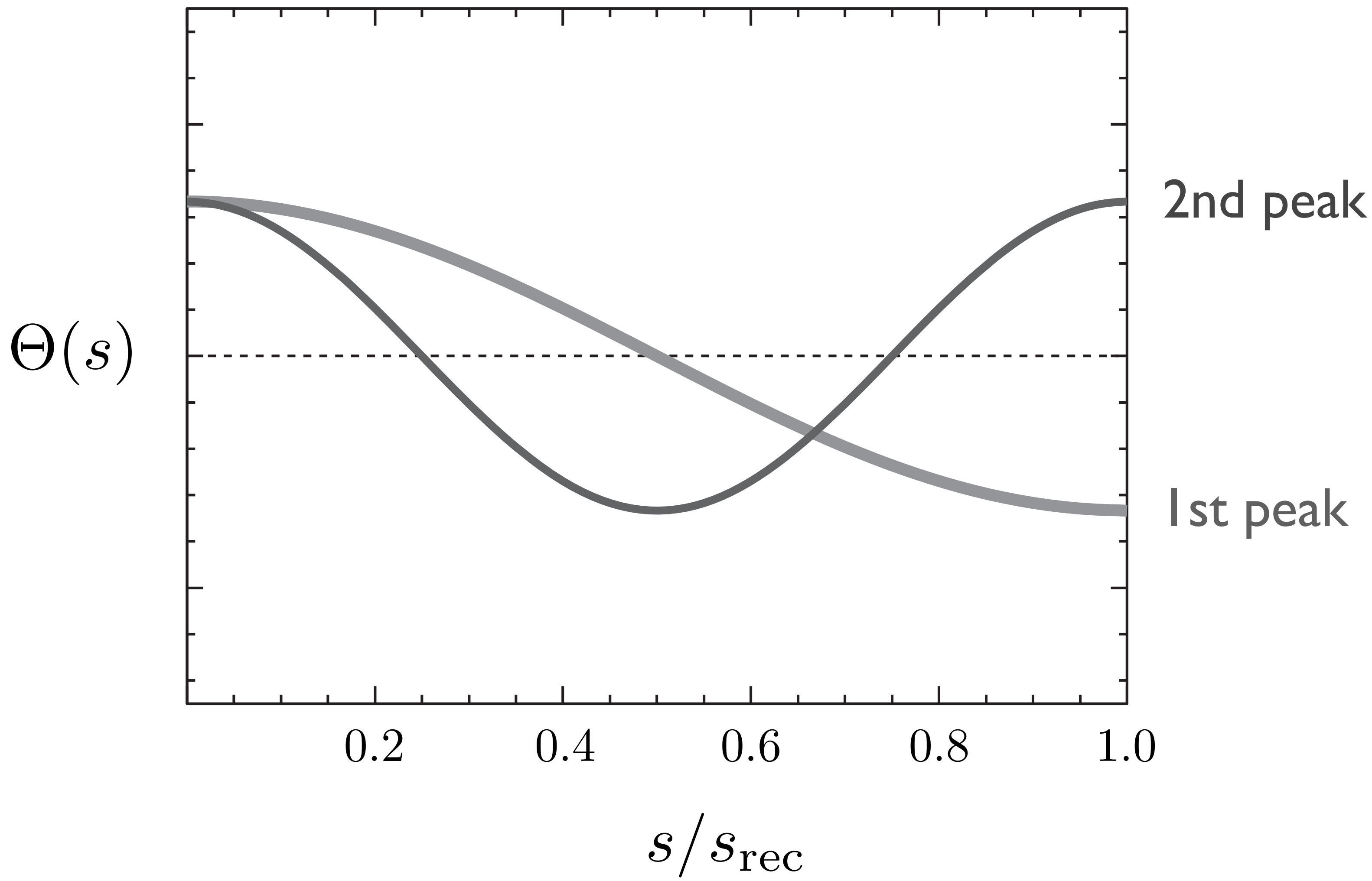
coherent superposition

vs.



noise

Acoustic Peaks



At recombination, we have

$$\Theta_{\rm rec} = \frac{2}{5}\zeta(0)\cos(ks_{\rm rec})$$

peaks at $k_n = \frac{n\pi}{s_{\rm rec}}$ become the peaks of the C_ℓ .

$\ell_{\rm peaks}$ measures Ω_k

Including Gravity

Recall $$ds^2 = (1+2\Psi)dt^2 - a^2(t)(1-2\Phi)d\boldsymbol{x}^2$$

The curvature perturbation is a local perturbation of the scale factor $a \mapsto a(1-\Phi)$

... and the Hubble rate $\frac{a'}{a} \mapsto \frac{a'}{a} - \Phi'$

- Continuity equation:

$$\Theta' = -\frac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}_\gamma + \Phi' \quad (1)$$

perturbed expansion

- Euler equation:

$$\boldsymbol{v}_\gamma' = -\boldsymbol{\nabla}\Theta - \boldsymbol{\nabla}\Psi \quad (2)$$

gravitational force

Including Gravity

Combining (1) and (2), we get

pressure gravity

$$\Theta'' + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi + \Phi''$$

Forced Simple Harmonic Oscillator

We are still assuming:

- no anisotropic stress $\Psi \approx \Phi$
- matter-dominated $\Psi \approx const.$

so, we can write

$$(\Theta + \Psi)'' + c_s^2 k^2 (\Theta + \Psi) = 0$$

effective temperature

Solution: $(\Theta + \Psi)_{\rm rec} = -\frac{1}{5}\zeta(0)\cos(k s_{\rm rec})$

Including Baryons

Baryons add mass to the photon-baryon fluid.

Conservation applies to the total momentum density

$$= (\rho_\gamma + P_\gamma)\boldsymbol{v}_\gamma + (\rho_b + P_b)\boldsymbol{v}_b \equiv \frac{4}{3}(1+R)\rho_\gamma \boldsymbol{v}_\gamma$$

- Euler equation:

inertial mass — gravitational mass

$$\left[(1+R)\boldsymbol{v}_\gamma\right]' = -\boldsymbol{\nabla}\Theta - (1+R)\boldsymbol{\nabla}\Psi \qquad (2)$$

- Continuity equation:

stays the same: (1)

Including Baryons

Combining (1) and (2), we get

$$\left[(1+R)\Theta'\right]' + \frac{k^2}{3}\Theta = -\frac{k^2}{3}(1+R)\Psi - \left[(1+R)\Phi'\right]'$$

We are still assuming:
- no anisotropic stress $\Psi \approx \Phi$
- matter-dominated $\Psi \approx const.$

On subhorizon scales, $k \gg \frac{R'}{R} = \frac{a'}{a} = aH$, we can write

$$\left[\Theta + (1+R)\Psi\right]'' + c_s^2 k^2 \left[\Theta + (1+R)\Psi\right] = 0$$

$$c_s^2 = \frac{1}{3(1+R)}$$

Solution: $\left[\Theta + (1+R)\Psi\right](\tau) = -\frac{1}{5}(1+3R)\zeta(0)\cos(ks)$

Including Baryons

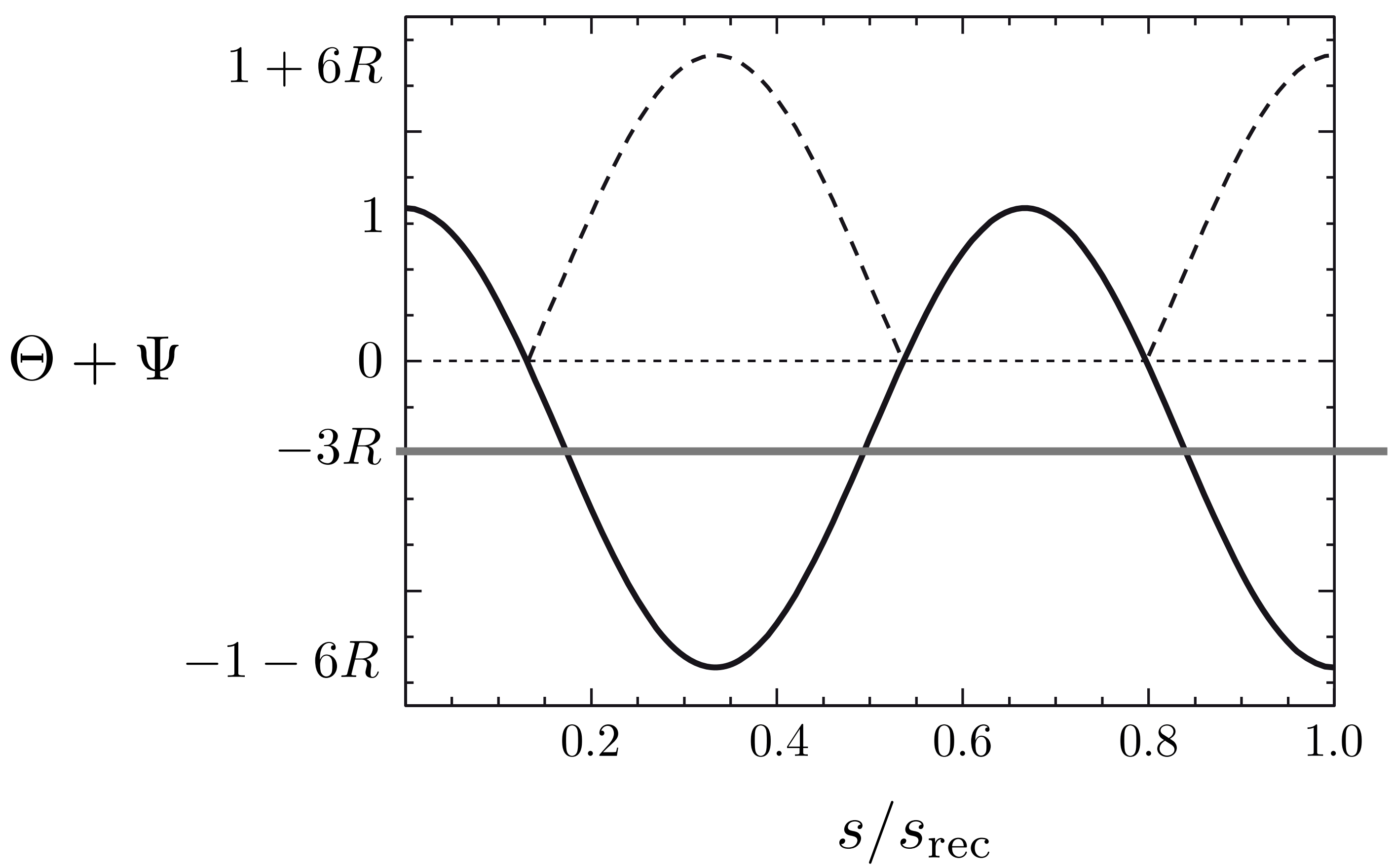
At recombination, we have

$$(\Theta + \Psi)_{\rm rec} = -\frac{1}{5}\zeta(0)\Big[(1+3R)\cos(ks_{\rm rec}) - 3R\Big]$$

shifted equilibrium

larger amplitude

smaller frequency

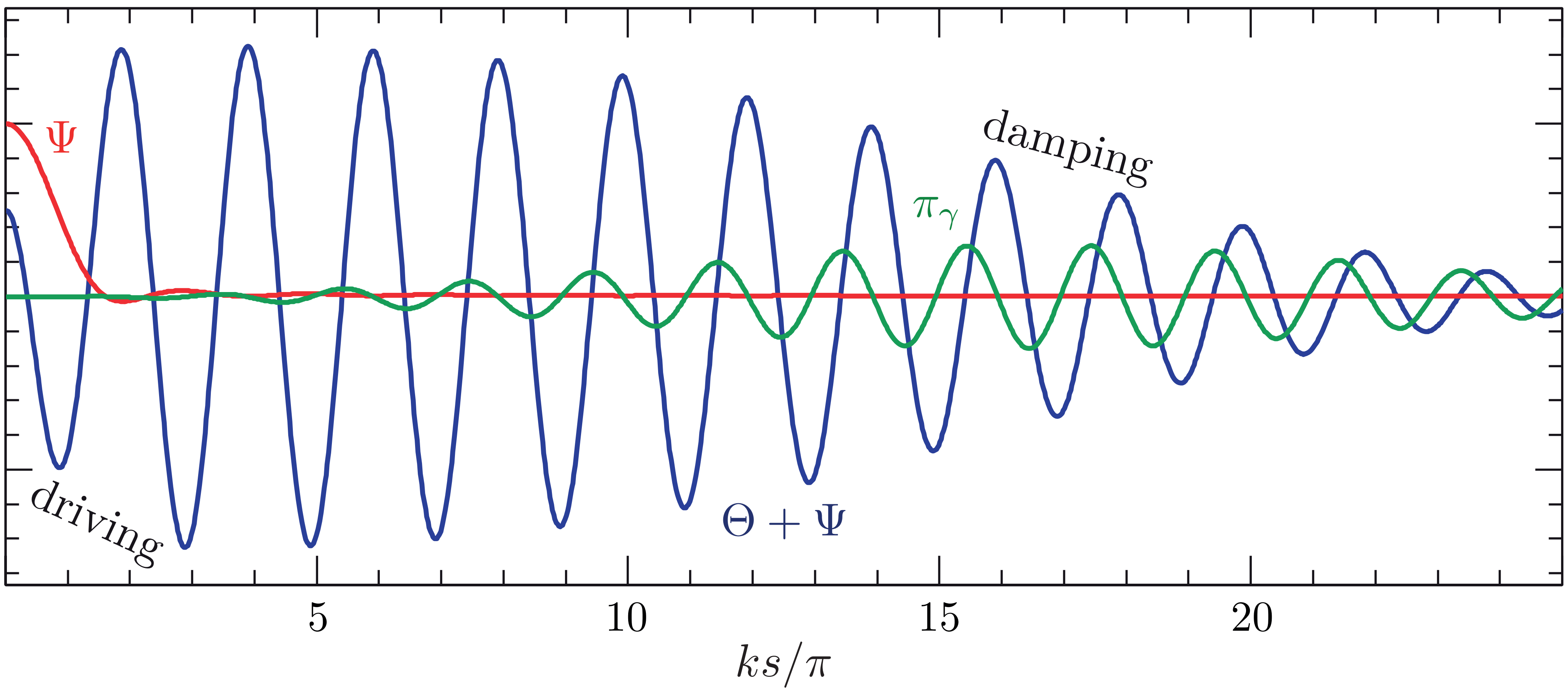


$\dfrac{\mathcal{A}_{\rm odd}}{\mathcal{A}_{\rm even}}$ measures Ω_b

Including Radiation

During the radiation era, we have $k^2\Phi = 4\pi G a^2 \bar{\rho}_r \delta_r$

$\propto a^{-2}$ (for $k^2\Phi$), $\propto a^{-4}$ (for $\bar{\rho}_r$), $const.$ (for δ_r)



Modes that enter the horizon during the radiation era have boosted amplitude: $\dfrac{\mathcal{A}_{\text{peaks}}}{\mathcal{A}_{\text{plateau}}}$ measures Ω_m

Including Damping

On scales smaller than the mean free path, the tight-coupling approximation breaks down and the fluctuations experience diffusion damping.

This can be incorporated as an effective viscosity in the oscillator equation:

$$c_s^2\left[c_s^{-2}\Theta'\right]' + \frac{k^2}{k_D^2}\Theta' + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2[c_s^{-2}\Phi']'$$

↓ WKB approximation

Solution: $(\Theta + \Psi)_{\rm rec} \propto \cos(k s_{\rm rec}) \times e^{-(k/k_D)^2}$

↑ damping tail

Summary

We obtained the solution for a single Fourier mode: *

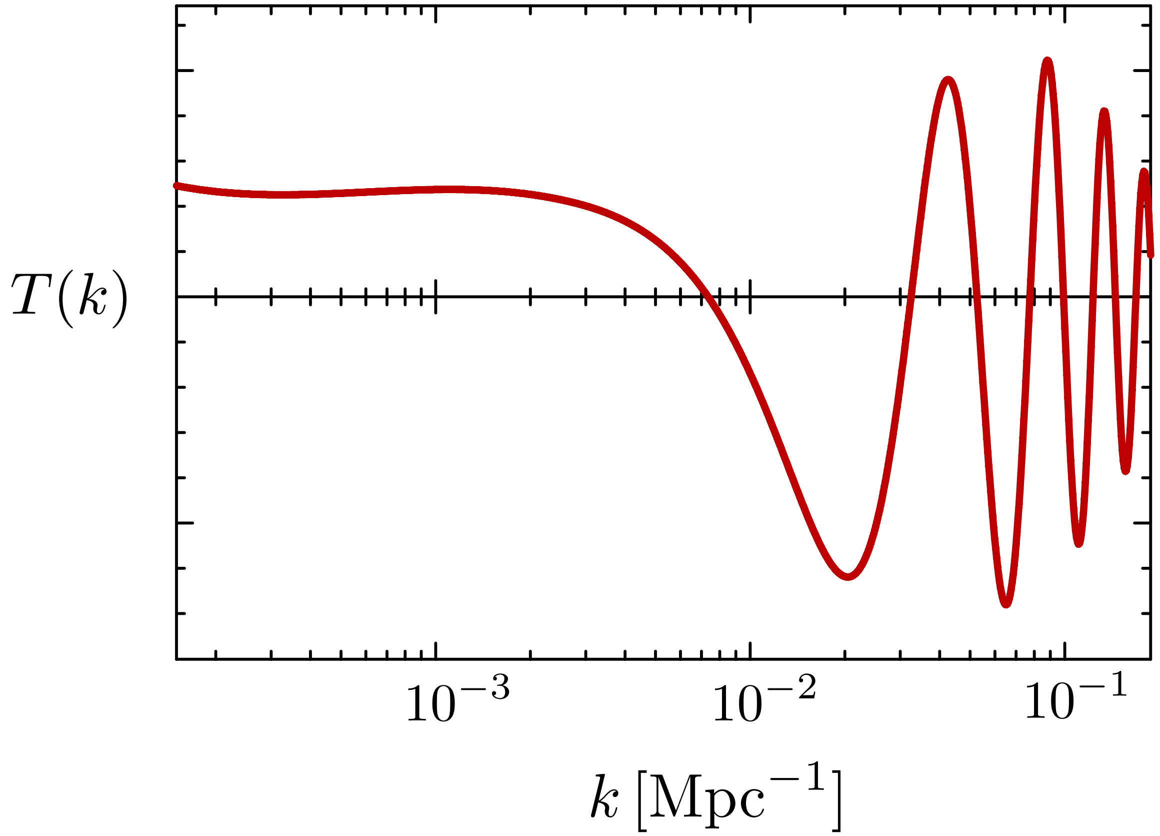
initial condition

$$\Theta_0(\boldsymbol{k}) = \zeta_{\boldsymbol{k}}(0) \times T(k)$$

transfer function

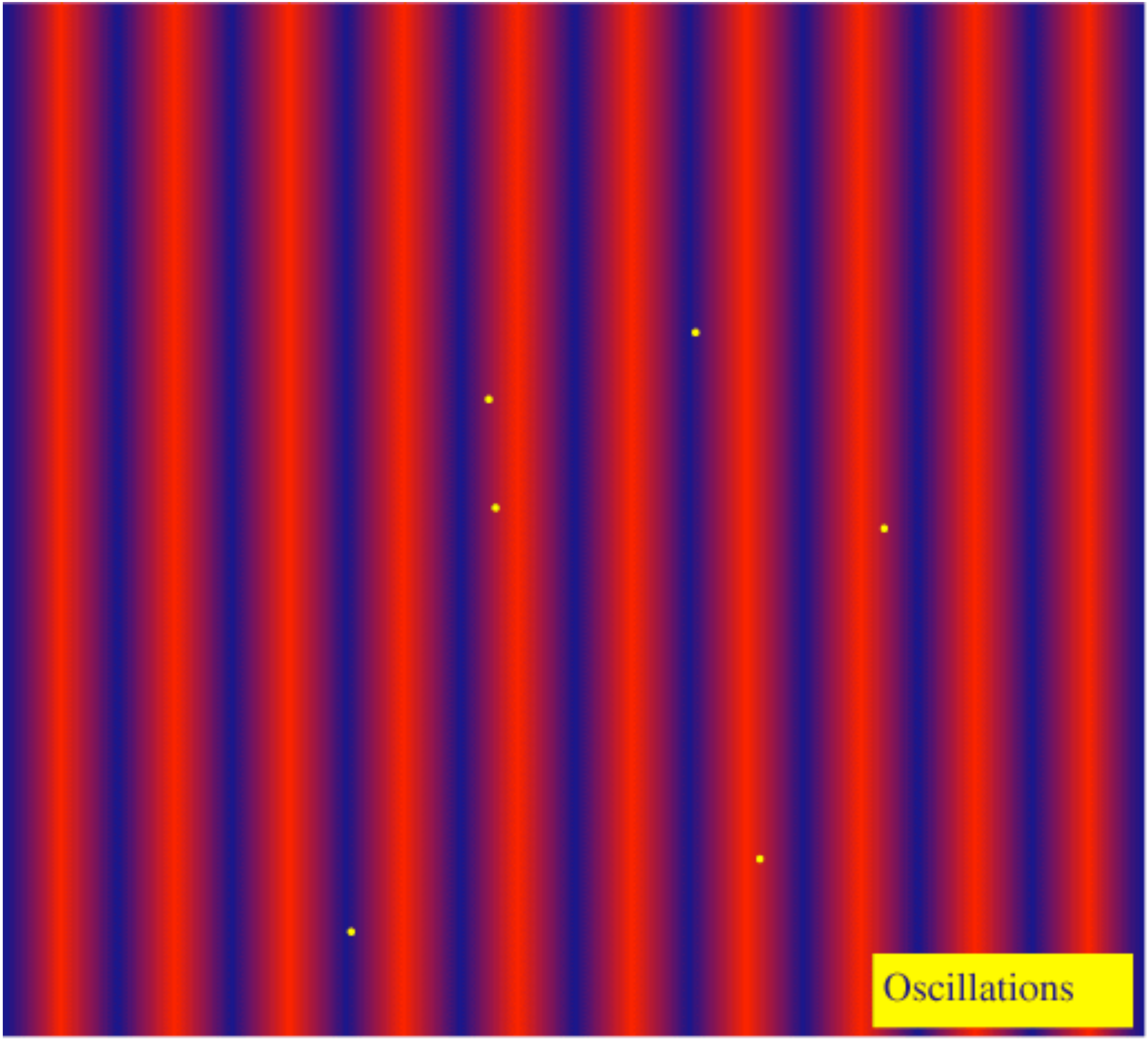
* Should also include Doppler and ISW terms.

CMBFast, CAMB



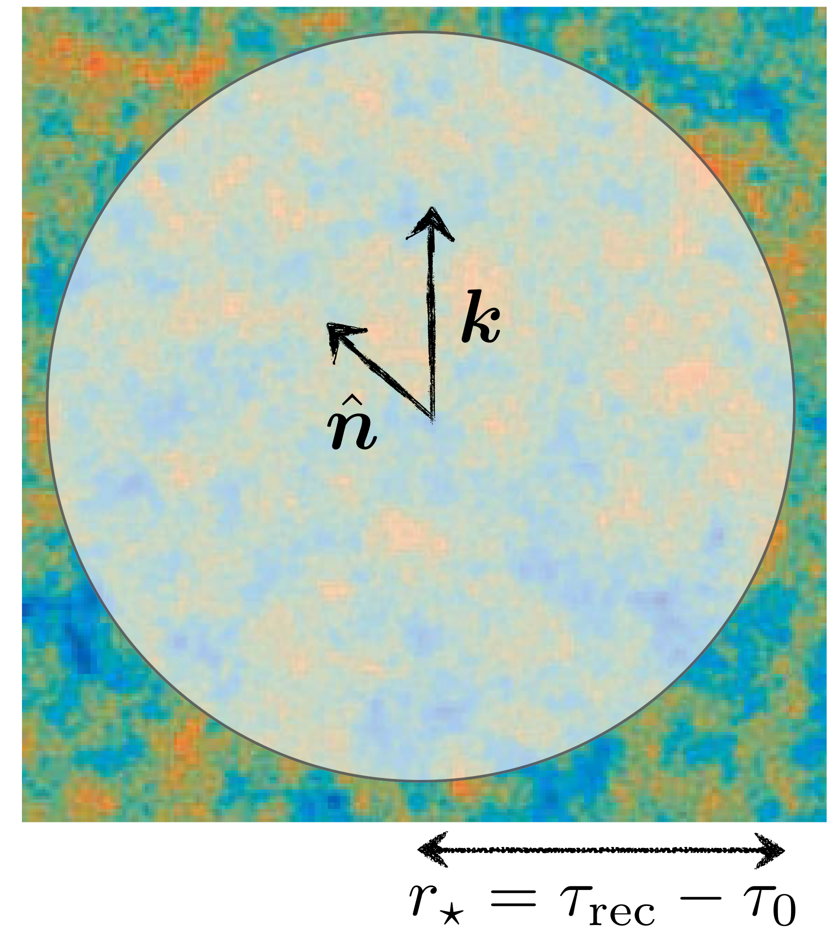
Next, we sum over Fourier modes and project onto the sky.

Projection


Oscillations

The real space temperature field is

$$\Theta_0(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \Theta_0(\boldsymbol{k})$$



Assuming instantaneous recombination, the CMB anisotropies are

$$\Theta_0(\hat{\boldsymbol{n}}) = \int \mathrm{d}r \; \Theta_0(\boldsymbol{x} = r\hat{\boldsymbol{n}}) \, \delta(r - r_\star)$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \, e^{i(kr_\star)\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}} \, \Theta_0(\boldsymbol{k}) \;\equiv \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}})$$

Ex: *Using* $e^{i(kr_\star)\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}} = 4\pi \sum_{\ell m} i^\ell \, j_\ell(kr_\star)\, Y^*_{\ell m}(\hat{\boldsymbol{k}})\, Y_{\ell m}(\hat{\boldsymbol{n}})$

show that

initial conditions

$$C_\ell = 4\pi \int \mathrm{d}\ln k \;\Delta_\zeta^2(k)\,\Delta_\ell^2(k)$$

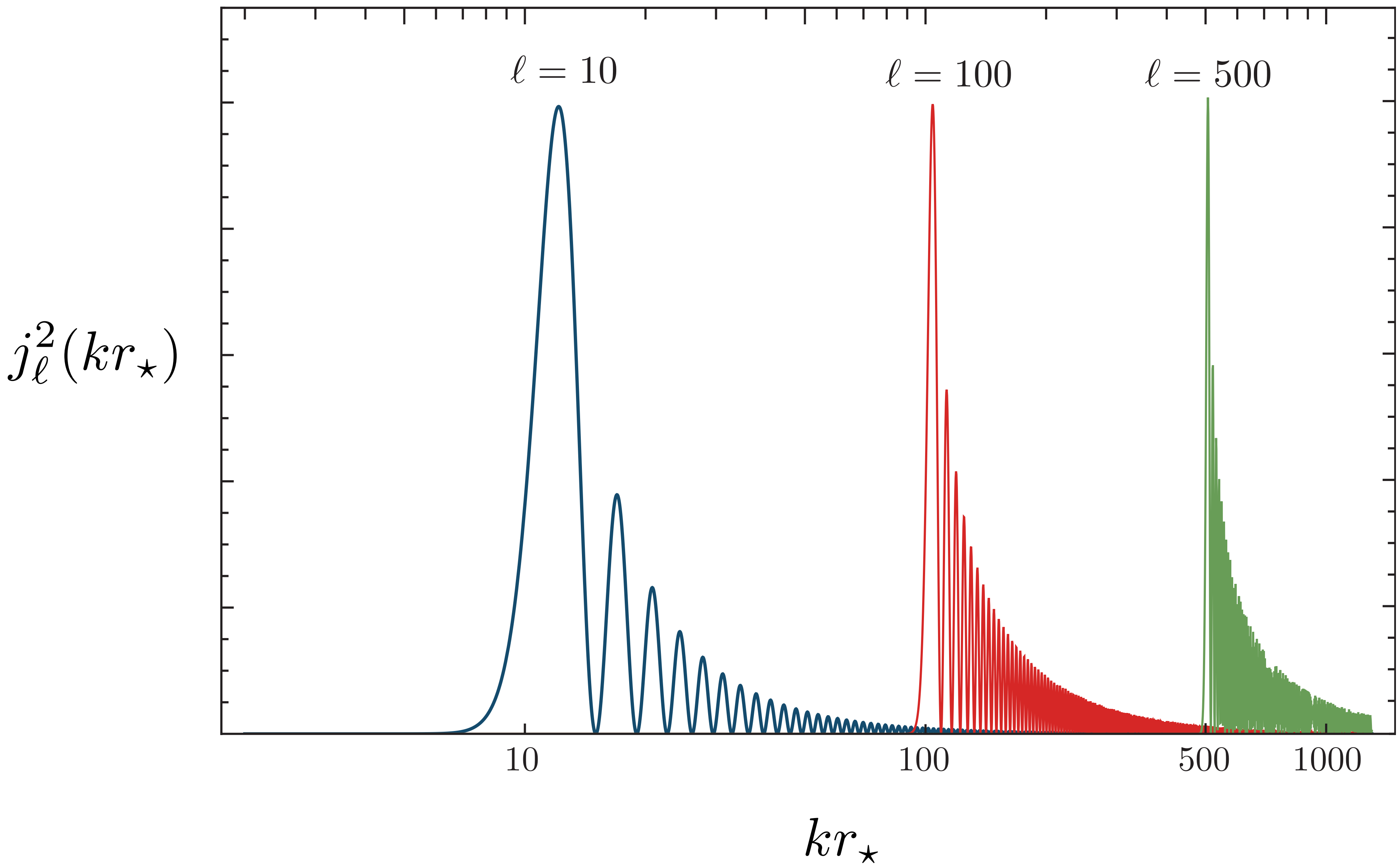
evolution

where $$\Delta_\ell(k) = T(k) \times j_\ell(kr_\star)$$

projection

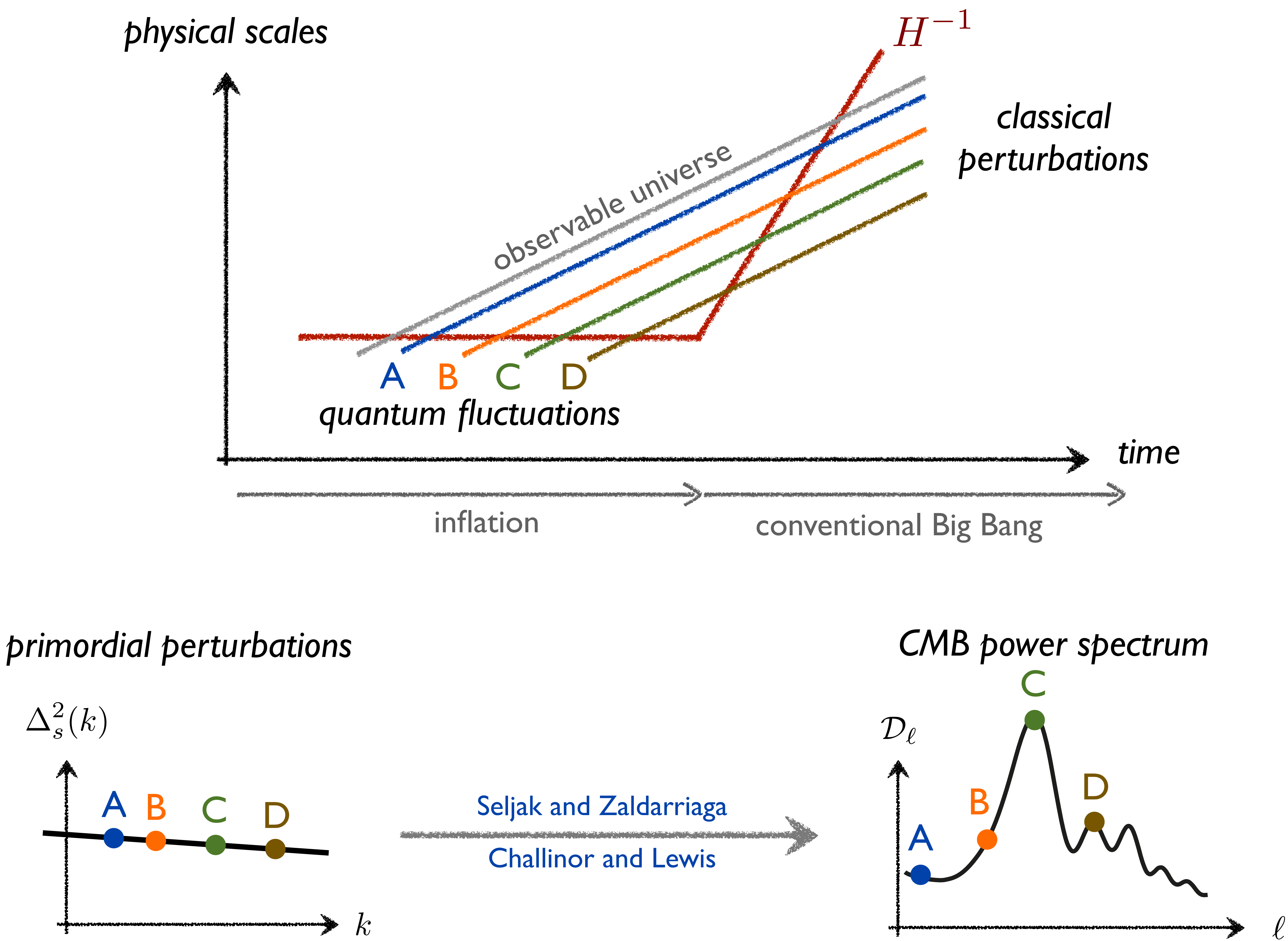
Bessel Projection

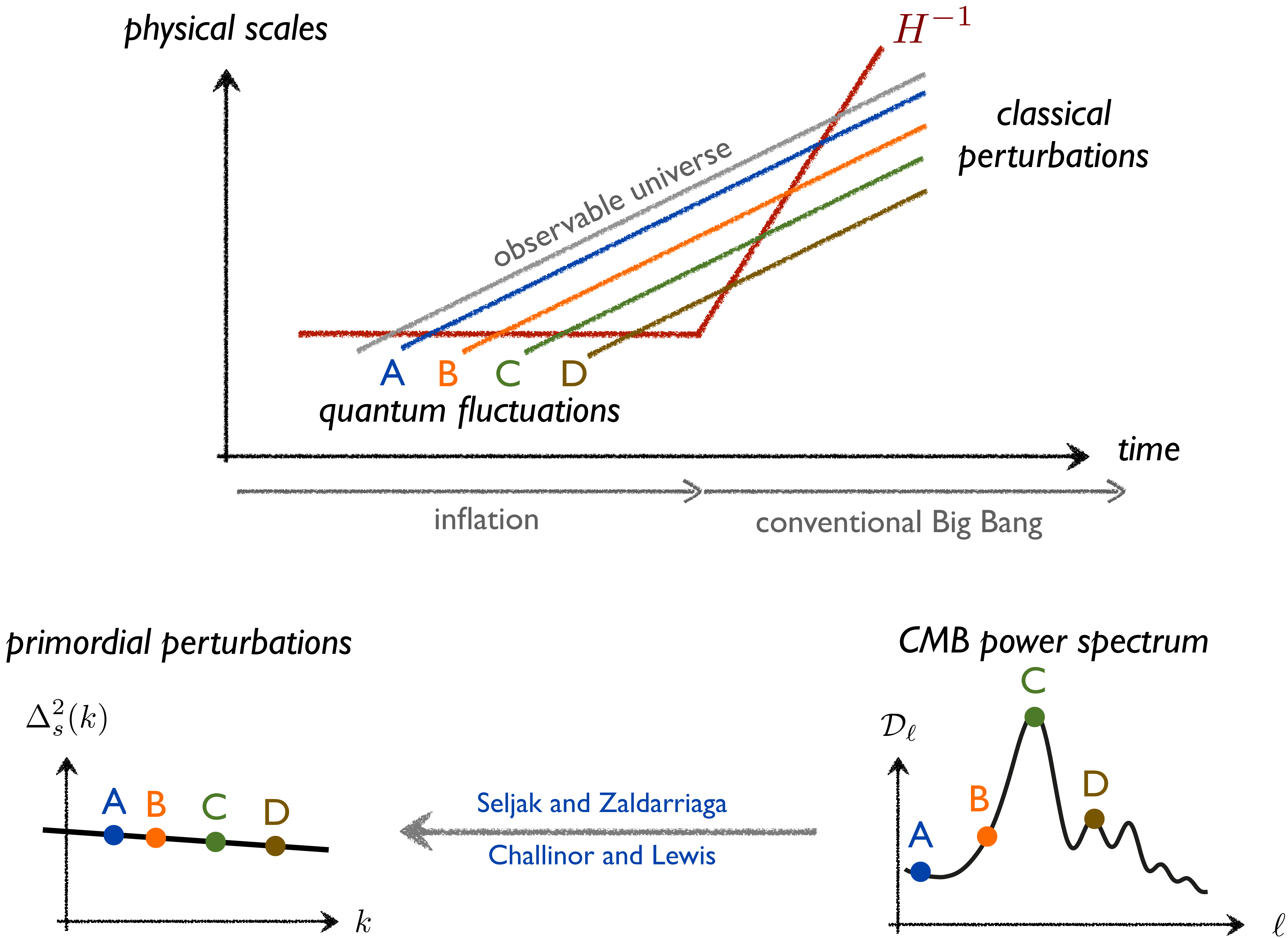
" $j_\ell^2(kr_\star)$ acts like $\delta(\ell - kr_\star)$ "



$$C_\ell \approx 4\pi\, \Delta_\zeta^2(k)\, \Delta_\ell^2(k) \Big|_{kr_\star = \ell}$$

Summary


physical scales
H^{-1}
classical
perturbations
observable universe
A
B
C
D
quantum fluctuations
time
inflation
conventional Big Bang
primordial perturbations
$\Delta_s^2(k)$
A
B
C
D
k
Seljak and Zaldarriaga
Challinor and Lewis
CMB power spectrum
$\mathcal{D}_\ell$
C
B
D
A
ℓ



physical scales
H^{-1}
observable universe
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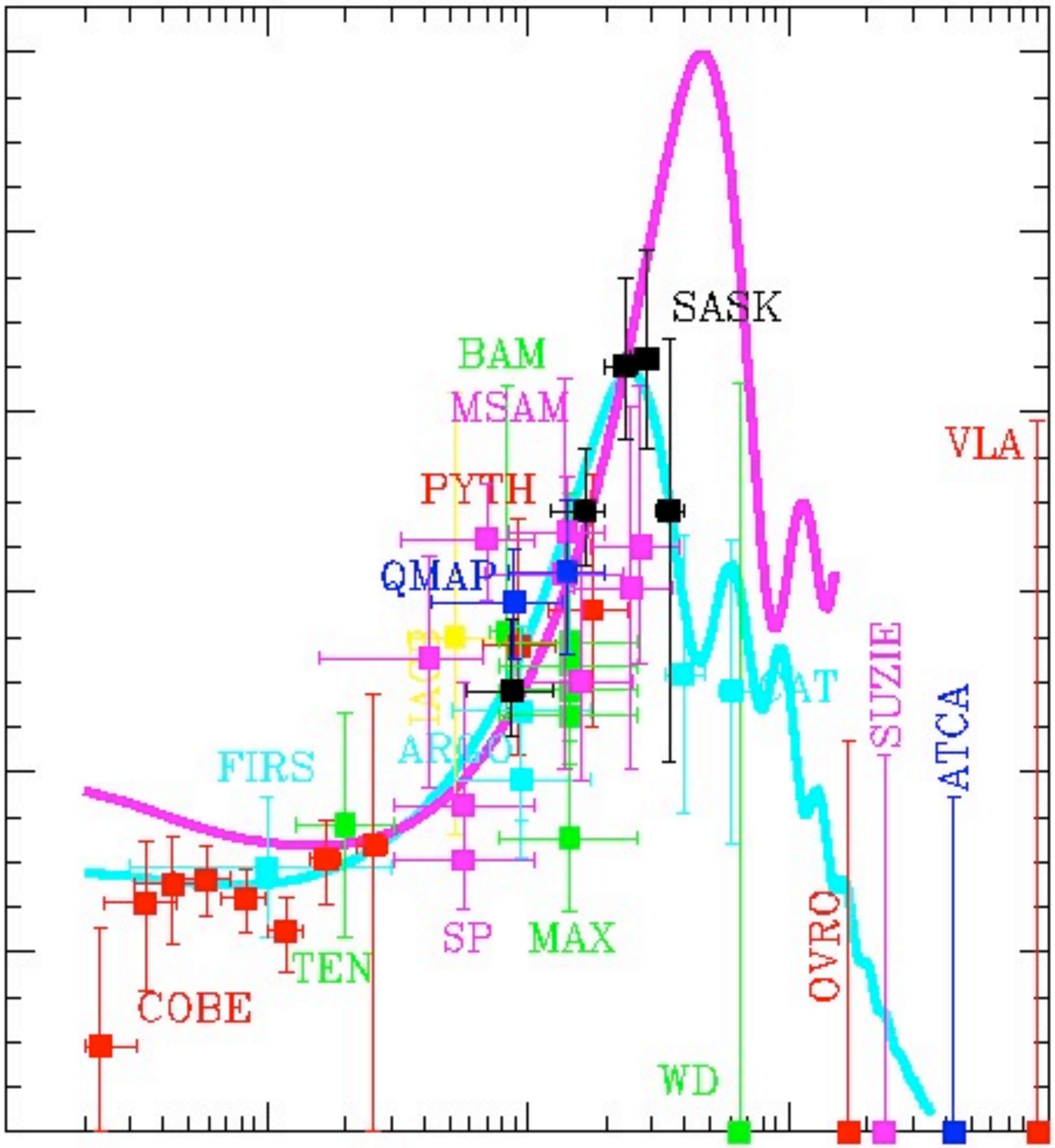
A B C D
quantum fluctuations
time
inflation
conventional Big Bang
primordial perturbations
$\Delta_s^2(k)$
A B C D
k
Seljak and Zaldarriaga
Challinor and Lewis
CMB power spectrum
$\mathcal{D}_\ell$
A B C D
ℓ

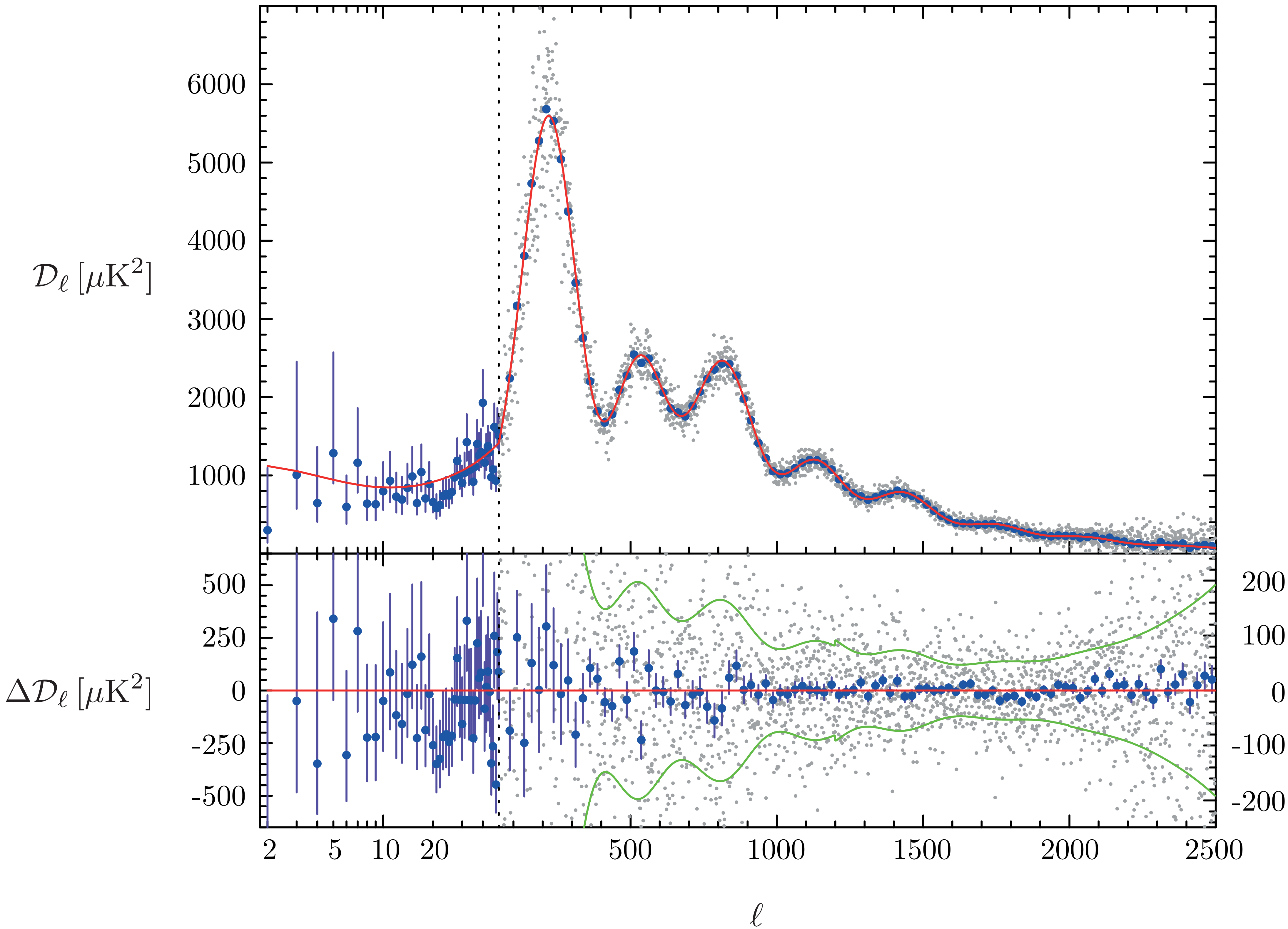
CMB Data

15 years ago



Tegmark

Today



Planck (Paper 16)

6-Parameter Fit

Baseline ΛCDM Model

4 parameters for the ***background***:

$$\Omega_b = 0.045 \pm 0.001$$ baryons

$$\Omega_m = 0.315 \pm 0.016$$ dark matter

$$\Omega_\Lambda = 0.685 \pm 0.018$$ dark energy

$$\tau = 0.089 \pm 0.014$$ optical depth

evolution

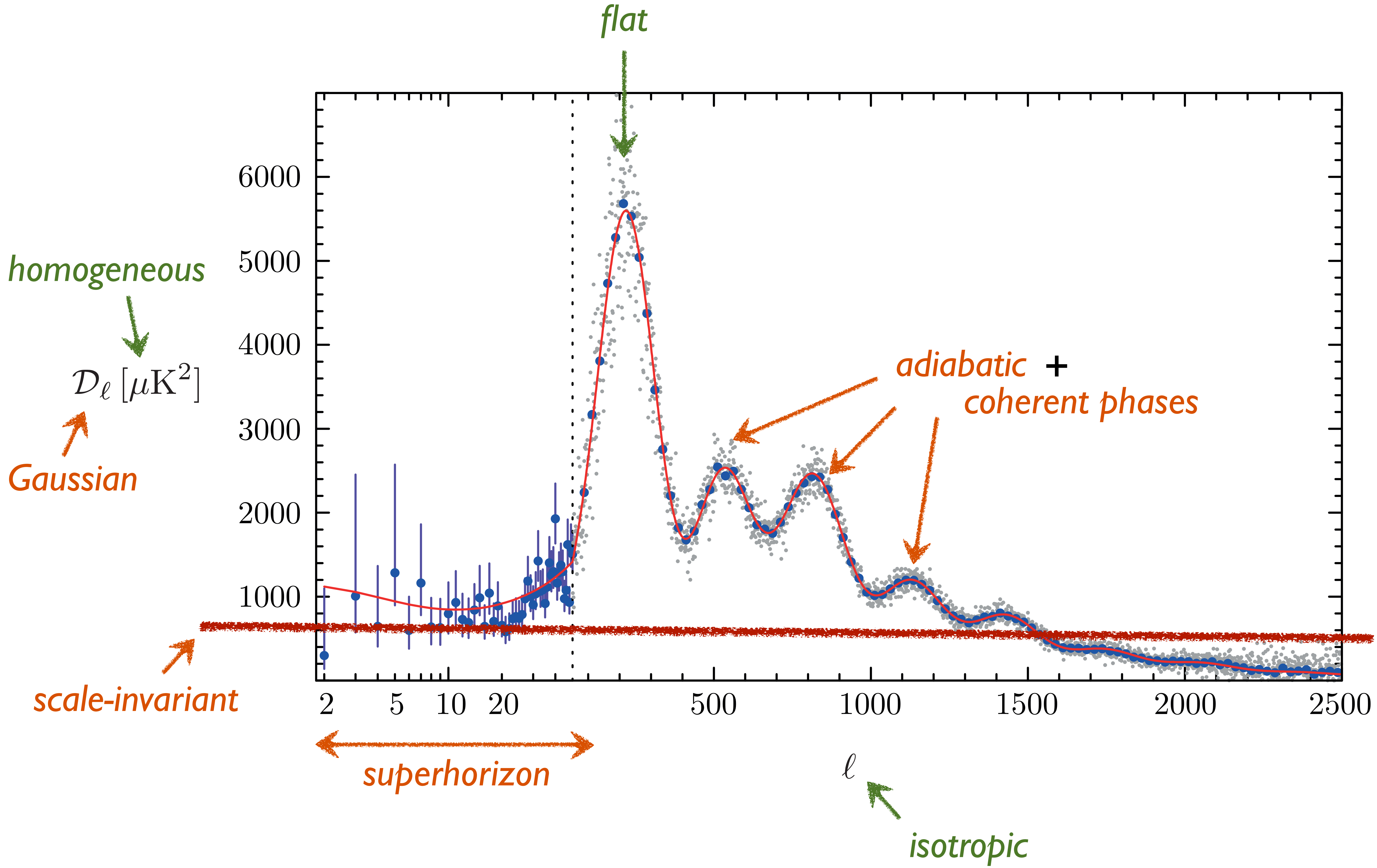
2 parameters for the ***perturbations***:

$$10^9 A_s = 2.20 \pm 0.11$$ amplitude

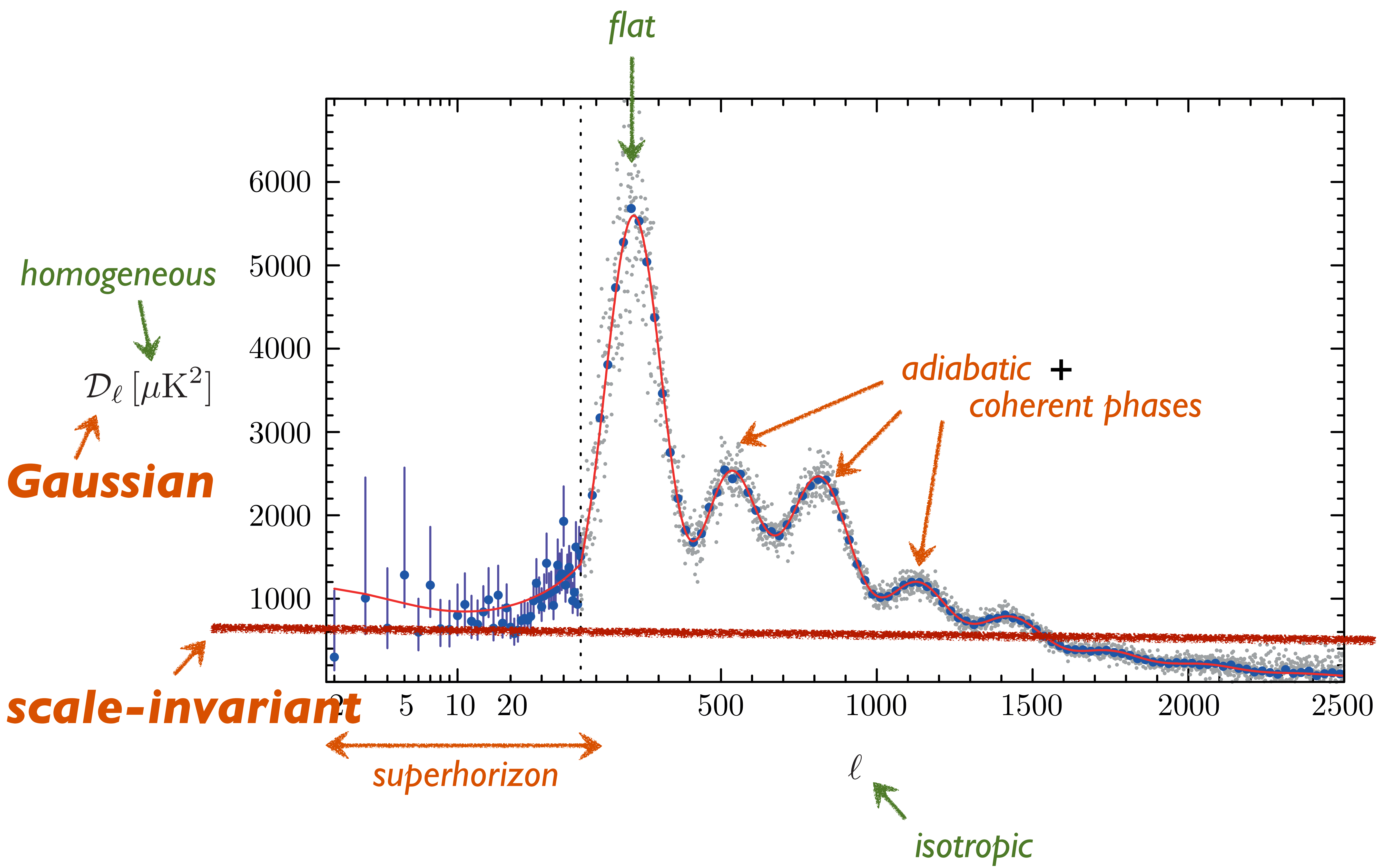
$$n_s = 0.960 \pm 0.014$$ spectral index

initial conditions

Inflation after Planck



flat
homogeneous
Gaussian
scale-invariant
adiabatic +
coherent phases
superhorizon
isotropic
$\mathcal{D}_\ell$ [μK^2]
ℓ
6000
5000
4000
3000
2000
1000
2
5
10
20
500
1000
1500
2000
2500



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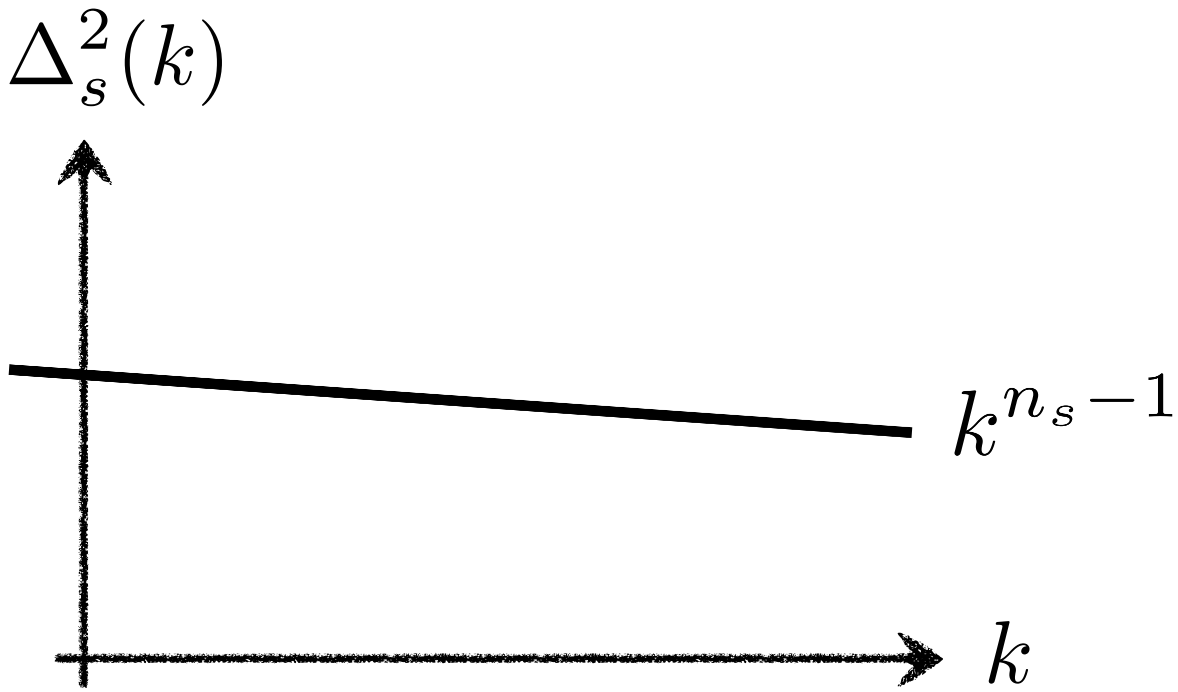
ℓ
isotropic

Scale-Invariance

We expect the fluctuations from inflation to be nearly (but not exactly!) scale-invariant:

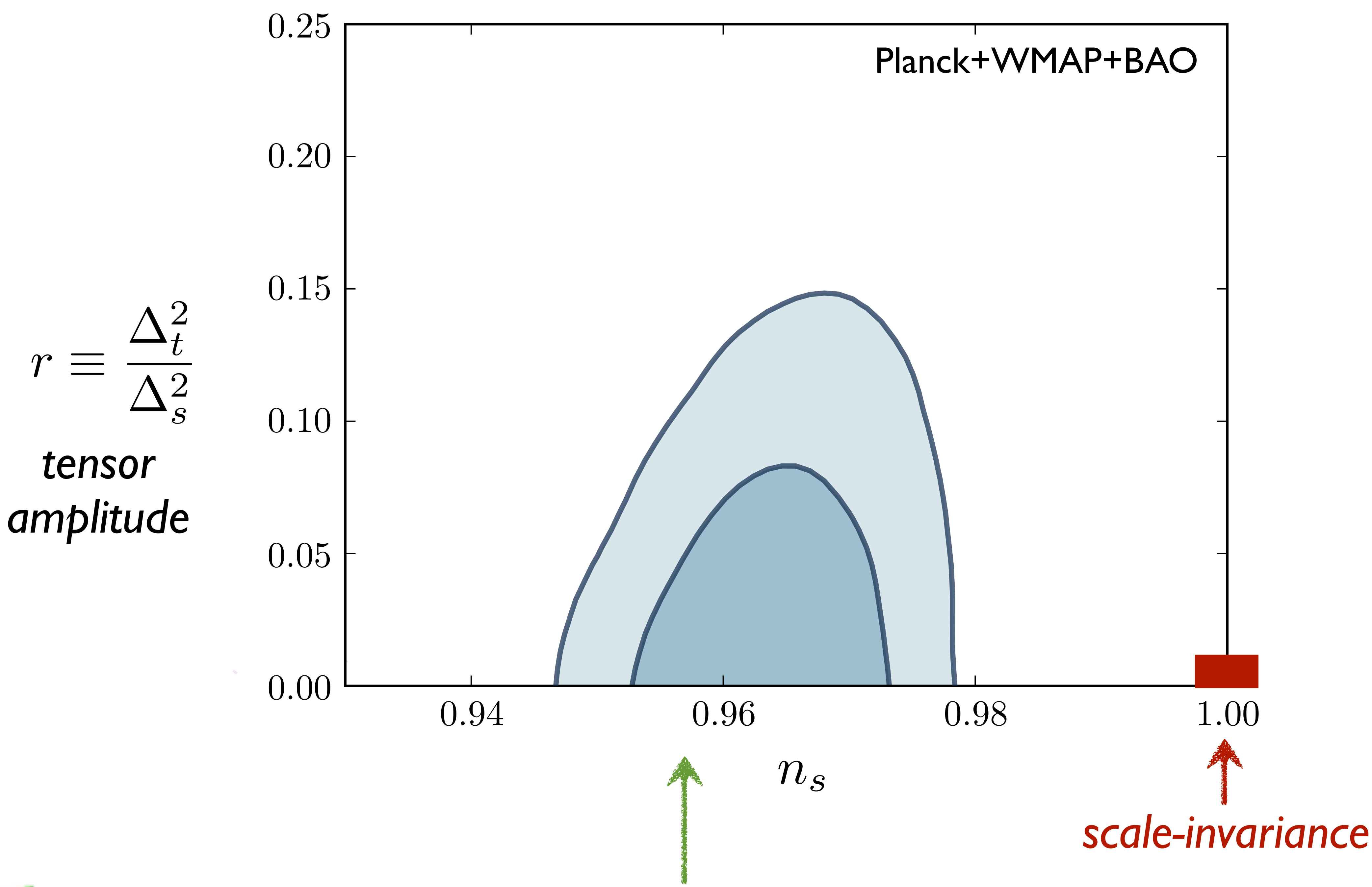


$$n_s - 1 \approx 4\frac{\dot{H}}{H^2} - \frac{\ddot{H}}{\dot{H}H} \sim 0$$

The deviation from scale-invariance measures the dynamics during inflation:

$$H(t)$$

Scale-Invariance

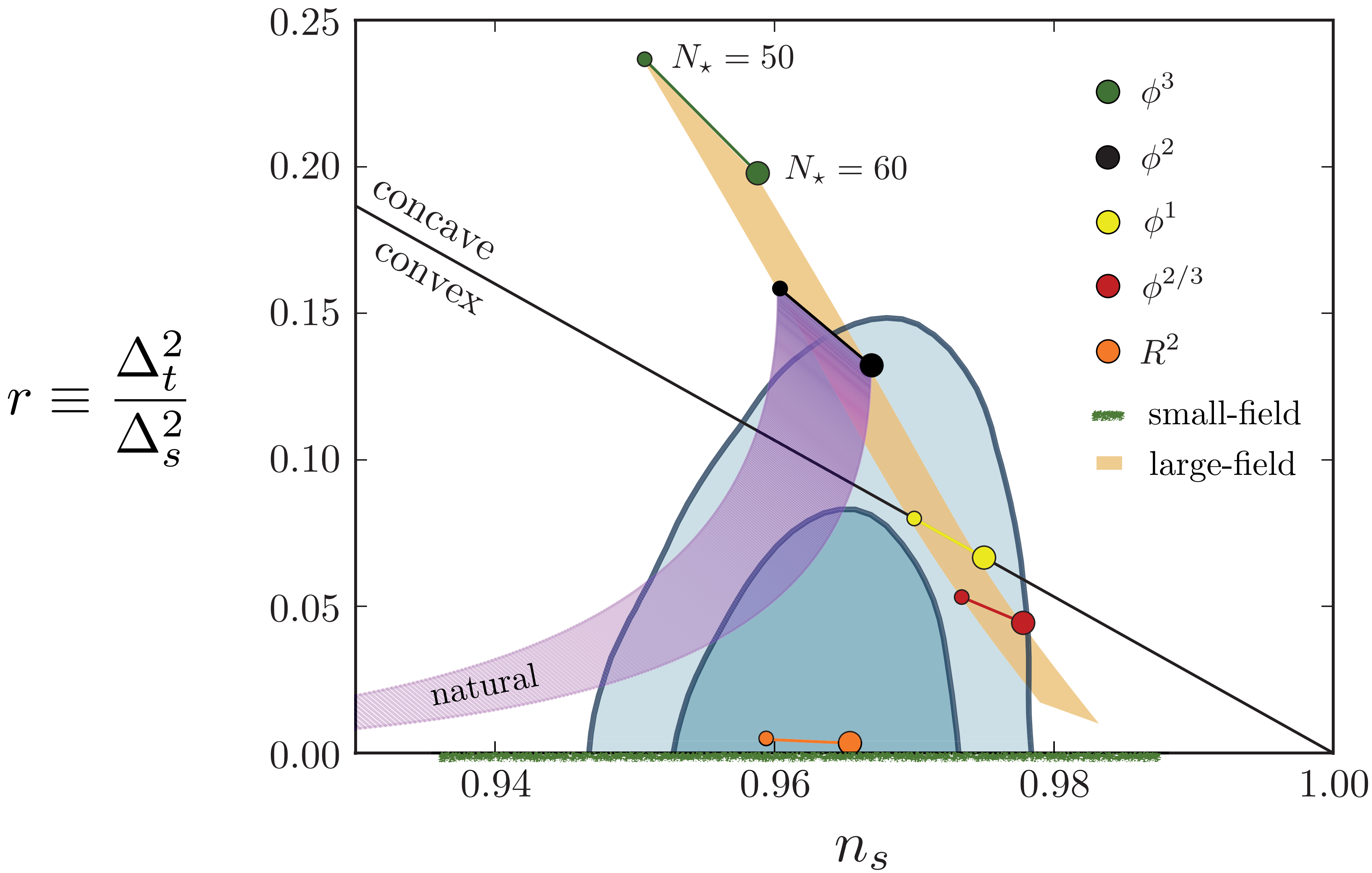


✓ Planck has detected the expected percent-level deviation !!!

$$n_s = 0.961 \pm 0.011$$

Planck (Paper 22)

Scale-Invariance



Many inflationary models are being tested. Some are falsified.

Planck (Paper 22)

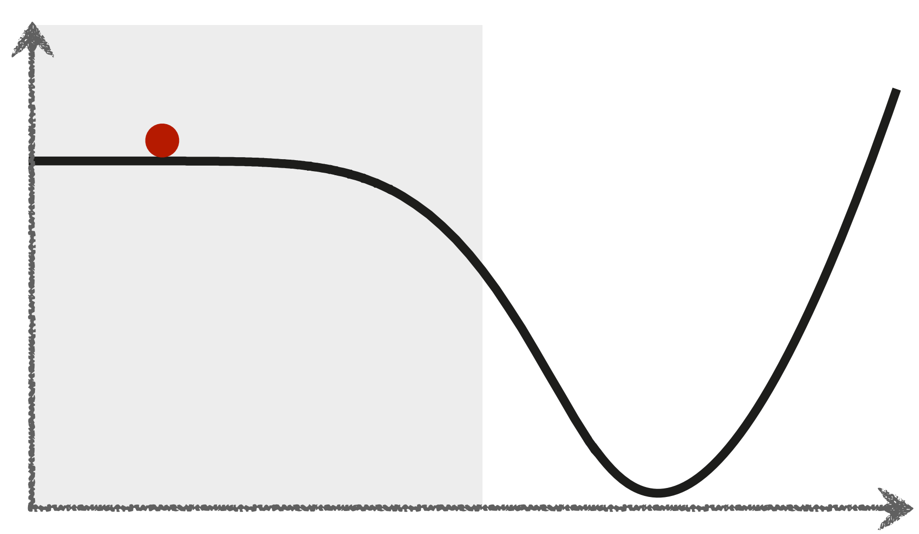
Gaussianity

The CMB is Gaussian to better than 0.1%.

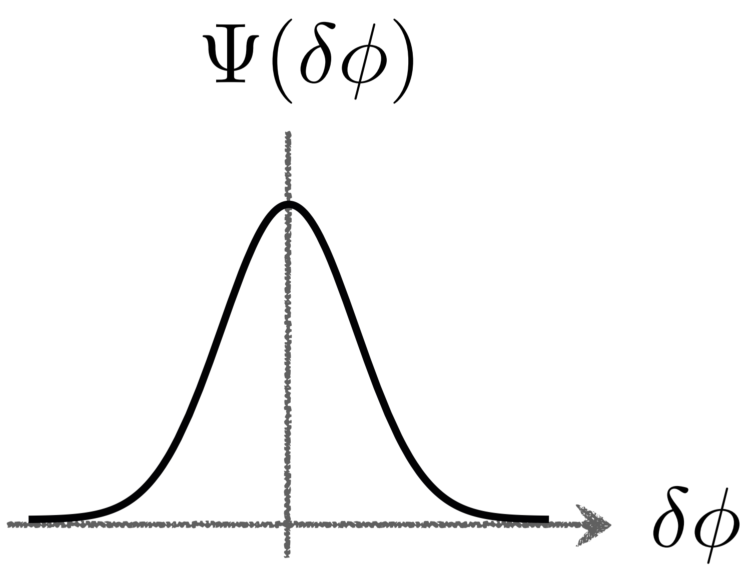
Planck (Paper 24)

- Slow-roll inflation predicts Gaussian fluctuations:

Inflation only occurs on the flat part of the potential where the self-interactions of the field are small.



The wavefunction of a free field is Gaussian:

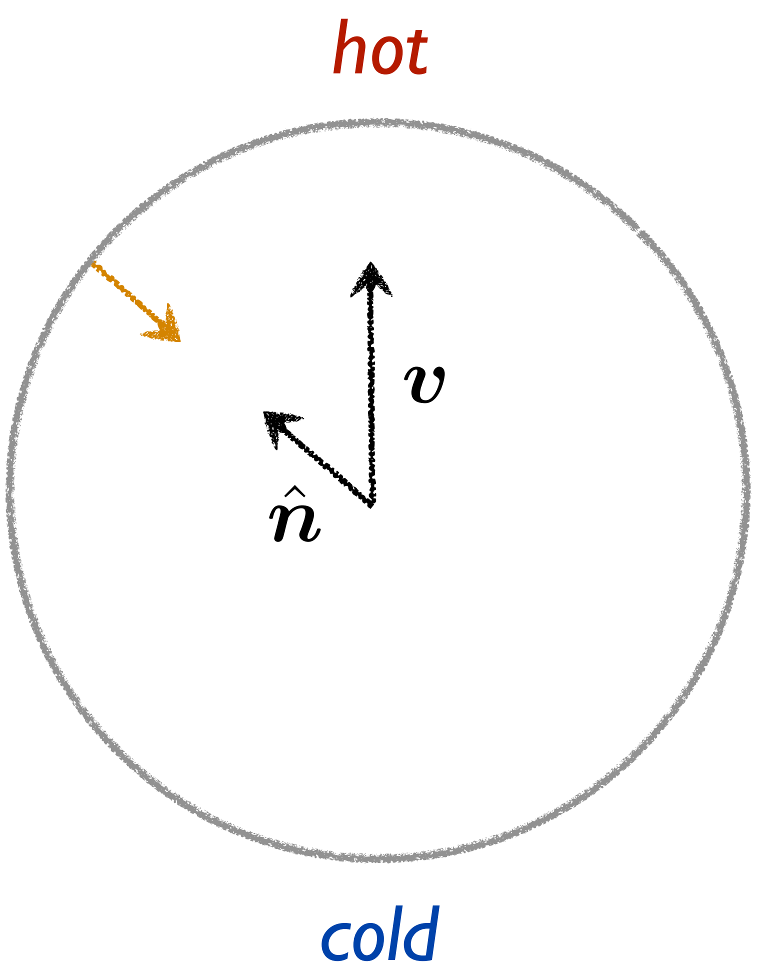


- Extensions of slow-roll models can produce non-Gaussian fluctuations from interactions in the inflaton sector or couplings to other sectors.

see Leonardo's lecture

Appendix

The biggest effect is the motion of the solar system



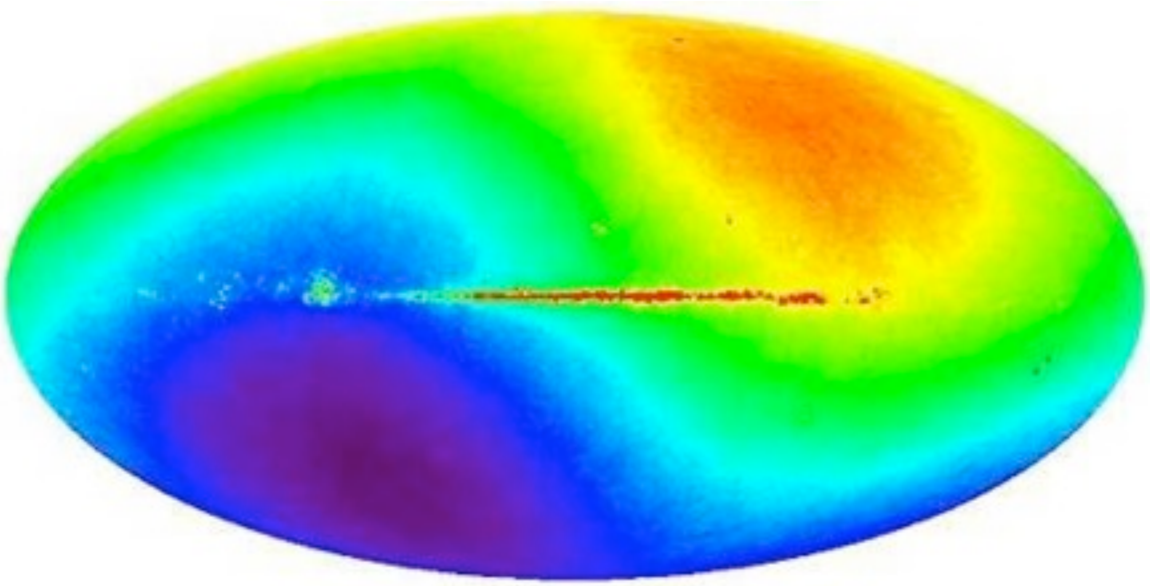
The observed photon momentum is Doppler-shifted

$$p_0 = p\,(1 + \hat{\boldsymbol{n}} \cdot \boldsymbol{v})$$

CMB rest frame

This corresponds to a large temperature dipole

$$\Theta_0(\hat{\boldsymbol{n}}) = \frac{T_0(\hat{\boldsymbol{n}}) - T}{T} = \frac{p_0(\hat{\boldsymbol{n}}) - p}{p} = \hat{\boldsymbol{n}} \cdot \boldsymbol{v}$$



Fitting the CMB dipole, we find

$$v \approx 368\,\mathrm{km/s}$$

After removing the dipole, we are left with primordial anisotropy.

After decoupling, the photons travel along geodesics in an inhomogeneous spacetime

Newtonian gauge

$$\mathrm{d}s^2 = (1+2\Psi)\mathrm{d}t^2 - a^2(t)(1-2\Phi)\mathrm{d}\boldsymbol{x}^2$$

Ψ: gravitational potential; Φ: curvature perturbation

Ex: *Using the geodesic equation, show that*

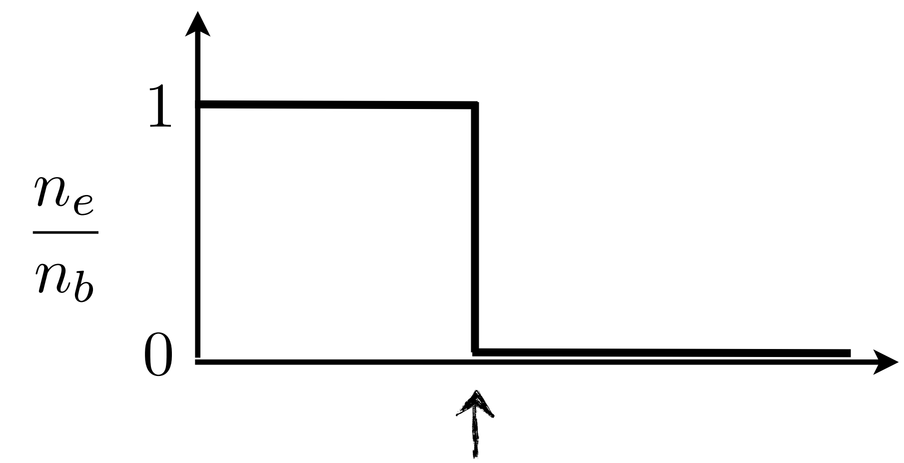
$$\boxed{\frac{1}{p}\frac{dp}{dt} = -\frac{1}{a}\frac{da}{dt} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} + \frac{\partial\Phi}{\partial t}}$$

REDSHIFT ($-\frac{1}{a}\frac{da}{dt}$) LENSING ($-\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}$) GRAVITATIONAL REDSHIFT ($\frac{\partial\Phi}{\partial t}$)

Using $\displaystyle \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = \left(\frac{\partial\Psi}{\partial t} + \frac{dx^i}{dt}\frac{\partial\Psi}{\partial x^i}\right) - \frac{\partial\Psi}{\partial t} = \frac{d\Psi}{dt} - \frac{\partial\Psi}{\partial t}$

we get $$\boxed{\frac{d}{dt}\ln(ap) = -\frac{d\Psi}{dt} + \frac{\partial(\Psi+\Phi)}{\partial t}} \quad (1)$$

For simplicity, we will assume
instantaneous recombination :



all photons are emitted at $t_{\rm rec}$

Integrate eq. (1) from $t_{\rm rec}$ to t_0 :

$$\ln(ap)_0 = \ln(ap)_{\rm rec} + (\Psi_{\rm rec} - \Psi_0) + \int_{t_{\rm rec}}^{t_0} {\rm d}t\,(\dot{\Psi} + \dot{\Phi})$$

$ap \propto a\bar{T}(1+\Theta)$

with

$(a\bar{T})_0 = (a\bar{T})_{\rm rec}$

$$\Theta_0 = \Theta_{\rm rec} + (\Psi_{\rm rec} - \Psi_0) + \int_{t_{\rm rec}}^{t_0} {\rm d}t\,(\dot{\Psi} + \dot{\Phi})$$

unobservable

$\equiv 0$

Hence, we get

intrinsic ΔT

gravitational redshift

$$\Theta_0 = \Big[(\Theta + \Psi) + \hat{\boldsymbol{n}} \cdot \boldsymbol{v}_e \Big]_{\rm rec} + \int_{t_{\rm rec}}^{t_0} {\rm d}t \, (\dot{\Psi} + \dot{\Phi})$$

SW — SACHS-WOLFE

D — DOPPLER

ISW

Two steps: I) Compute sources at recombination.

II) Project onto the sky.