Winter School on Stochastic Analysis and Control of Fluid Flows

Thiruvananthapuram (India)

December 3th - December 20th 2012

This volume contains in extenso the lectures to *Winter School on Stochastic Analysis and Control of Fluid Flows.*

The following topics will be treated.

1. Review on basic results on controllability and stabilization of Navier–Stokes equations
2. Stabilization by noise of Navier–Stokes equations
3. Stabilization of stochastic Navier–Stokes equations

The main references

1. V. Barbu, *Stabilization of Navier–Stokes Flows*, Springer, New York, 2012.
2. V. Barbu, *ESAIM COCV* 17 (2011), 117-130.
3. V. Barbu, *SIAM J. Control & Optimiz.*
4. V. Barbu, G. Da Prato, *SIAM J. Control & Optimiz.*, vol. 49 (1–20), 2011.
5. V. Barbu, *Scientiae Math. Japonica*, 2002.

Lecture 1.
Navier–Stokes equations.
Controllability and stabilization

The Navier–Stokes equations

$$
\begin{aligned}
& y_t(x,t) = \nu\Delta y(x,t) + (y,\cdot\nabla)y(x,t) = f(x,t) + \nabla p(x,t), && \\
& && x\in\Omega,\ t\in(0,T) \\
& (\nabla\cdot y)(x,t) = 0 && \forall(x,t)\in\Omega\times(0,T) \\
& y = 0 && \text{on } \partial\Omega\in(0,T) \\
& y(x,0) = y_0(x), && x\in\Omega
\end{aligned}
\tag{1}
$$

describe the non slip motion of a viscous, incompressible, Newtonian fluid in an open domain $\Omega\subset R^n$, $n=2,3$. Here $y=(y_1,y_2,...,y_n)$ is the velocity field, p is the pressure, f is the density of an external force and $\nu_0>0$ is the viscosity.

We have used the following notation

$$
\nabla\cdot y = \operatorname{div} y = \sum_{i=1}^n D_i y_i,\ \ D_i = \frac{\partial}{\partial x_i},\ \ i=1,...,n
$$

$$
(y\cdot\nabla)y = \left\{\sum_{i=1}^n y_i D_i y_j\right\}_{j=1}^n
$$

If the force $f=f_e$ is independent of f then the motion of the fluid is governed by the *stationary* (*steady–state*) Navier–Stokes equation

$$
\begin{aligned}
& -\nu_0\Delta y(x) + (y\cdot\nabla)y(x) = f_e(x) + \nabla p(x), && x\in\Omega \\
& \nabla\cdot y = 0 && \text{in } \Omega \\
& y = 0 && \text{on } \partial\Omega.
\end{aligned}
\tag{2}
$$

A *steady–state* (*equilibrium*) *solution* to the Navier–Stokes equation (6.1) with $f\equiv f_e$ is a solution (y_e,p_e) to stationary equation (6.2).

The linearized equation

$$
\begin{aligned}
& y_t - \nu_0\Delta y = f + \nabla p && \text{in } \Omega\times(0,T) \\
& \nabla\cdot y = 0 && \text{in } \Omega\times(0,T) \\
& y = 0 && \text{in } \Omega \\
& y(x,0) = y_0(x) && \text{in } \Omega
\end{aligned}
\tag{3}
$$

is called the *Stokes equation*.

The boundary value problem (7.1) can be written as an infinite dimensional Cauchy problem in appropriate function space on Ω. To this end we shall introduce the following spaces

$$
H = \{y\in(L^2(\Omega))^n;\ \nabla\cdot y = 0,\ y\cdot\nu = 0 \text{ on } \partial\Omega\} \tag{4}
$$

$$
V = \{y\in(H_0^1(\Omega))^n;\ \nabla\cdot y = 0\}. \tag{5}
$$

Here ν is the outward normal to $\partial\Omega$.

The space H is a closed subspace of $(L^2(\Omega))^n$ and it is a Hilbert space with the scalar product

$$(y,z)=\int_\Omega y\cdot z\,dx$$

and the norm

$$|y|=\left(\int_\Omega |y|^2 dx\right)^{1/2} \tag{6}$$

(We shall denote by the same symbol $|\cdot|$ the norm in R^n, $(L^2(\Omega))^n$ and H, respectively.)

The norm of the space V will be denoted by $\|\cdot\|$, i.e.,

$$\|y\|=\left(\int_\Omega |\nabla y(x)|^2 dx\right)^{1/2}. \tag{7}$$

We shall denote by $P:(L^2(\Omega))^n \longrightarrow H$ the orthogonal projection of $(L^2(\Omega))^n$ onto H (the Leray projector) and set

$$a(y,z)=\int_\Omega \nabla y\cdot\nabla z\,dx,\ \forall y,z\in V. \tag{8}$$

$$A=-P\Delta,\ D(A)=(H^2(\Omega))^n\cap V. \tag{9}$$

Equivalently,

$$(Ay,z)=a(y,z),\ \forall y,z\in V. \tag{9'}$$

The *Stokes operator* A is self–adjoint in H, $A\in L(V,V')$ (V' is the dual of V) and

$$(Ay,y)=\|y\|^2,\ \forall y\in V. \tag{10}$$

Finally, consider the trilinear functional

$$b(y,z,w)=\int_\Omega \sum_{i,j=1}^n y_i D_i z_j w_j\,dx,\ \forall y,z,w\in V \tag{11}$$

and denote by $B:V\longrightarrow V'$ the operator defined by

$$By=P(y\cdot\nabla)y \tag{12}$$

or, equivalently,

$$(By,w)=b(y,y,w),\ \forall w\in V. \tag{11'}$$

Then, taking in account that $P(\nabla p)=0$, problem (6.1) can be written as

$$\begin{aligned}&\frac{dy}{dt}(t)+\nu_0 Ay(t)+By(t)=Pf(t),\ t\in(0,T)\\&y(0)=y_0.\end{aligned} \tag{6.1'}$$

(We have assumed of course that $y_0\in H$.)

Similarly, equation (6.2) can be rewritten as

$$\nu_0 Ay + By = Pf_e. \tag{6.2$'$}$$

Let $f \in L^2(0,T;V')$ and $y_0 \in H$. The function $y : [0,T] \longrightarrow H$ is said to be a weak solution to equation (6.1) if

$$y \in L^2(0,T;V') \cap C_w([0,T];H) \cap W^{1,1}([0,T];V') \tag{13}$$

$$\begin{aligned} &\frac{d}{dt}(y(t),\psi)+\nu_0 a(y(t),\psi)+b(y(t),y(t),\psi)=(f(t),\psi), \text{ a.e. } t\in(0,T), \\ &y(0) = y_0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\psi\in V. \end{aligned} \tag{14}$$

(Here $(\cdot,\cdot)$ is, as usually, the pairing between V, V' and the scalar product of H.)

Equation (14) can be, equivalently, written as

$$\begin{aligned} &\frac{dy}{dt}(t) + \nu_0 Ay(t) + By(t) = f(t), \text{ a.e. } t \in (0,T) \\ &y(0) = y_0 \end{aligned} \tag{15}$$

where $\frac{dy}{dt}$ is the strong derivative of function $y : [0,T] \longrightarrow V'$.

The function y is said to be *strong solution* to (6.1) if $y \in W^{1,1}([0,T];H) \cap L^2(0,T;D(A))$ and (15) holds with $\frac{dy}{dt} \in L^1(0,T;H)$ the strong derivative of function $y : [0,T] \longrightarrow H$.

Before proceeding with the existence for problem (6.1), we pause briefly to present some fundamental properties of the trilinear functional b (see [4], [6]).

Proposition 1 *Let $1 \le n \le 4$. Then*

$$b(y,z,w) = -b(y,w,z) \quad \forall y,z,w \in V \tag{16}$$

$$|b(y,z,w)| \le C\|y\|_{m_1}\|z\|_{m_2+1}\|w\|_{m_3}, \ \forall u \in V_{m_1}, \ v \in V_{m_2}, \ w \in V_{m_3} \tag{17}$$

where $m_i \ge 0, \ i = 1,2,3$ and

$$\begin{aligned} &m_1 + m_2 + m_3 \ge \frac{n}{2} \quad \text{if } m_i \ne \frac{n}{2}, \quad \forall i = 1,2,3, \\ &m_1 + m_2 + m_3 \ge \frac{n}{2} \quad \text{if } m_i = \frac{n}{2}, \quad \text{for some } i = 1,2,3. \end{aligned} \tag{18}$$

Here $V_{m_i} = \{u \in (H^{m_i}(\Omega))^n; \ \nabla \cdot u = 0\}$.

In particular, it follows that B is continuous from V to V'. Indeed, we have

$$(By - Bz, w) = b(y, y-z, w) + b(y-z, z, w), \ \forall w \in V$$

and this yields

$$|(By - Bz, w)| \le C(\|y\|\|y-z\|\|w\| + \|y-z\|\|z\|\|w\|).$$

Hence

$$\|By - Bz\|_{V'} \le C\|y-z\|(\|y\| + \|z\|), \ \forall y,z \in V. \tag{19}$$

For each $N > 0$, define the operator $B_N : V \longrightarrow V'$ by

$$B_N y = \begin{cases} By & \text{if } \|y\| \leq N \\ \dfrac{N^2}{\|y\|^2} By & \text{if } \|y\| > N \end{cases} \tag{20}$$

and consider the operator $\Gamma_N : D(\Gamma_N) \subset H \longrightarrow H$

$$\Gamma_N = \nu_0 A + B_N,\ D(\Gamma_N) = D(A). \tag{21}$$

Let us show that Γ_N is well defined. Indeed, we have

$$|\Gamma_N y| \leq \nu_0 |Ay| + |B_N y|,\ \forall y \in D(A).$$

On the other hand, by (17) for $m_1 = 1,\ m_2 = \dfrac{1}{2},\ m_3 = 0$, we have for $\|y\| \leq N$

$$|(B_N y, w)| = |b(y,y,w)| \leq C\|y\|^{3/2}|Ay|^{1/2}|w|$$

because $\|y\|_{3/2} \leq \|y\|^{1/2}|Ay|^{1/2}$. This yields

$$|B_N y| \leq C|Ay|^{1/2}\|y\|^{3/2},\ \forall y \in D(A).$$

Similarly, we get for $\|y\| > N$

$$|B_N y| \leq \frac{CN^2}{\|y\|^2}|Ay|^{1/2}\|y\|^{3/2} \leq C|Ay|^{1/2}\|y\|^{3/2}.$$

This yields

$$|\Gamma_N y| \leq \nu|Ay| + C|Ay|^{1/2}\|y\|^{3/2},\ \forall y \in D(A), \tag{22}$$

as claimed.

Lemma 2 *There is α_N such that $\Gamma_N + \alpha_N I$ is m–accretive in $H \times H$.*

For each $N > 0$, consider the equation

$$\begin{aligned} &\frac{dy}{dt} + \nu_0 Ay + B_N y = f,\ t \in (0,T) \\ &y(0) = y_0. \end{aligned} \tag{23}$$

Proposition 3 *Let $y_0 \in D(A)$ and $f \in W^{1,1}([0,T];H)$ be given. Then there is a unique solution $y_N \in W^{1,\infty}([0,T];H) \cap L^\infty(0,T;D(A)) \cap C([0,T];V)$ to equation (23). Moreover, $\dfrac{d^+}{dt}y_N(t)$ exists for all $t \in [0,T)$ and*

$$\frac{d^+}{dt}y_N(t) + \nu_0 Ay_N(t) + B_N y_N(t) = f(t),\ \forall t \in [0,T). \tag{24}$$

Proof. Proposition 3 follows by Theorem 1.5, Chapter 4 of [1]. Since $\Gamma_N y_N = \nu_0 A y_N + B_N y_N \in L^\infty(0,T;H)$, by (**??**) we infer that $Ay_N \in L^\infty(0,T;H)$. As $\dfrac{dy_N}{dt} \in L^\infty(0,T;H)$, we conclude also that $y_N \in C([0,T];V)$,as claimed.

Now, we are ready to formulate the main existence result for the strong solutions to the Navier–Stokes equation (6.1) ((6.1′)).

Theorem 4 *Let $n = 2, 3$ and $f \in W^{1,1}([0,T];H)$, $y_0 \in D(A)$, where $0 < T < \infty$. Then, there is a unique function $y \in W^{1,\infty}([0,T^*];H) \cap L^\infty(0,T^*;D(A)) \cap C([0,T^*];V)$ such that*

$$\begin{aligned} &\frac{dy(t)}{dt} + \nu_0 Ay(t) + By(t) = f(t), \quad a.e.\ t \in (0,T^*) \\ &y(0) = y_0 \end{aligned} \tag{25}$$

for some $T^* = T^*(\|y_0\|) \le T$. If $n = 2$ then $T^* = T$. Moreover, y is right differentiable and

$$\frac{d^+}{dt} y(t) + \nu_0 Ay(t) + By(t) = f(t),\ \forall t \in [0,T^*). \tag{26}$$

Proof. Let y_N be the solution to (23), i.e.,

$$\begin{aligned} &\frac{dy_N}{dt} + \nu_0 Ay_N + By_N = f, \quad \text{a.e. } t \in (0,T) \\ &y(0) = y_0. \end{aligned} \tag{27}$$

If multiply (27) by y_N and integrate on $(0,t)$, we get

$$|y_N(t)|^2 + \nu_0 \int_0^t \|y_N(s)\|^2 ds \le C\left(|y_0|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right),\ \forall N.$$

Next, we multiply (27) (scalarly in H) by $Ay_N(t)$. We get

$$\frac{1}{2}\frac{d}{dt}\|y_N(t)\|^2 + \nu_0 |Ay_N(t)|^2 \le |(B_N y_N(t), Ay_N(t))| + |f(t)||Ay_N|,\ \text{a.e. } t \in (0,T).$$

This yields

$$\begin{aligned} &\|y_N(t)\|^2 + \nu_0 \int_0^t |Ay_N(s)|^2 ds \\ &\quad \le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt + \int_0^t |(B_N y_N, Ay_N)| ds\right). \end{aligned} \tag{28}$$

On the other hand, for $n = 3$, by (17) we have (the case $n = 2$ will be treated separately below)

$$\begin{aligned} |(B_N y_N, Ay_N)| &< |b(y_N, y_N, Ay_N)| \le C\|y_N\|\|y_N\|_{3/2}|Ay_N| \\ &\le C\|y_N\|^{3/2}|Ay_N|^{3/2},\ \text{a.e. } t \in (0,T). \end{aligned}$$

(Everywhere in the following C is independent of N, ν.) Then, by (28), we have

$$\begin{aligned} &\|y_N(t)\|^2 + \nu_0 \int_0^t |Ay_N(s)|^2 ds \\ &\quad \le C \left(\|y_0\|^2 + \frac{1}{\nu} \int_0^T |f(t)|^2 dt + \int_0^t |Ay_N(s)|^{3/2} \|y_N(s)\|^{3/2} ds \right) \\ &\quad \le C \left(\|y_0\|^2 + \frac{1}{\nu} \int_0^T |f(t)|^2 dt + \frac{1}{\nu_0} \int_0^t \|y_N(s)\|^6 ds \right) + \frac{\nu}{2} \int_0^t |Ay_N(s)|^2 ds, \\ &\qquad \forall t \in [0, T]. \end{aligned}$$

Finally,

$$\begin{aligned} &\|y_N(t)\|^2 + \frac{\nu_0}{2} \int_0^t |Ay_N(s)|^2 ds \\ &\quad \le C_0 \left(\|y_0\|^2 + \frac{1}{\nu_0} \int_0^T |f(s)|^2 ds + \frac{1}{\nu_0} \int_0^t \|y_N(s)\|^6 ds \right). \end{aligned} \tag{29}$$

Next, we consider the integral inequality

$$\|y_N(t)\|^2 \le C_0 \left(\|y_0\|^2 + \frac{1}{\nu_0} \int_0^T |f(s)|^2 ds + \frac{1}{\nu_0} \int_0^t \|y_N(s)\|^6 ds \right). \tag{30}$$

We have

$$\|y_N(t)\|^2 \le \varphi(t), \ \forall t \in (0, T)$$

where

$$\begin{aligned} &\varphi' = \frac{1}{2} \varphi^3, \ \forall t \in (0, T), \\ &\varphi(0) = C_0 \left(\|y_0\|^2 + \frac{1}{\nu_0} \int_0^T |f(s)|^2 ds \right). \end{aligned}$$

This yields

$$\varphi(t) = \left(\frac{\nu_0 \varphi^3(0)}{\nu_0 - 3t\varphi^3(0)} \right)^{1/3}, \ \forall t \in \left(0, \frac{\nu_0}{3\varphi^3(0)} \right).$$

Hence

$$\|y_N(t)\|^2 \le \left(\frac{\nu_0 \varphi^3(0)}{\nu_0 - 3t\varphi^3(0)} \right)^{1/3}, \ \forall t \in (0, T^*), \tag{31}$$

where

$$T^* = \frac{\nu_0}{3C_0^3 \left(\|y_0\|^2 + \frac{1}{\nu_0} \int_0^T |f(s)|^2 ds \right)^3}.$$

Then, by (29) we get

$$\begin{aligned} &\|y_N(t)\|^2 + \frac{\nu_0}{2} \int_0^t |Ay_N(s)|^2 ds \\ &\quad \le C_1 \left(|y_0|^2 + \frac{1}{\nu_0} \int_0^T |f(t)|^2 dt \right) \left(\int_0^t \frac{ds}{T^* - s} + 1 \right), \ 0 < t < T^*. \end{aligned} \tag{32}$$

For $n = 2$, we have (see (17))

$$|(B_N y_N, Ay_N)| \le C|y_N|^{1/2}\|y_N\||Ay_N|^{3/2} \le \frac{\nu_0}{2}|Ay_N|^2 + \frac{C}{\nu_0}\|y_N\|^4.$$

This yields

$$\begin{aligned}&\|y_N(t)\|^2 + \frac{\nu_0}{2}\int_0^t |Ay_N(s)|^2 ds \\ &\le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt + \frac{1}{\nu_0}\int_0^t \|y_N(s)\|^4 ds\right).\end{aligned}$$

Then, by (29) and the Gronwall lemma, we obtain

$$\begin{aligned}&\|y_N(t)\|^2 + \frac{\nu_0}{2}\int_0^t |Ay_N(s)|^2 ds \\ &\le C\left(\|y_0\|^2 + \frac{1}{\nu_0}\int_0^T |f(t)|^2 dt\right), \ \forall t \in (0,T).\end{aligned} \tag{33}$$

By (33) we infer that, for N large enough, $\|y_N(t)\| \le N$ on $(0, T^*)$ if $n = 3$ or on the whole of $(0, T)$ if $n = 2$.

Hence $B_N y_N = By_N$ on $(0, T^*)$ (respectively on $(0, T)$) and so $y_N = y$ is a solution to (25). This completes the proof of the existence.

Uniqueness. If y_1, y_2 are two solutions to (25), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|y_1(t) - y_2(t)|^2 + \nu_0\|y_1(t) - y_2(t)\|^2 \\ &\le |(B(y)(t) - By_2(t), y_1(t) - y_2(t))| \\ &= |b(y_1(t), y_1(t), y_1(t) - y_2(t)) - b(y_2(t), y_2(t), y_1(t) - y_2(t))| \\ &= |b(y_1(t) - y_2(t), y_1(t), y_1(t) - y_2(t))| \\ &\le C\|y_1(t) - y_2(t)\|^2\|y_1(t)\|, \quad \text{a.e. } t \in (0, T^*).\end{aligned}$$

Hence $y_1 \equiv y_2$.

Hence

$$B_N y_j \longrightarrow B_N y_N = \eta_N \text{ strongly in } L^1(0, T; V').$$

We have shown therefore that for each $y_0 \in H$ and $f \in L^2(0, T; V')$ the equation

$$\begin{aligned}&\frac{dy_N}{dt} + \nu Ay_N + B_N y_N = f, \ \text{a.e. } t \in (0, T) \\ &y_N(0) = y_0\end{aligned} \tag{34}$$

has a solution $y_N \in L^2(0, T; V) \cap C([0, T]; H)$ with $\frac{dy_N}{dt} \in L^{4/3}(0, T; V')$ if $n = 3$, $\frac{dy_N}{dt} \in L^{\frac{2}{1+\varepsilon}}(0, T; V')$ if $n = 2$.

Now, we let $N \longrightarrow \infty$. Then, on a subsequence, again denoted N, we have

$$\begin{array}{lll} y_N & \longrightarrow \; y^* & \text{weak star in } L^\infty(0,T;H) \\ & & \text{weakly in } L^2(0,T;V) \\ \dfrac{dy_N}{dt} & \longrightarrow \; \dfrac{dy^*}{dt} & \text{weakly in } L^{4/3}(0,T;V') \text{ if } n=3 \\ & & \text{weakly in } L^{\frac{2}{1+\varepsilon}}(0,T;V') \text{ if } n=2 \\ Ay_N & \longrightarrow \; Ay^* & \text{weakly in } L^2(0,T;V') \\ B_N y_N & \longrightarrow \; \eta & \text{weakly in } L^{4/3}(0,T;V') \text{ if } n=3 \\ & & \text{weakly in } L^{\frac{2}{1+\varepsilon}}(0,T;V') \text{ if } n=2. \end{array}$$

We have

$$\begin{aligned} & \frac{dy^*}{dt} + \nu_0 A y^* + \eta = f, \ \text{ a.e. in } (0,T) \\ & y^*(0) = y_0. \end{aligned} \tag{35}$$

To conclude the proof it remains to be shown that $\eta(t) = By^*(t)$, a.e. $t \in (0,T)$.

References

[1] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer 2010.

[2] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press, Boston 1993.

[3] V. Barbu, S. Sritharan, Flow invariance preserving feedback controllers for the Navier–Stokes equations, *J. Math. Anal. Appl.* 255(2001), 281–307.

[4] P. Constantin, *Navier–Stokes Equations*, The University of Chicago Press, Chicago, 1988.

[5] O.A. Ladyzenskaja, *The Mathematical Theory of Viscous Incompressible Fluids*, Gordon and Breach 1969.

[6] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, SIAM Philadelphia, 1983.

2 Internal controllability of Navier–Stokes equations

Consider the controlled Navier–Stokes system

$$\begin{aligned} & y_t(x,t) - \nu_0 \Delta y(x,t) + (y \cdot \nabla) y(x,t) = m(x) u(x,t) + \\ & \qquad\qquad + f_e(x) + \nabla p(x,t), \;\; (x,t) \in Q \\ & (\nabla \cdot y)(x,t) = 0, \qquad \forall (x,t) \in Q = \Omega \times (0,T) \\ & y = 0, \qquad \text{on } \Sigma = \partial\Omega \times (0,T) \\ & y(x,0) = y_0(x), \qquad x \in \Omega. \end{aligned} \tag{36}$$

Here $y = (y_1, y_2, ..., y_n)$, $u = (u_1, u_2, ..., u_n)$, Ω is an open smooth domain of R^n, $\nabla \cdot y$ is the divergence of y and Δ is the Laplacian in the space variables $x = (x_1, ..., x_n)$. The function m is the characteristic function of an open subset ω of Ω and y_0 is a given divergence free vector field. These equations describe the motion of an incompressible fluid in R^n ($n = 2$ or 3); the vector field $y = y(x,t)$ is the velocity, $p = p(x,t)$ is the pressure, $u = u(x,t)$ (the control variable) is the density of external forces and ν_0 is a positive constant (the viscosity). Throughout in the sequel we shall take $\nu_0 = 1$. Recall the notations

$$H = \{y \in (L^2(\Omega))^n;\; \nabla \cdot y = 0, y \cdot \nu = 0 \text{ on } \partial\Omega\}$$

the space of solenoidal vectors on Ω (here $\nu = (\nu_1, ..., \nu_n)$ is the normal to $\partial\Omega$) and let

$$V = \{y \in (H_0^1(\Omega))^n;\; \nabla \cdot y = 0\}$$

$$b(y,z,w) = \sum_{i,j=1}^{n} \int_\Omega y_i D_i z_j w_j dx$$

$$(By, w) = b(y,y,w), \;\; \forall y, w \in V.$$

Then, as noticed earlier, we may rewrite equation (7.1) as

$$\begin{aligned} & \frac{dy}{dt}(t) + Ay(t) + By(t) = P(mu) + Pf_e, \quad t \in (0,T) \\ & y(0) = y_0 \qquad t \in (0,T) \end{aligned} \tag{7.1$'$}$$

where $A \in L(V, V')$ is the Stokes operator.

Let (y_e, p_e) be a steady–state (equilibrium) solution to (7.1), i.e.,

$$\begin{aligned} & -\Delta y_e + y_e \cdot \nabla y_e = \nabla p_e + f_0(x) \quad \text{in } \Omega \\ & \nabla \cdot y_e = 0 \quad \text{in } \Omega \\ & y_e = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Substituting y by $y + y_e$ into (7.1) we are lead to the null controllability of the equation

$$\begin{aligned} & y_t - \Delta y + (y \cdot \nabla) y + (y_e \cdot \nabla) y + (y \cdot \nabla) y_e = mu + \nabla p \quad \text{in } Q \\ & \nabla \cdot y = 0 \quad \text{in } Q \\ & y = 0 \quad \text{on } \Sigma \\ & y(x,0) = y_0(x) - y_e(x) = y^0(x), \quad x \in \Omega. \end{aligned} \tag{37}$$

Theorem 5 *Let Ω be a bounded and open subset of R^n, $n = 2, 3$, and let ω be an open subset of Ω. Let $(y_e, p_e) \in ((W^{2,4}(\Omega))^n \cap V) \times H^1(\Omega)$ be a steady–state solution to* (7.1)*. Then there is $\eta > 0$ such that for all $y_0 \in (H^2(\Omega))^n \cap V$ satisfying the condition*

$$\|y_0 - y_e\|_{(H^2(\Omega))^n} \leq \eta$$

there are $u \in H^1(0, T; (L^2(\Omega))^n)$, $y \in (L^\infty(0, T; (H^2(\Omega))^n \cap V)) \cap H^1(0, T; V)$, $p \in L^2(0, T; H^1(\Omega))$ which satisfy (7.1) *and*

$$y(x, T) \equiv y_e(x), \ \ a.e. \ x \in \Omega.$$

Proof. As seen earlier, it suffices to prove the local null controllability of equation (7.1). To this purpose we shall invoke the fixed point argument. Namely, we consider the set

$$K = \{w \in L^\infty(0, T; (H^2(\Omega))^n \cap V)) \cap H^1(0, T; V); \mu(w) \leq M\} \tag{38}$$

where

$$\mu(w) = \|w\|_{L^\infty(0,T;(H^2(\Omega))^n \cap V)} + \|w\|_{H^1(0,T;V)}.$$

We fix w in K and consider the solution

$$(y, p) \in ((H^{2,1}(Q))^n \cap V) \times L^2(0, T; H^1(\Omega))$$

to the linear problem

$$\begin{aligned} & y_t - \Delta y + ((w + y_e) \cdot \nabla) y + (y \cdot \nabla) y_e = mu + \nabla p \text{ in } Q \\ & \nabla \cdot y = 0 \ \text{ in } Q \\ & y(x, 0) = y_0 - y_e = y^0 \ \text{ in } \Omega; \ y = 0 \ \text{ on } \Sigma \end{aligned} \tag{39}$$

The main step is Lemma 6 below.

Lemma 6 *There is $M > 0$ such that for all $w \in K$ there are $u \in H^1(0, T; (L^2(\Omega))^n)$ and $y \in L^2(0, T; H^2(\Omega) \cap V) \cap H^1(0, T; (L^2(\Omega))^n)$, $p \in L^2(Q)$ satisfying* (7.4) *and such that*

$$y(T) \equiv 0 \tag{40}$$

$$\|u\|_{H^1(0,T;(L^2(\Omega))^n)} \leq \beta(1 + M)^2 |y^0|_2 \tag{41}$$

for some constant $\beta > 0$ independent of M, w and y^0.

Proof of Theorem 1. (continued) Define the map

$$\Phi : K \longrightarrow (L^2(Q))^n$$

by

$$\begin{aligned} \Phi(w) = \{ & y^{u,w} \in L^2(0, T; (H^2(\Omega))^n \cap V) \cap H^1(0, T; (L^2(\Omega))^n, \\ & y^{u,w}(T) \equiv 0, \ \|u\|_{H^1(0,T;(L^2(\Omega))^n} \leq \beta(1 + M)^2 |y^0|_2\}. \end{aligned}$$

Here $y^{u,w}$ is the solution to (1.4) and β is the constant arising in Lemma 6. Then, by Lemma 6, $\Phi(w) \neq \emptyset$, for each $w \in K$, and it follows by a standard device that Φ is upper–semicontinuous in $(L^2(Q))^n$, i.e., if $w_k \longrightarrow w$ strongly in $(L^2(Q))^n$ and $y_k \in \Phi(w_k)$ is strongly convergent to y in $(L^2(Q))^n$, then $y \in \Phi(w)$, i.e., $y = y^{u,w}$ for some u satisfying equation (7.6).

References

[1] V. Barbu, Controllability of parabolic and Navier–Stokes equations, *Scientiae Mathematicae Japonicae*, 56, No. 1 (2002), 143–211.

[2] V. Barbu, T. Havarneanu, C. Popa, S.S. Sritharan, Exact controllability of the magneto-hydrodynamic equations, *Comm Pure Appl. Math.*, 56 (2003), 732-783.

3 Stabilization of Navier–Stokes equations

Consider the Navier-Stokes equation in a domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, with the smooth boundary $\partial\mathcal{O}$,

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y = f_e + \nabla p && \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ &\nabla \cdot y = 0 && \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ &y = 0 && \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ &y(0) = y_0 && \text{on } \mathbb{R}^+ \times \mathcal{O}, \end{aligned} \tag{42}$$

where $f_e \in (L^2(\mathcal{O}))^d$, $\nabla \cdot f_e = 0$, $f_e \cdot n = 0$.

Here n is the normal to $\partial\mathcal{O}$.

Let $y_e \in (H^2(\mathcal{O}))^d$ be an equilibrium solution to (8.1), that is,

$$\begin{aligned} &-\nu\Delta y_e + (y_e \cdot \nabla)y_e = f_e + \nabla p_e \quad \text{in } \mathcal{O}, \\ &\nabla \cdot y_e = 0 \ \text{ in } \mathcal{O}, \qquad y_e = 0 \ \text{ on } \partial\mathcal{O}. \end{aligned} \tag{43}$$

3.1 Internal stabilization

Let $\mathcal{O}_0 \subset \mathcal{O}$ be an open subdomain of $\mathcal{O}$ and consider the controlled system associated with (8.1)

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y = f_e + \nabla p + 1_{\mathcal{O}_0} u \ \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\ &\nabla \cdot y \ \text{ in } \mathbb{R}^+ \times \mathcal{O}; \qquad y = 0 \ \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ &y(0) = y_0 \ \text{ in } \mathcal{O}, \end{aligned} \tag{44}$$

where the controller u is in $L^2(0, \infty; (L^2(\mathcal{O}))^d)$.

Problem 7 *Find the controller u in feedback form, that is $u(t) = \phi(y(t) - y_e)$ such that the solution to the corresponding solution y to the closed loop system* (8.3) *satisfies for all y_0 in a neighborhood of y_e*

$$\|y(t) - y_e\|_{(L^2(\mathcal{O}))^d} \le Ce^{-\gamma t}\|y_0 - y_e\|_{(L^2(\mathcal{O}))^d}, \ \forall t \ge 0, \tag{45}$$

where $\gamma > 0$.

If we set $y - y_e \to y$, Problem 7 reduces to find $u = \phi(y)$ such that the solution y to the equation

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y + (y_e \cdot \nabla)y + (y \cdot \nabla)y_e = \nabla p + 1_{\mathcal{O}_0} u, \ t \ge 0, \\ &\nabla \cdot y = 0 \ \text{ in } \mathbb{R}^+ \times \mathcal{O}, \\ &y = 0 \ \text{ on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ &y(0, x) = y_0(x) - y_e(x) = y^0(x), \ x \in \mathcal{O}. \end{aligned} \tag{46}$$

satisfies

$$\|y(t)\|_{(L^2(\mathcal{O}))^d} \le Ce^{-\gamma t}\|y^0\|_{(L^2(\mathcal{O}))^d}, \ \forall t \ge 0. \tag{47}$$

We use the standard formalism to represent the Navier–Stokes equations as infinite-dimensional differential equations (see, e.g., [2], [3]). That is we set

$$\begin{aligned} H &= \{y \in (L^2(\mathcal{O}))^d; \nabla \cdot y = 0 \text{ in } \mathcal{O},\ y \cdot n = 0 \text{ on } \partial\mathcal{O}\}, \\ Ay &= -P(\Delta y),\ \forall y \in D(A) = y \in H \cap (H_0^1(\mathcal{O}))^d \cap (H^2(\mathcal{O}))^d, \\ A_0 y &= P((y_e \cdot \nabla)y + (y \cdot \nabla)y_e), D(A_0) = H \cap (H_0^1(\mathcal{O}))^d, \\ By &= P((y \cdot \nabla)y), \end{aligned}$$

where $P : (L^2(\mathcal{O}))^d \to H$ is the Leray projector.

We may rewrite (8.5) as

$$\begin{aligned} &\frac{dy}{dt} + \nu Ay + A_0 y + By = P(1\!\!1_{\mathcal{O}_0} u),\ t \geq 0, \\ &y(0) = y^0, \end{aligned} \tag{48}$$

or, in a more compact form,

$$\begin{aligned} &\frac{dy}{dt} + \mathcal{A}y + By = P(1\!\!1_{\mathcal{O}_0} u),\ t \geq 0, \\ &y(0) = y^0, \end{aligned} \tag{49}$$

where $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is the so called Oseen–Stokes operator

$$\mathcal{A} = \nu A + A_0, \quad D(\mathcal{A}) = D(A). \tag{50}$$

Then the *internal stabilization problem* reduces to find a feedback controller $u = \phi(y)$ such that the corresponding solution y to (49), that is,

$$\begin{aligned} &\frac{dy}{dt} + \mathcal{A}y + By = P(1\!\!1_{\mathcal{O}_0} \phi(y)),\ \forall t \geq 0, \\ &y(0) = y_0, \end{aligned} \tag{51}$$

satisfies

$$|y(t)|_H \leq Ce^{-\gamma t}|y_0|_H,\ \forall t \geq 0, \tag{52}$$

for $\gamma > 0$ and all y_0 in a neighborhood of the origin. Here and everywhere in the following, $|\cdot|_H$ is the norm of the space H and $(\cdot,\cdot)_H$ is the corresponding scalar product.

3.2 Boundary stabilization

Consider the boundary control system associated with (8.1)

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y = f_e + \nabla p && \text{in } \mathbb{R}^+ \times \mathcal{O}, \\ &\nabla \cdot y = 0 && \text{on } \mathbb{R}^+ \times \mathcal{O}, \\ &y = u && \text{on } \mathbb{R}^+ \times \partial\mathcal{O}, \\ &y(0) = y_0 && \text{in } \mathcal{O}. \end{aligned} \tag{53}$$

Problem 8 *Find a boundary controller u in the feedback form $u = \psi(y - y_e)$ such that the corresponding solution y to* (8.5) *satisfies* (8.4) *for all y_0 in a neighborhood of y_e.*

Equivalently, the solution y to

$$\begin{aligned}&\frac{\partial y}{\partial t} - \nu\Delta y+(y\cdot\nabla)y+(y\cdot\nabla)y_e+(y_e\cdot\nabla)y = \nabla p \quad \text{in } \mathbb{R}^+\times\mathcal{O},\\ &y = u \ \text{ on } \mathbb{R}^+\times\partial\mathcal{O}, \quad y(0) = y^0 - y_e,\\ &\nabla\cdot y = 0 \ \text{ on } \mathbb{R}^+\times\mathcal{O},\end{aligned} \tag{54}$$

where $u = \psi(y)$, satisfies (8.4).

If u is tangential, that is $u\cdot n = 0$ on $\mathbb{R}^+\times\partial\mathcal{O}$, then the stabilization is said to be *tangential* while, if $u\cdot\tau = 0$ on $\mathbb{R}^+\times\partial\mathcal{O}$ (where τ is the tangent vector to $\partial\mathcal{O}$), the stabilization is called *normal.*

Denote by $D : (L^2(\partial\mathcal{O}))^d \to H$ the Dirichlet map defined by

$$\begin{aligned}&-\nu\Delta(Du)+(y_e\cdot\nabla)Du+(Du\cdot\nabla)y_e+kDu = \nabla p \ \text{ in } \mathcal{O},\\ &Du = u \ \text{ on } \partial\mathcal{O},\end{aligned} \tag{55}$$

where $k > 0$ is sufficiently large but fixed.

It turns out that D is well defined on the space of all $u \in (L^2(\partial\mathcal{O}))^d$ such that $u\cdot n = 0$ on $\partial\mathcal{O}$ and that D is continuous from $(H^s(\partial\mathcal{O}))^d \to (H^{s+\frac12}(\mathcal{O}))^d \cap H$ if $s \geq \frac12$. (See Theorem A.2.1 in [1].) Then, (55) reduces to

$$\begin{aligned}&\frac{dy}{dt} + \mathcal{A}(y - Du) + By = kDu,\ t \geq 0,\\ &y(0) = y^0.\end{aligned} \tag{56}$$

If we denote by $\widetilde{\mathcal{A}}$ the extension, by transposition, $\widetilde{\mathcal{A}} : H \to (D(\mathcal{A}^*))'$ with respect to H as pivot space of the original operator $\mathcal{A}$, that is, $(\widetilde{\mathcal{A}}y, z) = (y, \mathcal{A}^*z),\ \forall z \in D(\mathcal{A})$, we can write (56) as

$$\begin{aligned}&\frac{dy}{dt} + \widetilde{\mathcal{A}}y + By = kDu + \widetilde{\mathcal{A}}Du,\ t \geq 0,\\ &y(0) = y^0,\end{aligned} \tag{57}$$

and so, the tangential stabilization problem reduces to find a feedback controller $u = \psi(y)$ such that the solution y to (57) satisfies (8.4) for all y in a neighborhood of the origin.

It is obvious that the solution y to the Cauchy problem is taken here in a mild sense

$$\begin{aligned}y(t) = e^{-\mathcal{A}t}y^0 - \textstyle\int_0^t e^{-\widetilde{\mathcal{A}}(t-s)}(By(s) + kDu + \widetilde{\mathcal{A}}Du(s))ds,\\ t \geq 0.\end{aligned} \tag{58}$$

Of course, if $\frac{d}{dt}Du \in L^2_{\text{loc}}(0,\infty; H)$, we may rewrite (58) as

$$\begin{aligned}y(t) &= Du(t) + e^{-\mathcal{A}t}(y^0 - Du(0))\\ &\quad - \int_0^t e^{-\mathcal{A}(t-s)}\left(By(s) + kDu(s) - \frac{d}{ds}Du(s)\right)ds,\\ &\qquad \forall t \geq 0.\end{aligned} \tag{59}$$

The functional representation of system (54) with normal boundary controller is a more delicate problem.

3.3 Main results

Theorem 9 (Barbu & Triggiani 2004) *There is a feedback controller*

$$u = \sum_{i=1}^{M} (R(y - y_e), \psi_i)_{(L^2(\mathcal{O}_0))^d} \psi_i, \quad R \in (L^2(\mathcal{O})), \tag{60}$$

which stabilizes exponentially y_e for

$$\|y_0 - y_e\|_W \leq \rho, \quad W = (H^{\frac{1}{2}}(\mathcal{O}))^d.$$

Here M^* is dependent of the multiplicity of eigenvalues λ_j of the Oseen–Stokes operator $\operatorname{Re} \lambda_j \leq 0$, $j = 1, ..., N$. *The functions ψ_j are linear combinations of eigenfunctions φ_j^*.*

Theorem 10 (Barbu, Lasiecka & Triggiani 2006) *Under additional assusmptions of eigenfunctions φ_j to the Oseen–Stokes operator, there is a tangential boundary controller*

$$u = \sum_{i=1}^{M} \left(\int_{\partial\mathcal{O}} R(y - y_e) \psi_i dx \right) \psi_i \quad \text{on } \partial\mathcal{O}, \tag{61}$$

which stabilizes y_e for y_0 in a neighborhood of y_e.

Other stabilization results are due to A. Fursikov (2004, 2006), J.P. Raymond (2006, 2007), Badra (2008).

Normal stabilization

The existence of a normal feedback controller stabilizing y_e was studied so far for special domains $\mathcal{O} \subset \mathbb{R}^d$ and fluid flows. Most results refer to periodic fluid flows in $2 - D$ channel (V. Barbu [2007], R. Triggiani [2008], I. Munteanu [2010], Vazquez & Krstc [2001]-[2007], the later for a large viscosity coefficient ν).

Stochastic stabilization

The results obtained here refer to design a feedback controller of the form

$$u = \eta \sum_{i=1}^{N} (y - y_e, \widetilde{\phi}_i^*)_{(L^2(\mathcal{O}))^d} \phi_i \dot{\beta}_i,$$

which, inserted into (8.1), that is,

$$dy - \nu \Delta y dt + (y \cdot \nabla) y \, dt = \eta \sum_{i=1}^{N} (y - y_e, \widetilde{\phi}_i^*)_{(L^2(\mathcal{O}))^d} \phi_i d\dot{\beta}_i + f_e dt + \nabla p \quad \text{in } (0, \infty) \times \mathcal{O},$$

$$\nabla \cdot y = 0, \qquad y|_{\partial\mathcal{O}} = 0,$$

yields

$$\lim_{t \to \infty} y(t) = y_e \quad \text{with probability 1.}$$

3.4 General description of the spectral stabilization technique

$$\begin{aligned} &\frac{dy}{dt} + \mathcal{A}y + By = Du, \quad t \geq 0, \\ &y(0) = y_0, \end{aligned} \tag{62}$$

where $-\mathcal{A}$ generates a C_0-analytic semigroup in a Hilbert space $\mathcal{H}$, B is a nonlinear on $\mathcal{H}$, $D \in L(U, \mathcal{H})$.

Problem 11 *Find $u = \phi(y)$ such that the solution to the closed loop system*

$$\begin{aligned} &\tfrac{dy}{dt} + \mathcal{A}y + By - D\phi(y) = 0, \ t \geq 0, \\ &y(0) = y_0, \end{aligned} \tag{63}$$

satisfies for some $\gamma > 0$

$$|y(t)|_{\mathcal{H}} \leq Ce^{-\gamma t}|y_0|, \quad \forall t > 0, \tag{64}$$

for $y_0 \in \mathcal{V}$ a neighborhood of origin.

Let λ_j eigenvalues of $\mathcal{A}$, $\mathcal{A}\varphi_j = \lambda_j \varphi_j$, $\mathcal{A}^* \varphi_j^* = \overline{\lambda}_j \varphi_j^*$ and N the number of eigenvalues with Re $\lambda_j < 0$.

Let $X_u = \text{lin span}\{\varphi_j\}_{j=1}^N$, $X_s = \{\varphi_j\}_{j=N+1}^\infty$.

First, one stabilizes the linear part of (8.2), that is

$$\begin{aligned} &\frac{dy}{dt} + \mathcal{A}y = Du, \ t \geq 0, \\ &y(0) = y_0, \end{aligned} \tag{65}$$

$$\begin{aligned} &y = y_u + y_s, \ y_u = \sum_{i=1}^N y_i \varphi_i, \ y_s = (I - P_N)y, \\ &u = \sum_{j=1}^M u_j \psi_j, \\ &\frac{dy_j}{dt} + \sum_{i=1}^N a_{ij} y_i = \sum_{j=1}^M b_{ij} u_i, \ j = 1, ..., N, \end{aligned} \tag{66}$$

$$\begin{aligned} &\frac{dy_s}{dt} + \mathcal{A}_s y_s = (I - P_N) \sum_{j=1}^M u_j D\psi_j, \\ &a_{ij} = (\mathcal{A}_u \varphi_i, \varphi_j^*), \ \mathcal{A}_s = \mathcal{A}|_{X_s}, \ b_{ij} = (D\psi_j, \varphi_i^*). \end{aligned} \tag{67}$$

One proves first that (66) is exactly null controllable via Kalman criteria

$$\text{rank}[\widetilde{B}, \widetilde{B}A, ..., \widetilde{B}A^{N-1}] = N, \quad \widetilde{B} = \|b_{ij}\|_{i,j=1}^M.$$

Then, since $|e^{-A_s t}| \leq Ce^{-\gamma t}$, we see that

$$|y(t)| \leq Ce^{-\gamma t}|y_0|. \tag{68}$$

Next, we construct a feedback controller via the optimization problem

$$\text{Min}\left\{\int_0^\infty \left(|y(t)|^2+|u|^2 dt;\ y'+Ay = Du,\ y(0) = y_0,\ u = \sum_{i=1}^{M^*} u_i\psi_i\right)\right\}.$$

We get

$$\begin{aligned} &u = -D^*Ry = -\phi(y),\\ &\mathcal{A}^*R + R\mathcal{A} + RDD^*R = I. \end{aligned} \tag{69}$$

Next, we insert the feedback (69) into (65) and prove the stabilization.

3.5 Stochastic stabilization

Look for a stochastic feedback

$$u = \eta\sum_{i=1}^{N}(y,\phi_i)\psi_i\dot{\beta}_i$$

and show that it stabilizes in probability (62), that is,

$$\begin{aligned} &dy + (\mathcal{A}y + By)dt - \eta\sum_{i=1}^{N}(y,\phi_i)D\psi_i d\beta_i,\\ &y(0) = y_0. \end{aligned}$$

To this end, if $B \equiv 0$, we decompose the equation in

$$dy_u + \mathcal{A}_u y_u dt = \eta P_N \sum_{i=1}^{N}(y,\phi_i)D\psi_i d\beta,$$

$$dy_s + \mathcal{A}_s y_s dt = \eta(I - P_N)\sum_{i=1}^{N}(y,\phi_i)D\psi_i.$$

Assume that $|\varphi_i^*,\varphi_j) = \delta_{i,j}$,

$$y_u = \sum_{j=1}^{N} y_j\varphi_j,\quad \phi_i = \varphi_i^*,\quad (D\psi_i,\varphi_j^*) = \delta_{ij}.$$

Then

$$\begin{aligned} &dy_j + \lambda_j y_j ds = \eta y_j d\beta_j,\ j = 1, ..., N,\\ &y_j(t) = e^{-\lambda_j t}e^{-\frac{1}{2}\eta^2 t+\eta\beta_j(t)} \to 0,\ \mathbb{P}\text{-a.s.} \end{aligned}$$

if $\eta^2 > -\text{Re}\ \lambda_j$.

The stabilizing effect of the noise is better illustrated by the following simple example.

$$\begin{aligned} &dX + aX\,dt = \eta X\,d\beta\\ &dX + aX\,dt = \eta X\,d\beta,\ X(0) = x,\\ &X(t) = e^{-at-\frac{1}{2}\eta^2 t+\eta\beta(t)}x,\quad \forall t > 0,\ \mathbb{P}\text{-a.s.}, \end{aligned}$$

Hence, $X(t) \to 0$, $\mathbb{P}$-a.s. for $t \to 0$.

References

[1] V. Barbu, I. Lasiecka, R. Triggiani, Tangential boundary stabilization of Navier–Stokes equations, *Memoires American Math. Soc.*, vol. 181, Number 852, 2006.

[2] V. Barbu, R. Triggiani, *Indiana Univ. Math. Journal*, 53 (2004), 1443-1494. P

[3] J.P. Raymond, *SIAM J. Control & Optimiz.*, 45 (2006), 790-828.

[4] J.P. Raymond, *J. Math. Pures Appl.*, 87 (2007), 627-669.

Lecture 2.
The internal stabilization by noise of the linearized Navier-Stokes equation

2.1 Introduction

Consider the Navier-Stokes equation

$$\begin{aligned} & X_t - \nu_0 \Delta X + (X \cdot \nabla) X = f_e + \nabla p \quad \text{in } (0,\infty) \times \mathcal{O} \\ & \nabla \cdot X = 0, \quad X\big|_{\partial \mathcal{O}} = 0 \\ & X(0) = x_0, \qquad\qquad\qquad\qquad \text{in } \mathcal{O}, \end{aligned} \tag{1}$$

where $\mathcal{O}$ is an open and bounded subset of R^d, $d \geq 2$, with smooth boundary $\partial\mathcal{O}$. Here $f_e \in (L^2(\mathcal{O}))^d$ is given.

Let X_e be an equilibrium solution to (1), i.e.,

$$\begin{aligned} & -\nu_0 \Delta X_e + (X_e \cdot \nabla) X_e = f_e + \nabla p_e \quad \text{in } \mathcal{O} \\ & \nabla \cdot X_e = 0 \quad \text{in } \mathcal{O}, \quad X_e\big|_{\partial \mathcal{O}} = 0. \end{aligned} \tag{2}$$

If $X \longrightarrow X - X_e$ equation (1) reduces to

$$\begin{aligned} & X_t - \nu_0 \Delta X + (X \cdot \nabla) X_e + (X_e \cdot \nabla) X + (X \cdot \nabla) X = \nabla p \quad \text{in } (0,\infty) \times \mathcal{O}, \\ & \nabla \cdot X = 0 \quad \text{in } \mathcal{O}, \quad X\big|_{\partial \mathcal{O}} = 0, \quad \forall t \geq 0, \\ & X(0) = x \text{ in } \mathcal{O}, \end{aligned} \tag{3}$$

where $x = x_0 - X_e$.

Then, the linearized system around X_e associated with (1) is the Stokes-Oseen system

$$\begin{aligned} & X_t - \nu_0 \Delta X + (X \cdot \nabla) X_e + (X_e \cdot \nabla) X = \nabla p \quad \text{in } (0,\infty) \times \mathcal{O} \\ & \nabla \cdot X = 0 \text{ in } \mathcal{O}, \quad X\big|_{\partial \mathcal{O}} = 0, \quad t \geq 0 \\ & X(0) = x \quad \text{in } \mathcal{O}. \end{aligned} \tag{4}$$

If set $H=\{X \in (L^2(\mathcal{O}))^d;\ \nabla \cdot X = 0,\ X \cdot n\big|_{\partial\mathcal{O}}=0\}$, where ν is the normal to $\partial\mathcal{O}$ and $P : (L^2(\mathcal{O}))^d \to H$ is the Leray projector on H, we can rewrite system (4) as

$$\begin{aligned} & \dot{X}(t) + \mathcal{A} X(t) = 0,\ t \geq 0, \\ & X(0) = x, \end{aligned} \tag{5}$$

where $\mathcal{A} = \nu_0 A + A_0$, $A = -P\Delta$, $D(A) = (H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}))^d \cap H$, $A_0(X) = P((X \cdot \nabla)X_e + (X_e \cdot \nabla)X)$, $D(\mathcal{A}) = D(A)$.

Our purpose here is to stabilize (5) or, equivalently, the stationary solution X_e to (1), using a stochastic controller with support in an arbitrary open subset $\mathcal{O}_0 \subset \mathcal{O}$. To this aim we associate with (5) the control stochastic system

$$\begin{aligned} & dX(t) + \mathcal{A} X(t) dt = \sum_{i=1}^{N} V_i(t) \psi_i d\beta_i(t), \\ & X(0) = x, \end{aligned} \tag{6}$$

where $\{\beta_i\}_{i=1}^N$ is an independent system of real Brownian motions in a probability space $\{\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t\}_{t>0}$. The main results, Theorems 1 and 4 below, amounts to saying that, in the complexified space $\widetilde{H}$ associated with H, under appropriate assumptions on $\mathcal{A}$ (and, implicitly, on X_e), for each $\gamma > 0$ there exists $N \in \mathbb{N}$, $\{\psi_i\}_{i=1}^N \subset \widetilde{H}$ and an N-dimensional adapted process $\{V_i = V_i(t,\omega)\}_{i=1}^N$, $\omega \in \Omega$ such that $t \to e^{\gamma t} X(t,\omega)$ is convergent to zero in probability for $t \to \infty$. Moreover, it turns out that the stabilizable controller $\{V_i\}_{i=1}^{i=N}$ can be expressed as a linear feedback controller of the form

$$V_i(t) = \eta(X(t), \varphi_i^*)_{\widetilde{H}},\ \psi_i = P(m\phi_i),\ i = 1, ..., N, \tag{7}$$

where φ_i^* are the eigenfunctions of the dual Stokes-Oseen operator $\mathcal{A}^*$ corresponding to eigenvalues $\overline{\lambda}_j$ with $\operatorname{Re}\lambda_j \leq \gamma$, $\{\phi_i\}_{i=1}^N$ is a system of functions related to φ_i^* and $m = \mathcal{X}_{\mathcal{O}_0}$ is the characteristic function of $\mathcal{O}_0$.

We may view (6) as the deterministic system (5) perturbed by the white noise controller $\sum_{i=1}^N V_i(t)\psi_i\dot{\beta}_i$, i.e.,

$$\dot{X} + \mathcal{A}X = \sum_{i=1}^N V_i(t)\psi_i\dot{\beta}_i.$$

The proof uses some spectral techniques developed in [5, 6] (see also [4, 13, 14, 20, 21]) for stabilization of Navier-Stokes equations. The previous treatment for the stabilization of Navier-Stokes equations is a Riccati based approach which can be described in a few words as follows; one shows first that the unstable finite dimensional part of the Stokes-Oseen equation is stabilizable and one uses this to construct, via the algebraic infinite dimensional Riccati equations associated with the Stokes-Oseen operator, a stabilizable feedback controller. In this context, we note also that in [12] was developed a statistical approach to stabilization of Stokes-Oseen equation in order to treat the unpredictible fluctuations arising in feedback mechanism. This is related to some long-time behaviour results for solutions to Navier-Stokes equations perturbed by random kick-forces (see [16, 22]). However, the results obtained here are essentially stochastic not only because the stabilizable controller arises as multiplicative term of a Brownian N-dimensional motion but mainly because the asymptotic nature of stabilization results as well as the stochastic approach have no analogue in deterministic stabilization technique. As a matter of fact, it was known long time ago that one might use the multiplicative noise to stabilize differential systems (see [3]) and more recent results in this direction can be found in [1, 2, 7, 8, 9, 19]. (See also [11] for related results.) It must be said however that in the context of Navier-Stokes equations the results obtained here are new. The apparent advantage of the stochastic feedback controller (7) compared with deterministic stabilizable controllers constructed by spectral techniques (see [5, 6, 13, 14, 20, 21]) is that it avoids the infinite dimensional algebraic Riccati equations which are not numerically tractable by discretization with a larger number of grid points and so are inadequate to treat most fluid dynamic problems with a sufficient degree of resolution. One might suspect that the controller (7) is locally stabilizable as well for the Navier-Stokes equation (3), and we expect to study this problem in a forthcoming paper.

The plan of the paper is the following. The internal stabilization result, Theorem 1, is formulated in Section 2 and proven in Section 3. The boundary stabilization by noise is studied in Section 4.

Notations

Throughout in the following β_i, $i = 1, ...,$ are independent real Brownian motions in a probability space $\{\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t\}_{t>0}$ and we shall refer to [10, 17] for definition and basic results on stochastic analysis of differential systems and spaces of stochastic processes adapted to filtration $\{\mathcal{F}_t\}_{t>0}$. We shall denote by $\widetilde{H}$ the complexified space $H + iH$ with scalar product denoted by $\langle \cdot, \cdot \rangle$ and norm by $|\cdot|_{\widetilde{H}}$. The scalar product of H is denoted $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$. $C_W([0, T]; L^2(\Omega, \widetilde{H}))$ is the space of all adapted square-mean $\widetilde{H}$-valued continuous processes on $[0, T]$.

2.2 The main result

To begin with, let us briefly recall a few elementary spectral properties of the Stokes-Oseen operator $\mathcal{A}$.

Denote again by $\mathcal{A}$ the extension of $\mathcal{A}$ to the complex space $\widetilde{H}$. The operator $\mathcal{A}$ has a compact resolvent $(\lambda I - \mathcal{A})^{-1}$ and $-\mathcal{A}$ generates a C_0-analytic semigroup $e^{-\mathcal{A}t}$ in $\widetilde{H}$. Consequently, $\mathcal{A}$ has a countable number of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ with corresponding eigenfunctions φ_j each with finite algebraic multiplicity m_j. Of course, certain eigenfunctions φ_j might be generalized and so, in general, $\mathcal{A}$ is not diagonalizable, i.e., the algebraic multiplicity of λ_j might not coincide with its geometric multiplicity. Also, each eigenvalue λ_j will be repeated according to its algebraic multiplicity m_j.

We shall denote by N the number of eigenvalues λ_j with $\operatorname{Re}\lambda_j \leq \gamma$, $j = 1, ..., N$, where γ is a fixed positive number.

Denote by P_N the projector on the finite dimensional subspace

$$\mathcal{X}_u = \operatorname{lin\,span}\{\varphi_j\}_{j=1}^{N}.$$

We have $\mathcal{X}_u = P_N \widetilde{H}$ and

$$P_N = -\frac{1}{2\pi i} \int_{\Gamma} (\lambda I - \mathcal{A})^{-1} d\lambda, \tag{8}$$

where Γ is a closed smooth curve in $\mathbb{C}$ which is the boundary of a domain containing in interior the eigenvalues $\{\lambda_j\}_{j=1}^{N}$.

Let $\mathcal{A}_u = P_N \mathcal{A}$, $\mathcal{A}_s = (I - P_N)\mathcal{A}$. Then $\mathcal{A}_u$, $\mathcal{A}_s$ leave invariant the spaces $\mathcal{X}_u, \mathcal{X}_s = (I - P_N)\widetilde{H}$ and the spectra $\sigma(\mathcal{A}_u)$, $\sigma(\mathcal{A}_s)$ are given by (see [15])

$$\sigma(\mathcal{A}_u) = \{\lambda_j\}_{j=1}^{N}, \quad \sigma(\mathcal{A}_s) = \{\lambda_j\}_{j=N+1}^{\infty}.$$

Since $\sigma(\mathcal{A}_s) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \gamma\}$ and $\mathcal{A}_s$ generates an analytic C_0-semigroup on $\widetilde{H}$, we have

$$|e^{-\mathcal{A}_s t} x|_{\widetilde{H}} \leq C e^{-\gamma t} |x|_{\widetilde{H}}, \quad \forall x \in \widetilde{H},\ t \geq 0. \tag{9}$$

The eigenvalue λ_j is said to be *semi-simple* if for it the algebraic and geometrical multiplicity coincides, or, equivalently, λ_j is a simple pole for $(\lambda I - \mathcal{A})^{-1}$. If all the eigenvalues $\{\lambda_j\}_{j=1}^{N}$ of the matrix $\mathcal{A}_u$ are semi-simple, then $\mathcal{A}_u$ *is diagonalizable.*

Herein, we shall assume that the following hypothesis holds.

(A_1) *All the eigenvalues λ_j, $j = 1, ..., N$, are semi-simple.*

It should be said that hypothesis (A_1) is less restrictive as it might appear to be at first glance. Indeed, it follows by a standard argument involving the Sard-Smale theorem that the property of eigenvalues of the Stokes-Oseen operator to be simple (and, consequently, semi-simple) is generic in the class of coefficients X_e. So, "almost everywhere" (in the sense of a set of first category), hypothesis (A_1) holds.

Denote by $\mathcal{A}^*$ the adjoint operator and by P_N^* the adjoint of P_N. We have

$$P_N^* = -\frac{1}{2\pi i}\int_{\overline{\Gamma}} (\lambda I - \mathcal{A}^*)^{-1} d\lambda. \tag{10}$$

The eigenvalues of $\mathcal{A}^*$ are precisely the complex conjugates $\overline{\lambda}_j$ of eigenvalues λ_j of $\mathcal{A}$ and they have the same multiplicity. Denote by φ_j^* the eigenfunction of $\mathcal{A}^*$ corresponding to the eigenvalue λ_j. We have, therefore,

$$\mathcal{A}\varphi_j = \lambda_j\varphi_j, \quad \mathcal{A}^*\varphi_j^* = \overline{\lambda}_j\varphi_j^*, \quad j = 1, \ldots. \tag{11}$$

Since the eigenvalues $\{\lambda_j\}_{j=1}^N$ *are semi-simple, it turns out that the system consisting of* $\{\varphi_j\}_{j=1}^N$, $\{\varphi_j^*\}_{j=1}^N$ *can be chosen to form a bio-orthonormal sequence in* $\widetilde{H}$, i.e,

$$\langle \varphi_j, \varphi_i^* \rangle = \delta_{ij}, \quad i, j = 1, ..., N, \tag{12}$$

(see, e.g., [5]). We notice also that the functions φ_j and φ_j^* have the unique continuation property, i.e.,

$$\varphi_j \not\equiv 0, \quad \varphi_j^* \not\equiv 0 \ \text{ on } \mathcal{O}_0 \text{ for all } j = 1, ..., N, \tag{13}$$

(see, e.g., Lemma 3.7 in [5]). We shall assume also that the following condition holds:

(A_2) *The system* $\{\varphi_j^*\}_{j=1}^N$ *is linearly independent in* $(L^2(\mathcal{O}_0))^d$.

It should be noticed that hypothesis (A_2) automatically holds if X_e is analytic because in this case φ_j^* are analytic too and so (A_2) is the consequence of linear independence of $\{\varphi_j^*\}_{j=1}^N$ on $\mathcal{O}_0$. Also, in the case where the system $\{\varphi_j^*\}_{j=1}^N$ contains only one distinct eigenvalue (which might be multiple), hypothesis (A_2) is implied by the unique continuation property (13). It turns out via unique continuation arguments that (A_2) holds under more general conditions on X_e but the presentation of this result is beyond the goals of this work.

Consider the following stochastic perturbation of the linearized system (5) considered in the complex space

$$\begin{aligned} &dX + \mathcal{A}X dt = \eta \sum_{i=1}^N \langle X, \varphi_i^* \rangle\, P(m\phi_i) d\beta_i, \\ &X(0) = x, \end{aligned} \tag{14}$$

where $\eta \in \mathbf{R}$ and $m = \chi_{\mathcal{O}_0}$ is the characteristic function of the open subset $\mathcal{O}_0 \subset \mathcal{O}$. Here $\{\phi_i\}_{i=1}^N \subset \widetilde{H}$ is a system of functions to be made precise below. We may rewrite (14) as

$$X(t) = e^{-\mathcal{A}t}x + \eta\sum_{i=1}^N \int_0^t \langle X(s), \phi_i \rangle\, e^{-\mathcal{A}(t-s)} P(m\phi_i) d\beta_i(s),\ t \geq 0,\ \mathbb{P}\text{-a.s.} \tag{15}$$

which, by the standard existence theory (see [10]), has a unique solution $X \in C_W([0,T]; L^2(\Omega, \widetilde{H}))$, $\forall T > 0$.

The closed loop system (14) can be equivalently written as (see (4))

$$
\begin{aligned}
&dX(t) - \nu_0 \Delta X(t)dt + (X(t)\cdot\nabla)X_e dt + (X_e\cdot\nabla)X(t)dt \\
&\qquad = \eta m \sum_{i=1}^{N} \langle X(t), \varphi_i^* \rangle \phi_i d\beta_i(t) + \nabla p(t)dt \text{ in } (0,\infty)\times\mathcal{O},\ \mathbb{P}\text{-a.s.} \\
&\nabla\cdot X(t) = 0 \text{ in } \mathcal{O},\quad X(t)\Big|_{\partial\mathcal{O}} = 0,\ \forall t\ge 0,\ \mathbb{P}\text{-a.s.} \\
&X(0) = x \text{ in } \mathcal{O}.
\end{aligned}
\tag{16}
$$

Hence, in the space $(L^2(\mathcal{O}))^d$, the feedback controller $\{u_i = \eta m \langle X, \varphi_i^* \rangle \phi_i\}_{i=1}^N$ has the support in $\mathcal{O}_0$.

We shall define now $\phi_j,\ j = 1, ..., N$, as follows.

$$
\phi_j(\xi) = \sum_{i=1}^{N} \alpha_{ij}\varphi_i^*(\xi),\quad \xi\in\mathcal{O}, \tag{17}
$$

where α_{ij} are chosen such that

$$
\sum_{i=1}^{N} \alpha_{ij} \langle \varphi_i^*, \varphi_k^* \rangle_0 = \delta_{jk},\quad j,k = 1, ..., N.
$$

(Since, in virtue of hypothesis (A_2), the matrix $\{\langle \varphi_i^*, \varphi_j^* \rangle_0\}_{i,j=1}^N$ is not singular, this is possible.) With this choice, we have

$$
\langle \phi_j, \varphi_i^* \rangle_0 = \delta_{ij},\quad i,j = 1, ..., N. \tag{18}
$$

Here, we have used the notation $\langle u, v\rangle_0 = \int_{\mathcal{O}_0} u(\xi)\bar{v}(\xi)d\xi$.

Theorem 1 below is the main result.

Theorem 1 *Under hypotheses* (A_1), (A_2)*, the solution* X *to equation* (14)*, where* $\{\phi_i\}_{i=1}^N$ *are given by* (17)*, satisfies for* $|\eta|$ *sufficiently large*

$$
\mathbb{P}\left[\lim_{t\to\infty} e^{\gamma t}|X(t,x)|_{\widetilde{H}} = 0\right] = 1,\ \forall x\in H. \tag{19}
$$

Remark 2 As mentioned earlier, system (16) is written here in the complex space $\widetilde{H}$. If set $X_1(t) = \mathrm{Re}\, X(t)$, $X_2(t) = \mathrm{Im}\, X(t)$, it can be rewritten as a real system in (X_1, X_2). In this case, the feedback controller is an implicit stabilizable feedback controller with support in $\mathcal{O}_0$ for the real Stokes-Oseen equation (4). Of course, if $\lambda_j,\ j = 1, ..., N$, are real, then we may view $X(t)$ as a real valued function and so, in (19), $|X|_{\widetilde{H}} = |X|_H$.

2.3 Proof of Theorem 1

The idea is to decompose equation (14) in a finite dimensional system and an infinite dimensional exponentially stable system. To this end, we set $X_u = P_N X$, $X_s = (I - P_N)X$ and we shall rewrite equation (14) as

$$\begin{aligned}&dX_u(t)+\mathcal{A}_uX_u(t)dt=\eta P_N\sum_{i=1}^{N}\langle X_u(t),\varphi_i^*\rangle\, P(m\phi_i)d\beta_i(t),\ t\ge 0,\ \mathbb{P}\text{-a.s.},\\ &X_u(0)=P_Nx.\end{aligned}\tag{20}$$

$$\begin{aligned}&dX_s(t)+\mathcal{A}_sX_s(t)dt=\eta(I-P_N)\sum_{i=1}^{N}\langle X_u(t),\varphi_i^*\rangle\, P(m\phi_i)d\beta_i(t),\ \ t\ge 0,\ \ \mathbb{P}\text{-a.s.},\\ &X_s(0)=(I-P_N)x.\end{aligned}\tag{21}$$

Then, we may represent X_u as $X_u(t)=\sum_{i=1}^{N}y_i(t)\varphi_i$ and reduce so equation (20) via biorthogonal relations (12) and (18) to the finite dimensional complex system

$$\begin{aligned}&dy_j+\lambda_jy_jdt=\eta y_jd\beta_j,\ \ j=1,...,N,\ \ t\ge 0,\ \ \mathbb{P}\text{-a.s.}\\ &y_j(0)=y_j^0,\end{aligned}\tag{22}$$

where $y_j^0=\left\langle P_Nx,\varphi_j^*\right\rangle$.

Applying Ito's formula in (22) to $\varphi(y)=e^{\gamma t}|y|^2$, we obtain that

$$\begin{aligned}&\frac{1}{2}\,d\left(e^{2\gamma t}|y_j(t)|^2\right)+e^{2\gamma t}\left(\operatorname{Re}\lambda_j-\gamma\right)|y_j(t)|^2dt\\ &\quad=\frac{1}{2}\,\eta^2e^{2\gamma t}|y_j(t)|^2dt+\eta e^{2\gamma t}|y_j(t)|^2d\beta_j(t),\quad \text{for}\ \ j{=}1,...,N.\end{aligned}\tag{23}$$

Now, in (23) we take $z(t)=e^{2\gamma t}|y_j(t)|^2$ and get that

$$dz+2e^{2\gamma t}\left(\operatorname{Re}\lambda_j-\gamma\right)|y_j|^2dt=\eta^2e^{2\gamma t}|y_j|^2dt+2\eta e^{2\gamma t}|y_j|^2d\beta_j,\ \ j=1,...,N.$$

In the latter equation, we shall apply Itô's formula to the function

$$\phi(r)=(\varepsilon+r)^\delta,\ \text{where}\ 0<\delta<\frac{1}{2}\ \text{and}\ \varepsilon>0.$$

We have

$$\phi'(r)=\delta(\varepsilon+r)^{\delta-1},\ \ \phi''(r)=\delta(\delta-1)(\varepsilon+r)^{\delta-2},\ \ r>0$$

and we obtain therefore that

$$d\phi(z)=\phi'(z)dz+2\eta^2e^{4\gamma t}\phi''(z)|y_j|^4dt.$$

This yields

$$\begin{aligned}d\phi(z)=\ &-\delta e^{2\gamma t}(\varepsilon+z)^{\delta-1}[2(\operatorname{Re}\lambda_j-\gamma)|y_j(t)|^2dt-\eta^2|y_j|^2dt\\ &-2\eta|y_j|^2d\beta_j]+2\eta^2\delta(\delta{-}1)e^{4\gamma t}(\varepsilon{+}z)^{\delta-2}|y_j|^4dt.\end{aligned}$$

Now, in the latter equation, if replace z by $e^{2\gamma t}|y_j|^2$, we obtain that

$$\begin{aligned}&d((\varepsilon{+}e^{2\gamma t}|y_j|^2)^\delta)+2\delta(\varepsilon{+}e^{2\gamma t}|y_j|^2)^{\delta-1}e^{2\gamma t}(\operatorname{Re}\lambda_j{-}\gamma)|y_j(t)|^2dt\\ &\quad=2\eta^2(\delta{-}1)\delta e^{4\gamma t}(\varepsilon{+}e^{2\gamma t}|y_j|^2)^{\delta-2}|y_j|^2dt\\ &\quad+\eta^2\delta e^{2\gamma t}(\varepsilon{+}e^{2\gamma t}|y_j|^2)^{\delta-1}|y_j|dt\\ &\quad+2\eta\delta e^{2\gamma t}(\varepsilon+e^{2\gamma t}|y_j|^2)^{\delta-1}|y_j|d\beta_j,\ \ j{=}1,...,N.\end{aligned}\tag{24}$$

We set

$$\begin{aligned} K_\varepsilon^j(t) &= 2\delta\, e^{2\gamma t}(\varepsilon + e^{2\gamma t}|y_j|^2)^{\delta-1}(\mathrm{Re}\,\lambda_j - \gamma)|y_j(t)|^2 \\ &-\delta\eta^2 e^{2\gamma t}(\varepsilon + e^{2\gamma t}|y_j(t)|^2)^{\delta-1}|y_j(t)|^2 \\ &-2\delta(\delta-1)\eta^2 e^{4\gamma t}(\varepsilon + e^{2\gamma t}|y_j(t)|^2)^{\delta-2}|y_j(t)|^4,\ j{=}1,...,N. \end{aligned} \tag{25}$$

Now, taking into account (25), we may rewrite (24) as

$$(\varepsilon + e^{2\gamma t}|y_j|^2)^\delta + \int_0^t K_\varepsilon^j(s)ds = (\varepsilon + |y_j^0|^2)^\delta + M_\varepsilon^j(t),\ t \ge 0, j = 1, ..., N,\ \mathbb{P}\text{-a.s.}, \tag{26}$$

where M_ε^j is the following stochastic process

$$M_\varepsilon^j(t) = 2\delta\eta \int_0^t e^{2\gamma s}|y_j(s)|^2(\varepsilon + e^{2\gamma s}|y_j(s)|^2)^{\delta-1})d\beta_j(s),\ j = 1, ..., N.$$

Taking into account that

$$\lim_{\varepsilon\to 0} |y_j(s)|^2(\varepsilon + e^{2\gamma s}|y_j(s)|^2)^{\delta-1}e^{2\gamma s} = e^{2\gamma\delta s}|y_j(s)|^{2\delta},\ \mathbb{P}\text{-a.s.}$$

uniformly on $[0, T]$, we may pass to limit into the stochastic equation (26) to get that

$$e^{2\gamma\delta t}|y_j(t)|^{2\delta} + \int_0^t K_j(s)ds = |y_j^0|^{2\delta} + M_j(t),\ \mathbb{P}\text{-a.s.},\ t > 0, \tag{27}$$

where

$$K_j(t) = \lim_{\varepsilon\to 0} K_\varepsilon^j(t) = 2\delta(\mathrm{Re}\,\lambda_j - \gamma)e^{2\gamma\delta t}|y_j(t)|^{2\delta} + 2\delta(1-2\delta)\eta^2 e^{2\gamma\delta t}|y_j(t)|^{2\delta},$$

$$M_j(t) = 2\delta\eta \int_0^t e^{2\gamma\delta s}|y_j(s)|^{2\delta}d\beta_j(s),\ \mathbb{P}\text{-a.s.}$$

If in (27) we take the expectation E, we obtain that

$$e^{2\gamma\delta t}E|y_j(t)|^{2\delta} + E\int_0^t K_j(s)ds = |y_j^0|^{2\delta}, \quad \forall t \ge 0.$$

This yields

$$2\delta(\eta^2(1-2\delta) + \mathrm{Re}\,\lambda_j - \gamma)E\int_0^t e^{2\gamma\delta s}|y_j(s)|^{2\delta}ds \le |y_j^0|^{2\delta},\ j = 1, ..., N,$$

and, since $0 < \delta < \frac{1}{2}$, for all $j = 1, ..., N$, we get therefore, for η sufficiently large,

$$E\int_0^t e^{2\gamma\delta s}|y_j(s)|^{2\delta}ds \le C, \quad \forall t \ge 0,\ j == 1, ..., N.$$

This yields

$$E\int_0^\infty e^{2\gamma\delta s}|y_j(s)|^{2\delta}ds < \infty, \quad \forall j = 1, ..., N,$$

and, in particular, it follows that

$$\int_0^\infty e^{2\gamma\delta s}|y_j(s)|^{2\delta}ds < \infty,\ \mathbb{P}\text{-a.s.}, j = 1, ..., N. \tag{28}$$

It should be said however that the latter does not imply automatically that $e^{2\gamma\delta t}|y_j(t)|^{2\delta}$ is $\mathbb{P}$-a.s. convergent to zero as $t \to \infty$ and for this we need to invoke some more sophisticated stochastic argument.

We write

$$\int_0^t K_j(s)ds = I_j(t) - (I_j)_1(t),\quad j = 1, ..., N,\ \forall t \geq 0,\ \mathbb{P}\text{-a.s.},$$

where

$$I_j(t) = 2(1-2\delta)\delta\eta^2 \int_0^t e^{2\gamma\delta s}|y_j(s)|^{2\delta}ds$$
$$(I_j)_1(t) = 2\delta(\gamma - \operatorname{Re}\lambda_j)\int_0^t e^{2\gamma\delta s}|y_j(s)|^{2\delta}ds.$$

Then, we may rewrite (27) as

$$e^{2\gamma\delta t}|y_j(t)|^{2\delta} + I_j(t) = |y_j^0|^{2\delta} + (I_j)_1(t) + M_j(t),\ t \geq 0,\ \mathbb{P}\text{-a.s.} \tag{29}$$

Taking into account that, for each $\varepsilon > 0$ and $j = 1, ..., N$, $M_j(t)$ is a local martingale and $t \to I_j(t)$, $t \to (I_j)_1(t)$ are nondecreasing processes, we see by equation (29) that $t \to e^{2\gamma\delta t}|y_j(t)|^{2\delta}$ is a semi-martingale, as the sum of a local martingale and of an adapted finite variation process (see, e.g., [17]). Then, we may apply to equation (29) the following asymptotic result which is a variant of the *martingale convergence theorem* (see Theorem 7 in [18], p. 139 or Lemma 1 in [1]).

Lemma 3 *Let I and I_1 be nondecreasing adapted processes, Z be a nonnegative semi-martingale and M a local martingale such that $E(Z(t)) < \infty$, $\forall t \geq 0$, $I_1(\infty) < \infty$, $\mathbb{P}$-a.s. and*

$$Z(t) + I(t) = Z(0) + I_1(t) + M(t),\ \forall t \geq 0.$$

Then, there is $\lim_{t\to\infty} Z(t) < \infty$, $\mathbb{P}$-a.s. and $I(\infty) < \infty$, $\mathbb{P}$-a.s.

We are going to apply Lemma 3 to processes $Z(t) = e^{2\gamma\delta t}|y_j(t)|^2)^{2\delta}$, $I = I_j$, $I_1 = (I_j)_1$, $M = M_j$ defined above.

In virtue of (28), $(I_j)_1(\infty) < \infty$. This implies, in virtue of Lemma 3, that there exists the limit

$$\lim_{t\to\infty}(e^{2\gamma\delta t}|y_j(t)|^{2\delta}) < \infty,\ j = 1, ..., N,\ \mathbb{P}\text{-a.s.} \tag{30}$$

Since, by (28), $e^{2\gamma\delta t}|y_j|^{2\delta} \in L^1(0,\infty)$, $\mathbb{P}$-a.s., the limit in (30) is zero. It follows therefore that

$$\lim_{t\to\infty} e^{\gamma t}|y(t)| = 0,\ \mathbb{P}\text{-a.s.}, \tag{31}$$

where $|y|^2 = \sum_{j=1}^N |y_j|^2$. We have therefore that

$$\lim_{t\to\infty} e^{2\gamma t}|X_u(t)|^2_{\widetilde{H}} = 0,\ \mathbb{P}\text{-a.s.} \tag{32}$$

By (28) and (31), it follows also that

$$\int_0^\infty e^{2\gamma t}|y(t)|^2 dt < \infty, \ \mathbb{P}\text{-a.s.},$$

because, by (30), it follows that $e^{2\gamma\delta t}|y|^{2\delta} \in L^\infty(0,\infty)$ $\mathbb{P}$-a.s. This yields

$$\int_0^\infty e^{2\gamma t}|X_u(t)|^2_{\widetilde{H}} dt < \infty, \ \mathbb{P}\text{-a.s.} \tag{33}$$

Next, we come back to the infinite dimensional system (21). Since, as seen earlier, the operator $-\mathcal{A}_s$ generates a γ-exponentially stable C_0-semigroup on $\widetilde{H}$, by the Lyapunov theorem there is $Q \in L(\widetilde{H}, \widetilde{H})$, $Q = Q^* \geq 0$ such that

$$\operatorname{Re}\langle Qx, \mathcal{A}_s x - \gamma x\rangle = \frac{1}{2}\ |x|^2_{\widetilde{H}}, \ \forall x \in D(\mathcal{A}_s).$$

(We note that though Q is not positively definite in the sense that $\inf\{\langle Qx, x\rangle\, ;\, |x| = 1\} > 0$, we have nevertheless that $\langle Qx, x\rangle > 0$ for all $x \neq 0$.)

Applying Itô's formula in (21) to the function $\varphi(x) = \dfrac{1}{2}\ \langle Qx, x\rangle$, we obtain that

$$\begin{aligned} \frac{1}{2}\ d\langle QX_s(t), X_s(t)\rangle \ &+\ \frac{1}{2}\,|X_s(t)|^2_{\widetilde{H}} dt + \gamma\langle QX_s(t), X_s(t)\rangle\, dt = \frac{1}{2}\ \eta^2 \sum_{i=1}^N (QY_i(t), Y_i(t))_H dt \\ &+\ \eta \sum_{i=1}^N ((\operatorname{Re}(QX_s(t)), \operatorname{Re} Y_i(t))_H + (\operatorname{Im}(QX_s(t)), \operatorname{Im} Y_i(t))_H) d\beta_i(t), \end{aligned}$$

where Y_i are processes defined by

$$Y_i(t) = \langle X_u(t), \varphi_i^*\rangle\,(I - P_N)P(m\phi_i), \ i = 1, ..., N.$$

This yields

$$\begin{aligned} & e^{2\gamma t}\langle QX_s(t), X_s(t)\rangle + \int_0^t e^{2\gamma s}|X_s(s)|^2_{\widetilde{H}} ds \\ & \quad = \langle Q(I-P_N)x, (I-P_N)\, x > \\ & \quad +\eta^2 \sum_{i=1}^N \int_0^t e^{2\gamma s}\langle QY_i(s), Y_i(s)\rangle\, ds \\ & \quad +2\eta \sum_{i=1}^N \int_0^t e^{2\gamma s}((\operatorname{Re}(QX_s(s)), \operatorname{Re} Y_i(s))_H \\ & \quad +(\operatorname{Im}(QX_s(s)), \operatorname{Im} Y_i(s)))_H) d\beta_i(s), \ t \geq 0, \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{34}$$

We shall once again apply Lemma 3 to processes Z, I, M defined below

$$\begin{aligned} & Z(t) = e^{2\gamma t}\langle QX_s(t), X_s(t)\rangle\,, \\ & I(t) = \int_0^t e^{2\gamma s}|X_s(s)|^2_{\widetilde{H}} ds, \ I_1(t) = \eta^2 \sum_{i=1}^N \int_0^t e^{2\gamma s}\langle QY_i, Y_i\rangle\, ds, \\ & M(t) = 2\eta \sum_{i=1}^N \int_0^t e^{2\gamma s}((\operatorname{Re}(QX_s(s)), \operatorname{Re} Y_i(s))_H \\ & \qquad +(\operatorname{Im}(QX_s(s)), \operatorname{Im} Y_i(s)))_H) d\beta_j(s), \ \mathbb{P}\text{-a.s.}, \ t \geq 0. \end{aligned}$$

Since, by the first step of the proof (see (33)), $I_1(\infty) < \infty$, we conclude therefore that

$$\lim_{t\to\infty} e^{2\gamma t} \langle QX_s(t), X_s(t)\rangle = 0, \ \mathbb{P}\text{-a.s.},$$

and, since Q is positive definite in the sense that $\langle Qx, x\rangle = (Qx, x)_H > 0$ for all $x \in \widetilde{H}$, we have that

$$\lim_{t\to\infty} e^{\gamma t} |X_s(t)|_{\widetilde{H}} = 0, \ \mathbb{P}\text{-a.s.}$$

Recalling that $X = X_u + X_s$ and again invoking (32), the latter implies (19), thereby completing the proof of Theorem 1.

2.4 The tangential boundary stabilization by noise

We shall keep the notations of Section 3.

We come back to the Stokes-Oseen system with boundary controller, i.e.,

$$\begin{aligned}
&X_t - \nu_0 \Delta X + (X\cdot\nabla)X_e + (X_e\cdot\nabla)X = \nabla p && \text{in } (0,\infty)\times\mathcal{O},\\
&\nabla\cdot X = 0 && \text{in } (0,\infty)\times\mathcal{O},\\
&X\cdot\nu = 0, \quad X = u && \text{in } (0,\infty)\times\partial\mathcal{O},\\
&X(0) = x && \text{in } \mathcal{O}.
\end{aligned} \tag{35}$$

Our purpose here is to stabilize the null solutions to (35) by a noise boundary controller u of the form

$$u = \eta \sum_{i=1}^{N} \frac{\partial \widetilde{\phi}_i}{\partial\nu} \ \langle X, \varphi_i^*\rangle \, \dot\beta_i, \tag{36}$$

where N is, as above, the number of eigenvalues λ_j of the operator $\mathcal{A}$ with $\mathrm{Re}\,\lambda_j \le \gamma$ and $\widetilde{\phi}_i$ will be defined below. As in the previous case, φ_j^* are the eigenfunctions of $\mathcal{A}^*$ corresponding to $\overline{\lambda}_j$ (see (11)) and $\{\beta_i\}_{i=1}^N$ is an independent system of real Brownian motions in $\{\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t\}$.

Here, we shall assume that hypothesis (A_1) holds and also that

(A_3) *The system* $\left\{\dfrac{\partial\varphi_i^*}{\partial\nu}\right\}_{i=1}^N$ *is linearly independent in* $(L^2(\partial\mathcal{O}))^d$.

One might suspect that this property is generic in the class of equilibrium solutions X_e as is the case with the following weaker version of (A_3):

"*each* $\dfrac{\partial\varphi_j^*}{\partial\nu}$ *is not identically zero on* $\partial\mathcal{O}$."

We set

$$\mathcal{L}y = -\nu_0\Delta y + (X_e\cdot\nabla)y + (y\cdot\nabla)X_e \ \text{ in } \mathcal{O}. \tag{37}$$

The Stokes-Oseen system

$$\begin{aligned}
&X_t + \mathcal{L}X = \nabla p && \text{in } (0,\infty)\times\mathcal{O}\\
&\nabla\cdot X = 0 && \text{in } (0,\infty)\times\mathcal{O}\\
&X\cdot\nu = 0, \quad X = u && \text{in } (0,\infty)\times\partial\mathcal{O}\\
&X(0) = x && \text{in } \mathcal{O}
\end{aligned} \tag{38}$$

can be equivalently written as (see, e.g., [6])

$$\begin{aligned} &\frac{d}{dt}X(t) + \widetilde{A}X(t) = \widetilde{A}_\alpha Du(t), \quad t \geq 0, \\ &X(0) = x, \end{aligned} \tag{39}$$

where $y = Du$ is the solution to the equation

$$\begin{aligned} &\alpha y + \mathcal{L}y = \nabla p \ \text{ in } \mathcal{O} \\ &\nabla \cdot y = 0 \ \text{ in } \mathcal{O}, \quad y = u, \ y \cdot \nu = 0 \ \text{ on } \partial\mathcal{O} \end{aligned}$$

and $\alpha > 0$ is fixed and sufficiently large. ($D : (L^2(\partial\mathcal{O}))^d \to H$ is the Dirichlet map associated with the operator $\mathcal{L} + \alpha I$).

Indeed, subtracting the latter from (38), we obtain

$$\begin{aligned} &X_t + (\alpha + \mathcal{L})(X - Du) - \alpha X = \nabla p \ \text{ in } (0,\infty) \times \mathcal{O} \\ &\nabla \cdot X = 0, \quad X(0) = x \\ &X - Du = 0 \ \text{ on } (0,\infty) \times \partial\mathcal{O} \end{aligned}$$

and this can be expressed in the form (39), where $\widetilde{\mathcal{A}} : \widetilde{H} \to (D(A))'$ is the extension by transposition of $\mathcal{A} = P\mathcal{L}$ to all of $\widetilde{H}$ and with values in $(D(A))'$, defined by

$$\widetilde{\mathcal{A}}y(\varphi) = \int_{\mathcal{O}} y\overline{\mathcal{A}}^* \varphi d\xi = \langle y, \mathcal{A}^* \varphi \rangle, \quad \forall \varphi \in D(\mathcal{A}^*), \ y \in \widetilde{H}, \tag{40}$$

and $\widetilde{\mathcal{A}}_\alpha = \alpha I + \widetilde{\mathcal{A}}$. Here, $(D(A))' = (D(\mathcal{A}^*))'$ is the dual of the space $D(A)$ endowed with the graph norm in pairing induced by $\widetilde{H}$ as pivot space; we have $D(A) \subset \widetilde{H} \subset (D(A))'$ algebraically and topologically. It should be noticed that in this formulation, which is standard in boundary control theory, the right hand side of (39) is an element of $(D(A))' = (D(\mathcal{A}^*))'$, i.e., roughly speaking is a "pure" distribution on $\mathcal{O}$, which incorporates the boundary control u. We note also that (see [6]) the dual $D^*\mathcal{A}_\alpha^*$ of $\mathcal{A}_\alpha D$ is given by

$$D^*\mathcal{A}_\alpha^*\varphi = -\nu_0 \frac{\partial\varphi}{\partial\nu}, \quad \forall \varphi \in D(A). \tag{41}$$

Our aim here is to insert into the controlled system (39) a stochastic boundary controller of the form (36). Namely, we shall consider the stochastic differential equation

$$\begin{aligned} &dX(t) + \widetilde{\mathcal{A}}X(t)dt = \eta \sum_{i=1}^N \widetilde{\mathcal{A}}_\alpha D \left(\frac{\partial \widetilde{\phi}_i}{\partial\nu} \right) \langle X(t), \varphi_i^* \rangle \, d\beta_i(t), \ t \geq 0, \\ &X(0) = x. \end{aligned} \tag{42}$$

Here, $\{\widetilde{\phi}_i\}_{i=1}^N$ is given by (17), where α_{ij} are chosen such that

$$\sum_{i=1}^N \alpha_{ij} \left\langle \frac{\partial\varphi_i^*}{\partial\nu}, \frac{\partial\varphi_k^*}{\partial\nu} \right\rangle_1 = \delta_{jk}, \quad j,k = 1,...,N.$$

Here, $|\cdot|_1 = |\cdot|_{(L^2(\partial\mathcal{O}))^d}$ and $\langle u,v\rangle_1 = \displaystyle\int_{\partial\mathcal{O}} \frac{\partial u}{\partial\nu}\frac{\partial\bar{v}}{\partial\nu}\,d\xi$. By assumption (A$_3$), it is clear that the system $\{\widetilde{\phi}_i\}_{i=1}^N$ is well defined and

$$\left\langle \varphi_i^*, \widetilde{\phi}_j \right\rangle_1 = \delta_{ij}, \quad i,j=1,...,N. \tag{43}$$

We can, equivalently, write (42) as

$$X(t) = e^{-\widetilde{\mathcal{A}}t}x + \eta\int_0^t \sum_{i=1}^N e^{-\widetilde{\mathcal{A}}(t-s)}\left(\widetilde{\mathcal{A}}_\alpha D\left(\frac{\partial\widetilde{\phi}_i}{\partial\nu}\right)\right)\langle X(s),\varphi_i^*\rangle\, d\beta_i(s). \tag{44}$$

Equation (44) has a unique solution $X = X(t)$, which is an $\widetilde{H}$-valued continuous process which can be viewed as solution to problem (see [10], p. 244)

$$\begin{aligned}
&X_t - \nu_0\Delta X + (X\cdot\nabla)X_e + (X_e\cdot\nabla)X = \nabla p \ \text{ in } (0,\infty)\times\mathcal{O}\\
&\nabla\cdot X = 0 \ \text{ in } (0,\infty)\times\mathcal{O}\\
&X(0,\xi) = x(\xi) \ \text{ in } \mathcal{O}\\
&X = \sum_{i=1}^N \frac{\partial\widetilde{\phi}_i}{\partial\nu}\ \langle X,\varphi_i^*\rangle\,\dot{\beta}_i \ \text{ on } (0,\infty)\times\partial\mathcal{O}.
\end{aligned} \tag{45}$$

In other words, the boundary controller $u = X\Big|_{\partial\mathcal{O}}$ is a white noise on $\partial\mathcal{O}$. Moreover, since $\left(\dfrac{\partial\widetilde{\phi}_i}{\partial\nu}\cdot\nu\right)\cdot\nu = 0$ on $\partial\mathcal{O}$ (see, e.g., Lemma 3.3.1 in [6]) this stochastic controller is tangential, i.e., $X\cdot\nu = 0$ on $(0,\infty)\times\partial\mathcal{O}$.

Theorem 4 *Assume that hypothesis* (A$_1$), (A$_3$) *are satisfied. Then, for* $|\eta|$ *large enough we have for the solution* X *to* (42) (*equivalently,* (44))

$$\mathbb{P}\left[\lim_{t\to\infty} e^{\gamma t}\|X(t)\|_{(D(A))'}\right] = 0. \tag{46}$$

In particular, we have

$$\lim_{t\to\infty} e^{\gamma t}\langle X(t),\varphi\rangle = 0,\ \mathbb{P}\text{-}a.s.,\ \forall\varphi\in D(A). \tag{47}$$

Proof. We shall argue as in the proof of Theorem 1. Namely, as in the previous case, we shall decompose system (42) in two parts,

$$\begin{aligned}
&dX_u + \mathcal{A}_u X_u dt = \eta\widetilde{P}_N\sum_{i=1}^N \widetilde{\mathcal{A}}_\alpha\left(\frac{\partial\widetilde{\phi}_i}{\partial\nu}\right)\langle X,\varphi_i^*\rangle\, d\beta_i,\ \mathbb{P}\text{-a.s.}\\
&X_u(0) = P_N x
\end{aligned} \tag{48}$$

$$\begin{aligned}
&dX_s + \mathcal{A}_s X_s dt = \eta(I-\widetilde{P}_N)\sum_{i=1}^N \widetilde{\mathcal{A}}_\alpha\left(\frac{\partial\widetilde{\phi}_i}{\partial\nu}\right)\langle X,\varphi_i^*\rangle\, d\beta_i,\ \mathbb{P}\text{-a.s.}\\
&X_s(0) = (I-P_N)x.
\end{aligned} \tag{49}$$

Here $\widetilde{P}_N : (D(A))' \to \mathcal{X}_u = \text{lin span}\{\varphi_j\}_{i=1}^N$ is the projector defined as in (8) and $\mathcal{A}_u = \widetilde{P}_N \mathcal{A}\big|_{\mathcal{X}_u}$, $\mathcal{A}_s = (I - \widetilde{P}_N)\mathcal{A}\big|_{\mathcal{X}_s}$. The operator $\widetilde{\mathcal{A}}_s$ is the extension of $\mathcal{A}_s$ to all of $\widetilde{H}$, i.e., $\widetilde{\mathcal{A}}_s : \widetilde{H} \to (D(A))'$ is defined by (40).

We represent the solution X_u to (48) as $X_u = \sum_{j=1}^N y_j \varphi_j$ and taking into account (12), (43), we obtain for $\{y_j\}_{j=1}^N$ the finite dimensional stochastic system

$$\begin{aligned} &dy_j + \lambda_j y_j dt = \eta y_j d\beta_j, \quad j = 1, ..., N, \\ &y_j(0) = y_j^0. \end{aligned} \tag{50}$$

System (50) will be treated in the same way as system (22). In fact, we get by exactly the same argument as in the proof of Theorem 1, that (see (31), (33))

$$\lim_{t\to\infty} e^{\gamma t}|y(t)| = 0, \qquad \mathbb{P}\text{-a.s.} \tag{51}$$

$$\int_0^\infty e^{2\gamma t}|y(t)|^2 dt < \infty, \qquad \mathbb{P}\text{-a.s.} \tag{52}$$

where $|y|^2 = \sum_{j=1}^N y_j^2$, $y = X_u$.

Now, coming back to system (49), we shall write it as

$$\begin{aligned} &dX_s + \widetilde{\mathcal{A}}_s X_s dt = \eta \sum_{i=1}^N Y_i(t) d\beta_i, && t \geq 0, \\ &Y_i(t) = (I - \widetilde{P}_N)\widetilde{\mathcal{A}}_s D\left(\frac{\partial \widetilde{\phi}_i}{\partial \nu}\right) \langle X_u(t), \varphi_i^* \rangle, && i = 1, ..., N. \end{aligned} \tag{53}$$

By (9), it follows that

$$\|e^{-\mathcal{A}_s t} x\|_{(D(A))'} \leq C e^{-\gamma t} \|x\|_{(D(A))'}, \quad \forall x \in (D(A))'$$

and so, by the Lyapunov theorem, there is a self-adjoint, continuous and positive operator $Q = L((D(A))', (D(A))')$ such that

$$\text{Re}\, \left\langle Qx, \widetilde{\mathcal{A}}_s x \right\rangle_* = \gamma \langle Qx, x \rangle_* + \frac{1}{2}\, \|x\|^2_{(D(A))'}, \quad \forall x \in (D(A))' \tag{54}$$

where $\langle \cdot, \cdot \rangle_*$ is the natural scalar product in $(D(A))'$.

Applying Itô's formula in (53), we obtain that

$$\begin{aligned} &\frac{1}{2} d \langle QX_s(t), X_s(t) \rangle_* + \frac{1}{2} \|X)_s(t)\|^2_{(D(A))'} dt + \gamma \langle QX_s(t), X_s(t) \rangle_* dt \\ &\quad = \frac{1}{2} \eta^2 \sum_{i=1}^N (QY_i(t), Y_i(t))_H dt + \eta \sum_{i=1}^N (\langle \text{Re}(QX_s), Y_i \rangle_* + \langle \text{Im}(QX_s), \text{Im } Y_i \rangle_*) d\beta_i. \end{aligned}$$

This yields

$$e^{2\gamma t}\left\langle QX_s(t),X_s(t)\right\rangle_* + \int_0^t e^{2\gamma s}\|X_s(s)\|^2_{(D(A))'}ds$$
$$= \left\langle Q(I-P_N)x,(I-P_N)x\right\rangle_* + 2\eta\sum_{i=1}^N\int_0^t e^{2\gamma s}\left\langle QY_i(s),Y_i(s)\right\rangle_* ds$$
$$+2\eta\sum_{i=1}^N\int_0^t e^{2\gamma s}(\left\langle \mathrm{Re}(QX_s),\mathrm{Im}\,Y_i\right\rangle_* + \left\langle \mathrm{Im}(QX_s),\mathrm{Im}\,Y_i\right\rangle_*)d\beta_i.$$

Then, applying Lemma 3 exactly as in the proof of Theorem 1, we infer that

$$\mathbb{P}\left[\lim_{t\to\infty}\left\langle QX_s(t),X_s(t)\right\rangle_* e^{2\gamma t}=0\right]=1$$

and, since $\left\langle Qx,x\right\rangle_*=0$ implies $x=0$, we infer that

$$\mathbb{P}\left[\lim_{t\to\infty}e^{\gamma t}|X_s(t)|_{(D(A))'}=0\right]=1,$$

as claimed. This completes the proof.■

References

[1] J.A.D. Apleby, X. Mao, A. Rodkina, Stabilization and destabilization of nonlinear differential equations by noise, *IEEE Transactions on Automatic Control*, vol. 53 (2008), 683-691.

[2] J.A.D. Apleby, X. Mao, A. Rodkina, Stochastic stabilization of functional differential equations, *Systems & Control Letters*, vol. 54 (2005), 1069-1081.

[3] L. Arnold, H. Craul, V. Wihstutz, Stabilization of linear systems by noise, *SIAM J. Control Optimiz.*, vol. 21 (1983), 451-461.

[4] V. Barbu, Feedback stabilization of Navier-Stokes equations, *ESAIM COCV*, 9 (2003), 197–206.

[5] V. Barbu, R. Triggiani, Internal stabilization of Navier-Stokes euations with finite dimensional controllers, *Indiana Univ. Math. J.*, vol. 53 (2004), 1443–1494.

[6] V. Barbu, I. Lasiecka, R. Triggiani, Boundary stabilization of Navier-Stokes equations, *Memoires Amer. Math. Soc.*, 2006.

[7] T. Caraballo, H. Craul, J.A. Langa, J.C. Robinson, Stabilization of linear PDEs by Stratonovich noise, *Systems & Control Letters*, vol. 53 (2004), 41-50.

[8] T. Caraballo, K. Liu, X. Mao, On stabilization of partial differential equations by noise, *Nagoya Math. J.*, vol. 101 (2001), 155-170.

[9] S. Cerrai, Stabilization by noise for a class of stochastic reaction-diffusion equations, *Probab. Theory Rel. Fields*, 133 (2000), 190-214.

[10] G. Da Prato, *An Introduction to Infinite Dimensional Analysis*, Springer-Verlag, Berlin, 2006.

[11] H. Ding, M. Krstic, R.J. Williams, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, *IEEE Transactions on Automatic Control*, vol. 46 (2001), 1237-1253.

[12] J. Duan, A. Fursikov, Feedback stabilization for Oseen Fluid Equations. A stochastic approach, *J. Math. Fluids Mech.*, 7 (2005), 574–610.

[13] A. Fursikov, Real processes of the $3-D$ Navier-Stokes systems and its feedback stabilization from the boundary, M.S. Agranovic and M.A. Shubin (eds.), *AMS Translations, Partial Differential Equations, M. Vîshnik Seminar*, vol. 206, 2002, 95-123.

[14] A. Fursikov, Stabilization for the $3-D$ Navier-Stokes systems by feedback boundary control, *Discrete and Con6tinuous Dynamical Systems*, 10 (2004), 289-314.

[15] T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, New York, Berlin, 1966.

[16] S. Kuksin, A. Shirikyan, Ergodicity for the randomly forced $2D$ Navier-Stokes equations, *Math. Phys. Anal. Geom.*, 4 (2001), 147–195.

[17] T. Kurtz, *Lectures on Stochastic Analysis*, Wisconsin 2007 (Lecture Notes Online).

[18] R. Lipster, A.N. Shiraev, *Theory of Martingals*, Dordrecht, Kluwer, 1989.

[19] X.R. Mao, Stochastic stabilization and destabilization, *Systems & Control Letters*, vol. 23 (2003), 279-290.

[20] J.P. Raymond, Feedback boundary stabilization of the two dimensional Navier-Stokes equations, *SIAM J. Control Optimiz.*, vol. 45, 3 (2006), 790-828.

[21] J.P. Raymond, Feedback boundary stabilization of the three dimensional incompressible Navier-Stokes equations, *J. Math. Pures et Appl.*, vol. 87 (2007), 627-669.

[22] A. Shirikyan, Exponential mixing $2D$ Navier-Stokes equations perturbed by an unbounded noise, *J. Math. Fluids Mech.,* 6 (2004), 169–193.

Lecture 3.
Internal stabilization by noise of the Navier–Stokes equation

3.1 Introduction

Consider the Navier-Stokes equation

$$\begin{cases} X_t - \nu_0 \Delta X + (X \cdot \nabla) X = f_e + \nabla p, \quad \text{in } (0,\infty) \times \mathcal{O}, \\ \nabla \cdot X = 0, \quad \text{in } (0,\infty) \times \mathcal{O}, \\ X = 0, \quad \text{on } (0,\infty) \times \partial\mathcal{O}, \\ X(0) = x_0, \quad \text{in } \mathcal{O}. \end{cases} \tag{3.1}$$

where $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial\mathcal{O}$. Here $f_e \in (L^2(\mathcal{O}))^d$ is given.Let X_e be an equilibrium solution to (3.1), i.e.,

$$\begin{cases} -\nu_0 \Delta X_e + (X_e \cdot \nabla) X_e = f_e + \nabla p_e, \quad \text{in } \ \mathcal{O}, \\ \nabla \cdot X_e = 0 \quad \text{in } \mathcal{O}, \\ X_e = 0 \quad \text{on } (0,\infty) \times \partial\mathcal{O}. \end{cases} \tag{3.2}$$

If we replace X by $X - X_e$ equation (3.1) reduces to

$$\begin{cases} X_t - \nu_0 \Delta X + (X \cdot \nabla) X_e + (X_e \cdot \nabla) X + (X \cdot \nabla) X = \nabla p \quad \text{in } (0,\infty) \times \mathcal{O}, \\ \nabla \cdot X = 0 \ \text{ in } \mathcal{O}, \\ X = 0 \quad \text{on } (0,\infty) \times \partial\mathcal{O}, \\ X(0) = x \quad \text{in } \mathcal{O}, \end{cases} \tag{3.3}$$

where $x = x_0 - X_e$. If we set

$$H = \left\{ X \in (L^2(\mathcal{O}))^d : \ \nabla \cdot X = 0, \, X \cdot \nu\big|_{\partial\mathcal{O}} = 0 \right\},$$

where ν is the normal to $\partial\mathcal{O}$ and denote by $P : (L^2(\mathcal{O}))^d \to H$ the Leray projector on H, we can rewrite system (3.3) as

$$\begin{cases} \dot{X}(t) + \mathcal{A} X(t) + B(X(t)) = 0, \quad t \geq 0, \\ X(0) = x, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} &A = -P\Delta, \quad D(A) = (H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}))^d \cap H, \\ &A_0 x = P((x \cdot \nabla) X_e + (X_e \cdot \nabla) x), \quad B(x) = P((x \cdot \nabla) x), \\ &\mathcal{A} = \nu_0 A + A_0, \quad D(\mathcal{A}) = D(A) \end{aligned}$$

We have

$$\langle B(x), y \rangle_H = b(x, x, y), \quad \forall \, x, y \in D(A),$$

where $\langle \cdot, \cdot \rangle_H$ is the scalar product induced by H as pivot space and

$$b(x, z, y) = \sum_{j,k=1}^{d} \int_{\mathcal{O}} x_j D_j z_k y_k, \quad \forall\, x, y, z \in D(A).$$

We recall that for large values of the Reynolds number $\frac{1}{\nu_0}$ the stationary solution X_e to (3.1) is unstable. i.e. the corresponding flow is turbulent. Our purpose here is to stabilize (3.4) or, equivalently, the stationary solution X_e to (3.1), using a stochastic controller with support in an arbitrary open subset $\mathcal{O}_0 \subset \mathcal{O}$. To this aim we associate with (3.4) the controlled stochastic system

$$\begin{cases} dX(t) + (\mathcal{A}X(t) + B(X(t))dt = \sum_{j=1}^{N} V_j(t)\psi_j d\beta_j(t), \\ X(0) = x, \end{cases} \tag{3.5}$$

where $\{\beta_j\}_{j=1}^N$ is an independent system of real Brownian motions in a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t>0})$.

The main result, Theorems 3.2 below, amounts to saying that, in the complexified space $\widetilde{H}$ associated with H, under appropriate assumptions on $\mathcal{A}$ (and, implicitly, on X_e), for each $\gamma > 0$ there exist $N \in \mathbb{N}$, $\{\psi_j\}_{j=1}^N \subset \widetilde{H}$, and an N-dimensional adapted process $\{V_j = V_j(t, \omega)\}_{j=1}^N$, $\omega \in \Omega$, such that for all x in a sufficiently small neighbourhood of the origin, $t \to e^{\frac{\gamma t}{4}} X(t, \omega)$ is decaying to zero for $t \to \infty$ in a set Ω_x^* of positive probability which is precisely estimated. Moreover, it turns out that the stabilizable controller arising in the right hand side of (3.5) is a linear feedback controller of the form

$$V_j(t) = \eta \langle X(t), \varphi_j^* \rangle_{\widetilde{H}}, \quad \psi_j = P(m\phi_j), \quad j = 1, ..., N, \tag{3.6}$$

where $|\eta| > 0$ and φ_j^* are the eigenfunctions of the dual Stokes-Oseen operator $\mathcal{A}^*$ corresponding to eigenvalues $\overline{\lambda}_j$ with $\operatorname{Re} \lambda_j \leq \gamma$, $\{\phi_j\}_{j=1}^N$ is a system of functions related to φ_j^* and $m = \mathbb{1}_{\mathcal{O}_0}$ is the characteristic function of $\mathcal{O}_0$ where $\mathcal{O}_0$ is a given arbitrary open subset of $\mathcal{O}$.

We may view (3.5) as the deterministic system (3.4) perturbed by the white noise controller $\sum_{j=1}^N V_j(t)\psi_j\dot{\beta}_j$ with the support in $\mathcal{O}_0$.

This work is a continuation of [2] where such a result is proved for the linearized Navier-Stokes equation associated with (3.3). The previous treatment of internal stabilization of Navier-Stokes equations ([1],[4]) is based on the stabilization by a linear feedback provided by the solution of an algebraic infinite dimensional Riccati equation associated with the Stokes-Oseen operator $\mathcal{A}$. (This approach was also used in [5],[6],[15],[16],[20],[21] for boundary stabilization of Navier-Stokes equations.)

The main advantage of this stochastic based stabilization technique with respect to the Riccati-feedback based approach in above mentioned works, is that it avoids the difficult computation problems related to infinite dimensional Riccati equations. Also a nice features of this feedback control which has a stabilizing influence with high probability if applied in a small neighbourhood of a stationary solution is that besides its simplicity it is robust in the class of finite dimensional Gaussian multiplicative perturbations.

It should be said also that stabilization by noise of the dynamic PDEs was already used in the literature and we refer to [6],[7],[8],[9],[10],[12],[14] for related results. However, there is not overlap with existing literature and methods used here are different and may be viewed as a combination of spectral stabilization techniques ([1],[4]) with that of noise stabilization. In particular in [9] is studied the stabilization of some classes of PDE using Stratonovich noise which has a special interest in construction of an approximating stabilizing controller.

Notations

Throughout in the following β_j, $j = 1, ..., N$ are independent real Brownian motions in a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t>0})$ and we shall refer to [12, 14] for definition and basic results on stochastic analysis of differential systems and spaces of stochastic processes adapted to filtration $\{\mathcal{F}_t\}_{t>0}$. The scalar product of H is denoted $\langle\cdot,\cdot\rangle_H$ and the norm $|\cdot|_H$. We shall denote by $\widetilde{H}$ the complexified space $H + iH$ with scalar product denoted by $\langle\cdot,\cdot\rangle_{\widetilde{H}}$ and norm by $|\cdot|_{\widetilde{H}}$. $C_W([0,T]; L^2(\Omega, \widetilde{H}))$ is the space of all adapted square-mean $\widetilde{H}$-valued continuous processes on $[0,T]$.

3.2 The main result

To begin with, let us briefly recall a few elementary spectral properties of the Stokes-Oseen operator $\mathcal{A}$. Denote again by $\mathcal{A}$ the extension of $\mathcal{A}$ to the complex space $\widetilde{H}$. The operator $\mathcal{A}$ has a compact resolvent $(\lambda I - \mathcal{A})^{-1}$ and $-\mathcal{A}$ generates a C_0-analytic semigroup $e^{-\mathcal{A}t}$ in $\widetilde{H}$. Consequently, $\mathcal{A}$ has a countable number of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ with corresponding eigenfunctions φ_j each with finite algebraic multiplicity m_j. Of course, certain eigenfunctions φ_j might be generalized and so, in general, $\mathcal{A}$ is not diagonalizable, i.e., the algebraic multiplicity of λ_j might not coincide with its geometric multiplicity. Also, each eigenvalue λ_j will be repeated according to its algebraic multiplicity m_j.

We shall denote by N the number of eigenvalues λ_j with $\operatorname{Re}\lambda_j \le \gamma$, $j = 1, ..., N$, where γ is a fixed positive number.

Denote by P_N the projector on the finite dimensional subspace

$$\mathcal{X}_u = \operatorname{lin\,span}\{\varphi_j\}_{j=1}^N.$$

We have $\mathcal{X}_u = P_N\widetilde{H}$ and

$$P_N = -\frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathcal{A})^{-1} d\lambda, \tag{3.7}$$

where Γ is a closed smooth curve in $\mathbb{C}$ which is the boundary of a domain containing in interior the eigenvalues $\{\lambda_j\}_{j=1}^N$.

Let $\mathcal{A}_u = P_N\mathcal{A}$, $\mathcal{A}_s = (I - P_N)\mathcal{A}$. Then $\mathcal{A}_u$, $\mathcal{A}_s$ leave invariant the spaces $\mathcal{X}_u$ and $\mathcal{X}_s = (I - P_N)\widetilde{H}$ and the spectra $\sigma(\mathcal{A}_u)$, $\sigma(\mathcal{A}_s)$ are given by (see [10])

$$\sigma(\mathcal{A}_u) = \{\lambda_j\}_{j=1}^N, \quad \sigma(\mathcal{A}_s) = \{\lambda_j\}_{j=N+1}^\infty.$$

Since $\sigma(\mathcal{A}_s) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \gamma\}$ and $\mathcal{A}_s$ generates an analytic C_0-semigroup on $\widetilde{H}$, we have

$$|e^{-\mathcal{A}_s t}x|_{\widetilde{H}} \le Ce^{-\gamma t}|x|_{\widetilde{H}}, \quad \forall x \in \widetilde{H},\ t \ge 0. \tag{3.8}$$

The eigenvalue λ_j is said to be *semi-simple* if its algebraic and geometrical multiplicity coincides, or, equivalently, λ_j is a simple pole for $(\lambda I - \mathcal{A})^{-1}$. If all eigenvalues $\{\lambda_j\}_{j=1}^N$ of the matrix $\mathcal{A}_u$ are semi-simple, then $\mathcal{A}_u$ *is diagonalizable.*

Herein, we shall assume that the following hypothesis holds.

(H_1) *All eigenvalues* λ_j, $j = 1, ..., N$, *are semi-simple.*

As regards Hypothesis (H_1), it should be said that it follows by a standard argument involving the Sard-Smale theorem that the property of eigenvalues of the Stokes-Oseen operator to be simple (and, consequently, semi-simple) is generic in the class of coefficients X_e. (See [3], p. 159.) So, one might say that "almost everywhere" (in the sense of a set of first category), hypothesis (H_1) holds.

Denote by $\mathcal{A}^*$ the adjoint operator and by P_N^* the adjoint of P_N. We have

$$P_N^* = -\frac{1}{2\pi i} \int_{\overline{\Gamma}} (\lambda I - \mathcal{A}^*)^{-1} d\lambda. \tag{3.9}$$

The eigenvalues of $\mathcal{A}^*$ are precisely the complex conjugates $\overline{\lambda}_j$ of eigenvalues λ_j of $\mathcal{A}$ and they have the same multiplicity. Denote by φ_j^* the eigenfunction of $\mathcal{A}^*$ corresponding to the eigenvalue $\overline{\lambda_j}$. We have, therefore,

$$\mathcal{A}\varphi_j = \lambda_j \varphi_j, \quad \mathcal{A}^* \varphi_j^* = \overline{\lambda}_j \varphi_j^*, \quad j \in \mathbb{N}. \tag{3.10}$$

Since the eigenvalues $\{\lambda_j\}_{j=1}^N$ *are semi-simple, it turns out that the system consisting of* $\{\varphi_j\}_{j=1}^N$, $\{\varphi_j^*\}_{j=1}^N$ *can be chosen to form a bi-orthonormal sequence in* $\widetilde{H}$, i.e,

$$\langle \varphi_j, \varphi_k^* \rangle_{\widetilde{H}} = \delta_{jk}, \quad j, k = 1, ..., N, \tag{3.11}$$

where δ_{jk} is the Kronecker symbol (see, e.g., [4]). We notice also that the functions φ_j and φ_j^* have the unique continuation property, i.e.,

$$\varphi_j \not\equiv 0, \quad \varphi_j^* \not\equiv 0 \ \text{ on } \mathcal{O}_0 \text{ for all } j = 1, ..., N, \tag{3.12}$$

(see, e.g., Lemma 3.7 in [4]).

We have also the following property which will be proven in Appendix.

Lemma 3.1 *The system* $\{\varphi_1^*, ..., \varphi_N^*\}$ *is linearly independent in* $(L^2(\mathcal{O}_0))^d$.

If the eigenvalues λ_j are the same then Lemma 3.1 follows by the unique continuation property (3.12).

Consider the following stochastic perturbation of the system (3.4) considered in the complex space

$$\begin{cases} dX + (\mathcal{A}X + B(X))dt = \eta \displaystyle\sum_{j=1}^N \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j) d\beta_j, \\ X(0) = x, \end{cases} \tag{3.13}$$

where $|\eta| > 0$ and $m = 1\!\!1_{\mathcal{O}_0}$ is the characteristic function of the open subset $\mathcal{O}_0 \subset \mathcal{O}$. Here $\{\phi_j\}_{j=1}^N \subset \widetilde{H}$ is a system of functions to be precised in (3.15). This is a closed loop system with a stochastic linear feedback controller associated with (3.4).

In two dimensions the stochastic differential equation (3.13) has a global solution $X \in C_W([0,T]; L^2(\Omega, \widetilde{H}))$ for all $T > 0$ (see e.g. [14]).

The closed loop system (3.13) can be equivalently written as

$$\begin{cases} dX(t)-\nu_0\Delta X(t)dt+(X(t)\cdot\nabla)X_e dt+(X_e\cdot\nabla)X(t)dt+(X(t)\cdot\nabla)X(t)dt \\ \qquad = \eta m \displaystyle\sum_{j=1}^N \langle X(t), \varphi_j^*\rangle_{\widetilde{H}} \phi_j d\beta_j(t) + \nabla p(t)dt \text{ in } (0,\infty)\times\mathcal{O}, \ \mathbb{P}\text{-a.s.} \\ \nabla\cdot X(t) = 0 \text{ in } \mathcal{O}, \quad X(t)\Big|_{\partial\mathcal{O}} = 0, \ \forall t \geq 0, \ \mathbb{P}\text{-a.s.} \\ X(0) = x \text{ in } \mathcal{O}. \end{cases} \tag{3.14}$$

Hence, in the space $(L^2(\mathcal{O}))^d$, the feedback controller $\{u_j = \eta m\langle X, \varphi_j^*\rangle_{\widetilde{H}}\phi_j\}_{j=1}^N$ has the support in $\mathcal{O}_0$.

We shall define now $\phi_j, \ j = 1, ..., N$, as follows.

$$\phi_j(\xi) = \sum_{l=1}^N \alpha_{lj}\varphi_l^*(\xi), \quad \xi \in \mathcal{O}, \tag{3.15}$$

where α_{lj} are chosen in such a way that

$$\sum_{l=1}^N \alpha_{lj}\langle\varphi_l^*, \varphi_k^*\rangle_0 = \delta_{jk}, \quad j,k = 1, ..., N.$$

(Since, in virtue of Lemma 3.1 the Gram matrix $\{\langle\varphi_l^*, \varphi_k^*\rangle_0\}_{l,k=1}^N$ is not singular, this is possible.) With this choice, we have

$$\langle\phi_j, \varphi_k^*\rangle_0 = \delta_{kj}, \quad k,j = 1, ..., N. \tag{3.16}$$

Here, we have used the notation $\langle u, v\rangle_0 = \int_{\mathcal{O}_0} u(\xi)\bar{v}(\xi)d\xi$.

In the following we shall denote by A^α, $\alpha \in (0,1)$, the fractional power of order α of A, by $D(A^\alpha)$ its domain and set $|x|_\alpha = |A^\alpha x|$ for all $x \in D(A^\alpha)$. Moreover, we shall denote by W the space $D(A^{\frac{1}{4}})$ if $d = 2$ and $D(A^{\frac{1}{4}+\epsilon})$ if $d = 3$ where $\epsilon > 0$ is small.

Theorem 1 below is the main result of the paper.

Theorem 3.2 *Let $d = 2, 3$, $X_e \in C^2(\overline{\mathcal{O}})$ and*

$$|\eta| \geq \max_{1\leq j\leq N} \sqrt{6\gamma - 2\text{Re}\,\lambda_j}. \tag{3.17}$$

Then, there is $C^* > 0$, independent of ω such that, for each $x \in W$, $|x|_W \leq (C^*)^2$ there is $\Omega_x^* \subset \Omega$ with

$$\mathbb{P}(\Omega_x^*) \geq 1 - 2\left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{2(\eta N)^2}}, \tag{3.18}$$

the solution $X(t,x)$ to (3.13) *satisfies*

$$\lim_{t\to\infty}\left(e^{\frac{\gamma t}{4}}|X(t,x)|_{\widetilde{H}}\right) = 0, \quad \mathbb{P}\text{-a.s. in } \Omega_x^*. \tag{3.19}$$

In particular, Theorem 3.2 implies that if $|x|_W \leq \rho_0 < (C^*)^{-2}$ then $X = X(t,x)$ is exponentially decaying to 0 on a set Ω_x^* of probability greater than

$$1 - 2\left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{2(\eta N)^2}}.$$

The constant C^* depends of X_e only. The optimal η for which $\mathbb{P}(\Omega_x^*)$ is maximal is of course that which follows by (3.17), i.e.,

$$|\eta| = \max_{1\leq j\leq N} \sqrt{6\gamma - 2\mathrm{Re}\,\lambda_j},$$

and we see that $\mathbb{P}(\Omega_x^*) \to 1$ as $|x|_W \leq \rho_0 \to 0$.

For the linearized Navier–Stokes equation, that is if one takes $B = 0$, the exponential decay in (3.13) occurs with probability one. In fact, as seen from the proof of Theorem 3.2 the constant C^* comes out from estimates on the nonlinear inertial term B and so, it is zero if this term is absent from the equation.

Remark 3.3 As mentioned earlier, system (3.14) is written here in the complex space $\widetilde{H}$. If set $X_1(t) = \mathrm{Re}\,X(t)$, $X_2(t) = \mathrm{Im}\,X(t)$, it can be rewritten as a real system in (X_1, X_2). In this case, the feedback controller is an implicit stabilizable feedback controller with support in $\mathcal{O}_0$ for the real Navier–Stokes equation (3.3). Of course, if λ_j, $j = 1, ..., N$, are real, then we may view $X(t)$ as a real valued function and so, in (3.18), $|X|_{\widetilde{H}} = |X|_H$.

In particular, by Theorem 3.2 we have

Corollary 3.4 *Under the assumptions of Theorem* 3.2 *the feedback controller*

$$\eta m \sum_{j=1}^{N} \langle X - X_e, \varphi_j^* \rangle_{\widetilde{H}} \phi_j \tag{3.20}$$

stabilizes exponentially the stationary solution X_e, $\mathbb{P}$-a.e in Ω_x^*.

3.3 Proof of Theorem 3.2

The idea of the proof is to transform equation (3.13) in a deterministic equation with random coefficients via substitution

$$y(t) = \prod_{j=1}^{N} e^{-\beta_j(t)\Gamma_j}\, X(t), \quad t \geq 0, \tag{3.21}$$

where $\Gamma_j : \widetilde{H} \to \widetilde{H}$ is the linear operator

$$\Gamma_j x := \eta \langle x, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j), \quad x \in \widetilde{H},\ j = 1, ..., N \tag{3.22}$$

and $e^{s\Gamma_j} \in L(\widetilde{H}, \widetilde{H})$ is the C_0-group generated by Γ_j, i.e.,

$$\frac{d}{ds}\, e^{s\Gamma_j} x - \Gamma_j e^{s\Gamma_j} x = 0, \quad \forall\, s \in \mathbb{R},\ x \in \widetilde{H}. \tag{3.23}$$

We have by (3.22) and by (3.15) that

$$\Gamma_j \Gamma_k x = \eta^2 \langle x, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j)\delta_{jk}, \quad \forall\, j,k = 1, ..., N \tag{3.24}$$

and therefore the operators $\Gamma_1, ..., \Gamma_N$ commute, because the Leray operator P is self-adjoint.

Then, by [13, p. 176] equation (3.13) reduces to

$$\begin{cases} \dfrac{dy(t)}{dt} + \mathcal{A}y(t) + \dfrac{1}{2} \displaystyle\sum_{j=1}^{N} \Gamma_j^2 y(t) + F(t)y(t) \\ \qquad e^{-\sum_{j=1}^{N} \beta_j(t)\Gamma_j} B\left(e^{\sum_{j=1}^{N} \beta_j(t)\Gamma_j} y(t)\right) = 0, \quad \forall\, t \geq 0, \ \mathbb{P}\text{-a.s.} \\ y(0) = x, \end{cases} \tag{3.25}$$

where

$$F(t)y(t) = e^{-\sum_{j=1}^{N} \beta_j(t)\Gamma_j} \mathcal{A}\left(e^{\sum_{j=1}^{N} \beta_j(t)\Gamma_j} y(t)\right) - \mathcal{A}y(t).$$

By a solution to (3.25) we mean a function $y \in C([0,\infty); D(A^{\frac{1}{4}})) \cap L^2(0,\infty; D(A))$ which fulfills (3.25) $\mathbb{P}$-a.s. in the mild sense (see Lemma 3.7 below).

Conversely, if y is a solution to (3.25), then it is an adapted process and so

$$X(t) = \prod_{j=1}^{N} e^{\beta_j(t)\Gamma_j}\, y(t), \quad t \geq 0, \tag{3.26}$$

belongs to $C_W([0,T]; L^2(\Omega,\mathbb{P}; D(A^{\frac{1}{4}})) \cap L^2(\Omega,\mathbb{P}, C([0,T]; D(A^{\frac{3}{4}})))$ and satisfies equation (3.13).

Then we shall confine in the following to study existence and exponential convergence in probability to solutions y to equation to (3.25).

We notice first that, as easily follows by (3.22) and (3.24), we have

$$\begin{aligned} e^{s\Gamma_j} y = \eta^{-1}\Gamma_j y(e^{\eta s} - 1) + y = (e^{\eta s} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j) + y, \\ \forall\, s > 0, \ j = 1, ..., N, \ y \in H. \end{aligned} \tag{3.27}$$

respectively

$$\begin{aligned} e^{-s\Gamma_j} y = \eta^{-1}\Gamma_j y(e^{-\eta s} - 1) + y = (e^{-\eta s} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j) + y, \\ \forall\, s > 0, \ j = 1, ..., N, \ y \in H. \end{aligned}$$

This yields

$$F(t)y = \sum_{j=1}^{N} \left(e^{\beta_j(t)} - 1\right) \langle y, \varphi_j^* \rangle_{\widetilde{H}} (\mathcal{A}P(m\phi_j) - \lambda_j P(m\phi_j)). \tag{3.28}$$

We consider the operator

$$\mathcal{A}_\Gamma y := \mathcal{A}y + \frac{1}{2} \sum_{j=1}^{N} \Gamma_j^2 y, \quad \forall\, y \in D(\mathcal{A}) \tag{3.29}$$

and notice that the C_0-semigroup $e^{-\mathcal{A}_\Gamma t}$ generated by $-\mathcal{A}_\Gamma$ on $\widetilde{H}$ is analytic. The operator $\mathcal{A}_\Gamma + F(t)$ generates an evolution operator $U(t,\tau)$ on $\widetilde{H}$, that is

$$\begin{cases} \dfrac{d}{dt}\, U(t,\tau) + (\mathcal{A}_\Gamma + F(t))U(t,\tau) = 0, \quad 0 \le \tau \le t \\ \\ U(\tau,\tau) = I. \end{cases}$$

Lemma 3.5 *Let γ the number fixed at the beginning of Section* 2.
We have for $\eta \ge \max_{1\le j\le N} \sqrt{6\gamma - 2\mathrm{Re}\ \lambda_j}$

$$\|U(t,\tau)\|_{L(\widetilde{H},\widetilde{H})} \le Ce^{-\gamma(t-\tau)}(1+\eta^2)|x| \left(1 + \int_\tau^t e^{-\gamma(\tau+2s)}\zeta(s)ds\right), \qquad (3.30)$$
$$\forall\, t \ge \tau, \quad \mathbb{P}\text{-a.s.},$$

where C is independent of ω and $\zeta(t) = \sum_{j=1}^N e^{\beta_j(t)}$.

Proof. We shall use as in [2], [4] the spectral decomposition of the system

$$\begin{cases} \dfrac{dy}{dt} + \mathcal{A}_\Gamma y + F(t)y = 0, \quad t \ge \tau \\ \\ y(\tau) = x. \end{cases} \qquad (3.31)$$

in the direct sum $\mathcal{X}_u \oplus \mathcal{X}_s$ of γ-unstable and γ-stable spaces of the operator $\mathcal{A}$. Namely we set

$$y_u = P_N y, \quad y_s = (I - P_N)y$$

and, since by (3.28), $P_N F(t)y = 0$, we may rewrite system (3.31) as

$$\begin{cases} \dfrac{dy_u}{dt} + \mathcal{A}_u y_u + \dfrac{1}{2}\, P_N \displaystyle\sum_{j=1}^N \Gamma_j^2 y_u = 0, \quad t \ge \tau \\ y_u(\tau) = P_N x. \end{cases} \qquad (3.32)$$

and

$$\begin{cases} \dfrac{dy_s}{dt} + \mathcal{A}_s y_s + \dfrac{1}{2}\,(I-P_N)\displaystyle\sum_{j=1}^N \Gamma_j^2 y_u + (I-P_N)F(t)y_u = 0,\ t \ge \tau \\ y_s(\tau) = (I-P_N)x. \end{cases} \qquad (3.33)$$

We have $y = y_u + y_s$, $y_u = \sum_{j=1}^N y_j\varphi_j$ and by (3.10)

$$\mathcal{A}_u\varphi_j = \lambda_j\varphi_j, \quad j = 1, ..., N.$$

Recalling that in virtue of (3.24)

$$\Gamma_j^2 y = \eta\Gamma_j y = \eta^2\langle y, \varphi_j^*\rangle_{\widetilde{H}} P(m\phi_j),$$

we may rewrite (3.32) as

$$\begin{cases} \dfrac{dy_j}{dt} + \lambda_j y_j + \dfrac{1}{2}\,\eta^2 y_j \langle P(m\phi_j), \varphi_j^* \rangle_{\widetilde{H}} = 0, \quad t \ge \tau,\ j = 1, ..., N, \\ y_j(\tau) = \langle x, \varphi_j^* \rangle_{\widetilde{H}}. \end{cases}$$

Taking into account (3.16) it follows that

$$\begin{cases} \dfrac{dy_j}{dt} + \lambda_j y_j + \dfrac{1}{2}\,\eta^2 y_j = 0, \quad t \ge \tau,\ j = 1, ..., N, \\ y_j(\tau) = \langle x, \varphi_j^* \rangle_{\widetilde{H}}. \end{cases}$$

This yields,

$$y_j(t) = e^{-(\lambda_j + \frac{1}{2}\,\eta^2)t} \langle x, \varphi_j^* \rangle_{\widetilde{H}}, \quad j = 1, ..., N,\ t \ge 0.$$

Hence for $\eta^2 \ge 6\gamma - 2 \text{ Re } \lambda_j,\ j = 1, ..., N$, we have

$$|y_u(t)|_{\widetilde{H}} \le Ce^{-3\gamma(t-\tau)} |x|_{\widetilde{H}}, \quad \forall\, t \ge \tau. \tag{3.34}$$

Now coming back to system (3.33) we shall rewrite it as

$$\begin{cases} \dfrac{dy_s}{dt} + \mathcal{A}_s y_s + \dfrac{1}{2}\,\eta^2 \displaystyle\sum_{j=1}^{N} y_j (I - P_N) P(m\phi_j) \\ \qquad + \displaystyle\sum_{j=1}^{N} \left(e^{\beta_j(t)} - 1\right) y_j (I - P_N)(\mathcal{A}P(m\phi_j) - \lambda_j P(m\phi_j)) = 0,\ t \ge \tau \\ y_s(\tau) = (I - P_N)x. \end{cases} \tag{3.35}$$

Then, by (3.34) and (3.8) we have that

$$\begin{aligned} |y_s(t)|_{\widetilde{H}} &\le |e^{-\mathcal{A}_s(t-\tau)}(I - P_N)x|_{\widetilde{H}} \\ &+ \frac{1}{2}\,\eta^2 \int_\tau^t \sum_{j=1}^{N} \left(e^{\beta_j(s)} - 1\right) |e^{-\mathcal{A}_s(t-s)} y_j(s)(I - P_N) \\ &\times |P(m\phi_j) + \mathcal{A}P(m\phi_j) - \lambda_j P(m\phi_j)|_{\widetilde{H}} ds \\ &\le Ce^{-\gamma(t-\tau)}|x| + C\,\frac{\eta^2}{2}\,|x|_H \int_0^t \sum_{j=1}^{N} |e^{-3\gamma s} e^{-\gamma(t-s)} \zeta(s) ds \\ &\le Ce^{-\gamma(t-\tau)}(1 + \eta^2)|x|_H \int_\tau^t e^{-\gamma(\tau+2s)} \zeta(s) ds, \quad \forall\, t \ge 0,\ \mathbb{P}\text{-a.s.} \end{aligned}$$

for some constant C independent of x and $\omega \in \Omega$. This completes the proof of (3.30). □

Now, we fix η.

Lemma 3.6 *We have*

$$\int_\tau^\infty e^{\gamma(t-\tau)} |U(t, \tau)x|_W^2 dt \le C|x|_W^2 \left(1 + \int_\tau^\infty e^{-\gamma(\tau+2t)} \zeta(t) dt\right)^2, \ \forall\, x \in W, \tag{3.36}$$

where C is independent of $\omega \in \Omega,\ 0 \le \epsilon < \frac{1}{4}$.

Proof. We set

$$z(t) := e^{\frac{\gamma}{2}(t-\tau)}U(t,\tau)x, \quad 0 < \tau < t.$$

Then by Lemma 3.5 we have

$$\int_\tau^\infty |z(t)|^2_{\widetilde{H}}\, dt \le C|x|^2 \left(1 + \int_\tau^\infty e^{-\gamma(\tau+2t)}\zeta(t)dt\right)^2, \quad \forall\, x \in H$$

while

$$\frac{dz}{dt} + \nu_0 Az + A_0 z + \frac{1}{2}\sum_{j=1}^N \Gamma_j^2 z + F(t)z = \frac{\gamma}{2}\, z, \quad t \ge \tau.$$

Multiplying the latter by z and $A^{\frac{1}{2}+2\epsilon}z$ (scalarly in $\widetilde{H}$) we have the standard estimates for $d = 2, 3$

$$|\langle A_0 z, z\rangle| = |b(z, X_e, z)| \le C|z|_{\frac{1}{4}+\varepsilon}\, |X_e|_1\, |z|_{\widetilde{H}} \le C|z|_{\frac{1}{4}+\varepsilon}\, |z|_{\widetilde{H}}$$

and

$$\begin{aligned} |\langle A_0 z, A^{\frac{1}{2}+2\varepsilon}z\rangle| &= |b(z, X_e, A^{\frac{1}{2}+2\varepsilon}z)| + |b(X_e, z, A^{\frac{1}{2}+2\varepsilon}z)| \\ &\le C(|z|_{\frac{1}{4}}\, |X_e|_1\, |A^{\frac{1}{2}+2\varepsilon}z|_{\widetilde{H}} + |X_e|_1\, |z|_{\frac{1}{2}}\, |A^{\frac{1}{2}+2\varepsilon}z|_{\widetilde{H}}) \le C|z|^2_{\frac{1}{2}+2\varepsilon}. \end{aligned}$$

We get that

$$\frac{1}{2}\,\frac{d}{dt}\,|z(t)|^2_H + \nu_0|z(t)|^2_{\frac{1}{2}} \le C(|z(t)|_{\frac{1}{2}}|z(t)|_{\widetilde{H}} + |z(t)|^2_{\widetilde{H}}) + |\langle F(t)z, z(t)\rangle|$$

(here $|\cdot| = |\cdot|_{\widetilde{H}}$) and

$$\frac{1}{2}\,\frac{d}{dt}\,|z(t)|^2_{\frac{1}{4}+\epsilon} + \nu_0|z(t)|^2_{\frac{3}{4}+\epsilon} \le C(|z(t)||z(t)|_{\frac{1}{2}+2\varepsilon} + |z(t)|^2_{\frac{1}{2}+\varepsilon}) + \left|\langle F(t)z, A^{\frac{1}{2}+2\varepsilon}z(t)\rangle\right|.$$

This yields, via interpolatory inequality

$$|z(t)|_\alpha \le |z(t)|^{\frac{4\alpha}{3}}_{\frac{3}{4}}\, |z(t)|^{1-\frac{4\alpha}{3}}, \quad \text{for } \alpha = \frac{1}{4}, \frac{1}{2}$$

and since by (3.28) $|\langle F(t)z, A^{\frac{1}{2}}z(t)\rangle| \le C|z|$, we get

$$\frac{d}{dt}\,|z(t)|^2_{\frac{1}{4}+\epsilon} + |z(t)|^2_{\frac{3}{4}+\epsilon} \le C|z(t)|^2, \quad t > \tau$$

which yields

$$\int_\tau^\infty |z(t)|^2_{\frac{3}{4}+\epsilon}\, dt \le C|x|^2_{\frac{1}{4}+\epsilon}\left(1 + \int_\tau^\infty e^{-\gamma(\tau+2s)}\zeta(t)dt\right)^2,$$

which is just (3.30) for $d = 3$. The case $d = 2$ follows completely similarly by multiplying the equation by $A^{\frac{1}{2}}z$. (Here and everywhere in the following, C is a positive constant independent of ω.) □

We come back to (3.25) and set

$$G(t, y) := e^{-\sum_{j=1}^N \beta_j(t)\Gamma_j} B(e^{\sum_{j=1}^N \beta_j(t)\Gamma_j} y), \quad \forall\, y \in \widetilde{H},\ t \ge 0.$$

Recalling (3.22) and (3.27) we see that

$$\begin{aligned} B(e^{\beta_j(t)\Gamma_j}y) &= B(y) + \langle y, \varphi_j^* \rangle_{\widetilde{H}}^2 (e^{\eta\beta_j} - 1)^2 B(P(m\phi_j)) \\ &= (e^{\eta\beta_j(t)} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} [B_1(y, P(m\phi_j)) + B_2(y, P(m\phi_j))], \end{aligned} \tag{3.37}$$

where $B(y) = P((y \cdot \nabla)y)$ and

$$B_1(y,z) = P((y \cdot \nabla)z), \quad B_2(y,z) = P((z \cdot \nabla)y), \quad \forall\, y, z \in D(\mathcal{A}). \tag{3.38}$$

Then by (3.27),(3.28) and (3.37) we have for all $j, k = 1, ..., N$

$$\begin{aligned} e^{-\beta_k(t)\Gamma_k} B(e^{\beta_j(t)\Gamma_j}y) &= e^{-\beta_k(t)\Gamma_k}[B(y) + \langle y, \varphi_j^* \rangle_{\widetilde{H}}^2 (e^{\eta\beta_j} - 1)^2 B(P(m\phi_j)) \\ &+ (e^{\eta\beta_j(t)} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} [B_1(y, P(m\phi_j)) + B_2(y, P(m\phi_j))] \end{aligned}$$

But in virtue of (3.28) we have

$$e^{-\beta_k(t)\Gamma_k} y = (e^{-\eta\beta_k(t)} - 1)\langle y, \varphi_k^* \rangle_{\widetilde{H}} P(m\phi_k) + y$$

Therefore

$$\begin{aligned} e^{-\beta_k(t)\Gamma_k} B(e^{\beta_j(t)\Gamma_j}y) &= B(e^{\beta_j(t)\Gamma_j}y) + (e^{-\eta\beta_j(t)} - 1)\langle B(e^{\beta_j(t)\Gamma_j}y), \varphi_k^* \rangle_{\widetilde{H}} P(m\phi_k) \\ &= B(y) + \langle y, \varphi_j^* \rangle_{\widetilde{H}}^2 (e^{\eta\beta_j(t)} - 1)^2 B(P(m\phi_j)) \\ &+ (e^{\eta\beta_j(t)} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} [B_1(y, P(m\phi_j)) + B_2(y, P(m\phi_j))] \\ &+ (e^{-\eta\beta_j(t)} - 1)\langle B(e^{\beta_j(t)\Gamma_j}y), \varphi_k^* \rangle_{\widetilde{H}} P(m\phi_k). \end{aligned}$$

Taking into account that φ_j^*, φ_k^* are smooth we may write the previous relation as

$$e^{-\beta_k(t)\Gamma_k} B(e^{\beta_j(t)\Gamma_j}y) = B(y) + \Theta_{j,k}(t,y), \quad j,k = 1, ..., N. \tag{3.39}$$

where

$$\begin{aligned} |\Theta_{j,k}(t,y)|_\alpha &\le C(1+\delta(t))(|\langle y, \varphi_j^* \rangle_{\widetilde{H}}|^2 + |B_1(y, P(m\phi_j)|_\alpha^2 \\ &+ |B_2(P(m\phi_j), y)|_\alpha^2 + |\langle B(y), \varphi_j^* \rangle_{\widetilde{H}}|), \\ &\qquad \forall\, t \ge 0,\ y \in D(\mathcal{A}),\ j,k = 1, ..., N, \end{aligned} \tag{3.40}$$

where $0 < \alpha < 1$ (recall that $|x|_\alpha = |A^\alpha x|$) and

$$\delta(t) = \sup_{1 \le j \le N} \max\{e^{-4\eta\beta_j(t)}, e^{4\eta\beta_j(t)}\}. \tag{3.41}$$

To conclude, we have by (3.37)–(3.41) that

$$G(t,y) = B(y) + \Theta(t,y), \quad \forall\, t \ge 0,\ y \in D(\mathcal{A}). \tag{3.42}$$

Here for each $\alpha \in (0,1)$

$$\begin{aligned} &|\Theta(t,y)|_\alpha \le C(1+\delta^N(t)) \\ &\times(\max_{1 \le j \le N}\{|B_1(y, P(m\phi_j)|_\alpha^2 + |B_2(P(m\phi_j), y)|_\alpha^2\} + |B(y)|_{\widetilde{H}}), \end{aligned} \tag{3.43}$$

where δ is given by (3.41) and C is independent of t, y and ω.

We write (3.25) as

$$\frac{dy(t)}{dt} + \mathcal{A}_\Gamma y(t) + G(t, y(t)) + F(t)y(t) = 0, \quad \forall\, t \geq 0,\ \mathbb{P}\text{-a.s.}.$$

We set $z(t) = e^{\frac{1}{2}\gamma t} y(t)$ and rewrite it as

$$\begin{cases} \dfrac{dz(t)}{dt} + (\mathcal{A}_\Gamma - \dfrac{1}{2}\gamma) z(t) + e^{-\frac{\gamma}{2}t} G(t, z(t)) + F(t)z(t) = 0 \\ z(0) = x. \end{cases} \tag{3.44}$$

Equivalently,

$$z(t) = S(t,0)x - \int_0^t S(t,s) e^{-\frac{\gamma}{2}s} G(s, z(s))ds, \quad \forall\, t \geq 0, \tag{3.45}$$

where

$$S(t,\tau) = U(t,\tau) e^{-\frac{1}{2}\gamma(t-\tau)}.$$

We have seen earlier in Lemma 3.5 that $S(t,\tau)$ is exponentially stable in H.

Lemma 3.7 *There is $\Omega_x \subset \Omega$, with*

$$\mathbb{P}(\Omega_x) \geq 1 - \left(C^* |x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{8(\eta N)^2}},$$

with $C^* > 0$ independent of ω and x such that for each $x \in X$ with $|x|_W \leq C^*$ equation (3.45) *has a unique solution*

$$z \in C([0,\infty); W)) \cap L^2(0,\infty; Z).$$

Here $W = D(A^{\frac{1}{4}})$, $Z = D(A^{\frac{3}{4}})$ if $d = 2$ and $W = D(A^{\frac{1}{4}+\epsilon})$, $Z = D(A^{\frac{3}{4}+\epsilon})$ if $d = 3$.

Proof. We shall proceed as in the proof of [6, Theorem 5.1]. Namely, we rewrite (3.45) as

$$z(t) = S(t,0)x + \mathcal{N}z(t) := \Lambda z(t), \quad t \geq 0,$$

where $\mathcal{N} : L^2(0,\infty; Z)$ is the integral operator

$$\mathcal{N}z(t) = -\int_0^t S(t,s) e^{-\frac{\gamma}{2}s} G(s, z(s))ds.$$

We shall prove first the following estimate

$$|\mathcal{N}z|_{L^2(0,\infty;Z)} \leq C \int_0^\infty e^{-\frac{\gamma}{2}t} |G(t, z(t))|_W dt. \tag{3.46}$$

Indeed, for any $\zeta \in L^2(0,\infty; Z')$ (Z' is the dual of Z), we have via Fubini's theorem

$$\begin{aligned} \int_0^\infty \langle \mathcal{N}z(t), \zeta(t)\rangle dt &= \int_0^\infty dt \left\langle \int_0^t S(t,s) e^{-\frac{\gamma}{2}s} G(s, z(s))ds, \zeta(t) \right\rangle \\ &\leq \int_0^\infty dt \int_0^t |S(t,s) e^{-\frac{\gamma}{2}s} G(s, z(s))|_Z ds\, |\zeta(t)|_{Z'} \\ &= \int_0^\infty d\tau \int_\tau^\infty |S(t,\tau) e^{-\frac{\gamma}{2}\tau} G(\tau, z(\tau))|_Z\, |\zeta(t)|_{Z'} dt \\ &\leq \int_0^\infty d\tau \left(\int_\tau^\infty |S(t,\tau) e^{-\frac{\gamma}{2}\tau} G(\tau, z(\tau))|_Z^2\, dt \right)^{\frac{1}{2}} |\zeta|_{L^2(0,\infty;Z')}. \end{aligned}$$

Now we set

$$I := \int_0^\infty d\tau \left(\int_\tau^\infty |S(t,\tau)e^{-\frac{\gamma}{2}\tau}G(\tau,z(\tau))|_Z^2\,dt \right)^{\frac{1}{2}}.$$

By Lemma 3.6, we have

$$\int_\tau^\infty |S(t,\tau)x|_{\frac{3}{4}}^2 dt \le C|x|_W^2 \left(1 + \int_\tau^\infty e^{-\gamma(\tau+2t)}\zeta(t)dt\right)^2, \quad \forall\, x \in W.$$

Next, we apply this for $x = e^{-\frac{\gamma}{2}\tau}G(\tau,z(\tau))$ and get

$$\begin{aligned}&\int_\tau^\infty |S(t-\tau)e^{-\frac{\gamma}{2}\tau}G(\tau,z(\tau))|_Z^2 dt \\ &\qquad \le C|G(\tau,z(\tau))|_W^2 e^{-\gamma\tau}\left(1 + \int_\tau^\infty e^{-\gamma(\tau+2t)}\zeta(t)dt\right)^2, \quad \forall\, x \in W\end{aligned}$$

and, therefore,

$$I \le C\int_0^\infty |G(\tau,z(\tau))|_W e^{-\frac{\gamma}{2}\tau}d\tau \left(1 + \int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)^2,$$

as claimed.

Next, by (3.46) and Lemma 3.6, we have

$$\begin{aligned}&|\Lambda z|_{L^2(0,\infty;Z)} \\ &\le C\left(|x|_W + \left(1 + \int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)\int_0^\infty e^{-\frac{\gamma}{2}\tau}|G(\tau,z(\tau))|_W\,d\tau\right).\end{aligned} \tag{3.47}$$

On the other hand, by (3.42), (3.43), we have

$$|G(t,y)|_W \le |By|_W + |\Theta(t,y)|_W.$$

By [6, Lemma 5.4], we deduce also that

$$|By|_W \le C|y|_Z^2, \quad \forall\, y \in Z$$

and, similarly, by (3.40) we have

$$|\Theta(t,y)|_W \le C(1+\delta^N(t))|y|_W^2, \quad \forall\, y \in Z.$$

Then, (3.47) yields

$$|\Lambda z|_{L^2(0,\infty;Z)} \le C_1^*\left(|x|_W + \int_0^\infty (1+\delta^N(t))e^{-\frac{\gamma}{2}t}|z(t)|_Z^2 dt\right), \quad \mathbb{P}\text{-a.s.}, \tag{3.48}$$

where C_1^* is a positive constant independent of ω. By (3.41), we have

$$\sup_{t\ge 0}(1+\delta^N(t)(\omega))e^{-\frac{\gamma}{2}t} = 1 + \sup_{t\ge 0}\max_{0\le j\le N}\{e^{4\eta N\beta_j(t)-\frac{\gamma}{2}t}\} = 1 + \mu(\omega),\ \omega \in \Omega. \tag{3.49}$$

Similarly, we have

$$\int_0^\infty e^{-2\gamma t}\zeta(t)dt \le \frac{1}{\gamma}\sup_{1\le j\le N}\sup_{t\ge 0} e^{\beta_j(t)-\frac{\gamma}{2}t} \le \frac{1}{\gamma}\,\mu(\omega).$$

So, (3.48) yields

$$|\Lambda z|_{L^2(0,\infty;Z)} \le C_1^*\left(|x|_W + (1+\mu(\omega)^2)|z|^2_{L^2(0,\infty;Z)}\right), \quad \mathbb{P}\text{-a.s.} \tag{3.50}$$

In order to estimate the right hand side of (3.50) we need the following lemma.

Lemma 3.8 *Let $\beta(t),\ t\ge 0$ be a real Brownian motion in some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then, for each $\lambda>0$, we have*

$$\begin{aligned}\mathbb{P}\big(\sup_{t>0} e^{\beta(t)-\lambda t}\ge r\big) &= \mathbb{P}\big(e^{\sup_{t>0}(\beta(t)-\lambda t)}\ge r\big)\\ &= \mathbb{P}\big(\sup_{s>0}(\beta(s)-\lambda s)\ge \log r\big) = r^{-2\lambda}.\end{aligned} \tag{3.51}$$

Proof. Fix $T>0$. By Girsanov's theorem, $\widetilde{\beta}(t) := \beta(t)-\lambda t,\ t\le T$ is a Brownian motion in $(\Omega,\mathcal{F},\widetilde{\mathbb{P}})$, where

$$d\widetilde{\mathbb{P}} = e^{\lambda\beta(T)-\frac{1}{2}\lambda^2 T}d\mathbb{P}.$$

We have

$$\mathbb{P}\left(\sup_{0\le t\le T} e^{\beta(t)-\lambda t}\ge r\right) = \mathbb{P}\left(\sup_{0\le t\le T} e^{\widetilde{\beta}(t)}\ge r\right).$$

Setting $M_T = \sup_{0\le t\le T} e^{\widetilde{\beta}(t)}$ we have

$$\mathbb{P}(M_T\ge r) = \int_\Omega 1\!\!1_{[r,+\infty)}(M_T)d\mathbb{P} = \int_\Omega 1\!\!1_{[r,+\infty)}(M_T)e^{-\lambda\beta(T)+\frac{1}{2}\lambda^2 T}d\widetilde{\mathbb{P}}.$$

Replacing in the latter identity $\beta(t)$ by $\widetilde{\beta}(t)+\lambda t$ yields

$$\mathbb{P}(M_T\ge r) = \int_\Omega 1\!\!1_{[r,+\infty)}(M_T)e^{-\lambda\widetilde{\beta}(T)-\frac{1}{2}\lambda^2 T}d\widetilde{\mathbb{P}}.$$

Because $\widetilde{\beta}$ is a Brownian motion with respect to $\widetilde{\mathbb{P}}$ we can compute the integral above by using the well known expression of the law of $(M_t,\widetilde{\beta}(t))$, see e.g. [18, (8.2) page 9]. We obtain that

$$\mathbb{P}\big(M_T\ge r\big) = \frac{2}{\sqrt{2\pi T^3}}\int_r^\infty db\int_{-\infty}^b (b-a)e^{-\lambda a-\frac{1}{2}\lambda^2 T}\ \ e^{-\frac{(2b-a)^2}{2T}}da.$$

It follows that

$$\mathbb{P}(M_T\ge r) = \frac{1}{2}\,e^{-2\lambda r}\mathrm{Erfc}\left(\frac{r-\lambda T}{\sqrt{2T}}\right) + \frac{1}{2}\,e^{2\lambda r}\mathrm{Erfc}\left(\frac{r+\lambda T}{\sqrt{2T}}\right),$$

where

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{+\infty} e^{-t^2}dt.$$

For $T \to \infty$ we obtain (3.51). $\square$

Proof of Lemma 3.7 (continued). By (3.51), it follows that

$$\mathbb{P}\left(\sup_{t\geq 0} e^{4\eta N\beta_j(t)-\frac{\gamma}{2}\,t} \leq r\right) \geq 1 - r^{-\frac{\gamma}{8(N\eta)^2}}, \quad j=1,...,N, \tag{3.52}$$

and, therefore, by (3.49),

$$\mathbb{P}(1+\mu \leq r) \geq 1 - (r-1)^{-\frac{\gamma}{8(N\eta)^2}}, \quad \forall\, r \geq 1. \tag{3.53}$$

We set

$$\mathcal{U}(\omega) := \{z \in L^2(0,\infty;Z) : \ |z|_{L^2(0,\infty;Z)} \leq R(\omega)\},$$

where $R : \Omega \to \mathbb{R}^+$ is a random variable such that

$$\begin{aligned}&\frac{2C_1^*|x|_W}{1+\sqrt{1-4(C_1^*)^2|x|_W(1+\mu)^2}}\\ &\qquad \leq R(\omega) \leq \frac{2C_1^*|x|_W}{1-\sqrt{1-4(C_1^*)^2|x|_W(1+\mu)^2}}, \quad \omega\in\Omega.\end{aligned} \tag{3.54}$$

Then, as easily follows from (3.50) and (3.54) for

$$|x|_W \leq \rho_1(\omega) := [8(1+\mu(\omega)^2)(C_1^*)^2]^{-1}, \tag{3.55}$$

we have

$$\Lambda\mathcal{U}(\omega) \subset \mathcal{U}(\omega).$$

Now, we shall apply the Banach fixed point theorem to Λ on the set $\mathcal{U}(\omega)$. Let $z_1, z_2 \in \mathcal{U}(\omega)$. Arguing as in the proof of (3.50), we find that

$$\begin{aligned}&|\mathcal{N}z_1 - \mathcal{N}z_2|_{L^2(0,\infty;Z)} \leq C_1^*\int_0^\infty e^{-\gamma t}|G(t,z_1)-G(t,z_2)|_{\frac14}\,dt\left(1+\int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)\\
&\leq C_1^*C_2^*\int_0^\infty (1+\delta(t))e^{-\gamma t}|z_1(t)-z_2(t)|_Z(|z_1(t)|_Z+|z_2(t)|_Z)dt\left(1+\int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)\\
&\leq C_1^*C_2^*\left(\int_0^\infty |z_1(t)-z_2(t)|_Z^2dt\right)^{\frac12}\left(\int_0^\infty e^{-\gamma t}(|z_1(t)|_Z^2+|z_2(t)|_Z^2)dt\right)^{\frac12}(1+\mu(\omega))^2\\
&\leq 2C_1^*C_2^*(1+\mu(\omega))^2R(\omega)|z_1-z_2|_{L^2(0,\infty;Z)},\end{aligned}$$

where C_1^*, C_2^* are independent of ω.

Now, if we choose x such that, besides (3.55), to have also

$$|x|_W \leq \frac{\sqrt{2}+1}{2\sqrt{2}(C_1^*)^2C_2^*(1+\mu)^2} =: \rho_2(\omega),$$

we see that there is $R = R(\omega)$ satisfying (3.54) and such that

$$2C_1^*C_2^*(1+\mu)^2R < 1.$$

Now, we take

$$|x|_W \le \rho(\omega) := \min\{\rho_1(\omega), \rho_2(\omega)\} = ((C^*)^2(1+\mu)^2)^{-1}, \tag{3.56}$$

where C^* is a suitable chosen constant independent of ω. Then, for x satisfying (3.56), $\mathcal{N}$ is a contraction on $\mathcal{U}(\omega)$ and maps $\mathcal{U}(\omega)$ on itself.

We set

$$\Omega_x = \{\omega \in \Omega : \ |x|_W \le \rho(\omega)\}. \tag{3.57}$$

Hence, for each $\omega \in \Omega_x$, equation (3.45) has a unique solution z satisfying conditions in Lemma 3.7. On the other hand, by (3.53) and (3.57), we see that

$$\mathbb{P}(\Omega_x) \ge 1 - \left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{8(\eta N)^2}},$$

as claimed. □

Lemma 3.9 *Let z be the solution to* (3.44) *given by Lemma* 3.7*. Then*

$$\lim_{t\to\infty} |z(t)|_{\widetilde{H}} = 0, \quad \mathbb{P}\text{-a.s. in } \Omega_x. \tag{3.58}$$

Proof. By (3.44), it follows as in the proof of Lemma 3.6 that

$$\frac{1}{2}\,\frac{d}{dt}\,|z(t)|^2_{\widetilde{H}} + \frac{\nu_0}{2}\,|z(t)|^2_{\frac{1}{2}} \le C_1|z(t)|^2_{\widetilde{H}} + e^{-\gamma t}\,|\langle G(t,z(t)),z(t)\rangle + \langle F(t,z(t)),z(t)\rangle|\,.$$

Taking into account that

$$|e^{-\gamma t}\langle G(t,z(t)),z(t)\rangle| = e^{-\gamma t}|\langle \Theta(t,z(t)),z(t)\rangle| \le C_2|z(t)|^2_Z$$

and that $z \in L^2(0,\infty;D(A^{\frac{3}{4}}))$, we infer that

$$\frac{d}{dt}\,|z(t)|^2_{\widetilde{H}} \in L^\infty(0,\infty),$$

and, together with $z \in L^2(0,\infty;\widetilde{H})$), this implies (3.58) as claimed. □

Proof of Theorem 3.2 (continued). By Lemma 3.9 we have that

$$\lim_{t\to\infty} |y(t)|_{\widetilde{H}} e^{\frac{1}{2}\gamma t} = 0, \quad \forall\, \omega \in \Omega_x. \tag{3.59}$$

Then, as seen earlier,

$$X(t) = \prod_{j=1}^N e^{\beta_j(t)\Gamma_j}\, y(t), \quad \mathbb{P}\text{-a.s.}$$

is the solution to (3.13). Then,by (3.27) and (3.28), we see that

$$|X(t)|_{\widetilde{H}}\, e^{\frac{\gamma t}{4}} \le C_1^*\left(1 + \max_{1\le j\le N}\left\{e^{N\eta\beta_j(t)-\frac{\gamma t}{4}}, e^{-N\eta\beta_j(t)-\frac{\gamma t}{4}}\right\}\right)|y(t)|_{\widetilde{H}}\, e^{\frac{\gamma t}{2}}. \tag{3.60}$$

We set

$$\Omega_x^r = \left\{\omega \in \Omega : \ \sup_{t\ge 0}\max_{1\le j\le N}\left\{e^{N\eta\beta_j(t)-\frac{\gamma t}{4}}, e^{-N\eta\beta_j(t)-\frac{\gamma t}{4}}\right\} \le r\right\},$$

where $r > 0$. By Lemma 3.8 (see (3.52)), we have

$$\mathbb{P}(\Omega_x^r) \geq 1 - r^{-\frac{\gamma}{2(\eta N)^2}}. \tag{3.61}$$

This yields

$$\mathbb{P}(\Omega_x \cap \Omega_x^r) \geq 1 - \left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{2(\eta N)^2}} - r^{-\frac{\gamma}{2(\eta N)^2}}, \tag{3.62}$$

for any $r > 0$. We set $\Omega_x^* = \Omega_x \cap \Omega_x^r$, where

$$r = \left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{\frac{1}{4}}$$

and, by (3.61), (3.62), we get (**??**) and

$$\lim_{t\to\infty} |X(t)|_{\widetilde{H}} e^{\frac{\gamma t}{4}} = 0 \quad \mathbb{P}\text{-a.s. in } \Omega_x^*.$$

This completes the proof of Theorem 3.2. □

3.4 Final remarks

3.4.1 Stochastic stabilization versus deterministic stabilization

By the same proofs as that of Theorem 3.2, it follows that the deterministic feedback controller

$$u = -\eta \sum_{j=1}^{N} \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j), \tag{3.63}$$

where η is sufficiently large, stabilizes exponentially system (3.4) in a neighborhood $\{x \in H : |x|_{\frac{1}{4}} < \rho\}$. Here ϕ_j are chosen as in (3.16). Apparently the feedback controller (3.63) is simpler than its stochastic counterpart (3.6) above while the stabilization performances are comparable. It should be said, however, that the controller (3.63) though stabilizable is not robust while the stochastic one designed here is. In fact, it is easily seen that (3.63) is very sensitive to structural perturbations in system (3.1) because small variations of the spectral system $\{\varphi_j^*\}$ might break the orthogonality condition (3.16) from which ϕ_j are determined. In this way, the deterministic linear closed loop equation

$$dX + AXdt = -\eta \sum_{j=1}^{N} \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j)dt$$

might became unstable even for $\eta > 0$ and large. On the contrary, this does not happen for the stochastic system

$$dX + AXdt = -\eta \sum_{j=1}^{N} \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j)d\beta_j, \tag{3.64}$$

because its unstable part, that is, $X = \sum_{j=1}^{N} X_j \phi_j$, where

$$dX_j + \lambda_j X_j dt = -\eta \sum_{j=1}^{N} X_j \langle \phi_j, \varphi_j^* \rangle_0 P(m\phi_j) d\beta_j, \ \mathrm{Re}\ \lambda_j \leq \gamma, \ j = 1, ..., N, \tag{3.65}$$

still remains exponentially stable with probability one to small perturbations of $\{\varphi_j^*\}$. Indeed, in this case, instead of (3.16), we have

$$|\langle \phi_j, \varphi_k^* \rangle - \delta_{kj}| \leq \epsilon, \quad \forall\ j,k = 1, ..., N$$

and, therefore,

$$\sum_{j=1}^{N} \sum_{i=1}^{N} |\langle \phi_j, \varphi_j^* \rangle_0|^2 \, |X_j|^2 \geq \mu \sum_{j=1}^{N} |X_j|^2$$

which, as seen earlier in [2], implies the stability of (3.65) for sufficiently large $|\eta|$.

As mentioned in Introduction, one might design, starting from (3.63), a robust stabilizable controller via infinite dimensional Riccati equations associated with the linear system but this involves, however, hard numerical computation.

3.4.2 Giving up to assumption (H1)

One might design a feedback stochastic feedback controller of the above form in absence of assumption (H1).

Indeed, if we replace $\{\varphi_j\}_1^N$ by its Schmidt's orthogonalization $\{\widetilde{\varphi}_j\}_1^N$, we still have $\mathcal{X}_u =$ lin span $\{\widetilde{\varphi}_j\}_1^N$ and $\mathcal{X}_s =$ lin span $\{\widetilde{\varphi}_j\}_{N+1}^{\infty}$.

Consider the feedback controller

$$u = \eta \sum_{j=1}^{N} \langle X, \phi_j^* \rangle_{\widetilde{H}} P(m\widetilde{\Phi}_j) \dot{\beta}_j \tag{3.66}$$

where $\{\widetilde{\Phi}_j\}$ are determined by

$$\langle \widetilde{\Phi}_j, \widetilde{\varphi}_k \rangle_0 = \delta_{kj}, \quad j,k = 1, ..., N. \tag{3.67}$$

By Lemma 3.1, it follows that system $\{\widetilde{\varphi}_j\}_1^N$, is independent on $\mathcal{O}_0$ and so such a system $\{\widetilde{\Phi}_j\}_1^N$, always exists. Then, the proof of Theorem 3.2 applies with minor modifications to show that the controller u defined by (3.66) is exponentially stabilizable in the sense of Theorem 3.2. The details are omitted.

3.5 Appendix. Proof of Lemma 3.1

Consider the Stokes–Oseen operator

$$\mathcal{L}\varphi = -\nu_0 \Delta\varphi + (y_e \cdot \nabla)\varphi + (\varphi \cdot \nabla) y_e, \quad \text{in } \mathcal{O}$$

and mention first the following unique continuation result.

Lemma 3.10 *Assume $y_e \in C^2(\overline{\mathcal{O}})$ and let $\varphi \in C^2(\overline{\mathcal{O}})$ be the solution to the problem*

$$\begin{cases} \mathcal{L}\varphi = \lambda\varphi + \nabla p, \quad \text{in } \mathcal{O}, \\ \\ \nabla \cdot \varphi = 0, \quad \text{in } \mathcal{O}, \ \ \varphi = 0, \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{3.68}$$

such that $\varphi \equiv \nabla q$ on $\mathcal{O}_0$ where $q \in C^1(\overline{\mathcal{O}})$ and $\mathcal{O}_0$ is an open subset of $\mathcal{O}$. Then $\varphi \equiv 0$.

A simple proof of Lemma 3.1 for $d = 2$ can be given by reducing (3.68), via the vorticity transformation $\psi = \text{curl } \varphi = D_2\varphi_1 - D_1\varphi_2$, to

$$-\nu_0\Delta\psi + y_e \cdot \nabla\psi + \varphi \cdot \nabla(\text{curl } y_e) = \lambda\psi, \quad \text{in } \mathcal{O}$$

and via the stream function ϕ to

$$-\nu_0\Delta^2\phi + y_e \cdot \nabla\phi + \nabla^\perp\phi \cdot \Delta y_e - \lambda\Delta\phi = 0, \quad \text{in } \mathcal{O}. \tag{3.69}$$

(Here $\varphi = \nabla^\perp\phi = \{D_2\phi, -D_1\phi\}$.)

Then, if $\varphi \equiv \nabla q$ in $\mathcal{O}_0$, it follows that $\Delta\phi = 0$ in $\mathcal{O}_0$, which implies that $\Delta\phi = 0$ in $\mathcal{O}$.

To prove this (we are indebted to D. Tataru for suggesting us this simple device), we set $P(x, D)u = -\nu_0\Delta u + y_e \cdot \nabla u - \lambda u$ and we write (3.69) as

$$P(x, D)u = -\Delta y_e \cdot \nabla^\perp\phi, \ \ u = \Delta\phi.$$

Then, we apply the Carleman inequality (see [17, Theorem 8.3.1])

$$\sum_{|\alpha|\le 2} \tau^{2(2-|\alpha|)} \int |D^\alpha u|^2 e^{2\tau\chi} dx \le K\tau \int |P(x, D)u|^2 e^{2\tau\chi}, \ \forall u \in C_0^\infty(\mathcal{O}), \ \tau > 0,$$

where χ is a smooth function such that $\nabla\chi(x_0) \neq 0$, $x_0 \in \partial\mathcal{O}_0$ and the surface $\{x;\ \chi(x) = \chi(x_0)\}$ is strongly pseudoconvex in x_0. Then, arguing as in the proof of Theorem 8.9.1 in [17], it follows that $\Delta\phi \equiv 0$ in $\mathcal{O}$, which implies that $\psi = \text{curl}\,\varphi = 0$ in $\mathcal{O}$. Hence, $\Delta\varphi_1 = \Delta\varphi_2 = 0$ in $\mathcal{O}$ and so $\varphi \equiv 0$ in $\mathcal{O}$.

The case $d = 3$ follows in a similar way by reducing (3.68) to a fourth-order equation of the form (3.69) via the transformation $\psi = \text{curl}\,\varphi = \nabla \times \varphi$.

Let $\{\varphi_j\}_{j=1}^N$ be eigenfunctions corresponding to eigenvalues λ_j, i.e.,

$$\begin{cases} \mathcal{L}\varphi_j = \lambda_j\varphi_j + \nabla p_j, \quad \text{in } \mathcal{O}, \\ \nabla \cdot \varphi_j = 0, \quad \text{in } \mathcal{O}, \\ \varphi_j = 0, \quad \text{on } \partial\mathcal{O}. \end{cases} \tag{3.70}$$

One must prove that each system $\{\varphi_1, ..., \varphi_m\}$, $1 \le m \le N$, is linearly independent in $\mathcal{O}_0$. As mentioned earlier this is immediate if all φ_j are eigenfunctions corresponding to the same eigenvalue λ_j and so, it suffices to prove this for distinct eigenvalues λ_j. For $m = 1$ this follows by Lemma 3.10. Let $m = 2$ and let φ_1, φ_2 two eigenfunctions with corresponding eigenvalues λ_1, λ_2. Assume that $\alpha_1\varphi_1 + \alpha_2\varphi_2 \equiv 0$ on $\mathcal{O}_0$ for $\alpha_1, \alpha_2 \neq 0$ and argue from this to a contradiction. We have

$$\mathcal{L}(\lambda_2\varphi_1 - \lambda_1\varphi_2) = \lambda_1\lambda_2(\varphi_1 - \varphi_2) + \lambda_2\nabla p_1 - \lambda_1\nabla p_2 = \nabla p, \quad \text{in } \mathcal{O}. \tag{3.71}$$

Replacing φ_1 by $\frac{\alpha_1}{\lambda_2}\varphi_1$ and φ_2 by $-\frac{\alpha_2}{\lambda_1}\varphi_2$, we see that $\lambda_2\varphi_1 - \lambda_1\varphi_2 \equiv 0$ in $\mathcal{O}_0$ and so, by (3.71) we see that $\varphi_1 = \alpha\nabla p$ in $\mathcal{O}_0$ for some α. Then, by Lemma 3.10, we infer that $\varphi_1 \equiv 0$ in $\mathcal{O}$ which is, of course, absurd. We shall treat now the case $m = 3$. We have as above, besides (3.71), that

$$\mathcal{L}(\lambda_3\varphi_1 - \lambda_1\varphi_3) = \lambda_1\lambda_3(\varphi_1 - \varphi_3) + \nabla q, \quad \text{in } \mathcal{O}$$

and therefore

$$\begin{aligned}\mathcal{L}((\lambda_2 - \lambda_3)\varphi_1 - \lambda_1\varphi_2 + \lambda_1\varphi_3)& \\ = \lambda_1\lambda_2(\varphi_1 - \varphi_2) - \lambda_1\lambda_3(\varphi_1 - \varphi_3) + \nabla q,& \quad \text{in } \mathcal{O}.\end{aligned} \tag{3.72}$$

If $\alpha_1\varphi_1 + \alpha_2\varphi_2 + +\alpha_3\varphi_3 \equiv 0$ in $\mathcal{O}_0$, then replacing $\varphi_1, \varphi_2, \varphi_3$ by $\frac{\alpha_1}{\lambda_2-\lambda_3}\varphi_1$, $-\frac{\alpha_2}{\lambda_1}\varphi_2$, $\frac{\alpha_3}{\lambda_1}\varphi_3$ respectively, we obtain that

$$(\lambda_2 - \lambda_3)\varphi_1 - \lambda_1\varphi_2 + \lambda_1\varphi_3 \equiv 0, \quad \text{in } \mathcal{O}_0,$$

which, in virtue of (3.72) and Lemma 3.10, implies

$$(\lambda_2 - \lambda_3)\varphi_1 - \lambda_2\varphi_2 + \lambda_3\varphi_3 \equiv \nabla q, \quad \text{in } \mathcal{O}_0.$$

This yields $\widetilde{\alpha}_1\varphi_1 + \widetilde{\alpha}_2\varphi_2 = \nabla q$ in $\mathcal{O}_0$ for $\overline{\alpha}_1, \overline{\alpha}_2 \neq 0$, which, in virtue of the previous step, is once again absurd. The argument works for all $m \in \mathbb{N}$ and this concludes the proof of Lemma 3.1. □

References

[1] V. Barbu, *Feedback stabilization of Navier-Stokes equations*, Control, Optimization and Calculus of Variations, 9, 197–206, 2003.

[2] V. Barbu, *The internal stabilization by noise of the linearized Navier-Stokes equations*, ESAIM Control, Optimization and Calculus of Variations, vol. 17, 2011, 117-130.

[3] V. Barbu, *Stabilization of Navier–Stokes Flows*, Communications and Control Engineering, Springer-Verlag, 2010.

[4] V. Barbu, R. Triggiani, *Internal stabilization of Navier-Stokes equations with finite dimensional controllers,* Indiana Univ. Math. J., vol. 53, 1443–1494, 2004.

[5] V. Barbu, I. Lasiecka, R. Triggiani, *Boundary stabilization of Navier-Stokes equations,* Memoirs Amer. Math. Soc., 2006.

[6] V. Barbu, I. Lasiecka, R. Triggiani, *Abstract setting for tangential boundary stabilization of Navier-Stokes equations by high and low-gain feedback controllers,* Nonlinear Analysis, 64, 2704–2746, 2006.

[7] T. Caraballo, H. Crauel, J. Langa, J. Robinson, *The effect of noise on the Chafee-Infante equation: a nonlinear case study.* Proc. Amer. Math. Soc. 135 (2007), no. 2, 373–382 (electronic), 2007.

[8] T. Caraballo, J. Langa, T. Taniguchi, *The exponential behaviour and stabilizability of stochastic 2D-Navier-Stokes equations.* J. Differential Equations 179 (2002), no. 2, 714–737, 2002.

[9] T. Caraballo, K. Liu, X. Mao, *On stabilization of partial differential equations by noise,* Nagoya Math. J., vol. 101, 155-170, 2001.

[10] T. Caraballo, J. Robinson, *Stabilisation of linear PDEs by Stratonovich noise.* Systems Control Lett. 53, no. 1, 41–50, 2004.

[11] S. Cerrai, *Stabilization by noise for a class of stochastic reaction-diffusion equations*, Probab. Theory Rel. Fields, 133, 190-214, 2000.

[12] G. Da Prato, *An Introduction to Infinite Dimensional Analysis*, Springer-Verlag, Berlin, 2006.

[13] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions,* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.

[14] G. Da Prato, J. Zabczyk, *Ergodicity for Infinite Dimensional Systems,* London Mathematical Society Lecture Notes, n.229, Cambridge University Press, 1996.

[15] A. Fursikov, *Real processes of the* $3-D$ *Navier-Stokes systems and its feedback stabilization from the boundary,* M.S. Agranovic and M.A. Shubin (eds.), AMS Translations, Partial Differential Equations, M. Vîshnik Seminar, vol. 206, 95-123, 2002.

[16] A. Fursikov, *Stabilization for the* $3-D$ *Navier-Stokes systems by feedback boundary control,* Discrete and Continuous Dynamical Systems, 10, 289-314, 2004.

[17] L. Hörmander, *Linear partial differential operators.* Springer Verlag, Berlin-New York, 1976.

[18] I. Karatzas, S. Shreve, *Brownian motion and stochastic calculus.* Second edition. Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 1991.

[19] T. Kato, *Perturbation Theory of Linear Operators*, Springer-Verlag, New York, Berlin, 1966.

[20] J.P. Raymond, Feedback boundary stabilization of the two dimensional Navier-Stokes equations, *SIAM J. Control Optimiz.*, vol. 45, 3 (2006), 790-828.

[21] J.P. Raymond, Feedback boundary stabilization of the three dimensional incompressible Navier-Stokes equations, *J. Math. Pures et Appl.*, vol. 87 (2007), 627-669.

4 Lecture 4. The internal stabilization of stochastic parabolic equations with linearly multiplicative Gaussian noise

4.1 Introduction

Consider the stochastic nonlinear controlled parabolic equation

$$\begin{aligned}
&dX(t) - \Delta X(t)dt + a(t,\xi)X(t)dt + b(t,\xi)\cdot\nabla_\xi X(t)dt \\
&\quad + f(X(t))dt = X(t)dW(t) + \mathbb{1}_{\mathcal{O}_0}u(t)dt \text{ in } (0,\infty)\times\mathcal{O}, \\
&X = 0 \text{ on } (0,\infty)\times\partial\mathcal{O}, \quad X(0) = x \text{ in } \mathcal{O}.
\end{aligned} \tag{4.1}$$

Here, $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\mathcal{O}$ and $W(t)$ is a Wiener process of the form

$$W(t) = \sum_{k=1}^{\infty} \mu_k e_k(\xi)\beta_k(t), \quad t \geq 0,\ \xi \in \mathcal{O}, \tag{4.2}$$

where μ_k are real numbers, $\{e_k\} \subset C^2(\overline{\mathcal{O}})$ is an orthonormal system in $L^2(\mathcal{O})$ and $\{\beta_k\}_{k=1}^\infty$ are independent Brownian motions in a stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$.

We assume throughout this work that

$$\sum_{k=1}^{\infty} \mu_k^2 |e_k|_\infty^2 < \infty, \tag{4.3}$$

where $|\cdot|_\infty$ denotes the $L^\infty(\mathcal{O})$-norm.

The function $a : [0,\infty)\times\mathcal{O} \to \mathbb{R}$, $b : [0,\infty)\times\mathcal{O} \to \mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ are assumed to satisfy

$$a \in C([0,\infty)\times\overline{\mathcal{O}}),\ b \in C^1([0,\infty)\times\overline{\mathcal{O}}) \tag{4.4}$$

$$\sup\{|a(t)|_\infty + |\nabla b(t)|_\infty;\ t \geq 0\} < \infty \tag{4.5}$$

$$f \in \mathrm{Lip}(\mathbb{R}),\ f(0) = 0. \tag{4.6}$$

Finally, $\mathcal{O}_0$ is an open subdomain of $\mathcal{O}$ with smooth boundary, $\mathbb{1}_{\mathcal{O}_0}$ is its characteristic function and $u = u(t,\xi)$ is an adapted controller with respect to the filtration $\{\mathcal{F}_t\}$.

The main problem we address here is the design of a feedback controller $u = F(X)$ such that the corresponding closed loop system (4.1) is asymptotically stable in probability, that is,

$$\lim_{t\to\infty} X(t) = 0,\ \mathbb{P}\text{-a.s.}$$

The construction of this stabilizing feedback controller is given in Theorem 1. It should be said, in this context, that a stronger property, the exact controllability of (4.1) in finite time, is in general still an open problem. (See, however, [5], [10], [12] for partial results.) A similar result is established (see Theorem **??**) for the $2-D$ Navier–Stokes equations with multiplicative noise.

Notation

By $L^2(\mathcal{O})$ we denote the space of all Lebesgue square integrable functions on $\mathcal{O}$ with the norm $|\cdot|_2$ and the scalar product $\langle\cdot,\cdot\rangle$. The scalar product of $L^2(\mathcal{O}_0)$ is denoted by $\langle\cdot,\cdot\rangle_0$. If Y is a Banach space with the norm $\|\cdot\|_Y$, we denote by $L^p(0,T;Y)$, $1\le p\le\infty$, the space of all Bochner measurable functions $u:(0,T)\to Y$ with $\|u\|_Y\in L^p(0,T)$. By $C([0,T];Y)$, we denote the space of all continuous Y-valued functions on $[0,T]$. We denote also by $H^k(\mathcal{O})$, $k=1,2$, the standard Sobolev space of functions on $\mathcal{O}$, $H_0^1(\mathcal{O})=\{y\in H^1(\mathcal{O});\ y=0 \text{ on } \partial\mathcal{O}\}$.

Given an $\mathcal{F}_t$-adapted process $u\in L^2(0,T;L^2(\Omega,L^2(\mathcal{O})))$, a continuous $\mathcal{F}_t$-adapted process $X:[0,T]\to L^2(\mathcal{O})$ is said to be a solution to (4.1) if

$$X\in L^2(\Omega;L^\infty(0,T;L^2(\mathcal{O})))\cap C([0,T];L^2(\Omega;L^2(\mathcal{O}))) \tag{4.7}$$

and

$$\begin{aligned} X(t,\xi)=&\int_0^t(\Delta X(s,\xi)-a(t,\xi)X(s,\xi)-b(s,\xi)\cdot\nabla_\xi X(s,\xi)\\ &+f(X(s,\xi)))ds+\int_0^t \mathbb{1}_{\mathcal{O}_0}u(s,\xi)ds+\int_0^t X(s,\xi)dW(s),\\ &\qquad\qquad \xi\in\mathcal{O},\ t\in(0,T),\ \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.8}$$

Taking into account assumptions (4.3)–(4.6), we may conclude (see, e.g., [9], p. 208) that (4.1) has a unique solution X satisfying (4.7), (4.8). In fact, for every $x\in L^2(\mathcal{O})$, the operator $B(x)y=x\sum_{j=1}^{\infty}\mu_j\langle y,e_j\rangle e_j$, $y\in L^2(\mathcal{O})$ is in $L_2(H,H)$ (the space of Hilbert–Schmidt operators) and we may represent XdW as $B(X)dW^*$, where $W^*(t)$ is the cylindrical Wiener process $\{\beta_j(t)e_j\}_{j\in\mathbb{N}}$. Then, the stochastic Itô integral arising in (4.8) is a standard one (see [9], p. 102).

4.2 The stabilization of equation (4.1)

We set $\mathcal{O}_1=\mathcal{O}\setminus\overline{\mathcal{O}}_0$ and denote by $A_1:D(A_1)\subset L^2(\mathcal{O}_1)\to L^2(\mathcal{O}_2)$ defined by

$$A_1y=-\Delta y,\ y\in D(A_1)=H_0^1(\mathcal{O}_1)\cap H^2(\mathcal{O}_1), \tag{4.9}$$

or, equivalently,

$$\langle A_1y,z\rangle_1=\int_{\mathcal{O}_1}\nabla y\cdot\nabla z\,d\xi,\ \forall y,z\in H_0^1(\mathcal{O}_1), \tag{4.10}$$

where $\langle\cdot,\cdot\rangle_1$ is the duality on $H_0^1(\mathcal{O}_1)\times H^{-1}(\mathcal{O}_1)$ induced by $L^2(\mathcal{O}_1)$ as pivot space.

Denote by $\lambda_1^*(\mathcal{O}_1)$ the first eigenvalue of the operator A_1, that is,

$$\lambda_1^*(\mathcal{O}_1)=\inf\left\{\int_{\mathcal{O}_1}|\nabla y|^2d\xi;\ y\in H_0^1(\mathcal{O}_1),\ \int_{\mathcal{O}_1}y^2d\xi=1\right\}. \tag{4.11}$$

Consider in (4.1) the feedback controller

$$u = -\eta X,\ \eta \in \mathbb{R}^+, \tag{4.12}$$

and the corresponding closed loop system

$$\begin{aligned} &dX - Xdt + aXdt + b\cdot\nabla X dt + f(X)dt = Xdt - \eta \mathbb{1}_{\mathcal{O}_0} X dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathcal{O}, \\ &X(0) = x \text{ in } \mathcal{O}, \quad X = 0 \text{ on } (0,\infty)\times\partial\mathcal{O}. \end{aligned} \tag{4.13}$$

Theorem 1 is the main result.

Theorem 1 *Assume that*

$$\begin{aligned} &\lambda_1^*(\mathcal{O}_1) - \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2|e_j|_\infty^2 - \|f\|_{\mathrm{Lip}} \\ &\qquad - \sup\left\{-a(t,\xi) + \frac{1}{2}\,\mathrm{div}_\xi b(t,\xi);\ (t,\xi)\in\mathbb{R}^+\times\mathcal{O}\right\} > 0. \end{aligned} \tag{4.14}$$

Then, for each $x \in L^2(\mathcal{O})$ *and for* η *sufficiently large (independent of* x*), the feedback controller* (4.12) *exponentially stabilizes in probability equation* (4.1)*. More precisely, there is* $\gamma > 0$ *such that the solution* X *to* (4.13) *satisfies*

$$\lim_{t\to\infty} e^{\gamma t}|X(t)|_2^2 = 0,\ \mathbb{P}\text{-}a.s. \tag{4.15}$$

$$e^{\gamma t}\mathbb{E}|X(t)|_2^2 + \mathbb{E}\int_0^\infty e^{\gamma t}|X(t)|_2^2 dt \le C|x|_2^2. \tag{4.16}$$

We recall that, by the classical Rayleigh–Faber–Krahn perimetric inequality in dimension $d \ge 2$, we have

$$\lambda_1^*(\mathcal{O}_1) \ge \left(\frac{\omega_d}{|\mathcal{O}_1|}\right)^{\frac{2}{d}} J_{\frac{d}{2}-1,1}, \tag{4.17}$$

where $|\mathcal{O}_1| = \mathrm{Vol}(\mathcal{O}_1),\ \omega_d = \pi^{\frac{d}{2}}/\left(\Gamma\left(\frac{d}{2}+1\right)\right)$, and $J_{m,1}$ is the first positive zero of the Bessel function $I_m(r)$.

In particular, by Theorem 1, we conclude that, if $|\mathcal{O}_1|$ is sufficiently small, then the feedback controller (4.12) is exponentially stabilizable. More precisely, we have

Corollary 2 *Assume under hypotheses* (4.3)–(4.6) *that*

$$|\mathcal{O}_1| \le \omega_d^{\frac{2}{d}} J_{\frac{d}{2}-1,1}\left(\frac{1}{2}\sum_{Lip}^{\infty}\mu_j^2|e_j|_\infty^2 + \sup_{\mathbb{R}^+\times\mathcal{O}}\left\{-a + \frac{1}{2}\,\mathrm{div}_\xi b\right\} + \|f\|_{\mathrm{Lip}}\right)^{-\frac{1}{2}}. \tag{4.18}$$

Then, for each $x \in L^2(\mathcal{O})$*, the feedback controller* (4.12) *stabilizes system* (4.1) *in sense of* (4.15), (4.16)*.*

Remark 3 One might suspect that system (4.1) is stabilizable and even exact null controllable in probability by controllers u with support in an arbitrary open subset $\mathcal{O}_0 \subset \mathcal{O}$, as is the case in the deterministic case (see, e.g., [2], [7], [3]), but so far this is an open problem. (More will be said about this in Section 5 below.)

Roughly speaking, Theorem 1 implies, in particular, that the stochastic perturbation destabilizing effect in system $dX - \Delta X dt = X dW$ can be compensated by a linear stabilizing feedback controller with support in a subdomain $\mathcal{O}_0$ satisfying (4.18).

An example. The stochastic equation

$$\begin{aligned} &dX - X_{\xi\xi}dt + (aX + bX_\xi)dt = \mu X d\beta + V dt,\ 0 < \xi < 1, \\ &X(t,0) = X(t,1) = 0,\ t \geq 0, \end{aligned}$$

where β is a Brownian motion and $\mu \in \mathbb{R}$, $a \in C([0,T] \times \mathbb{R})$, $b \in C^1([0,1] \times \mathbb{R})$, is exponentially stabilizable in probability by any feedback controller $V = -\eta \mathbb{1}_{[a_1,a_2]} X$, where $\eta > 0$ is sufficiently large and $0 < a_1 < a_2 < 1$ are such that

$$\pi \inf \left\{ \frac{1}{a_1}, \frac{1}{1-a_2} \right\} > \frac{\mu^2}{2} + \sup_{(t,\xi)\in\mathbb{R}^+\times(0,1)} \left\{ -a(t,\xi) + \frac{1}{2} b_\xi(t,\xi) \right\}.$$

4.3 Proof of Theorem 1

The main ingredient of the proof is the following lemma.

Lemma 4 *For each $\varepsilon > 0$ there is $\eta_0 = \eta_0(\varepsilon)$ such that*

$$\int_{\mathcal{O}} |\nabla y(\xi)|^2 d\xi + \eta \int_{\mathcal{O}_0} y^2(\xi) d\xi \geq (\lambda_1^*(\mathcal{O}_1) - \varepsilon)|y|_2^2,\ \forall y \in H_0^1(\mathcal{O}),\ \eta \geq \eta_0. \tag{4.19}$$

The proof is well known (see [1], [4]), but we outline it for the sake of completeness. Denote by ν_1 the first eigenvalue of the self-adjoint operator

$$A^\eta y = Ay + \eta \mathbb{1}_{\mathcal{O}_0} y,\ \forall y \in D(A^\eta) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}),$$

where $A = -\Delta$, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and $\eta \in \mathbb{R}^+$.

We have by the Rayleigh formula

$$\nu_1^\eta = \inf \left\{ \int_{\mathcal{O}} |\nabla y|^2 d\xi + \eta \int_{\mathcal{O}_0} |y|^2 d\xi;\ |y|_2 = 1 \right\} \leq \lambda_1^*(\mathcal{O}_1) \tag{4.20}$$

because any function $y \in H_0^1(\mathcal{O}_1)$ can be extended by zero to $H_0^1(\mathcal{O})$ across the smooth boundary $\partial\mathcal{O}_1 = \partial\mathcal{O}_0$. Let $\varphi_1^\eta \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ be such that

$$A^\eta \varphi_1^\eta = \nu_1^\eta \varphi_1^\eta,\quad |\varphi_1^\eta|_2 = 1.$$

We have by (4.20) that

$$\int_{\mathcal{O}} |\nabla \varphi_1^\eta|^2 d\xi + \eta \int_{\mathcal{O}_0} |\varphi_1^\eta|^2 d\xi = \nu_1^\eta \leq \lambda_1^*(\mathcal{O}),\ \forall \eta > 0.$$

Then, on a subsequence, again denoted η, we have for $\eta \to \infty$, that $\nu_1^\eta \to \nu^*$ and

$$\begin{aligned} \varphi_1^\eta &\to \varphi_1 \quad \text{weakly in } H_0^1(\mathcal{O}), \text{ strongly in } L^2(\mathcal{O}) \\ \int_{\mathcal{O}_0} |\varphi_1^\eta|^2 d\xi &\to 0. \end{aligned}$$

We have, therefore, $\varphi_1 \in H_0^1(\mathcal{O}_1)$, $|\varphi_1|_2 = 0$ and, since $A^\eta \varphi_1^\eta|_{\mathcal{O}_1} = A_1 \varphi_1^\eta$, we have also that $A_1 \varphi_1 = \nu^* \varphi_1$. Moreover, by (4.20) we see that $\nu^* \leq \lambda_1^*(\mathcal{O}_1)$. Since $\lambda_1^*(\mathcal{O}_1)$ is the first eigenvalue of A_1, we have that $\nu^* = \lambda_1^*(\mathcal{O}_1)$ and

$$\lim_{\eta \to \infty} \inf \left\{ \int_{\mathcal{O}} |\nabla y|^2 d\xi + \eta \int_{\mathcal{O}_0} y^2 d\xi;\ |y|_1 = 1 \right\} = \lambda_1^*(\mathcal{O}_1).$$

This yields (4.19), as claimed.

Proof of Theorem 1 (continued). For simplicity, we take $f = 0$. By applying Itô's formula in (4.13), which by virtue of (4.7) is possible, we obtain that

$$\begin{aligned} &\frac{1}{2}\, d(e^{\gamma t}|X(t)|_2^2) + \int_{\mathcal{O}} e^{\gamma t} |\nabla X(t,\xi)|^2 d\xi\, dt + \int_{\mathcal{O}} (a(t,\xi) - \gamma \\ &-\frac{1}{2}\, \mathrm{div}_\xi b(t,\xi)) e^{\gamma t} X^2(t,\xi) d\xi = \frac{1}{2} \int_{\mathcal{O}} e^{\gamma t} \sum_{k=1}^\infty |(Xe_k)(t,\xi)|^2 d\xi\, dt \\ &-\eta \int_{\mathcal{O}_0} e^{\gamma t} |X(t,\xi)|^2 d\xi\, dt + \int_{\mathcal{O}} e^{\gamma t} \sum_{k=1}^\infty (Xe_k)(t,\xi) X(t,\xi) d\beta_k(t),\ \mathbb{P}\text{-a.s.},\ t \geq 0. \end{aligned}$$

Equivalently,

$$\frac{1}{2}\, e^{\gamma t}|X(t)|_2^2 + \int_0^t K(s) ds = \frac{1}{2}\, |x|_2^2 + M(t),\ t \geq 0,\ \mathbb{P}\text{-a.s.}, \tag{4.21}$$

where

$$\begin{aligned} K(t) = &\int_{\mathcal{O}} e^{\gamma t} (|\nabla X(t,\xi)|^2 + (a(t,\xi) - \gamma - \frac{1}{2}\, \mathrm{div}_\xi b(t,\xi)) |X(t,\xi)|^2 \\ &-\frac{1}{2} \sum_{k=1}^\infty X^2(t,\xi) e_k^2(\xi)) d\xi + \eta \int_{\mathcal{O}_0} e^{\gamma t} |X(t,\xi)|^2 d\xi, \end{aligned} \tag{4.22}$$

$$M(t) = \int_0^t \int_{\mathcal{O}} \sum_{k=1}^\infty e^{\gamma s} X^2(s,\xi) e_k(s,\xi) d\beta_k(s),\ t \geq 0. \tag{4.23}$$

By Lemma 4 and by (4.14), we see that, for $\eta \geq \eta_0$ sufficiently large and $0 < \gamma \leq \gamma_0$ sufficiently small, we have

$$K(t) \geq \varepsilon_0 \int_{\mathcal{O}} e^{\gamma t} |X(t,\xi)|^2 d\xi,\ \forall t > 0,\ \mathbb{P}\text{-a.s.}, \tag{4.24}$$

where $\varepsilon_0 > 0$. Taking expectation into (4.21), we obtain that

$$\frac{1}{2}\, \mathbb{E}[e^{\gamma t}|X(t)|_2^2] + \varepsilon_0 \int_0^t e^{\gamma s} \mathbb{E}|X(s)|_2^2 ds \leq \frac{1}{2}\, |x|_2^2,\ \forall t \geq 0. \tag{4.25}$$

Since $t \to \int_0^t K(s)$ is an a.s. nondecreasing stochastic process and $t \to M(t)$ is a continuous local martingale, we infer by (4.21), (4.25), and by virtue of [11], p. 139, that there exist

$$\lim_{t\to\infty}(e^{\gamma t}|X(t)|_2^2) < \infty,\ K(\infty) < \infty,$$

which imply (4.15), (4.16), as claimed.

4.4 Stabilization of Navier–Stokes equations with multiplicative noise

We consider here the stochastic Navier–Stokes equation

$$\begin{aligned}
&dX(t) - \nu\Delta X(t)dt + (a(t)\cdot\nabla)X(t)dt\\
&\qquad +(X(t)\cdot\nabla)b(t)dt + (X(t)\cdot\nabla)X(t)dt\\
&\qquad = X(t)dW(t) + \nabla p(t)dt + \mathbb{1}_{\mathcal{O}_0}u(t)dt\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathcal{O},\\
&\nabla\cdot X(t) = 0 \quad \text{in } (0,\infty)\times\mathcal{O},\\
&X(t) = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O},\ X(0) = x \quad \text{in } \mathcal{O},
\end{aligned} \tag{4.26}$$

where $\nu > 0$, $a, b \in (C^1((0,\infty)\times\overline{\mathcal{O}}))^2$, $\nabla\cdot a = \nabla\cdot b = 0$, $a\cdot\vec{n} = b\cdot\vec{n} = 0$ on $\partial\mathcal{O}$. Here $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^2$ and $\mathcal{O}_0$ is an open subset of $\mathcal{O}$. The boundaries $\partial\mathcal{O}$ and $\partial\mathcal{O}_0$ are assumed to be smooth. We set

$$H = \{y \in (L^2(\mathcal{O}))^2;\ \nabla\cdot y = 0,\ y\cdot\vec{n} = 0 \text{ on } \partial\mathcal{O}\},$$

where $\vec{n}$ is the normal to $\partial\mathcal{O}$. We denote by $\langle\cdot,\cdot\rangle_H$ the scalar product of H and by $|\cdot|_H$ the norm. The Wiener process $W(t)$ is of the form (4.2), where $\{e_k\} \subset (C^2(\overline{\mathcal{O}}))^2$ is an orthonormal basis in H, and $\mu_k \in \mathbb{R}$. As in the previous case, the main objective here is the design of a stabilizable feedback controller u for equation (4.26).

We use the standard notations

$$\begin{aligned}
H &= \{y \in (L^2(\mathcal{O}))^d;\ \nabla\cdot y = 0 \text{ in } \mathcal{O},\ y\cdot\vec{n} = 0 \text{ on } \partial\mathcal{O}\},\\
V &= \{y \in (H_0^1(\mathcal{O}))^d;\ \nabla\cdot y = 0 \text{ in } \mathcal{O}\},\\
A &= -\nu\Pi\Delta,\ D(A) = (H^2(\mathcal{O}))^d \cap V,
\end{aligned}$$

where Π is the Leray projector on H. Consider the Stokes operator A_1 on $\mathcal{O}_1 = \mathcal{O}\setminus\mathcal{O}_0$, that is,

$$\langle A_1 y, \varphi\rangle = \nu\sum_{i=1}^d \int_{\mathcal{O}_1} \nabla y_i\cdot\nabla\varphi_i d\xi,\ \forall\varphi \in V_1,$$

where $V_1 = \{y \in (H_0^1(\mathcal{O}_1))^d;\ \nabla\cdot y = 0 \text{ in } \mathcal{O}_1\}$. Denote again by $\lambda_1^*(\mathcal{O}_1)$ the first eigenvalue of A_1, that is,

$$\lambda_1^*(\mathcal{O}_1) = \inf\left\{\nu\sum_{i=1}^d \int_{\mathcal{O}} |\nabla\varphi_i|^2 d\xi,\ \varphi \in V_1,\ \int_{\mathcal{O}_1} |\varphi|^2 d\xi = 1\right\}. \tag{4.27}$$

Also, in this case, we have (see Lemma 1 in [4]), for $\eta \geq \eta_0(\varepsilon)$ and $\varepsilon > 0$,

$$\langle Ay, y\rangle_H + \eta\,\langle \Pi(\mathbb{1}_{\mathcal{O}_0} y), y\rangle_H \geq (\lambda_1^*(\mathcal{O}_1) - \varepsilon)|y|_H^2,\ \forall y \in V. \tag{4.28}$$

We consider in system (4.26) the linear feedback controller

$$u = -\eta X,\ \eta > 0. \tag{4.29}$$

We set

$$\gamma^*(t) = \sup\left\{\int_{\mathcal{O}} |y_i D_i b_j y_j d\xi|;\ |y|_H = 1\right\} < \infty,$$

where $b = \{b_1, b_2\}$.

The closed loop system (4.26) with the feedback controller (4.29) has a unique strong solution in the sense of (4.7), (4.8). (See, e.g., [8], p. 281.)

We have

Theorem 5 *Assume that*

$$\lambda_1^*(\mathcal{O}_1) > \frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2 |e_j|_\infty^2 + \sup_{t\in\mathbb{R}^+} \gamma^*(t). \tag{4.30}$$

Then, for each $x \in H$ and η sufficiently large independent of x, the solution X to the closed loop system (4.26) *with the feedback controller* (4.29) *satisfies*

$$\mathbb{E}[e^{\gamma t}|X(t)|_H^2] + \int_0^\infty e^{\gamma t}\mathbb{E}|X(t)|_H^2 dt < C|x|_H^2, \tag{4.31}$$

$$\lim_{t\to\infty} e^{\gamma t}|X(t)|_H^2 = 0,\ \mathbb{P}\text{-a.s.}, \tag{4.32}$$

for some $\gamma > 0$.

The proof is essentially the same as that of Theorem 1, and so it will be sketched only. Taking into account that

$$\langle (X\cdot\nabla)X, X\rangle_H + \langle (a(t)\cdot\nabla)X, X\rangle_H = 0,\ t > 0, \mathbb{P}\text{-a.s.},$$

we obtain by (4.26), (4.29), via Itô's formula, that

$$\begin{aligned}\frac{1}{2}\,e^{\gamma t}|X(t)|_H^2 &+ \int_0^t e^{\gamma s}\Big(\langle AX(s), X(s)\rangle_H + \langle X(s)\cdot\nabla b(s), X(s)\rangle_H\\ &-\frac{1}{2}\sum_{j=1}^{\infty}|X(s)e_j|_H^2 + \eta\,\langle \mathbb{1}_{\mathcal{O}_0}X(s), X(s)\rangle_H\Big)ds\\ &= \frac{1}{2}\,|x|_H^2 + \int_0^t e^{\gamma s}\sum_{j=1}^{\infty}\langle X(s)e_j, X(s)\rangle_H\, d\beta_j(s),\ t \geq 0.\end{aligned} \tag{4.33}$$

Then, by virtue of (4.28) and (4.30), we have, by (4.33), that

$$\frac{1}{2}\,e^{\gamma t}|X(t)|_H^2 + I(t) = \frac{1}{2}\,|x|_H^2 + M^*(t),\ t \geq 0,\ \mathbb{P}\text{-a.s.},$$

where $I(t)$ is a nondecreasing process, which satisfies

$$\mathbb{E}[I(t)] \geq \varepsilon_0 \int_0^t e^{\gamma s}\mathbb{E}|X(s)|_H^2 ds, \ \forall t \geq 0,$$

for η sufficiently large, and $M^*(t) = \int_0^t e^{\gamma s} \sum_{j=1}^{\infty} \langle X(s)e_j, X(s)\rangle_H \, d\beta_j(s)$ is a continuous local martingale. As in the previous case, this implies via [11] that $\lim_{t\to\infty} e^{\gamma t}|X(t)|_H^2$ exists $\mathbb{P}$-a.s. and, therefore, (4.31) and (4.32) hold.

Remark 6 By (4.27), we see that $\lambda_1^*(\mathcal{O}_1) > \nu\mu_1$, where μ_1 is the first eigenvalue on $\mathcal{O}$ of the Stokes operator $-\Pi\Delta$ and we have also that

$$\lambda_1^*(\mathcal{O}_1) \geq C\nu \left(\sup_{x\in\mathcal{O}_1} \ \mathrm{dist}(x, \partial\mathcal{O}) \right)^{-2}.$$

4.5 Final remarks

In order to make clear the novelty of the above results and the principal difficulties related to the internal stabilization of equations (4.1), we note that, via the substitution $y = e^{W(t)}X$, equation (4.1) reduces to a parabolic equation of the form

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \widetilde{a}(t)y + \widetilde{b}(t)\cdot\nabla y + \frac{1}{2}\sum_{i=1}^{\infty} \mu_k^2 e_k^2 y = \mathbb{1}_{\mathcal{O}_0} u, \ \mathbb{P}\text{-a.s.}, \\ &y = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O}, \end{aligned} \tag{4.34}$$

with random coefficients $\widetilde{a}, \widetilde{b}$.

If $\widetilde{a}$ and $\widetilde{b}$ are independent functions of t, then (4.34) can be stabilized by a controller $u = \sum_{j=1}^N u_j(t)\psi_j$, where ψ_j are linear combinations of eigenfunctions of the dual operator $y \to -\Delta y + \widetilde{a}y - \mathrm{div}(by)$ (see [7] or [3] for the case of Navier–Stokes equations).

For deterministic equations of the form (4.34) with smooth time dependent coefficients, a similar result was recently proved in [6] (for the Navier–Stokes equations), but it cannot be applied, however, to the random equation (4.34) since it does not provide an adapted stabilizable controller $u = u(t)$. (The reason is that the argument in [6] relies on exact $\mathbb{P}$-a.s. controllability of (4.34) via an adapted controller u which so far is still an open problem.)

By the new procedure we use here, we circumvert this basic difficulty by constructing an explicit feedback adapted controller and avoiding so the exact controllability of the stochastic equation with multiplicative noise which is equivalent to an open problem: observability inequality for the dual stochastic equation.

References

[1] S. Aniţa, Internal stabilization of diffusion equation, *Nonlinear Studies*, 8 (2001), 193-202.

[2] V. Barbu, Controllability of parabolic and Navier–Stokes equations, *Scientiae Mathematicae Japonicae*, vol. 56 (1) (2002), 143-211.

[3] V. Barbu, *Stabilization of Navier-Stokes Flows*, Springer, London. Dordrecht. Heidelberg, Berlin, 2011.

[4] V. Barbu, C. Lefter, Internal stabilizability of the Navier–Stokes equations, *Systems & Control Letters*, 48 (2003), 161-167.

[5] V. Barbu, A. Rascanu, G. Tessitore, Carleman estimates and controllability of linear stochastic heat equations, Appl. Math. Optimiz. 47 (2003), 97-1209.

[6] V. Barbu, S.S. Rodriguez, A. Shirikyan, Internal exponential stabilization to a nonstationary solution for $3-D$ Navier–Stokes equations, SIAM J. Control Optim. vol. 49 (2011), 1454-1478.

[7] V. Barbu, R. Triggiani, Internal stabilization of Navier–Stokes equations with finite-dimensional controllers, Indiana Univ. Math. J. 53 (2004), 1443-1494.

[8] G. Da Prato, J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[9] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 2008.

[10] D. Goreac, Approximate controllability for linear stochastic differential equations in infinite dimensions, Appl. Math. Optim. 53 (2009), 105-132.

[11] R.S. Lipster, A. Shiryaev, *Theory of Martingales*, Kluwer Academic, Dordrecht, 1989.

[12] S. Tang, Z. Zhang, Null controllability for forward and backward stochastic parabolic equations, SIAM J. Control Optim. 48 (2009), 2191-2216.

Lecture 5. Stabilization of Navier–Stokes equations by oblique boundary feedback controllers

5.1 Introduction

Consider the Navier–Stokes system

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)y = \nabla p + f_e && \text{in } (0,\infty)\times\mathcal{O},\\ &\nabla\cdot\ y = 0 && \text{in } (0,\infty)\times\mathcal{O},\\ &y = v && \text{on } (0,\infty)\times\partial\mathcal{O},\\ &y(0) = y_0 && \text{in } \mathcal{O}, \end{aligned} \tag{5.1}$$

in a bounded open domain $\mathcal{O}\subset\mathbb{R}^d$, $d=2,3$, with the boundary $\partial\mathcal{O}$ which is assumed to be a finite union of $d-1$ dimensional C^2– connected manifolds . Here $\nu>0$, f_e is a given smooth function and v is a boundary input. If y_e is an equilibrium solution to (5.1) then (5.1) can be, equivalently, written as

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y_e\cdot\nabla)y + (y\cdot\nabla)y_e + (y\cdot\nabla)y = \nabla p\\ & && \text{in } (0,\infty)\times\mathcal{O},\\ &\nabla\cdot\ y = 0 && \text{in } (0,\infty)\times\mathcal{O},\\ &y = u && \text{on } (0,\infty)\times\partial\mathcal{O},\\ &y(0) = y_0 - y_e && \text{in } \mathcal{O}. \end{aligned} \tag{5.2}$$

Let Γ be a connected component of $\partial\mathcal{O}$. Our main concern here is the design of an oblique boundary feedback controller with support in Γ which stabilizes exponentially the equilibrium state y_e, or, equivalently, the zero solution to (5.2). The main step toward this end is the stabilization of the linear system corresponding to (5.2) or, more generally, of the Oseen–Stokes system

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)a + (b\cdot\nabla)y = \nabla p && \text{in } (0,\infty)\times\mathcal{O},\\ &\nabla\cdot\ y = 0 && \text{in } (0,\infty)\times\mathcal{O},\\ &y = u && \text{on } (0,\infty)\times\partial\mathcal{O}, \end{aligned} \tag{5.3}$$

where $a,b\in(C^2(\overline{\mathcal{O}}))^d$, $\nabla\cdot a = \nabla\cdot b = 0$ in $\mathcal{O}$. Besides its significance as first order linear approximation of (5.2), this system models the dynamics of a Stokes flow with inclusion of a convection acceleration $(b\cdot\nabla)y$ and also the disturbance flow induced by a moving body in a Stokes fluid flow.

In its complex form, the main result of this work, Theorem 1, amounts to saying that, if the unstable eigenvalues of system (5.3) are semi-simple and a certain unique continuation type property for eigenfunctions of the dual linearized system holds then there is a boundary feedback controller of the form

$$u(t,x)=\eta \mathbb{1}_\Gamma \sum_{j=1}^{N}\mu_j\left(\int_{\mathcal{O}} y(t,x)\overline{\varphi}_j^*(x)dx\right)(\phi_j(x)+\alpha(x)\vec{n}(x)),\qquad t\geq 0,\ x\in\partial\mathcal{O}, \tag{5.4}$$

which stabilizes exponentially system (5.1). Here, $\mathbb{1}_\Gamma$ is the characteristic function of Γ as subset of $\partial\mathcal{O}$, $\phi_j\subset(C^2(\Gamma))^d$ are suitably chosen functions and $\{\varphi_j^*\}_{j=1}^N$ is an eigenfunction system for the adjoint $\mathcal{L}^*$ of the Stokes–Oseen operator

$$\begin{aligned}\mathcal{L}\varphi &= -\nu\Delta\varphi+(a\cdot\nabla)\varphi+(\varphi\cdot\nabla)b,\ \varphi\in D(\mathcal{L}),\\ D(\mathcal{L}) &= \{\varphi\in(H^2(\mathcal{O}))^d\cap(H_0^1(\mathcal{O}))^d;\ \nabla\cdot\varphi=0\ \text{ in }\ \mathcal{O}\},\end{aligned} \tag{5.5}$$

that is,

$$(\mathcal{L}^*\psi)_j=-\nu\Delta\psi_j-\sum_{i=1}^{N}(D_i(a_i\psi_j)-\psi_iD_jb_i),\quad j=1,..,d. \tag{5.6}$$

It turns out (see Theorem 5) that this feedback controller also stabilizes the Navier–Stokes system (5.2) in a neighborhood of the origin.

In (5.4), N is the number of the eigenvalue λ_j of $\mathcal{L}$ with $\operatorname{Re}\lambda_j\leq 0$ and $\alpha\in C^2(\partial\mathcal{O})$ is an arbitrary function with zero circulation on Γ, that is,

$$\int_\Gamma \alpha(x)dx=0. \tag{5.7}$$

If α is identically zero then the controller (5.4) is tangential but in general it is oblique to domain $\mathcal{O}$ which makes it more effective for control actuation. As a matter of fact, we shall see below (see Corollary 2) that, with exception of a set of Lebesgue measure arbitrarily small, the controller u given by (5.4) can be chosen in a direction close to $\vec{n}$, that is, "almost normal". It should be mentioned that, in literature, only in a few situations normal stabilizing controllers for equation (5.1) were designed, and this happened mostly for periodic flows in $2-D$ channels only (see, e.g., [1], [2], [3], [25], [26], [27]). However, we notice that there is a large body of remarkable results obtained in recent years on boundary stabilization of system (5.1) and here the works [10], [11], [13], [14], [17], [18], [20], [21] should be primarily cited. (See, also, [7], [14], [15], [23].) In this context we should also mention the work [22] which contains several significant results and techniques related to boundary stabilization of Oseen-Stokes system with non tangential controllers. The approach used in these works can be described in a few words as follows; one decomposes system (5.1) in a finite-dimensional unstable part which is exactly controllable and an infinite-dimensional part which is exponentially stable and proves so its stabilization by an open loop boundary controller with finite-dimensional structure. Then, one designs in a standard way a stabilizing feedback controller via the algebraic Riccati equation associated with an infinite horizon quadratic optimal control problem. (In [17], [18], one uses a somehow different stabilization technique based on the existence of an asymptotically stable invariant manifold for the Euler-Lagrange system associated with Stokes-Oseen operator which is, however, equivalent to a Riccati based approach.) Our construction of boundary stabilizing controller for (5.1) is an alternative to the Riccati equation based approach which though provides a robust controller it is, however, quite difficult to treat from computational point of view. This construction resembles the form of stabilizing noise controllers recently designed in the author's works [4], [5], [6], [8], [9], which seem to be, however, more robust to stochastic perturbations. It should be mentioned however that our stabilization results are conditional (see Hypotheses (H1) and (H2) below.)

The plan of the paper is the following. In Section 2, we present the main stabilization results which will be proved in Section 3. In Section 4, we shall give an example to stabilization of Stokes–Oseen periodic flows in a $2-D$ channel.

In the following we shall use the standard notation for spaces of functions on $\mathcal{O} \subset \mathbb{R}^d$. In particular, $C^k(\overline{\mathcal{O}})$, $k = 0, 1, ...$, is the space of k-differentiable functions on $\overline{\mathcal{O}}$ and $H^k(\mathcal{O})$, $k > 0$, $H_0^1(\mathcal{O})$ are Sobolev spaces on $\mathcal{O}$.

5.2 The main result

Notation

Everywhere in the following, $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^d$, $d = 2, 3$, its boundary $\partial\mathcal{O}$ is a finite union of $d-1$ dimensional C^2- connected manifolds and Γ is a connected component of $\partial\mathcal{O}$.

We set $H = \{y \in (L^2(\mathcal{O}))^d;\ \nabla \cdot y = 0 \ \text{ in } \ \mathcal{O}, \quad y \cdot \vec{n} = 0 \ \text{ on } \ \partial\mathcal{O}\}$ and denote by $\Pi : (L^2(\mathcal{O}))^d \to H$ the Leray projector on H. We consider the operator $A : D(A) \subset H \to H$, $\mathcal{A} : D(\mathcal{A}) \subset H \to H$,

$$\begin{aligned} Ay &= -\nu\Pi(\Delta y),\ \forall y \in D(A) = (H_0^1(\mathcal{O}))^d \cap (H^2(\mathcal{O}))^d \cap H, && (5.8)\\ \mathcal{A}y &= \Pi(-\nu\Delta y + (y \cdot \nabla)a + (b \cdot \nabla)y) && (5.9)\\ &= Ay + \Pi((y \cdot \nabla)a + (b \cdot \nabla)y),\ \forall y \in D(\mathcal{A}) = D(A). \end{aligned}$$

We denote by $\widetilde{H}$ the complexified space $\widetilde{H} = H + iH$ and consider the extension $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ to $\widetilde{H}$, that is, $\widetilde{\mathcal{A}}(y + iz) = \mathcal{A}y + i\mathcal{A}z$ for all $y, z \in D(\mathcal{A})$.

The scalar product of H and of $\widetilde{H}$ are denoted by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\widetilde{H}}$, respectively. The corresponding norms are denoted by $|\cdot|_H$ and $|\cdot|_{\widetilde{H}}$, respectively.

For simplicity, we denote in the following again by $\mathcal{A}$ the operator $\widetilde{\mathcal{A}}$ and the difference will be clear from the content. The operator $\mathcal{A}$ has a compact resolvent $(\lambda I - \mathcal{A})^{-1}$ (see, e.g., [7], p 92). Consequently, $\mathcal{A}$ has a countable number of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ with corresponding eigenfunctions φ_j each with finite algebraic multiplicity m_j. In the following, each eigenvalue λ_j is repeated according to its algebraic multiplicity m_j.

Note also that there is a finite number of eigenvalues $\{\lambda_j\}_{j=1}^N$ with $\operatorname{Re}\lambda_j \leq 0$ and that the spaces $X_u = \text{lin span}\{\varphi_j\}_{j=1}^N = P_N\widetilde{H}$, $X_s = (I - P_N)\widetilde{H}$ are invariant with respect to $\mathcal{A}$. Here, P_N is the algebraic projection of $\widetilde{H}$ on X_u and is defined by

$$P_N = \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda I - \mathcal{A})^{-1} d\lambda,$$

where Γ_0 is a closed curve which contains in interior the eigenvalues $\{\lambda_j\}_{j=1}^N$.

If we set $\mathcal{A}_u = \mathcal{A}|_{X_u}$, $\mathcal{A}_s = \mathcal{A}|_{X_s}$, then we have

$$\sigma(\mathcal{A}_u) = \{\lambda_j : \operatorname{Re}\lambda_j \leq 0\}, \quad \sigma(\mathcal{A}_s) = \{\lambda_j : \operatorname{Re}\lambda_j > 0\}. \tag{5.10}$$

We recall that the eigenvalue λ_j is called semi–simple if its algebraic multiplicity m_j coincides with its geometric multiplicity m_j^g. In particular, this happens if λ_j is simple and it turns out that the property of the eigenvalues λ_j to be all simple is generic (see [7], p. 164). The dual operator $\mathcal{A}^*$ has the eigenvalues $\overline{\lambda}_j$ with the eigenfunctions φ_j^*, $j = 1, \dots$.

For the time being, the following hypotheses will be assumed.

(H1) *The eigenvalues* λ_j, $j = 1, ..., N$, *are semi-simple.*

This implies that

$$\mathcal{A}\varphi_j = \lambda_j\varphi_j, \quad \mathcal{A}^*\varphi_j^* = \overline{\lambda}_j\varphi_j^*, \quad j = 1, ..., N, \tag{5.11}$$

or, equivalently,

$$\mathcal{L}\varphi_j = \lambda_j\varphi_j + \nabla p_j, \quad \mathcal{L}^*\varphi_j^* = \overline{\lambda}_j\varphi_j^* + \nabla p_j^*, \quad j = 1, ..., N, \tag{5.12}$$

and so we can choose systems $\{\varphi_j\}, \{\varphi_j^*\}$ in such a way that

$$\langle \varphi_j, \varphi_k^* \rangle_{\widetilde{H}} = \delta_{jk}, \quad j, k = 1, ..., N. \tag{5.13}$$

The next hypothesis is a unique continuation type assumption on the normal derivatives $\dfrac{\partial \varphi_j^*}{\partial n}$, $j = 1, ..., N$.

(H2) *The system* $\left\{\dfrac{\partial \varphi_j^*}{\partial n}\right\}_{j=1}^{N}$ *is linearly independent on* Γ.

In the special case where the unstable spectrum $\mathcal{A}$ has only one distinct eigenvalue λ_1 (eventually multivalued), hypothesis (H2) is implied by the following weaker assumption

(H2)$'$ $\dfrac{\partial \varphi^*}{\partial n}$ *is not identically zero on* Γ,

where φ^* is any eigenfunction corresponding to the unstable eigenvalue $\bar{\lambda}_1$.

Since any linear combination of this system of eigenfunctions is again an eigenfunction corresponding to $\bar{\lambda}_1$, it is clear that in this case (H2) is implied by (H2)$'$.

It is not known whether (H2)$'$ is always satisfied, but likewise hypothesis (H1), if $\Gamma = \partial\mathcal{O}$, it holds, however, for "almost all a, b" in the generic sense (see [12]). In Section 4, it is presented a significant example, where (H2)$'$ holds.

5.2.1 The main stabilization result

Consider the feedback boundary controller

$$u = \eta \mathbb{1}_\Gamma \sum_{j=1}^{N} \mu_j \left\langle P_N y, \varphi_j^* \right\rangle_{\widetilde{H}} (\phi_j + \alpha\vec{n}), \tag{5.14}$$

where

$$\mu_j = \frac{k+\lambda_j}{k+\lambda_j-\nu\eta} \quad j=1,...,N, \tag{5.15}$$

$$\phi_j = \sum_{\ell=1}^{N} a_{j\ell}\,\frac{\partial \varphi_\ell^*}{\partial n}\,, \quad j=1,...,N, \tag{5.16}$$

and $a_{\ell j} \in \mathbb{C}$ are chosen in such a way that

$$\sum_{\ell=1}^{N} a_{\ell j} \int_\Gamma \frac{\partial \varphi_\ell^*}{\partial n}\,\frac{\partial \overline{\varphi}_i^*}{\partial n}\,dx = \delta_{ij} - \frac{1}{\nu}\,\langle \alpha, p_i^*\rangle_0 \text{ for } i,j=1,...,N. \tag{5.17}$$

By virtue of hypothesis (H2), such a system $\{a_{\ell j}\}_{\ell,j=1}^N$ exists because the Gram matrix

$$\left\| \int_\Gamma \frac{\partial \varphi_\ell^*}{\partial n}\frac{\partial \overline{\varphi}_i^*}{\partial n} \right\|_{i,\ell=1}^N = Z_0$$

is not singular.

By (5.13), (5.17), we have

$$\int_\Gamma \phi_j\,\frac{\partial \overline{\varphi}_i^*}{\partial n}\,dx = \delta_{ij} - \frac{1}{\nu}\,\langle \alpha, p_i^*\rangle_0\,, \quad i,j=1,...,N. \tag{5.18}$$

Here $\{p_i^*\}$ are given by (5.12) and $\langle\cdot,\cdot\rangle_0$ is the scalar product in $L^2(\Gamma)$.

Theorem 1 *Assume that* $d=2,3$, (H1), (H2) and (5.7) *hold, and that* $\mathrm{Re}\,\lambda_j \le 0$ *for* $j=1,...,N$, $\mathrm{Re}\ \lambda_j > 0$ *for* $j>N$. *Let* $k>0$ *sufficiently large and* $\eta>0$ *be such that*

$$\mathrm{Re}\,\lambda_j + \eta\nu + \eta^2\nu^2(\mathrm{Re}\,\lambda_j + k - \eta\nu)|k+\lambda_j-\eta|^{-2} > 0 \textit{ for } j=1,...,N. \tag{5.19}$$

Then the feedback controller (5.14) *stabilizes exponentially system* (5.3)*, that is, the solution* y *to the closed loop system*

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)a + (b\cdot\nabla)y = \nabla p && \textit{in } (0,\infty)\times\mathcal{O}, \\ &\nabla\cdot y = 0 && \textit{in } (0,\infty)\times\mathcal{O}, \\ &y = \eta 1\!\!1_\Gamma \sum_{j=1}^{N} \mu_j \left\langle P_N y, \varphi_j^*\right\rangle_{\widetilde{H}} (\phi_j + \alpha\vec{n}) && \textit{on } (0,\infty)\times\partial\mathcal{O}, \end{aligned} \tag{5.20}$$

satisfies for some $\gamma>0$ *the estimate*

$$|y(t)|_{\widetilde{H}} \le Ce^{-\gamma t}|y(0)|_{\widetilde{H}}, \quad \forall t\ge 0. \tag{5.21}$$

It is easily seen that (5.19) holds for k sufficiently large and $\eta>0$ such that $\mathrm{Re}\,\lambda_j + \eta\nu > 0$, $\forall j=1,...,N$.

If λ_j are complex, then the controller (5.14) is complex valued too and plugged into system (5.3) leads to a real closed loop system in the state variables $(\mathrm{Re}\,y, \mathrm{Im}\,y)$. In order to circumvent such a situation, we shall construct in Section 3.3 a real stabilizing feedback controller of the form (5.14) which has a similar stabilization effect. (See Theorem 6.)

A problem of major interest is whether the controller u can be chosen "almost" normal, that is, its normal component $u_{\vec{n}}$ is close to the normal $\vec{n}$. We have

Corollary 2 *There is a stabilizing controller u of the form* (5.14), (5.16), *with* $\|a_{ij}\|_{i,j=1}^{N} = Z_0^{-1}$ *and* $\alpha = \lambda\alpha^*$, *where* $\lambda \in \mathbb{C}$ *is arbitrary and* $\alpha^* \in C^2(\Gamma)$ *satisfies* (5.7) *and*

$$|\alpha^*(x)| \neq 0, \quad a.e.\ x \in \Gamma. \tag{5.22}$$

The exact significance of this result is that, for each $\varepsilon > 0$, there is a Lebesgue measurable subset Γ_ε such that $m(\Gamma \setminus \Gamma_\varepsilon) \leq \varepsilon$ and, on Γ_ε, the normal component $\lambda\alpha^*\vec{n}$ of the controller u is $\neq 0$ and arbitrarily large with respect to the tangential component represented by ϕ_j. (Here m is the Lebesgue measure on Γ.)

Proof of Corollary 2. We set

$$\begin{aligned} X &= \left\{\psi \in L^2(\Gamma);\ \int_\Gamma \psi\, dx = 0\right\}, \\ Y &= \left\{\sum_{j=1}^{N} \gamma_j \left(p_j^* - \frac{1}{m(\Gamma)} \int_\Gamma p_j^* dx\right);\ \gamma_j \in \mathbb{C}\right\}, Y_1 = X \cap Y^\perp. \end{aligned}$$

Then $Y_1 = \{\psi \in X;\ \langle X, p_j^* \rangle_0 = 0,\ \forall j = 1,...,N\}$, $L^2(\Gamma) = Y \oplus Y_1 \oplus \mathbb{C}$, and so, any $\psi \in C^2(\Gamma)$ can be written as

$$\psi = \widetilde{\alpha} + \sum_{j=1}^{N} \gamma_j \left(p_j^* - \frac{1}{m(\Gamma)} \int_\Gamma p_j^* dx\right) + \gamma_0, \tag{5.23}$$

for some $\widetilde{\alpha} \in Y_1$ and $\gamma_j \in \mathbb{C}$, $j = 0, 1, ..., N$. We note that there are $\psi^* \in C^2(\Gamma)$ and $\gamma_j^* \in \mathbb{C}$ such that

$$|\psi^*(x) - \sum_{j=1}^{N} \gamma_j^* \left(p_j^*(x) - \frac{1}{m(\Gamma)} \int_\Gamma p_j^* dx\right) - \gamma_0^*| > 0, \text{ a.e. } x \in \Gamma. \tag{5.24}$$

Otherwise, for each $\widetilde{\psi} \in C^2(\Gamma)$ and $\{\widetilde{\gamma}_j\}_{j=0}^{N} \subset \mathbb{C}$, there is a Lebesgue measurable subset $\widetilde{\Gamma} \subset \Gamma$ such that $m(\widetilde{\Gamma}) > 0$ and

$$\widetilde{\psi} - \sum_{j=1}^{N} \widetilde{\gamma}_j \left(p_j^* - \frac{1}{m(\Gamma)} \int_\Gamma p_j^* dx\right) - \widetilde{\gamma}_0 = 0, \quad \text{in } \widetilde{\Gamma},$$

which, by virtue of arbitrarity of $\widetilde{\psi}$ and $\widetilde{\gamma}_j$, is absurd. Indeed, it suffices to fix $\widetilde{\gamma}_j \in \mathbb{C}$, $j = 0, 1, ..., N$, and take $\widetilde{\psi} \in C^2(\Gamma)$ in such a way that

$$\inf_\Gamma |\widetilde{\psi}| > \sup_\Gamma \left|\sum_{j=1}^{N} \widetilde{\gamma}_j \left(p_j^* - \frac{1}{m(\Gamma)} \int_\Gamma p_j^* dx\right) - \widetilde{\gamma}_0\right|$$

to arrive to a contradiction. Then, for the corresponding $\widetilde{\alpha}$ in (5.23), denoted α^*, we have (5.22). Now, we see that, for each $\lambda \in \mathbb{C}$, $\alpha = \lambda\alpha^*$, we have $\langle \alpha, p_i^* \rangle_0 = 0$, $\forall i = 1, ..., N$, and so, by (5.17) we have $\|a_{ij}\|_{i,j=1}^{N} = Z_0^{-1}$, as claimed.

Remark 3 The idea of the proof already used in the previous works mentioned above is to decompose system (5.20) or the space $\widetilde{H}$ in a finite differential system corresponding to unstable eigenvalue $\{\lambda_j\}_{j=1}^N$ and an infinite and stable differential system. For this, Hypothesis (H1) is not absolutely necessary (see, for instance, [7], [16]) and it was assumed only for convenience in order to get a simple diagonal form for the finite-dimensional unstable system and implicitly for the stabilizing feedback. As regards (H2), one might suspect too that it can be replaced by the weaker assumption (H2)′ eventually modifying the form of stabilizing controller.

Remark 4 As easily follows from the proof in (5.14), the function α can be replaced by a system of functions $\{\alpha_j\}_{j=1}^N$ satisfying (5.7) with the corresponding modification of (5.17).

The above construction works also in more general case where Γ is a smooth part (not necessarily connected) of $\partial\mathcal{O}$ but in this case $\mathbb{1}_\Gamma$ should be replaced by a C^2- function on $\partial\mathcal{O}$ with compact support in Γ and in condition (5.7) α should be replaced by $\mathbb{1}_\Gamma\alpha$.

5.2.2 Stabilization of system (5.1) ((5.2))

In the boundary stabilization of Navier-Stokes equation (5.3) with finite dimensional controllers due to compatibility of the boundary trace of state y with the boundary control there are two feasible regularity levels for the solution y, namely $(H^{\frac{1}{2}-\varepsilon}(\mathcal{O}))^d$ for $d=2$ and $(H^{\frac{1}{2}+\varepsilon}(\mathcal{O}))^d$ for $d=3$. (See [7], [11], [13].) However, in $3-D$ the high topological level $(H^{\frac{1}{2}+\varepsilon}(\mathcal{O}))^d$ is not appropriate for some technical reasons related to properties of the inertial term $(y\cdot\nabla)y$ and so, unlikewise the linear case, the treatment should be confined to $d=2$.

Consider the Sobolev spaces $W=(H^{\frac{1}{2}-\varepsilon}(\mathcal{O}))^2\cap\widetilde{H}$, $Z=(H^{\frac{3}{2}-\varepsilon}(\mathcal{O}))^2\cap\widetilde{H})$, where $0<\varepsilon<\frac{1}{2}$, with the norms denoted by $\|\cdot\|_W, \|\cdot\|_Z$. The main stabilization result for system (5.3) is Theorem 5 below.

Theorem 5 *Let $d=2$ and $a=b=y_e$. Then, under the assumptions of Theorem 1, the feedback boundary controller (5.14) stabilizes exponentially system (5.2) in a neighborhood $\mathcal{W}=\{y_0\in W;\ \|y_0\|_W<\rho\}$. More precisely, the solution $y\in C([0,\infty);W)\cap L^2(0,\infty;Z)$ to the closed loop system*

$$\begin{aligned}&\frac{\partial y}{\partial t}-\nu\Delta y+(y\cdot\nabla)y_e+(y_e\cdot\nabla)y+(y\cdot\nabla)y=\nabla p \ \text{in } (0,\infty)\times\mathcal{O},\\ &y=\eta\mathbb{1}_\Gamma\sum_{j=1}^N\mu_j\left\langle P_Ny,\varphi_j^*\right\rangle_{\widetilde{H}}(\phi_j+\alpha\vec{n}) \ \text{on } (0,\infty)\times\partial\mathcal{O},\end{aligned}\tag{5.25}$$

satisfies for $y(0)\in\mathcal{W}$ and ρ sufficiently small

$$\|y(t)\|_W\le Ce^{-\gamma t}\|y(0)\|_W,\quad \forall t\ge 0,\tag{5.26}$$

for some $\gamma>0$.

In particular, it follows that the boundary feedback controller

$$u=\eta\sum_{j=1}^N\mu_j\left\langle P_N(y-y_e),\varphi_j^*\right\rangle_{\widetilde{H}}(\phi_j+\alpha\vec{n})\tag{5.27}$$

stabilizes exponentially the equilibrium solution y_e to (5.1) in a neighborhood $\{y_0\in W;\ \|y_0-y_e\|_W<\rho\}$.

5.3 Proofs

5.3.1 Proof of Theorem 1

We set

$$U^0 = \left\{ u \in (L^2(\partial\mathcal{O}))^d; \int_{\partial\mathcal{O}} u(x)\cdot\vec{n}(x)dx = 0 \right\}.$$

Then, for $k>0$ sufficiently large, there is a unique solution $y \in (H^{\frac{1}{2}}(\mathcal{O}))^d$ to the equation

$$\begin{aligned} &-\nu\Delta y + (y\cdot\nabla)a + (b\cdot\nabla)y + ky = \nabla p \ \text{ in } \ \mathcal{O}, \\ &\nabla\cdot y = 0 \ \text{ in } \ \mathcal{O}, \quad y = u \ \text{ on } \ \partial\mathcal{O}. \end{aligned}$$

(See, e.g., [24], p. 365.) We set $y = Du$ and note that (see, e.g. [11], p. 102),

$$D \in L((H^s(\partial\mathcal{O}))^d \cap U^0; (H^{s+\frac{1}{2}})\mathcal{O}))^d), \ \text{for } s \geq -\frac{1}{2}\cdot$$

In terms of $\mathcal{A}$ and of the Dirichlet map D, system(5.3) can be written as (see [22])

$$\begin{aligned} &\Pi\frac{d}{dt}\,y(t) + \mathcal{A}(y(t) - Du(t)) = k\Pi Du, \quad t\geq 0, \\ &y(0) = y_0. \end{aligned} \tag{5.28}$$

Equivalently,

$$\begin{aligned} &\frac{d}{dt}\,z(t) + \mathcal{A}z(t) = -\Pi\left(D\,\frac{du}{dt}\,(t) - kDu(t)\right), \quad t\geq 0, \\ &z(0) = y_0 - Du(0), \end{aligned} \tag{5.29}$$

$$z(t) = y(t) - Du(t), \quad t \geq 0. \tag{5.30}$$

In the following, we fix $k>0$ sufficiently large and $\eta>0$ such that (5.19) holds. In particular, for this choice of k and η, we also have

$$\lambda_i + k - \nu\eta \neq 0 \ \text{ for } \ i = 1,2,...,N. \tag{5.31}$$

We note first that in terms of z the controller (5.14) can be, equivalently, expressed as

$$u(t) = \eta 1\!\!1_\Gamma \sum_{j=1}^{N} \left\langle P_N z(t), \varphi_j^*\right\rangle_{\widetilde{H}} (\phi_j + \alpha\vec{n}). \tag{5.32}$$

Indeed, by (5.30) and (5.32), we have

$$\begin{aligned} u(t) \;\; = \;\; &\eta 1\!\!1_\Gamma \sum_{j=1}^{N} \left\langle P_N y(t), \varphi_j^*\right\rangle)_{\widetilde{H}} (\phi_j + \alpha\vec{n}) \\ &-\eta 1\!\!1_\Gamma \sum_{j=1}^{N} \left\langle u(t), D^*\varphi_j^*\right\rangle_0 (\phi_j + \alpha\vec{n}), \end{aligned} \tag{5.33}$$

where D^* is the adjoint of D.

On the other hand, if we set $\psi = D\mathbb{1}_\Gamma(\phi_j + \alpha\vec{n})$ and recall that

$$\mathcal{L}^*\varphi_i^* - \overline{\lambda}_i\varphi_i^* = \nabla p_i^* \ \text{ in } \ \mathcal{O}, \quad \varphi_i^* = 0 \text{ on } \ \partial\mathcal{O}, \qquad \nabla\cdot\varphi_j^* = 0,$$
$$\mathcal{L}\psi + k\psi = \nabla\widetilde{p} \ \text{ in } \ \mathcal{O}, \qquad \psi = \mathbb{1}_\Gamma(\phi_j + \alpha\vec{n}) \ \text{ on } \partial\mathcal{O}, \quad \nabla\cdot\psi = 0,$$

we get by (5.18) via Green's formula

$$\begin{aligned}\langle\phi_j+\alpha\vec{n}, D^*\varphi_i^*\rangle_0 = \int_{\mathcal{O}}\psi\cdot\overline{\varphi}_i^* dx = -\frac{\nu}{\lambda_i+k}\int_\Gamma(\phi_j+\alpha\vec{n})\cdot\frac{\partial\overline{\varphi}_i^*}{\partial n}\,dx \\ -\frac{1}{k+\lambda_i}\ \langle\alpha, p_i^*\rangle_0 = -\frac{\nu}{\lambda_i+k}\ \delta_{ij},\ \forall i,j = 1,...,N,\end{aligned} \tag{5.34}$$

because $\vec{n}\cdot\dfrac{\partial\varphi_i^*}{\partial n} = 0$, a.e. on $\partial\mathcal{O}$ (see [11], Lemma 3.3). Then, by (5.33), (5.34), we see that

$$\langle u(t), D^*\varphi_i^*\rangle_0 = \frac{-\eta\nu}{k+\lambda_i-\nu\eta}\ \langle P_N y, \varphi_i^*\rangle_{\widetilde{H}} \tag{5.35}$$

and, substituting into (5.33), we get (5.14) as claimed.

Now, by (5.32) and (5.29), we obtain that

$$\begin{aligned}&\frac{dz}{dt}+\mathcal{A}z = -\eta\sum_{j=1}^N\left\langle P_N\left(\frac{d}{dt}z(t)-kz(t)\right),\varphi_j^*\right\rangle_{\widetilde{H}}\Pi D(\mathbb{1}_\Gamma(\phi_j+\alpha\vec{n})),\\ &z(0) = z_0 = y_0 - Du(0).\end{aligned} \tag{5.36}$$

It is convenient to decompose system (5.36) into a finite dimensional part corresponding to the unstable spectrum $\{\lambda_j, j = 1, ...N\}$ of $\mathcal{A}$ and an infinite dimensional one which corresponds to the stable spectrum $\{\lambda_j, j > N\}$. Namely, we write (5.36) as

$$\frac{dz_u}{dt} + \mathcal{A}_u z_u = -\eta P_N\sum_{j=1}^N\left\langle P_N(\frac{dz}{dt} - kz), \varphi_j^*\right\rangle_{\widetilde{H}}\Pi D(\mathbb{1}_\Gamma(\phi_j + \alpha\vec{n})), \tag{5.37}$$

$$\frac{dz_s}{dt} + \mathcal{A}_s z_s = -\eta(I - P_N)\sum_{j=1}^N\left\langle P_N(\frac{dz}{dt} - kz), \varphi_j^*\right\rangle_{\widetilde{H}}\Pi D(\mathbb{1}_\Gamma(\phi_j + \alpha\vec{n})), \tag{5.38}$$

where $z = z_u + z_s,\ z_u \in X_u,\ z_s \in X_s$ and P_N is the algebraic projection on X_u defined in Section 2.1. If we represent z_u as

$$z_u = \sum_{j=1}^N z_j\varphi_j,$$

and recall (5.35), we can rewrite (5.37) as

$$z_j' + \lambda_j z_j = \frac{\eta\nu}{k+\lambda_j}\ (z_j' - kz_j), \quad t \geq 0. \tag{5.39}$$

Equivalently,

$$z_j' + \frac{(k+\lambda_j)\lambda_j + k\eta\nu}{k+\lambda_j-\eta\nu}\, z_j = 0,\ j = 1, ..., N. \tag{5.40}$$

By (5.19) we have

$$\operatorname{Re} \frac{(k+\lambda_j)\lambda_j + k\eta\nu}{k+\lambda_j-\eta\nu} > 0 \quad \text{for } j = 1, ..., N.$$

Then, by (5.39) there is $\gamma_0 > 0$ such that

$$|z_j(t)| \le e^{-\gamma_0 t}|z_j(0)|, \quad j = 1, ..., N. \tag{5.41}$$

On the other hand, by (5.38) we have

$$\frac{dz_s}{dt} + \mathcal{A}_s z_s = -\eta(I-P_N)\sum_{j=1}^{N}(z_j' - kz_j)\Pi D(\mathbb{1}_G(\phi_j + \alpha\vec{n})), \tag{5.42}$$

and since

$$\|e^{-\mathcal{A}_s t}\|_{L(\widetilde{H},\widetilde{H})} \le Ce^{-\gamma_1 t}, \quad \forall t \ge 0,$$

for some $\gamma_1 > 0$, we see by (5.40), (5.42) that

$$|z_s(t)|_{\widetilde{H}} \le C\exp(-\gamma_0 t)|z_s(0)|_{\widetilde{H}}, \quad \forall t \ge 0, \tag{5.43}$$

which together with (5.41) yields

$$|z(t)|_{\widetilde{H}} \le C\exp(-\gamma_0 t)|z(0)|_{\widetilde{H}}, \quad \forall t \ge 0.$$

Now, recalling (5.30) and (5.32), we obtain estimate (5.21), thereby completing the proof.

5.3.2 Proof of Theorem 5

The system (5.2) with the feedback controller

$$u = Fy = \eta\mathbb{1}_\Gamma \sum_{j=1}^{N} \mu_j \left\langle P_N y, \varphi_j^* \right\rangle_{\widetilde{H}} (\phi_j + \alpha\vec{n})$$

can be written as (see(5.28))

$$\begin{aligned}
&\Pi\frac{dy}{dt} + \mathcal{A}(y - DFy) + By = k\Pi\, DFy, \ t > 0,\\
&y(0) = y_0,
\end{aligned}$$

where $By = \Pi(y\cdot\nabla)y)$. Setting $z = y - DFy$ we rewrite it as (see(5.36))

$$\begin{aligned}
&\frac{dz}{dt} + \mathcal{A}z + B((I-DF)^{-1}z)\\
&= -\eta\sum_{j=1}^{N}\left\langle P_N\left(\frac{d}{dt}z(t) - kz(t)\right), \varphi_j^*\right\rangle_{\widetilde{H}} \Pi D(\mathbb{1}_\Gamma(\phi_j + \alpha\vec{n})).
\end{aligned}$$

We set, as in previous case,

$$z = z_u + z_s,\ z_u \in X_u,\ z_s \in X_s \text{ and } z_u = \sum_{j=1}^{N} z_j \varphi_j.$$

Recalling (5.39), (5.40) and (5.42) we get for $j = 1, ..., N$

$$z_j' + \frac{(k+\lambda_j)\lambda_j + k\eta\nu}{k+\lambda_j-\eta\nu} z_j + \frac{k+\lambda_j}{k+\lambda_j-\eta\nu} \left\langle B((I-DF)^{-1}(z), \varphi_j^*)\right\rangle_{\widetilde{H}} = 0$$

and $z_j' - kz_j = K_j(z)$, where

$$K_j(z) = -\frac{1}{k+\lambda_j-\eta\nu}\left((k+\lambda_j)^2 z_j + (k+\lambda_j)\left\langle B((I-DF)^{-1}z, \varphi_j^*)\right\rangle_{\widetilde{H}}\right).$$

Then, we may write the above system as

$$\frac{dz_u}{dt} + \widetilde{\mathcal{A}}_u z_u + \sum_{j=1}^{N} \frac{k+\lambda_j}{k+\lambda_j-\eta\nu}\left\langle B((I-DF)^{-1}(z), \varphi_j)\right\rangle_{\widetilde{H}} P_N\varphi_j = 0, \tag{5.44}$$

$$\begin{aligned} &\frac{dz_s}{dt} + \mathcal{A}_s z_s + (I-P_N)B((I-DF)^{-1})z \\ &\quad = -\eta(I-P_N)\sum_{j=1}^{N} K_j(z)\Pi D(\mathbb{1}_\Gamma(\phi_j + \alpha\vec{n})). \end{aligned} \tag{5.45}$$

Here $\widetilde{\mathcal{A}}_u \in L(X_u, X_u)$ is the operator defined by

$$\widetilde{\mathcal{A}}_u z_u = \sum_{j=1}^{N} \frac{(k+\lambda_j)\lambda_j + k\eta\nu}{k+\lambda_j-\eta\nu} z_j\varphi_j.$$

By virtue of (5.41), (5.43) both operators $\widetilde{\mathcal{A}}_u$, $\mathcal{A}_s$ are exponentially stable on spaces X_u, respectively X_s and, therefore, so is the operator

$$\mathcal{C}(z) = \widetilde{\mathcal{A}}_u z_u + \mathcal{A}_s z_s\,, z = z_u + z_s$$

on the space $\widetilde{H}$. We set

$$\begin{aligned} \mathcal{B}(z) = &\sum_{j=1}^{N} \frac{k+\lambda_j}{k+\lambda_j+\eta\nu} B((I-DF)^{-1}(z), \varphi_j)_{\widetilde{H}} P_N\varphi_j \\ &+ (I-P_N)B((I-DF)^{-1}z) + \eta(I-P_N)\sum_{j=1}^{N} K_j(z)\Pi D(\mathbb{1}_\Gamma(\phi_j + \alpha\vec{n})) \end{aligned}$$

and rewrite (5.44),(5.45) as

$$\frac{dz}{dt} + \mathcal{C}z + \mathcal{B}(z) = 0,\ t \geq 0,\ z(0) = z_0 = y_0 - DFy_0.$$

Equivalently,

$$z(t) = e^{-t\mathcal{C}} z_0 - \int_0^t e^{-(t-s)\mathcal{C}} \mathcal{B}(z(s)) ds, t \geq 0. \tag{5.46}$$

We recall that (see [11], [13])

$$\|B(z_1) - B(z_2)\|_W \leq C(\|z_1\|_Z + \|z_2\|_Z)\|z_1 - z_2\|_Z, \forall z_1, z_2 \in Z. \tag{5.47}$$

On the other hand, we have

$$|Dz|_{s+\frac{1}{2}} \leq \frac{C}{k-c}\ |z|_s, \quad \forall z \in (H^s(\partial\mathcal{O}))^d,\ s \geq -\frac{1}{2},$$

where c, C are independent of k, and this yields, for $s = 1 - \varepsilon$,

$$\|DFy\|_Z \leq \frac{C\eta}{k-c}\ \|y\|_Z, \quad \forall y \in Z.$$

This implies that, for k large enough and η as in condition (5.19), the operator $(I - DF)^{-1}$ is Lipschitz on the space Z and this implies that the local Lipschitz property (5.47) extends to the operator $\mathcal{B}$. Then arguing exactly as in the proof of Theorem 5.1 in [11] (see, also, [12] and [13]) we conclude that the integral equation (5.46) has for $||z_0||_W \leq \rho$ sufficiently small, a unique solution $z \in C([0,\infty); W) \cap L^2(0,\infty; Z)$ which has the exponential decay

$$||z(t)||_W \leq Me^{-\gamma t}||z_0||_W, \forall t > 0$$

which completes the proof.

5.3.3 Real stabilizing feedback controllers

We shall construct here a real stabilizing feedback controller of the form (5.14). To do this we replace the system of functions $\{\varphi_j\}$ by that obtained taking the real and imaginary parts of the this one. Namely, we consider the following system of functions in the space H

$$\psi_{2j-1} = \operatorname{Re}\varphi_j, \quad \psi_{2j} = \operatorname{Im}\varphi_j,$$

and similarly for the adjoint system

$$\psi^*_{2j-1} = \operatorname{Re}\varphi^*_j, \quad \psi^*_{2j} = \operatorname{Im}\varphi^*_j.$$

For the sake of simplicity, we assume that all unstable eigenvalues λ_j, $j = 1, ..., N$ are simple mentioning however, that the general case can be treated completely similar.

We set $\widetilde{X}_u = \text{lin span}\{\psi_j; j = 1.,..N\}$. It should be mentioned that the dimension of this space is still N and denote again by P_N the algebraic projection of H on $\widetilde{X}_u$. Then we decompose the space as $H = \widetilde{X}_u \oplus \widetilde{X}_s$ and note that the real operator $\mathcal{A}$ leaves invariant both spaces $\widetilde{X}_s$ and $\widetilde{X}_u$ and since $\widetilde{X}_s + i\widetilde{X}_s = X_s$ we infer that the operator $\widetilde{\mathcal{A}}^*_s = \mathcal{A}|_{\widetilde{X}_s}$ generates an exponential stable semigroup on $\widetilde{X}_s \subset H$. We set also $\widetilde{\mathcal{A}}^*_u = \mathcal{A}|_{\widetilde{X}_u}$.

We have

$$\begin{aligned} \mathcal{A}\psi_{2j-1} &= \operatorname{Re}\lambda_{2j-1}\ \psi_{2j-1} - \operatorname{Im}\lambda_{2j-1}\ \psi_{2j}, \\ \mathcal{A}\psi_{2j} &= \operatorname{Im}\lambda_{2j-1}\ \psi_{2j-1} + \operatorname{Re}\lambda_{2j-1}\ \psi_{2j}, \end{aligned} \tag{5.48}$$

and, similarly for ψ_j^*, i.e.,

$$\begin{aligned}\mathcal{A}^*\psi_{2j-1}^* &= \operatorname{Re}\lambda_{2j-1}\ \psi_{2j-1}^* - \operatorname{Im}\lambda_{2j-1}\ \psi_{2j}^*,\\ \mathcal{A}^*\psi_{2j}^* &= \operatorname{Im}\lambda_{2j-1}\ \psi_{2j-1}^* + \operatorname{Re}\lambda_{2j-1}\ \psi_{2j}^*.\end{aligned} \tag{5.49}$$

Equivalently,

$$\begin{aligned}\mathcal{L}^*\psi_{2j-1}^* &= \operatorname{Re}\lambda_{2j-1}\ \psi_{2j-1}^* - \operatorname{Im}\lambda_{2j-1}\ \psi_{2j}^* + \nabla p_{2j-1}^*,\\ \mathcal{L}^*\psi_{2j}^* &= \operatorname{Im}\lambda_{2j-1}\ \psi_{2j-1}^* + \operatorname{Re}\lambda_{2j-1}\ \psi_{2j}^* + \nabla p_{2j}^*.\end{aligned} \tag{5.50}$$

Under assumption (H2), the following real version of this hypothesis holds.

(H2)* *The system* $\{\frac{\partial \psi_j^*}{\partial n},\ ,j=1,..,N\}$ *is linearly independent on* Γ.

Then, following (5.14) consider the real feedback controller

$$u^* = \eta \mathbb{1}_\Gamma \sum_{j=1}^N \left(\left\langle P_N y, \psi_j^* \right\rangle - \sum_{\ell=1}^N K_{j\ell} \left\langle P_N y, \psi_\ell^* \right\rangle \right) (\phi_j^* + \alpha \vec{n}), \tag{5.51}$$

where $K_{j\ell}$ are made precise later on, ϕ_j^* is of the form

$$\phi_j^* = \sum_{i=1}^N a_{ij}^* \frac{\partial \psi_i^*}{\partial n}, \quad j=1,...,N, \tag{5.52}$$

and a_{ij}^* are chosen in a such a way that (see (5.18)),

$$\left\langle \frac{\partial \psi_i^*}{\partial n}, \phi_j^* \right\rangle_0 = -\frac{1}{\nu}\ \langle p_i^*, \alpha \rangle_0 + \delta_{ij},\ i,j=1,...,N. \tag{5.53}$$

(As seen earlier, this choice is possible by virtue of (H2)*.)

Now, proceeding as in Section 3.1, we show that for $K_{j\ell}$ suitably chosen the feedback controller (5.51) can be put in the form

$$u = \eta \mathbb{1}_\Gamma \sum_{j=1}^N \left\langle P_N z, \psi_j^* \right\rangle (\phi_j^* + \alpha \vec{n}), \tag{5.54}$$

where z is given by (5.30). Indeed, in terms of y, (5.54) can be written as (see (5.33))

$$u = \eta \mathbb{1}_\Gamma \sum_{j=1}^N \left\langle P_N y, \psi_j^* \right\rangle (\phi_j^* + \alpha \vec{n}) - \eta \mathbb{1}_\Gamma \sum_{j=1}^N \left\langle u, D^*\psi_j^* \right\rangle_0 (\phi_j^* + \alpha \vec{n}). \tag{5.55}$$

This yields

$$\begin{aligned}\langle u, D^*\psi_i \rangle_0 &= \eta \sum_{j=1}^N \left\langle P_N y, \psi_j^* \right\rangle \left\langle \phi_j^* + \alpha \vec{n}, D^*\psi_i^* \right\rangle_0 \\ &\quad -\eta \sum_{j=1}^N \left\langle u, D^*\psi_j^* \right\rangle_0 \left\langle \phi_j^* + \alpha \vec{n}, D^*\psi_i^* \right\rangle_0,\ i=1,...,N.\end{aligned} \tag{5.56}$$

On the other hand, by (5.50), (5.53), we see that

$$
\begin{aligned}
&(\mathrm{Re}\,\lambda_{2i-1}+k)\left\langle \psi^*_{2i-1}, D(\mathbb{1}_\Gamma(\phi^*_j+\alpha\vec{n}))\right\rangle \\
&\quad -\mathrm{Im}\,\lambda_{2i-1}\left\langle \psi^*_{2i}, D(\mathbb{1}_\Gamma(\phi^*_j+\alpha\vec{n}))\right\rangle \\
&\quad = -\nu\left\langle \frac{\partial\psi^*_{2i-1}}{\partial n}, \phi^*_j+\alpha\vec{n}\right\rangle_0 - \left\langle p^*_{2i-1},\alpha\right\rangle_0 = -\nu\delta_{2i-1j}, \\
&(\mathrm{Re}\,\lambda_{2i-1}+k)\left\langle \psi^*_{2i}, D(\mathbb{1}_\Gamma(\phi^*_j+\alpha_k\vec{n}))\right\rangle \\
&\quad -\mathrm{Im}\,\lambda_{2i-1}\left\langle \psi^*_{2i-1}, D(\mathbb{1}_\Gamma(\phi^*_j+\alpha\vec{n}))\right\rangle \\
&\quad = -\nu\left\langle \frac{\partial\psi^*_{2i}}{\partial n}, \phi^*_j+\alpha\vec{n}\right\rangle_0 - \left\langle p^*_{2i},\alpha\right\rangle_0 = -\nu\delta_{2ij}.
\end{aligned}
\tag{5.57}
$$

We set

$$
\begin{aligned}
&\left\langle \psi^*_i, D(\mathbb{1}_\Gamma(\phi^*_j+\alpha\vec{n}))\right\rangle = \eta_{ij}, \\
&\gamma_i = ((\mathrm{Re}\,\lambda_i+k)^2+(\mathrm{Im}\,\lambda_i)^2)^{-1},\ \ i,j=1,...,N.
\end{aligned}
\tag{5.58}
$$

This yields

$$
\begin{aligned}
&(\mathrm{Re}\,\lambda_{2i-1}+k)\eta_{2i-1j}+\mathrm{Im}\,\lambda_{2i-1}\ \eta_{2ij} = -\nu\delta_{2i-1j} \\
&(\mathrm{Re}\,\lambda_{2i-1}+k)\eta_{2ij}-\mathrm{Im}\,\lambda_{2i-1}\ \eta_{2i-1j} = -\nu\delta_{2ij}.
\end{aligned}
\tag{5.59}
$$

Then, by (5.56), (5.58), we obtain

$$
\langle u, D^*\psi^*_i\rangle_0 = \eta\sum_{j=1}^N \left\langle P_N y, \psi^*_j\right\rangle \eta_{ij} - \eta\sum_{j=1}^N \left\langle u, D^*\psi^*_j\right\rangle_0 \eta_{ij}.
$$

Equivalently,

$$
\sum_{j=1}^N (\delta_{ij}+\eta\eta_{ij})\left\langle u, D^*\psi_j\right\rangle_0 = \eta\sum_{j=1}^N \left\langle P_N y, \psi^*_j\right\rangle \eta_{ij}. \tag{5.60}
$$

By (5.58), (5.59) we have

$$
\begin{aligned}
\eta_{2i-1j} &= -\nu\gamma_{2i-1}(\mathrm{Re}\,\lambda_{2i-1}+k)\delta_{2i-1j}-\mathrm{Im}\,\lambda_{2i-1}\delta_{2ij}) \\
\eta_{2ij} &= -\nu\gamma_{2i-1}(\mathrm{Im}\,\lambda_{2i-1}\delta_{2i-1j}+(\mathrm{Re}\,\lambda_{2i-1}+k)\delta_{2i}).
\end{aligned}
\tag{5.61}
$$

We set $K=\|K_{j\ell}\|^N_{j,\ell=1}$, where

$$
K = \eta(\|\delta_{ij}+\nu\eta\eta_{ij}\|^N_{i,j=1})^{-1}\times||\eta_{ij}||. \tag{5.62}
$$

By (5.61) we see that, for k sufficiently large, K is well defined. By (5.60), we have

$$
\left\langle u, D^*\psi_j\right\rangle_0 = \sum_{\ell=1}^N K_{j\ell}\left\langle P_N y, \psi^*_\ell\right\rangle,\ \ j=1,...,N.
$$

Substituting the latter into (5.56), we see that u is of the form (5.51), where $K_{j\ell}$ is given by (5.62).

Now, we rewrite system (5.28) as (see (5.36), (5.54))

$$\frac{dz}{dt} + \mathcal{A}z = -\eta \sum_{j=1}^{N} \left\langle P_N \left(\frac{dz}{dt} - kz \right), \psi_j^* \right\rangle \Pi D(\mathbb{1}_\Gamma(\phi_j^* + \alpha\vec{n})) \tag{5.63}$$

with the corresponding projection on X_u^* (see (5.37))

$$\frac{dz_u}{dt} + \widetilde{\mathcal{A}}_u^* z_u = -\eta P_N \sum_{j=1}^{N} \left\langle P_N \left(\frac{dz}{dt} - kz \right), \psi_j^* \right\rangle \Pi D(\mathbb{1}_\Gamma(\phi_j^* + \alpha\vec{n})). \tag{5.64}$$

We set $z_u = \sum_{i=1}^{N} z_i \psi_i$ and so, we may write (5.58) as

$$\sum_{i=1}^{N} (b_{i\ell} z_i' + a_{i\ell} z_i) = -\eta \sum_{j=1}^{N} \sum_{i=1}^{N} b_{ij}(z_i' - kz_i)\eta_{\ell j}, \tag{5.65}$$

where $b_{i\ell} = \langle \psi_i, \psi_\ell^* \rangle$, $a_{i\ell} = \langle \mathcal{A}\psi_i, \psi_\ell^* \rangle$ and $\eta_{\ell j}$ are given by (5.61).

We set $B = \|b_{i\ell}\|_{i,\ell=1}^N$, $A_0 = \|a_{i\ell}\|_{i,\ell=1}^N$, $E = \|\eta_{\ell j}\|_{\ell,j=1}^N$ and rewrite (5.64) as

$$Bz' + A_0 z + \eta EB(z' - kz) = 0, \;\; t \geq 0, \tag{5.66}$$

where $z = \{z_i\}_{j=1}^N$.

To study the stability of system (5.66) it is convenient to consider the limit case $k = \infty$. Taking into account (5.61), we see that for $k \to \infty$ $\eta_{ij} k \to \delta_{ij}$ and so system (5.66) reduces to

$$Bz' + A_0 z + \eta Bz = 0,$$

which is exponentially stable if $\eta > 0$ is sufficiently large because

$$|Bz(t)| \leq e^{-\eta t}|Bz(0)| + \int_0^t e^{-\eta(t-s)}|A_0 z(s)|ds$$

and B is invertible as consequence of the independence of the systems $\{\psi_j\}_{j=1}^N$ and $\{\psi_j^*\}_{j=1}^N$. Hence, system (5.66) and, consequently, (5.63) is exponentially stable for k and η sufficiently large.

Then, we have the following real version of Theorem 1.

Theorem 6 *Under assumptions* (H1), (H2) *and* (5.7) *for k and η sufficiently large, there is a boundary feedback controller u^* of the form* (5.62) *which stabilizes exponentially system* (5.3).

5.4 An example to the boundary stabilization of a periodic flow in a $2-D$ channel

The previous results remain true for the Navier-Stokes system in a $2-D$ channel $\mathcal{O} = \{(x, y) \in \mathbb{R} \times (0, 1)\}$ with periodic condition in direction x. We illustrate this on the standard problem of laminar flows in a two-dimensional channel with the walls located at $y = 0, 1$. (See e.g., [1], [2], [3], [5].) We assume that the velocity field $(u(t, x, y), v(t, x, y))$ and the pressure $p(t, x, y)$ are 2π periodic in x. Then, the dynamic of flow is governed by the system

$$\begin{aligned}
&u_t - \nu\Delta u + uu_x + vu_y = p_x, \quad x \in \mathbb{R},\ y \in (0,1),\\
&v_t - \nu\Delta v + uv_x + vv_y = p_y, \quad x \in \mathbb{R},\ y \in (0,1),\\
&u_x + v_y = 0,\\
&u(t, x + 2\pi, y) \equiv u(t, x, y), \quad v(t, x + 2\pi, y) \equiv v(t, x, y), \quad y \in (0,1).
\end{aligned} \tag{5.67}$$

Consider a steady-state flow with zero vertical velocity component, i.e., $(U(x, y), 0)$. We have $U(x, y) = U(y) = c(y^2 - y), \quad \forall y \in (0, 1)$ and take $c = -\dfrac{a}{2\nu}$, $a \in \mathbb{R}^+$.

The linearization of (5.67) around the steady-state flow $(U(y), 0)$ leads to the following system

$$\begin{aligned}
&u_t - \nu\Delta u + u_x U + vU' = p_x, \quad y \in (0,1),\ x, t \in \mathbb{R},\\
&v_t - \nu\Delta v + v_x U = p_y, \quad u_x + v_y = 0,\\
&u(t, x + 2\pi, y) \equiv u(t, x, y), \quad v(t, x + 2\pi, y) \equiv v(t, x, y).
\end{aligned} \tag{5.68}$$

A convenient way to treat this system is to represent u, v as Fourier series. Let us briefly recall this standard procedure. Denote by $L^2_\pi(Q)$, $Q = (0, 2\pi) \times (0, 1)$ the space of all the functions $u \in L^2_{\text{loc}}(R \times (0, 1))$ which are 2π-periodic in x. These functions are characterized by their Fourier series

$$u(x, y) = a_0(y) + \sum_{k \neq 0} a_k(y) e^{ikx}, \ a_k = \bar{a}_{-k}, \sum_{k \in \mathbb{Z}} \int_0^1 |a_k|^2 dy < \infty.$$

Similarly, there are defined the Sobolev spaces $H^1_\pi(Q),\ H^2_\pi(Q)$.

We set $H = \{(u, v) \in (L^2_\pi(Q))^2;\ u_x + v_y = 0,\ v(x, 0) = v(x, 1) = 0\}$. If $u_x + v_y = 0$, then the trace of v at $y = 0, 1$ is well defined as an element of $H^{-1}(0, 2\pi) \times H^{-1}(0, 2\pi)$ (see, e.g., [24]). We also set

$$V = \{(u, v) \in H \cap H^1_\pi(Q);\ u(x, 0) = u(x, 1) = v(x, 0) = v(x, 1) = 0\}.$$

The space H can be defined equally as

$$\begin{aligned}
H \ = \ \Big\{ & u = \sum_{k \in \mathbb{Z}} u_k(y) e^{ikx},\ v = \sum_{k \in \mathbb{Z}} v_k(y) e^{ikx},\ v_k(0) = v_k(1) = 0,\\
& \sum_{k \in \mathbb{Z}} \int_0^1 (|u_k|^2 + |v_k|^2) dy < \infty,\ iku_k(y) + v_k'(y) = 0,\\
& \text{a.e. } y \in (0, 1),\ k \in \mathbb{Z} \Big\}.
\end{aligned}$$

Let $\Pi : L^2_\pi(Q) \to H$ be the Leray projector and $\mathcal{A} : D(\mathcal{A}) \subset H \to H'$ the operator

$$\begin{aligned}
\mathcal{A}(u, v) &= \Pi\{-\nu\Delta u + u_x U + vU',\ -\nu\Delta v + v_x U\},\\
&\forall (u, v) \in D(\mathcal{A}) = (H^2((0, 2\pi) \times (0, 1)) \cap V.
\end{aligned} \tag{5.69}$$

We associate with (5.69) the boundary value conditions

$$\begin{aligned} &u(t,x,0)=u^0(t,x), \quad u(t,x,1)=u^1(t,x), \quad t\ge 0,\ x\in\mathbb{R},\\ &v(t,x,0)=v^0(t,x), \quad v(t,x,1)=v^1(t,x), \quad t\ge 0,\ x\in\mathbb{R}, \end{aligned} \tag{5.70}$$

and, for $k^*>0$ sufficiently large, we consider the Dirichlet map $D : X \to L^2_\pi(Q)$ defined by $D(u^*,v^*)=(\widetilde{u},\widetilde{v})$,

$$\begin{aligned} &-\nu\Delta\widetilde{u}+\widetilde{u}_xU+\widetilde{v}U'+k^*\widetilde{u}=p_x,\ x\in\mathbb{R},\ y\in(0,1),\\ &-\nu\Delta\widetilde{v}+\widetilde{v}_xU+k^*\widetilde{v}=p_y,\ x\in\mathbb{R},\ y\in(0,1),\\ &\widetilde{u}_x+\widetilde{v}_y=0,\ \widetilde{u}(x+2\pi,y)=\widetilde{u}(x,y),\ \widetilde{v}(x+2\pi,y)=\widetilde{v}(x,y),\\ &\widetilde{u}(x,y)=u^*(x,y),\ \widetilde{v}(x,y)=v^*(x,y),\ y=0,1. \end{aligned} \tag{5.71}$$

Here

$$\begin{aligned} X \ = \ &\Big\{(u^*,v^*)\in L^2((0,2\pi)\times\partial(0,1));\ u^*(x+2\pi,y)=u^*(x,y),\\ &v^*(x+2\pi,y)=v^*(x,y),\ \int_0^{2\pi} v^*(x,0)dx=\int_0^{2\pi} v^*(x,1)dx\Big\}. \end{aligned}$$

Then system (5.69) , (5.70) can be written as

$$\begin{aligned} &\Pi\ \frac{d}{dt}\ y(t)+\mathcal{A}(y(t)-DU^*(t))=k^*\Pi\, DU^*(t)\ t\ge 0,\\ &y(0)=(u_0,v_0), \end{aligned} \tag{5.72}$$

where $y=(u,v)$, $U^*=(u^*,v^*)$.We denote again by $\mathcal{A}$ the extension of $\mathcal{A}$ on the complexified space $\widetilde{H}$ and by λ_j,φ_j the eigenvalues and corresponding eigenvectors of the operator $\mathcal{A}$. By φ_j^*, we denote the eigenvector to the dual operator $\mathcal{A}^*$. Written into this form, which is exactly (5.28), it is clear that one can apply 1 provided hypotheses (H1), (H2) are satisfied for $\mathcal{A}$. We check below the unique continuation hypothesis (H2)$'$ which has also an interest in itself.

Lemma 7 *Assume that all eigenvalues λ_j, $j=1,2,...,N$, are semi-simple. Then we have*

$$\frac{\partial\varphi_j}{\partial n}\ (x,y)\not\equiv 0, \quad x\in(0,2\pi),\ y=0,1, \tag{5.73}$$

$$\frac{\partial\varphi_j^*}{\partial n}\ (x,y)\not\equiv 0, \quad x\in(0,2\pi),\ y=0,1. \tag{5.74}$$

Proof. If we represent $\varphi_j=(u^j,v^j)$, then (5.73) reduces to

$$\left|\frac{\partial}{\partial y}\ v^j(x,y)\right|+\left|\frac{\partial}{\partial y}\ u^j(x,y)\right|>0,\ x\in(0,2\pi),\ y=0,1. \tag{5.75}$$

We set $\lambda=\lambda_j$ and $\varphi_j=(u,v)$. This means that, if λ is semisimple, then

$$\begin{aligned} &-\nu\Delta u+u_xU+vU'=\lambda u+p_x,\ x\in\mathbb{R},\ y\in(0,1),\\ &-\nu\Delta v+v_xU=\lambda v+p_y,\ x\in\mathbb{R},\ y\in(0,1),\\ &u_x+v_y=0,\\ &u(x+2\pi,y)=u(x,y),\ v(x+2\pi,y)=v(x,y). \end{aligned} \tag{5.76}$$

If we represent u, v, p as Fourier series with coefficients u_k, v_k, p_k we reduce (5.76) to the system

$$\begin{aligned}
&-\nu u_k'' + (\nu k^2 + ikU)u_k + U'v_k = ikp_k + \lambda u_k,\ \ y \in (0,1),\\
&-\nu v_k'' + (\nu k^2 + ikU)v_k = p_k' + \lambda v_k,\ \ iku_k + v_k' = 0 \ \text{ in } \ (0,1),\\
&u_k(0) = u_k(1) = 0, \quad v_k(0) = v_k(1) = 0.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&-\nu v_k^{\text{iv}} + (2\nu k^2 + ikU)v_k'' - k(\nu k^3 + ik^2U + iU'')v_k - \lambda(v_k'' - k^2 v_k) = 0,\\
&\hspace{20em} y \in (0,1),\\
&v_k(0) = v_k(1) = 0,\ \ v_k'(0) = v_k'(1) = 0, \quad \forall k \neq 0.
\end{aligned} \tag{5.77}$$

Now, let us check (5.74) or, equivalently, (5.76). We have

$$\frac{\partial}{\partial n}\, u(x,y) = -i \sum_{k\neq 0} \frac{e^{ikx}}{k} v_k''(y), \quad \forall x, y \in 0, 1,$$

and so (5.76) reduces to

$$|v_k''(0)| + |v_k''(1)| > 0 \ \text{ for all } k. \tag{5.78}$$

Assume that $v_k''(0) = v_k''(1) = 0$ for all k and lead from this to a contradiction. To this end we set $W_k = v_k'' - k^2 v_k$ and rewrite (5.78) as

$$\begin{aligned}
-\nu W_k'' + (\nu k^2 + ikU - \lambda)W_k &= ikU''v_k \ \text{ in } \ (0,1),\\
W_k(0) = W_k(1) &= 0.
\end{aligned} \tag{5.79}$$

If we multiply (5.79) by $\overline{W}_k$, integrate on $(0,1)$ and take the real part, we obtain that

$$\int_0^1 (\nu |W_k'|^2 + (\nu k^2 - \operatorname{Re}\lambda)|W_k|^2)dy = 0,\ \forall k$$

and since $\operatorname{Re}\lambda = \operatorname{Re}\lambda_j \leq 0$ for all $j = 1, ..., N$, we get $W_k \equiv 0$, and so $v_k \equiv 0$. The contradiction we arrived at proves (5.78) and (5.73). To prove (5.74) one proceeds similarly with dual system of eigenfunction but since the proof is more delicate we refer to [19].

We suspect that the same argument applies to prove hypothesis (H2), that is, $\left\{\dfrac{\partial \varphi_j^*}{\partial n}\right\}_{j=1}^N$ is linearly independent on $\partial\mathcal{O}$ but this remains to be done. Of course, if all φ_j^*, $j = 1, ..., N$, are eigenvectors corresponding to the same eigenvalue, the independence follows by Lemma 7.

Then, following the general case (5.14), we can design a feedback controller $(\widetilde{u}, \widetilde{v})$ for system (5.68). (In terms of (5.70), $\widetilde{u} = (u^0, u^1)$, $\widetilde{v} = (v^0, v^1)$.) We set $\varphi_j^* = (u_j^*, v_j^*)$, $j = 1, ..., N$, where φ_j^* are eigenvectors of the dual operator $\mathcal{A}^*$ with corresponding eigenvalues $\overline{\lambda}_j$ and $\operatorname{Re}\lambda_j < 0$ for $j = 1, ..., N$. (Recall that eigenvalues λ_j are repeated according to their multiplicity.)

We consider the boundary feedback controller

$$\begin{aligned}\widetilde{u}(t,x,y) &= \eta\sum_{j=1}^{N}\mu_j v_j(t)\phi_j^1(x,y),\ x\in\mathbb{R},\ y=0,1,\\ \widetilde{v}(t,x,y) &= \eta\sum_{j=1}^{N}\mu_j v_j(t)(\phi_j^2(x,y)+aH(y)),\ x\in\mathbb{R},\ y=0,1,\\ v_j(t) &= \int_0^{2\pi}(u(t,x,y)\bar{u}_j^*(x,y)+v(t,x,y)\bar{v}_j^*(x,y))dx\,dy\\ &= \sum_{k\neq 0}(u_k(t,y)(\bar{u}_j^*)_k(y)+v_k(t,y)(\bar{v}_k^*)_k(y)).\end{aligned}\tag{5.80}$$

Here a is an arbitrary constant, H is a smooth function such that $H(0)=-1$, $H(1)=1$, μ_j are defined as in (5.15), ϕ_j^i, $i=1,2$, are of the form (see (5.16))

$$\phi_j^1=\sum_{j=1}^{N}a_{ij}\chi_i^1,\quad \phi_j^2=\sum_{i=1}^{N}a_{ij}\chi_i^2,$$

where

$$(\chi_i^1,\chi_i^2)=\chi_i,\ \chi_i(x,0)=-\frac{\partial\varphi_i^*}{\partial y}(x,0),\ \chi_i(x,1)=\frac{\partial\varphi_i^*}{\partial y}(x,1)$$

and a_{ij} are chosen as in (5.17). By Theorem 1, we have.

Corollary 8 *If there is at most one unstable semi-simple eigenvalue for* (5.76) (*eventually multiple*) *then, for each $a\in\mathbb{R}$ and $\eta>0$ suitably chosen, the feedback boundary controller* (5.80) *stabilizes exponentially system* (5.68).

We note also that, by Theorem 5, the feedback controller (5.80) stabilizes the Navier–Stokes equation (5.67).

References

[1] Aamo O.M., Krstic M., Bewley T.R., Control of mixing by boundary feedback in $2D$-channel, *Automatica*, 39 (2003), 1597-1606.

[2] Balogh A., Liu W.-L., Krstic M., Stability enhancement by boundary control in $2D$ channel flow, *IEEE Trans. Autom. Control*, 11 (2001), 1696-1711.

[3] Barbu V., Stabilization of a plane channel flow by wall normal controllers, *Nonlinear Anal. Theory – Methods Appl.*, 56 (2007), 145-168.

[4] Barbu V., The internal stabilization by noise of the linearized Navier–Stokes equation, *ESAIM COCV*, 17 (2011), 117-130.

[5] Barbu V., Stabilization of a plane periodic channel flow by noise wall normal controllers, *Syst. Control Lett.*, 50 (10) (2010), 608-618.

[6] Barbu V., Exponential stabilization of the linearized Navier–Stokes equation by pointwise feedback controllers, *Automatica*, 46 (12) (2010), 1890-1895.

[7] Barbu V., *Stabilization of Navier–Stokes Flows*, Communications and Control Engineering, Springer, London, 2010.

[8] Barbu V., Internal stabilization of the Oseen–Stokes equations by Stratonovitch noise, *Systems & Control Letters*, 59 (2010), 608-614.

[9] Barbu V., Da Prato G., Internal stabilization by noise of the Navier–Stokes equation, *SIAM J. Control Optim.*, 49 (1) (2010), 1-21.

[10] Barbu V., Lasiecka I., Triggiani R., Abstract setting for tangential boundary stabilization of Navier–Stokes equations by high and low-gain feedback controllers, *Nonlinear Anal.*, 64 (2006), 2704-2746.

[11] Barbu V., Lasiecka I., Triggiani R., Tangential boundary stabilization of Navier–Stokes equations, *Mem. Am. Math. Soc.*, 852 (2006), 1-145.

[12] Barbu V., Lasiecka I., The unique continuations property of eigenfunctions to Stokes–Oseen operator is generic with respect to coefficients, *Nonlinear Analysis* (2011), doi:10.1016/j.n.a 2011.07.056

[13] Barbu V., Triggiani R., Internal stabilization of Navier–Stokes equations with finite dimensional controllers, *Indiana Univ. Math. J.*, 53 (2004), 1443-1494.

[14] Badra M., Feedback stabilization of the $2-D$ and $3-D$ Navier–Stokes equations based on an extended system, *ESAIM COCV*, 15 (2009), 934-968.

[15] Badra M., Lyapunov functions and local feedback boundary stabilization of the Navier–Stokes equations, *SIAM J. Control Optim.*, 48 (2009), 1797-1830.

[16] Badra M., Takahashi T., Stabilization of parabolic nonlinear systems with finite dimensional feedback or dynamical controllers; application to the Navier–Stokes system, *SIAM J. Control Optimiz.*, 49 (2011), 420-463.

[17] Fursikov A.V., Real processes of the $3-D$ Navier–Stokes systems and its feedback stabilization from the boundary. In: Agranovic M.S., Shubin M.A. (eds.), AMS Translations. Partial Differential Equations, M. Vishnik Seminar, 2002, 95-123.

[18] Fursikov A.V., Stabilization for the $3-D$ Navier–Stokes systems by feedback boundary control, *Discrete Contin. Dyn. Syst.*, 10 (2004), 289-314.

[19] Munteanu I., Normal feedback stabilization of periodic flows in a $2-D$ channel, *J. Optim. Theory Appl.*, 152 (2012), 413-438.

[20] Raymond J.P., Feedback boundary stabilization of the two dimensional Navier–Stokes equations, *SIAM J. Control Optim.*, 45 (2006), 790-828.

[21] Raymond J.P., Feedback boundary stabilization of the three dimensional Navier–Stokes equations, *J. Math. Pures Appl.*, 87 (2007), 627-669.

[22] Raymond J.P., Stokes and Navier–Stokes equations with nonhomogeneous boundary conditions, *Ann. Inst. H. Poincaré Analyse Nonlinéaire*, 64 (2007), 921-951. Feedback boundary stabilization of the three dimensional inequations, *J. Math. Pures Appl.*, 87 (2007), 627-669.

[23] Raymond J.P., Thevenet L., Boundary feedback stabilization of the two dimensional Navier–Stokes equations with finite dimensional controllers, *Discrete and Continuous Dynamical Systems,* vol. 27 (2010), 1159-1187.

[24] Temam R., *Navier–Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1985.

[25] Triggiani R., Stability enhancement of a $2-D$ linear Navier–Stokes channel flow by a $2-D$, wall-normal boundary controller, *Discrete Contin. Dyn. Syst.*, Series B (DCDS-B) 8 (2) (2007), 279-314.

[26] Vazquez R., Krstic M., A closed-form feedback controller for stabilization of linearized Navier–Stokes equations: The $2D$ Poisseuille flow, *IEEE Trans. Automatica*, 2005, 10.1109/CDC.2005.1583349

[27] Vazquez R., Tvelat E., Coron J.M., Control for fast and stable Laminar-to-High-Reynolds-Number transfer in a $2D$ channel flow, *Discrete Contin. Dyn. Syst.*, SB 10 (2008).

Lecture 6.
Internal exact controllability and feedback stabilization of stochastic linear parabolic like equations with linearly multiplicative noise

6 Introduction

Consider the stochastic nonlinear controlled parabolic equation

$$\begin{aligned}
&dX(t) - \Delta X(t)dt + a(t,\xi)X(t)dt + b(t,\xi)\cdot\nabla_\xi X(t)dt \\
&\quad + f(X(t))dt = X(t)dW(t) + \mathbb{1}_{\mathcal{O}_0}u(t)dt \text{ in } (0,\infty)\times\mathcal{O}, \\
&X = 0 \text{ on } (0,\infty)\times\partial\mathcal{O}, \quad X(0) = x \text{ in } \mathcal{O}.
\end{aligned} \tag{6.1}$$

Here, $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\mathcal{O}$ and $W(t)$ is a Wiener process of the form

$$W(t) = \sum_{k=1}^{\infty} \mu_k e_k(\xi)\beta_k(t), \quad t \geq 0, \ \xi \in \mathcal{O}, \tag{6.2}$$

where μ_k are real numbers, $\{e_k\} \subset C^2(\overline{\mathcal{O}})$ is an orthonormal system in $L^2(\mathcal{O})$ and $\{\beta_k\}_{k=1}^{\infty}$ are independent Brownian motions in a stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$.

We assume throughout this work that

$$\sum_{k=1}^{\infty} \mu_k^2 |e_k|_\infty^2 < \infty, \tag{6.3}$$

where $|\cdot|_\infty$ denotes the $L^\infty(\mathcal{O})$-norm.

The function $a : [0,\infty)\times\mathcal{O} \to \mathbb{R}$, $b : [0,\infty)\times\mathcal{O} \to \mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ are assumed to satisfy

$$a \in C([0,\infty)\times\overline{\mathcal{O}}), \ b \in C^1([0,\infty)\times\overline{\mathcal{O}}) \tag{6.4}$$

$$\sup\{|a(t)|_\infty + |\nabla b(t)|_\infty; \ t \geq 0\} < \infty \tag{6.5}$$

$$f \in \mathrm{Lip}(\mathbb{R}), \ f(0) = 0. \tag{6.6}$$

Finally, $\mathcal{O}_0$ is an open subdomain of $\mathcal{O}$ with smooth boundary, $\mathbb{1}_{\mathcal{O}_0}$ is its characteristic function and $u = u(t,\xi)$ is an adapted controller with respect to the filtration $\{\mathcal{F}_t\}$.

The main problem we address here is the design of a feedback controller $u = F(X)$ such that the corresponding closed loop system (6.1) is asymptotically stable in probability, that is,

$$\lim_{t\to\infty} X(t) = 0, \ \mathbb{P}\text{-a.s.}$$

It should be said, in this context, that a stronger property, the exact controllability of (6.1) in finite time, is in general still an open problem. (See, however, [5], [11] for partial results.)

Notation

By $L^2(\mathcal{O})$ we denote the space of all Lebesgue square integrable functions on $\mathcal{O}$ with the norm $|\cdot|_2$ and the scalar product $\langle\cdot,\cdot\rangle$. The scalar product of $L^2(\mathcal{O}_0)$ is denoted by $\langle\cdot,\cdot\rangle_0$. If Y is a Banach space with the norm $\|\cdot\|_Y$, we denote by $L^p(0,T;Y)$, $1\le p\le\infty$, the space of all Bochner measurable functions $u:(0,T)\to Y$ with $\|u\|_Y\in L^p(0,T)$. By $C([0,T];Y)$, we denote the space of all continuous Y-valued functions on $[0,T]$. We denote also by $H^k(\mathcal{O})$, $k=1,2$, the standard Sobolev space of functions on $\mathcal{O}$, $H_0^1(\mathcal{O})=\{y\in H^1(\mathcal{O});\ y=0 \text{ on } \partial\mathcal{O}\}$.

Given an $\mathcal{F}_t$-adapted process $u\in L^2(0,T;L^2(\Omega,L^2(\mathcal{O})))$, a continuous $\mathcal{F}_t$-adapted process $X:[0,T]\to L^2(\mathcal{O})$ is said to be a solution to (6.1) if

$$X\in L^2(\Omega;L^\infty(0,T;L^2(\mathcal{O})))\cap C([0,T];L^2(\Omega;L^2(\mathcal{O}))) \tag{6.7}$$

and

$$\begin{aligned} X(t,\xi)=&\int_0^t(\Delta X(s,\xi)-a(t,\xi)X(s,\xi)\\ &-b(s,\xi)\cdot\nabla_\xi X(s,\xi)\\ &+f(X(s,\xi)))ds+\int_0^t \mathbb{1}_{\mathcal{O}_0}u(s,\xi)ds\\ &+\int_0^t X(s,\xi)dW(s),\\ &\qquad\qquad \xi\in\mathcal{O},\ t\in(0,T),\ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{6.8}$$

where the integral is considered in sense of Itô.

7 The stabilization of equation (6.1)

We set $\mathcal{O}_1 = \mathcal{O} \setminus \overline{\mathcal{O}}_0$ and denote by $A_1 : D(A_1) \subset L^2(\mathcal{O}_1) \to L^2(\mathcal{O}_2)$ defined by

$$A_1 y = -\Delta y,\ y \in D(A_1) = H^1_0(\mathcal{O}_1) \cap H^2(\mathcal{O}_1), \tag{7.1}$$

or, equivalently,

$$\langle A_1 y, z\rangle_1 = \int_{\mathcal{O}_1} \nabla y \cdot \nabla z \, d\xi,\ \forall y, z \in H^1_0(\mathcal{O}_1), \tag{7.2}$$

where $\langle \cdot, \cdot \rangle_1$ is the duality on $H^1_0(\mathcal{O}_1) \times H^{-1}(\mathcal{O}_1)$ induced by $L^2(\mathcal{O}_1)$ as pivot space.

Denote by $\lambda^*_1(\mathcal{O}_1)$ the first eigenvalue of the operator A_1, that is,

$$\lambda^*_1(\mathcal{O}_1) = \inf\left\{ \int_{\mathcal{O}_1} |\nabla y|^2 d\xi;\ y \in H^1_0(\mathcal{O}_1),\ \int_{\mathcal{O}_1} y^2 d\xi = 1 \right\}. \tag{7.3}$$

Consider in (6.1) the feedback controller

$$u = -\eta X,\ \eta \in \mathbb{R}^+, \tag{7.4}$$

and the corresponding closed loop system

$$\begin{aligned} &dX - X dt + aX dt + b \cdot \nabla X dt + f(X) dt \\ &\qquad = X dt - \eta \mathbb{1}_{\mathcal{O}_0} X dt \ \text{ in } (0, \infty) \times \mathcal{O}, \\ &X(0) = x \text{ in } \mathcal{O}, \quad X = 0 \text{ on } (0, \infty) \times \partial\mathcal{O}. \end{aligned} \tag{7.5}$$

Theorem 1 *Assume that*

$$\begin{aligned}\lambda_1^*(\mathcal{O}_1) - \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2|e_j|_\infty^2 - \|f\|_{\mathrm{Lip}} \\ - \sup\left\{-a(t,\xi) + \frac{1}{2}\,\mathrm{div}_\xi b(t,\xi);\ (t,\xi)\in\mathbb{R}^+\times\mathcal{O}\right\} > 0.\end{aligned} \tag{7.6}$$

Then, for each $x \in L^2(\mathcal{O})$ and for η sufficiently large (independent of x), the feedback controller (7.4) exponentially stabilizes in probability equation (6.1). More precisely, there is $\gamma > 0$ such that the solution X to (7.5) satisfies

$$\lim_{t\to\infty} e^{\gamma t}|X(t)|_2^2 = 0,\ \ \mathbb{P}\text{-}a.s. \tag{7.7}$$

$$e^{\gamma t}\mathbb{E}|X(t)|_2^2 + \mathbb{E}\int_0^\infty e^{\gamma t}|X(t)|_2^2 dt \le C|x|_2^2. \tag{7.8}$$

We recall that, by the classical Rayleigh–Faber–Krahn perimetric inequality in dimension $d \ge 2$, we have

$$\lambda_1^*(\mathcal{O}_1) \ge \left(\frac{\omega_d}{|\mathcal{O}_1|}\right)^{\frac{2}{d}} J_{\frac{d}{2}-1,1}, \tag{7.9}$$

where $|\mathcal{O}_1| = \mathrm{Vol}(\mathcal{O}_1)$, $\omega_d = \pi^{\frac{d}{2}}/\left(\Gamma\left(\frac{d}{2}+1\right)\right)$, and $J_{m,1}$ is the first positive zero of the Bessel function $I_m(r)$.

Then, by Theorem 1, we conclude that, if $|\mathcal{O}_1|$ is sufficiently small, then the feedback controller (7.4) is exponentially stabilizable.

Corollary 2 *Assume under hypotheses* (6.3)–(6.6) *that*

$$|\mathcal{O}_1| \le \omega_d^{\frac{2}{d}} J_{\frac{d}{2}-1,1} \left(\frac{1}{2} \sum_{Lip}^{\infty} \mu_j^2 |e_j|_\infty^2 + \sup_{\mathbb{R}^+ \times \mathcal{O}} \left\{ -a + \frac{1}{2} \operatorname{div}_\xi b \right\} + \|f\|_{\mathrm{Lip}} \right)^{-\frac{1}{2}}. \tag{7.10}$$

Then, for each $x \in L^2(\mathcal{O})$*, the feedback controller* (7.4) *stabilizes system* (6.1) *in sense of* (7.7)*,* (7.8)*.*

We note that, in particular, condition (7.10) is satisfied if the Hausdorff distance $d_H(\partial\mathcal{O}, \partial\mathcal{O}_0)$ is sufficiently small.

Remark 3 One might suspect that system (6.1) is stabilizable and even exact null controllable in probability by controllers u with support in an arbitrary open subset $\mathcal{O}_0 \subset \mathcal{O}$, as is the case in the deterministic case (see, e.g., [2], [7], [3]), but so far this is an open problem.

Roughly speaking, Theorem 1 implies, in particular, that the stochastic perturbation destabilizing effect in system $dX - \Delta X dt = X dW$ can be compensated by a linear stabilizing feedback controller with support in a subdomain $\mathcal{O}_0$ satisfying (7.10).

An example. The stochastic equation

$$\begin{aligned} &dX - X_{\xi\xi} dt + (aX + bX_\xi) dt = \mu X d\beta + V dt, \ 0 < \xi < 1, \\ &X(t,0) = X(t,1) = 0, \ t \ge 0, \end{aligned}$$

where β is a Brownian motion and $\mu \in \mathbb{R}$, $a \in C([0,T] \times \mathbb{R})$, $b \in C^1([0,1] \times \mathbb{R})$, is exponentially stabilizable in probability by any feedback controller $V = -\eta \mathbb{1}_{[a_1,a_2]} X$, where $\eta > 0$ is sufficiently large and $0 < a_1 < a_2 < 1$ are such that

$$\pi \inf \left\{ \frac{1}{a_1}, \frac{1}{1-a_2} \right\} > \frac{\mu^2}{2} + \sup_{(t,\xi) \in \mathbb{R}^+ \times (0,1)} \left\{ -a(t,\xi) + \frac{1}{2} b_\xi(t,\xi) \right\}.$$

8 Proof of Theorem 1

The main ingredient of the proof is the following lemma.

Lemma 1 *For each $\varepsilon > 0$ there is $\eta_0 = \eta_0(\varepsilon)$ such that*

$$\int_{\mathcal{O}} |\nabla y(\xi)|^2 d\xi + \eta \int_{\mathcal{O}_0} y^2(\xi) d\xi \geq (\lambda_1^*(\mathcal{O}_1) - \varepsilon)|y|_2^2, \quad \forall y \in H_0^1(\mathcal{O}),\ \eta \geq \eta_0. \tag{8.1}$$

The proof is well known (see [1], [4]), but we outline it for the sake of completeness. Denote by ν_1 the first eigenvalue of the self-adjoint operator $A^\eta y = Ay + \eta \mathbb{1}_{\mathcal{O}_0} y$, $\forall y \in D(A^\eta) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$, where $A = -\Delta$, $D(A) = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ and $\eta \in \mathbb{R}^+$.

We have by the Rayleigh formula

$$\nu_1^\eta = \inf\left\{\int_{\mathcal{O}} |\nabla y|^2 d\xi + \eta \int_{\mathcal{O}_0} |y|^2 d\xi;\ |y|_2 = 1\right\} \leq \lambda_1^*(\mathcal{O}_1) \tag{8.2}$$

because any function $y \in H_0^1(\mathcal{O}_1)$ can be extended by zero to $H_0^1(\mathcal{O})$ across the smooth boundary $\partial\mathcal{O}_1 = \partial\mathcal{O}_0$. Let $\varphi_1^\eta \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ be such that $A^\eta \varphi_1^\eta = \nu_1^\eta \varphi_1^\eta$, $|\varphi_1^\eta|_2 = 1$. We have by (8.2) that

$$\int_{\mathcal{O}} |\nabla \varphi_1^\eta|^2 d\xi + \eta \int_{\mathcal{O}_0} |\varphi_1^\eta|^2 d\xi = \nu_1^\eta \leq \lambda_1^*(\mathcal{O}),\ \forall \eta > 0.$$

Then, on a subsequence, again denoted η, we have for $\eta \to \infty$, that $\nu_1^\eta \to \nu^*$ and

$$\begin{aligned} \varphi_1^\eta &\to \varphi_1 \quad \text{weakly in } H_0^1(\mathcal{O}), \text{ strongly in } L^2(\mathcal{O}) \\ \int_{\mathcal{O}_0} |\varphi_1^\eta|^2 d\xi &\to 0. \end{aligned}$$

We have, therefore, $\varphi_1 \in H_0^1(\mathcal{O}_1)$, $|\varphi_1|_2 = 0$ and, since $A^\eta \varphi_1^\eta|_{\mathcal{O}_1} = A_1 \varphi_1^\eta$, we have also that $A_1 \varphi_1 = \nu^* \varphi_1$. Moreover, by (8.2) we see that $\nu^* \leq \lambda_1^*(\mathcal{O}_1)$. Since $\lambda_1^*(\mathcal{O}_1)$ is the first eigenvalue of A_1, we have that $\nu^* = \lambda_1^*(\mathcal{O}_1)$ and

$$\lim_{\eta\to\infty} \inf\left\{\int_{\mathcal{O}} |\nabla y|^2 d\xi + \eta \int_{\mathcal{O}_0} y^2 d\xi;\ |y|_1 = 1\right\} = \lambda_1^*(\mathcal{O}_1).$$

This yields (8.1), as claimed.

Proof of Theorem 1 (continued). By applying Itô's formula in (7.5), which by virtue of (6.7) is possible, we obtain that

$$\begin{aligned} &\frac{1}{2}\, d(e^{\gamma t}|X(t)|_2^2) + \int_{\mathcal{O}} e^{\gamma t} |\nabla X(t,\xi)|^2 d\xi\, dt + \int_{\mathcal{O}} (a(t,\xi) - \gamma \\ &-\frac{1}{2}\, \mathrm{div}_\xi b(t,\xi)) e^{\gamma t} X^2(t,\xi) d\xi + \int_{\mathcal{O}} e^{\gamma t} f(X(t,\xi)) X(t,\xi) d\xi \\ &= \frac{1}{2} \int_{\mathcal{O}} e^{\gamma t} \sum_{k=1}^\infty |(Xe_k)(t,\xi)|^2 d\xi\, dt - \eta \int_{\mathcal{O}_0} e^{\gamma t} |X(t,\xi)|^2 d\xi\, dt \\ &+ \int_{\mathcal{O}} e^{\gamma t} \sum_{k=1}^\infty (Xe_k)(t,\xi) X(t,\xi) d\beta_k(t),\ \mathbb{P}\text{-a.s.},\ t \geq 0. \end{aligned}$$

Equivalently,

$$\frac{1}{2}\,e^{\gamma t}|X(t)|_2^2+\int_0^t K(s)ds=\frac{1}{2}\,|x|_2^2+M(t),\ t\ge 0,\ \mathbb{P}\text{-a.s.},\tag{8.3}$$

where

$$\begin{aligned}K(t)=\int_{\mathcal{O}}&e^{\gamma t}(|\nabla X(t,\xi)|^2\\&+(a(t,\xi)-\gamma-\frac{1}{2}\,\mathrm{div}_\xi b(t,\xi))|X(t,\xi)|^2\\&+f(X(t,\xi))X(t,\xi)-\frac{1}{2}\sum_{k=1}^{\infty}X^2(t,\xi)e_k^2(\xi))d\xi\\&+\eta\int_{\mathcal{O}_0}e^{\gamma t}|X(t,\xi)|^2d\xi,\end{aligned}\tag{8.4}$$

$$M(t)=\int_0^t\int_{\mathcal{O}}\sum_{k=1}^{\infty}e^{\gamma s}X^2(s,\xi)e_k(s,\xi)d\beta_k(s),\ t\ge 0.\tag{8.5}$$

By Lemma 1 and by (6.6), (7.6), we see that, for $\eta\ge\eta_0$ sufficiently large and $0<\gamma\le\gamma_0$ sufficiently small, we have

$$K(t)\ge\varepsilon_0\int_{\mathcal{O}}e^{\gamma t}|X(t,\xi)|^2d\xi,\ \forall t>0,\ \mathbb{P}\text{-a.s.},\tag{8.6}$$

where $\varepsilon_0>0$. Taking expectation into (8.3), we obtain that

$$\frac{1}{2}\,\mathbb{E}[e^{\gamma t}|X(t)|_2^2]+\varepsilon_0\int_0^t e^{\gamma s}\mathbb{E}|X(s)|_2^2ds\le\frac{1}{2}\,|x|_2^2,\ \forall t\ge 0.\tag{8.7}$$

Since $t\to\int_0^t K(s)$ is an a.s. nondecreasing stochastic process and $t\to M(t)$ is a continuous local martingale, we infer that there exist

$$\lim_{t\to\infty}(e^{\gamma t}|X(t)|_2^2)<\infty,\ K(\infty)<\infty,$$

which imply (7.7), (7.8), as claimed.

Remark 2 By the proof, it is clear that Theorem 1 extends to more general nonlinear functions $f=f(t,\xi,x)$, as well as to smooth functions $\mu_k=\mu_k(t,\xi)$. Also, the Lipschitz condition (6.6) can be weakened to f continuous, monotonically increasing and with polynomial growth. Moreover, Δ can be replaced by any strongly elliptic linear operator in $\mathcal{O}$. The details are omitted.

9 Stabilization and exact controllability of Navier–Stokes equations with multiplicative noise

We consider here the stochastic Navier–Stokes equation

$$\begin{aligned} & dX(t) - \nu\Delta X(t)dt + (a(t)\cdot\nabla)X(t)dt \\ & \quad +(X(t)\cdot\nabla)b(t)dt + (X(t)\cdot\nabla)X(t)dt \\ & \quad = X(t)dW(t) + \nabla p(t)dt + \mathbb{1}_{\mathcal{O}_0}u(t)dt \\ & \qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathcal{O}, \\ & \nabla\cdot X(t) = 0 \quad \text{in } (0,\infty)\times\mathcal{O}, \\ & X(t) = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O},\ X(0) = x \quad \text{in } \mathcal{O}, \end{aligned} \tag{9.1}$$

where $\nu > 0$, $a, b \in (C^1((0,\infty)\times\overline{\mathcal{O}}))^2$, $\nabla\cdot a = \nabla\cdot b = 0$, $a\cdot\vec{n} = b\cdot\vec{n} = 0$ on $\partial\mathcal{O}$. Here $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^2$ and $\mathcal{O}_0$ is an open subset of $\mathcal{O}$. The boundaries $\partial\mathcal{O}$ and $\partial\mathcal{O}_0$ are assumed to be smooth. We set

$$H = \{y \in (L^2(\mathcal{O}))^2;\ \nabla\cdot y = 0,\ y\cdot\vec{n} = 0 \text{ on } \partial\mathcal{O}\},$$

where $\vec{n}$ is the normal to $\partial\mathcal{O}$. We denote by $\langle\cdot,\cdot\rangle_H$ the scalar product of H and by $|\cdot|_H$ the norm. The Wiener process $W(t)$ is of the form (6.2), where $\{e_k\} \subset (C^2(\overline{\mathcal{O}}))^2$ is an orthonormal basis in H, and $\mu_k \in \mathbb{R}$. As in the previous case, the main objective here is the design of a stabilizable feedback controller u for equation (9.1).

We use the standard notations

$$\begin{aligned} H &= \{y \in (L^2(\mathcal{O}))^d;\ \nabla\cdot y = 0 \text{ in } \mathcal{O},\ y\cdot\vec{n} = 0 \text{ on } \partial\mathcal{O}\}, \\ V &= \{y \in (H_0^1(\mathcal{O}))^d;\ \nabla\cdot y = 0 \text{ in } \mathcal{O}\}, \\ A &= -\nu\Pi\Delta,\ D(A) = (H^2(\mathcal{O}))^d \cap V, \end{aligned}$$

where Π is the Leray projector on H.

Consider the Stokes operator A_1 on $\mathcal{O}_1 = \mathcal{O}\setminus\mathcal{O}_0$, that is,

$$\langle A_1 y, \varphi\rangle = \nu\sum_{i=1}^d \int_{\mathcal{O}_1} \nabla y_i\cdot\nabla\varphi_i d\xi,\ \forall\varphi\in V_1,$$

where $V_1 = \{y \in (H_0^1(\mathcal{O}_1))^d;\ \nabla\cdot y = 0 \text{ in } \mathcal{O}_1\}$. Denote again by $\lambda_1^*(\mathcal{O}_1)$ the first eigenvalue of A_1, that is,

$$\lambda_1^*(\mathcal{O}_1) = \inf\left\{\nu\sum_{i=1}^d \int_{\mathcal{O}} |\nabla\varphi_i|^2 d\xi, \varphi\in V_1, \int_{\mathcal{O}_1} |\varphi|^2 d\xi = 1\right\}. \tag{9.2}$$

Also, in this case, we have (see Lemma 1 in [4]), for $\eta \ge \eta_0(\varepsilon)$ and $\varepsilon > 0$,

$$\langle Ay, y\rangle_H + \eta\,\langle\Pi(\mathbb{1}_{\mathcal{O}_0}y), y\rangle_H \ge (\lambda_1^*(\mathcal{O}_1) - \varepsilon)|y|_H^2,\ \forall y\in V. \tag{9.3}$$

We consider in system (9.1) the linear feedback controller

$$u = -\eta X, \ \eta > 0. \tag{9.4}$$

We set

$$\gamma^*(t) = \sup\left\{\int_{\mathcal{O}} |y_i D_i b_j y_j d\xi|; \ |y|_H = 1\right\} < \infty,$$

where $b = \{b_1, b_2\}$.

The closed loop system (9.1) with the feedback controller (9.4) has a unique strong solution in the sense of (6.7), (6.8). (See, e.g., [8], p. 281.)

We have

Theorem 1 *Assume that*

$$\lambda_1^*(\mathcal{O}_1) > \frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2|e_j|_\infty^2 + \sup_{t\in\mathbb{R}^+}\gamma^*(t). \tag{9.5}$$

Then, for each $x \in H$ and η sufficiently large independent of x, the solution X to the closed loop system (9.1) *with the feedback controller* (9.4) *satisfies*

$$\mathbb{E}[e^{\gamma t}|X(t)|_H^2] + \int_0^\infty e^{\gamma t}\mathbb{E}|X(t)|_H^2 dt < C|x|_H^2, \tag{9.6}$$

$$\lim_{t\to\infty} e^{\gamma t}|X(t)|_H^2 = 0,\ \mathbb{P}\text{-}a.s., \tag{9.7}$$

for some $\gamma > 0$.

The proof is essentially the same as that of Theorem 1, and so it will be sketched only. Taking into account that

$$\langle (X\cdot\nabla)X, X\rangle_H + \langle (a(t)\cdot\nabla)X, X\rangle_H = 0,\ t>0, \mathbb{P}\text{-a.s.},$$

we obtain by (9.1), (9.4), via Itô's formula, that

$$\begin{aligned}\frac{1}{2}\ e^{\gamma t}|X(t)|_H^2 &+ \int_0^t e^{\gamma s}\Big(\langle AX(s), X(s)\rangle_H + \langle X(s)\cdot\nabla b(s), X(s)\rangle_H \\ &-\frac{1}{2}\sum_{j=1}^{\infty}|X(s)e_j|_H^2 + \eta\,\langle \mathbb{1}_{\mathcal{O}_0}X(s), X(s)\rangle_H\Big) ds \\ &= \frac{1}{2}\ |x|_H^2 + \int_0^t e^{\gamma s}\sum_{j=1}^{\infty}\langle X(s)e_j, X(s)\rangle_H\, d\beta_j(s),\ t\ge 0.\end{aligned} \tag{9.8}$$

Then, by virtue of (9.3) and (9.5), we have, by (9.8), that

$$\frac{1}{2}\ e^{\gamma t}|X(t)|_H^2 + I(t) = \frac{1}{2}\ |x|_H^2 + M^*(t),\ t\ge 0,\ \mathbb{P}\text{-a.s.},$$

where $I(t)$ is a nondecreasing process, which satisfies

$$\mathbb{E}[I(t)] \ge \varepsilon_0\int_0^t e^{\gamma s}\mathbb{E}|X(s)|_H^2 ds,\ \forall t\ge 0,$$

for η sufficiently large, and $M^*(t) = \int_0^t e^{\gamma s}\sum_{j=1}^{\infty}\langle X(s)e_j, X(s)\rangle_H\, d\beta_j(s)$ is a continuous local martingale. As in the previous case, this implies via [10] that $\lim_{t\to\infty} e^{\gamma t}|X(t)|_H^2$ exists $\mathbb{P}$-a.s. and, therefore, (9.6) and (9.7) hold.

The exact controllability.

Consider the linear part of equation (9.1) in the special case

$$W(t)=\sum_{j=1}^{M}\mu_j(t)\beta_j(t), \tag{9.9}$$

where μ is an adapted process and $\mu_j \in L^\infty((0,T)\times\Omega)$. By the transformation $X = e^{\int_0^t \sum_{j=1}^{M}\mu_j d\beta_j} y$, we reduce equation (9.1) to

$$\begin{aligned}&\frac{\partial y}{\partial t}-\nu\Delta y+(a(t)\cdot\nabla)y+(y\cdot\nabla)b(t)+\widetilde{\mu}(t)y=1\!\!1_{\mathcal{O}_0}v \text{ in } (0,T)\times\mathcal{O},\\ &y(0)=x \text{ in } \mathcal{O},\\ &\nabla\cdot y=0,\ \ y=0 \text{ on } (0,T)\times\partial\mathcal{O}.\end{aligned} \tag{9.10}$$

We set $z(t)=e^{\int_0^t\widetilde{\mu}(s)ds}y(t)$. Then, (9.10) reduces to

$$\begin{aligned}&\frac{\partial}{\partial t}z-\nu\Delta z+(a(t)\cdot\nabla)z+(z\cdot\nabla)b(t)\\ &\qquad = 1\!\!1_{\mathcal{O}_0}e^{\int_0^t\widetilde{\mu}ds-\int_0^t\sum_{j=1}^{M}\mu_j d\beta_j}u=1\!\!1_{\mathcal{O}_0}v(t),\\ &z(0)=x \text{ in } \mathcal{O},\\ &\nabla\cdot z=0,\ \ z=0 \text{ on } (0,T)\times\partial\mathcal{O}.\end{aligned} \tag{9.11}$$

Here, $v(t)=e^{-\int_0^t\sum_{j=1}^{M}\mu_j d\beta_j}u$, $\widetilde{\mu}=\frac{1}{2}\sum_{j=1}^{M}\mu_j^2$.

On the other hand, we know (see Imanuvilov [9]) that (9.11) is exactly null controllable, that is, there is a controller $v\in L^2(0,T)\times\mathcal{O}$ such that $z(T)\equiv 0$. This means that (9.10) is exactly null controllable by the adopted (progressively measurable) controller

$$u(t,\xi)=e^{-\int_0^t\mu d\beta+\frac{1}{2}\int_0^t\mu^2 ds}v(t,\xi).$$

We have proved, therefore,

Theorem 2 *There is an adapted controller u such that the solution X to (9.1) satisfies*

$$X(T,\xi)\equiv 0,\quad \mathbb{P}\text{-a.s.},\ \ \xi\in\mathcal{O}. \tag{9.12}$$

We note that Theorem 2 remains true for the linearized equation (6.1).

Open problems

1° *Does Theorem* 2 *remain true* (in sense of local controllability) *for complete Navier–Stokes equation* (9.1)*?*

The answer is positive via the fixed point argument if $\mu = \mu(t)$ is deterministic.

2° *Does Theorem* 2 *remain true for more general noises?*

10 Final remarks

In order to make clear the novelty of the above results and the principal difficulties related to the internal stabilization of equations (6.1), we note that, via the substitution $y = e^{W(t)}X$, equation (6.1) reduces to a parabolic equation of the form

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + \widetilde{a}(t)y + \widetilde{b}(t)\cdot\nabla y + \frac{1}{2}\sum_{i=1}^{\infty}\mu_k^2 e_k^2 y = 1\!\!1_{\mathcal{O}_0}u, \ \mathbb{P}\text{-a.s.}, \\ &y = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O}, \end{aligned} \tag{10.1}$$

with random coefficients $\widetilde{a}, \widetilde{b}$.

If $\widetilde{a}$ and $\widetilde{b}$ are independent functions of t, then (10.1) can be stabilized by a controller $u = \sum_{j=1}^{N} u_j(t)\psi_j$, where ψ_j are linear combinations of eigenfunctions of the dual operator $y \to -\Delta y + \widetilde{a}y - \text{div}(by)$ (see [7] or [3] for the case of Navier–Stokes equations).

For deterministic equations of the form (10.1) with smooth time dependent coefficients, a similar result was recently proved in [6] (for the Navier–Stokes equations), but it cannot be applied, however, to the random equation (10.1) since it does not provide an adapted stabilizable controller $u = u(t)$. (The reason is that the argument in [6] relies on exact $\mathbb{P}$-a.s. controllability of (10.1) via an adapted controller u which so far is still an open problem.)

References

[1] S. Aniţa, Internal stabilization of diffusion equation, *Nonlinear Studies*, 8 (2001), 193-202.

[2] V. Barbu, Controllability of parabolic and Navier–Stokes equations, *Scientiae Mathematicae Japonicae*, vol. 56 (1) (2002), 143-211.

[3] V. Barbu, *Stabilization of Navier-Stokes Flows*, Springer, London. Dordrecht. Heidelberg, Berlin, 2011.

[4] V. Barbu, C. Lefter, Internal stabilizability of the Navier–Stokes equations, *Systems & Control Letters*, 48 (2003), 161-167.

[5] V. Barbu, A. Rascanu, G. Tessitore, Carleman estimates and controllability of linear stochastic heat equations, Appl. Math. Optimiz. 47 (2003), 97-1209.

[6] V. Barbu, S.S. Rodriguez, A. Shirikyan, Internal exponential stabilization to a nonstationary solution for $3-D$ Navier–Stokes equations, SIAM J. Control Optim. vol. 49 (2011), 1454-1478.

[7] V. Barbu, R. Triggiani, Internal stabilization of Navier–Stokes equations with finite-dimensional controllers, Indiana Univ. Math. J. 53 (2004), 1443-1494.

[8] G. Da Prato, J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[9] O. Imanuvilov, On the exact controllability of the Navier Stokes equations, *ESAIM COCV* 3 (1998), 97-131.

[10] R.S. Lipster, A. Shiryaev, *Theory of Martingales*, Kluwer Academic, Dordrecht, 1989.

[11] S. Tang, Z. Zhang, Null controllability for forward and backward stochastic parabolic equations, SIAM J. Control Optim. 48 (2009), 2191-2216.