Numerical methods for EIT

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Outline

1. The Calderón problem and linearization
2. The CGO-method for reconstruction in 2D
3. The CGO-method for reconstruction in 3D
4. The Calderón problem with partial data

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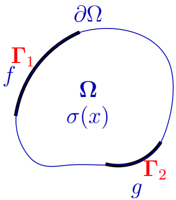
Calderón problem with partial data

Smooth bounded domain $\Omega \subset \mathbb{R}^d$, conductivity coefficient

$$0 < c \leq \sigma \leq C < \infty; \qquad \sigma \equiv 1 \text{ near } \partial\Omega.$$

Voltage potential u in Ω generated by boundary voltage potential f

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = f, \quad \text{supp} f \subset \Gamma_1.$$



Current field: $J = \sigma \nabla u$.
Normal component of current field at $\partial\Omega$:

$$g = \sigma \partial_\nu u|_{\Gamma_2}.$$

Partial Dirichlet to Neumann (voltage to current) map

$$\Lambda_\sigma : f \mapsto g.$$

Short story (with CGO solutions)

3D

1985 Kohn and Vogelius: Uniqueness for $\Gamma_1 = \Gamma_2$ possible small, but σ piecewise real analytic

2002 Bukhgeim and Uhlmann: Uniqueness for $\sigma \in C^2(\overline{\Omega})$ for $\Gamma_1 = \partial\Omega$ and Γ_2 half of $\partial\Omega$

2006 Kenig, Sjöstrand and Uhlmann: Uniqueness for $\sigma \in C^2(\overline{\Omega})$ and $\Gamma_1 \subset \partial\Omega$ being almost any subset and $\Gamma_2 \subset \partial\Omega \setminus \Gamma_1$

2007 Isakov: Uniqueness for $\Gamma_1 = \Gamma_2 = \Gamma$ possible small and $\partial\Omega \setminus \Gamma$ patr of sphere or plane, $\sigma \in C^2(\overline{\Omega})$

2006 Heck and Wang: stability

$$\|\sigma_1 - \sigma_0\| \leq w(\|\Lambda_{\sigma_1} - \Lambda_{\sigma_0}\|), \quad w(t) = C(\log|\log(t)|)^{-\alpha}.$$

2D

2010 Imanuvilov, Uhlmann and Yamamoto: Uniqueness for arbitrary small $\Gamma_1 = \Gamma_2$.

Tikhonov regularization revisited

For the inverse problem $Kx = y$ prior information can lead to different penalty terms:

$$J(x) = \|Kx - y\|_{L^2}^2 + \alpha P(x)$$

- $P(x) = \|x\|_{L^2}^2$: usual Tikhonov regulariztion
- $P(x) = \|\nabla x\|_{L^2}^2$: Sobolev space regularization
- $P(x) = \|\nabla x\|_{L^1}$: Total variation regularization
- $P(x) = \|x\|_{L^1}$: sparsity regularization

Sparsity regularization

Write $\sigma = \sigma_0 + \delta\sigma$. Suppose $\delta\sigma$ is sparse in ONB $\{\phi_j\}$:

$$\delta\sigma = \sum_j c_j \phi_j, \quad c_j = \langle \delta\sigma, \phi_j \rangle,$$

i.e. few $c_j \neq 0$.

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Take K measurements $f_k, g_k = \Lambda_\sigma g_k$ (Λ_σ is Neumann to Dirichlet map).
Functional to minimize.

$$J(\delta\sigma) = R(\delta\sigma) + P(\delta\sigma)$$

with

$$R(\delta\sigma) = \sum_{k=1}^{K} \|f_k - \Lambda_{\sigma_0+\delta\sigma} g_k\|^2_{L^2(\Gamma)}, \qquad P(\delta\sigma) = \sum_j \alpha_j |\langle \delta\sigma\, \phi_j \rangle|.$$

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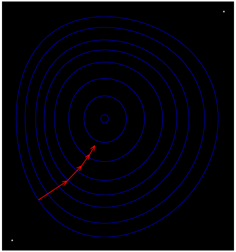
Solved iteratively using a conditional gradient method with soft thresholding.

Steepest descent

Idea is to iteratively take a step in direction of negative gradient.

Explicitly compute $R'[\delta\sigma](\eta) = \langle \nabla_s R, \eta \rangle$ where $\nabla_s R$ is known by solving forward problems.



Iterative soft thresholding

Iteratively solve the linearized problem

$$\zeta_{i+1} \equiv \text{Argmin}_{\delta\sigma} \left[\frac{1}{2} \| \delta\sigma - (\delta\sigma_i - s_i \nabla_s R(\delta\sigma_i)) \|^2 + s_i \sum_j \alpha_j |\langle \delta\sigma, \phi_j \rangle| \right] .$$

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Then do soft shrinkage/thresholding on coefficients in $\zeta_{i+1} = \sum_j d_j \phi_j$:

$$\delta\sigma_{i+1} = \sum_j \mathcal{S}_{s_i \alpha_j}(d_j) \phi_j,$$

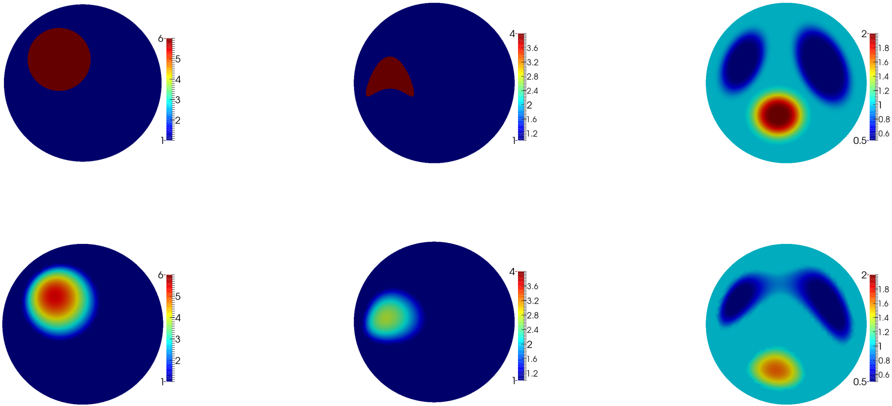
with

$$\mathcal{S}_\beta(t) \equiv \text{sign}(t) \max\{|t| - \beta, 0\}, \quad t \in \mathbb{R}.$$

Numerical choices

- Ball geometry, forward problems solved using FEM (the FEniCS project)
- Use ϕ_j as FEM basis functions; NB: NOT an ONB.
- ϕ_j is spatially well located
- Number of measurements $K = 10$ (Fourier basis functions)

Reconstruction full data



Distributed prior

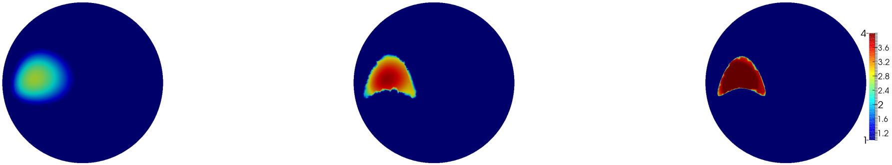
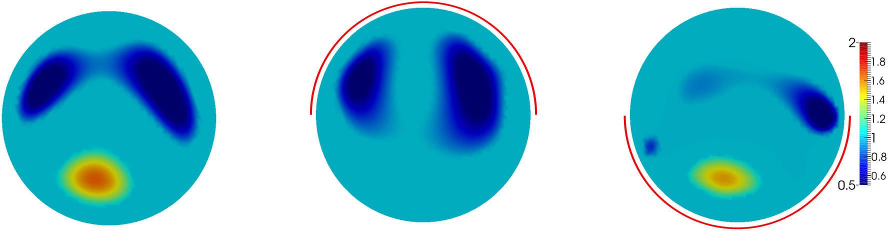


Figure: Sparse reconstruction of the phantom. Left: no prior; middle:10% overestimated support; right: exact support

Partial data



Partial data and prior

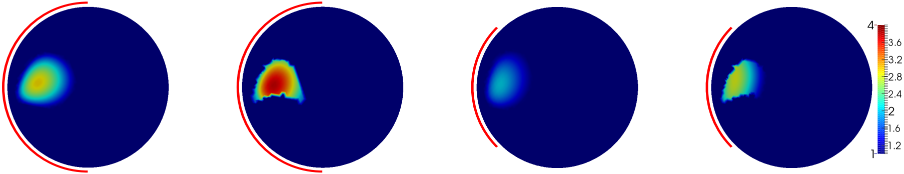


Figure: Sparse reconstruction with prior. From left 50% boundary data, no prior; 50% boundary data with 10% overestimated support; 25% boundary data, no prior; 25% boundary data with 10% overestimated support.

Conclusion and outlook

- Optimization allows flexible scheme for prior info and partial data
- Convergence theory is hard
- Vision: Stable reconstruction for partial data Calderón problem using CGO solutions
- Understand depth dependency in particular for partial data case

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Thank you

References

For more deatails consider the papers and in particular references therein:

K. Knudsen, M. Lassas, J. L. Mueller, S. Siltanen, D-bar method for Electrical Impedance Tomography with discontinuous conductivities. SIAM J. APPL MATH, 67(3): 893–913, 2007.

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