Numerical methods for EIT

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Outline

1. The Calderón problem and linearization
2. The CGO-method for reconstruction in 2D
3. The CGO-method for reconstruction in 3D
4. The Calderón problem with partial data

3. The CGO-method for reconstruction in 3D

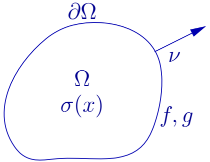
Mathematical model for EIT

Smooth bounded domain $\Omega \subset \mathbb{R}^3$, conductivity coefficient

$$0 < c \leq \sigma \leq C < \infty; \qquad \sigma \equiv 1 \text{ near } \partial\Omega.$$

Voltage potential u in Ω generated by boundary voltage potential f

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = f.$$



Current field: $J = \sigma \nabla u$.
Normal component of current field at $\partial\Omega$:

$$g = \nu \cdot J = \sigma \partial_\nu u|_{\partial\Omega}.$$

Dirichlet to Neumann (voltage to current) map

$$\Lambda_\sigma : f \mapsto g.$$

Complex Geometrical Optics solutions and the scattering transform

Let

$$q = \Delta\sigma^{1/2}/\sigma^{1/2} \Leftrightarrow (-\Delta + q)\sigma^{1/2} = 0.$$

Complex Geometrical Optics (CGO) solutions $\psi(x,\zeta),\ \zeta \in \mathbb{C}^n,\ \zeta \cdot \zeta = 0$:

$$(-\Delta + q)\psi(x,\zeta) = 0 \text{ in } \mathbb{R}^n,$$
$$\psi(x,\zeta) \approx e^{ix\cdot\zeta} \text{ for large } |x| \text{ or } |\zeta|.$$

Scattering transform for $\zeta^2 = (\xi + \zeta)^2 = 0$

$$\mathbf{t}(\xi,\zeta) = \left\langle (\Lambda_\sigma - \Lambda_1)\psi(x,\zeta)|_{\partial\Omega}, e^{ix\cdot(\overline{\zeta}+\xi)} \right\rangle$$
$$= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)dx.$$

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Assume no exceptional points, i.e. $\mathbf{t}(\xi,\zeta)$ defined for all ξ,ζ.

Choice of ζ

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Consequently

$$\zeta = -\frac{\xi}{2} + \kappa k + i(\kappa^2 + \frac{1}{4})^{1/2} k^{\perp}, \qquad k,\ k^{\perp} \in \mathbb{R}^3 : \quad \xi \cdot k = \xi \cdot k^{\perp} = k \cdot k^{\perp} = 0.$$

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- Works in 3 and more dimensions; not 2D
- In theory: $\kappa \to \infty$
- Numerically:
  - Consistent with theory: take κ large. Highly unstable.
  - For stability reasons fix $\kappa = 0 : |\zeta|^2 = |\xi|^2/2$.

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1. Solve for $\psi|_{\partial\Omega}$ boundary integral equation

$$\psi + \mathcal{S}_\zeta(\Lambda_\sigma - \Lambda_1)\psi = e^{ix\cdot\zeta}, \qquad x \in \partial\Omega$$

   and compute scattering transform

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3. Solve for σ

$$(-\Delta + q)\sigma^{1/2} = 0 \text{ in } \Omega, \quad \sigma^{1/2}|_{\partial\Omega} = 1.$$

Regularization by spectral truncation

Noise model

$$\Lambda_\sigma^\varepsilon = \Lambda_\sigma + E, \quad \|E\| < \varepsilon.$$

For fixed truncation R_ϵ define the regularized algorithm:

1. Solve the noisy boundary integral equation

$$\psi_\zeta^\varepsilon(x) + [\mathcal{S}_\zeta(\Lambda_\sigma^\varepsilon - \Lambda_1)\psi_\zeta^\varepsilon](x) = e^{ix\cdot\zeta}\ , \ x \in \partial\Omega, \quad |\zeta| < R_\varepsilon/2,$$

   and compute

$$\mathbf{t}^\varepsilon(\xi,\zeta) = \begin{cases} \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}[(\Lambda_\sigma^\varepsilon - \Lambda_1)\psi_\zeta^\varepsilon](x)\, dS(x), & |\xi| < R_\varepsilon \\ 0, & |\xi| \geq R_\varepsilon. \end{cases}$$

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2. Define q^ε through $\widehat{q^\varepsilon}(\xi) = \mathbf{t}^\varepsilon(\xi,\zeta)$.
3. Compute σ^ε by solving

$$(-\Delta + q^\varepsilon)(\sigma^\varepsilon)^{1/2} = 0 \quad \text{in } \Omega, \qquad (\sigma^\varepsilon)^{1/2} = 1 \quad \text{on } \partial\Omega.$$

We define the regularized inversion operator Γ_R by $\Gamma_R(\Lambda_\sigma) = \sigma^\varepsilon$.

Regularization strategy

Theorem: [Delbary, Hansen, K. 2013]
Γ_R is a regularization strategy for Λ with admissible parameter choice rule $R = R(\varepsilon) = C|\log \varepsilon|$.

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Consequence:
For any fixed σ

1. Reconstruction from exact data:

$$\lim_{R\to\infty} \Gamma_R(\Lambda_\sigma) = \sigma.$$

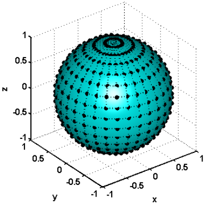
2. Reconstruction when $\epsilon \to 0$:

$$\lim_{\varepsilon\to 0} \sup\{\|\Gamma_{R(\varepsilon)}(\Lambda_\sigma + E) - \sigma\| \mid \|E\| < \varepsilon\} = 0.$$

Computation and representation of Λ_σ

Implementation in the case $\Omega = B(0,1)$.

Well-chosen grid points on the unit sphere
$x_{m,n} = (\sin\theta_m \cos\varphi_n, \sin\theta_m \sin\varphi_n, \cos\theta_m)$.



From grid values $f(x_{m,n})$ approximate $[(\Lambda_\sigma - \Lambda_1)f](x_{m,n})$:

$\rightarrow$ matrix approximation of $\Lambda_\sigma - \Lambda_1$.

Obtained by solving the forward problem:

1. Moment Method (globally varying σ) or
2. Boundary Element Method (piecewise constant σ).

The single layer potential

We need to solve on $\partial\Omega$

$$\psi + S_\zeta(\Lambda_\sigma - \Lambda_1)\psi = e^{ix\cdot\zeta}, \qquad x \in \partial\Omega$$

with S_ζ the single-layer potential with kernel G_ζ.

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Note that

$$G_\zeta(x) = G_0(x) + H_\zeta(x)$$

with

$$G_0(x) = 1/(4\pi|x|)$$

So

$$\psi + (S_0 + \mathcal{H}_\zeta)(\Lambda_\sigma - \Lambda_1)\psi = e^{ix\cdot\zeta}, \qquad x \in \partial\Omega$$

with S_0 the single-layer potential.

Numerical integration on the sphere

Quadrature points

$$x_{mn} = (\sin\theta_m \cos\varphi_n, \sin\theta_m \sin\varphi_n, \cos\theta_m), \quad 0 \le m \le N, 0 \le n \le 2N+1,$$

where

$\theta_m = \arccos t_m,$

t_m : increasing $N+1$ zeros of Legendre polynomial of degree $N+1$ P_{N+1}.

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Gauss-Legendre quadrature rule of order $N+1$ on $[-1,1]$

$$\int_{\partial\Omega} \psi\, ds \simeq \frac{\pi}{N+1} \sum_{m=0}^{N} \sum_{n=0}^{2N+1} \alpha_m \psi(x_{mn})\,,\ \psi \in C^0(\partial\Omega),$$

$$\text{Weigths: } \alpha_k = \frac{2(1-t_k^2)}{(N+1)^2 [P_N(t_k)]^2}.$$

Exact for spherical harmonics of degree less than or equal to $2N+1$.

Hyperinterpolation

We wish to expand functions using the projection operator on $L^2(\partial\Omega)$

$$T_N\phi = \sum_{n=0}^{N} \sum_{m=-n}^{n} \langle \phi, Y_n^m \rangle Y_n^m$$

Inner product approximated by quadrature rule defines the hyperinterpolation operator

$$L_N\phi = \frac{\pi}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} \sum_{k=0}^{N} \sum_{\ell=0}^{2N+1} \alpha_k \phi(x_{k\ell}) Y_n^{-m}(x_{k\ell}) Y_n^m \, , \; \phi \in C^0(\partial\Omega).$$

Well suited since singular part $S_0 Y_n^m$ can be calculated explicitly:

$$S_0 Y_n^m(x) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{Y_n^m(y)}{|x-y|} \, dS(y) = \frac{1}{2n+1} Y_n^m(x) \, , \; x \in \partial\Omega.$$

Discrete equation

Approximate $\psi^N \approx \psi$:

$$\psi + S_\zeta(\Lambda_\sigma - \Lambda_1)\psi = e^{ix\cdot\zeta}, \qquad x \in \partial\Omega$$

Discrete system

$$[I + S_\zeta L_N(\Lambda_\sigma - \Lambda_1)L_N]\psi^N = e^{ix\cdot\zeta}.$$

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Convergence rates: For any $s > 3/2$

$$\|\psi^N - \psi\|_{H^s(\partial\Omega)} \leq \frac{C}{N^{s-3/2}}\|e^{ix\cdot\zeta}\|_{H^s(\partial\Omega)}$$

where C may depend on $s, \zeta, \sigma, .$

The scattering transform (radial profile)

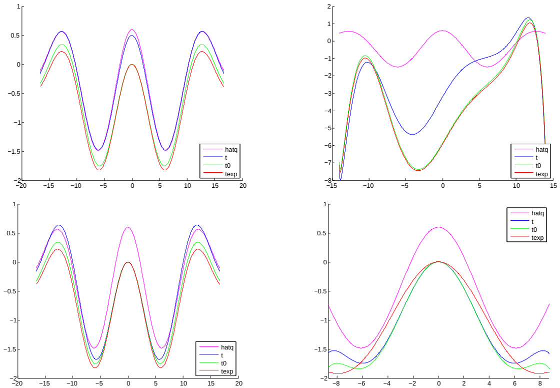


Figure: Reconstructed scattering transforms and the Fourier transform. Upper row: $|\zeta|$ large and fixed, lower row $|\zeta|$ minimal. Left column no noise, right column 0.1% noise.

Reconstruction of radial profile

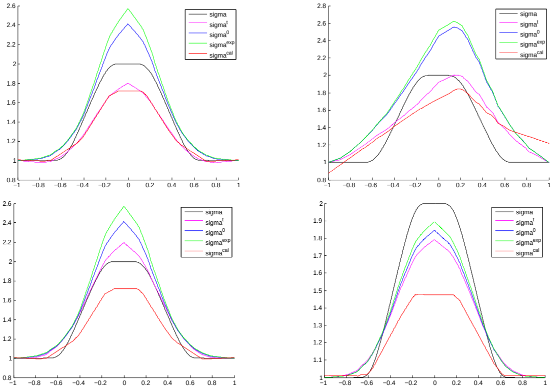
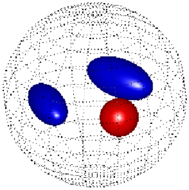
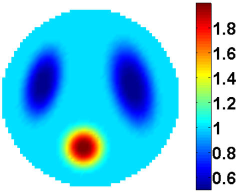


Figure: Reconstructions of radial conductivities. Upper row: $|\zeta|$ fixed, lower row $|\zeta|$ minimal. Left column no noise and truncation parameter $K = 8$, right column 0.1% noise and $K = 6$.

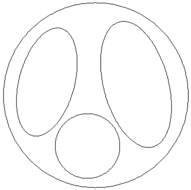
Phantom



3D phantom



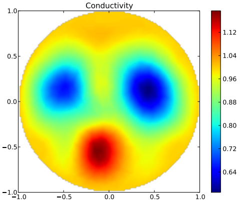
Profile



Support

Reconstruction from exact data, 2048 boundary points

Increasing regularization R



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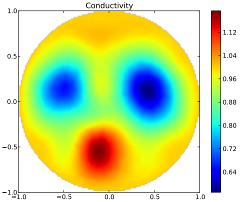

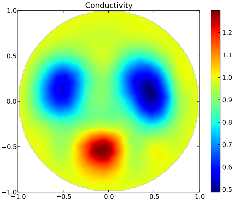

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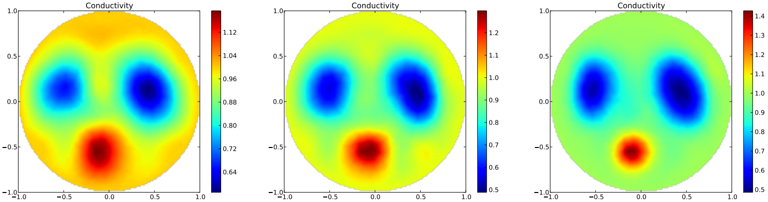

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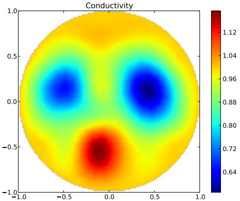

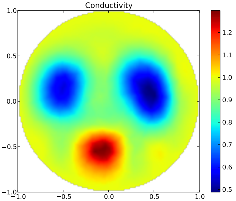

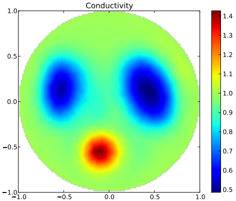

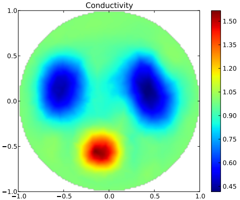

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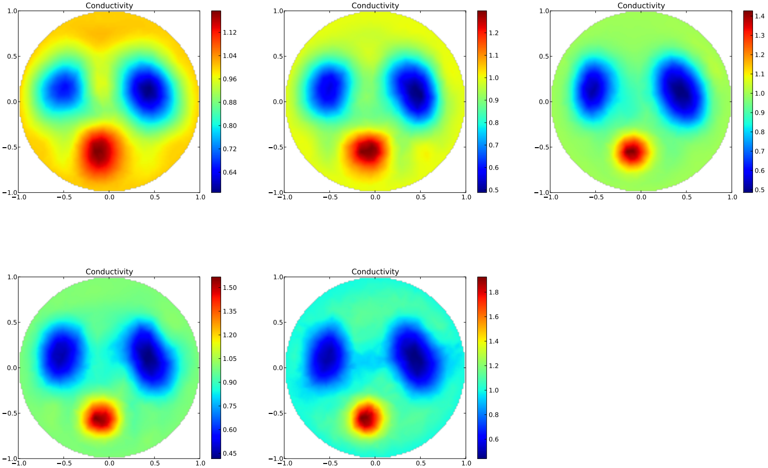

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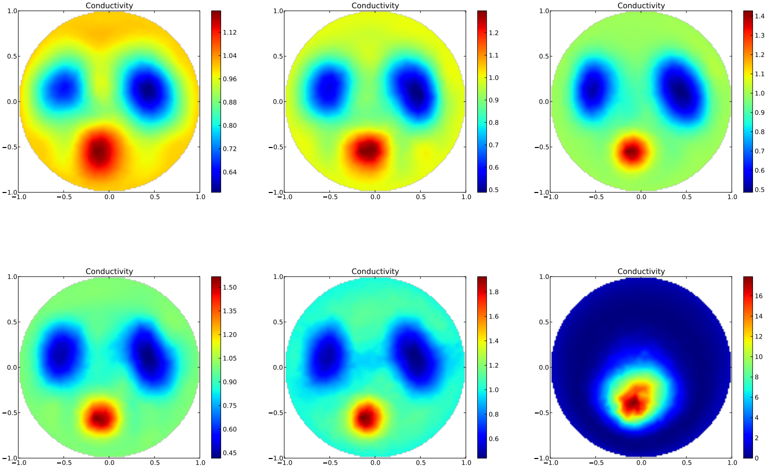


Large vs. minimal $|\zeta|$

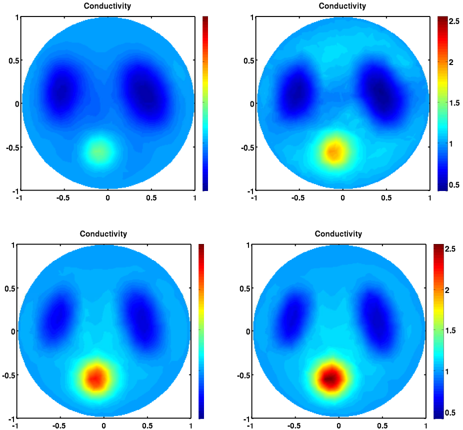
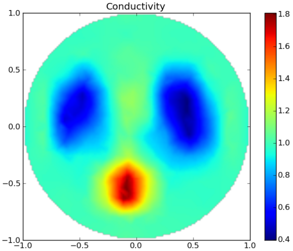


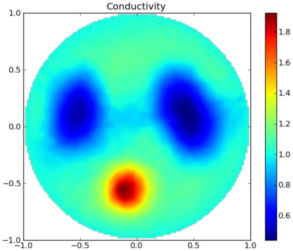
Figure: Cross sectional plot of reconstruction σ. Upper row $|\zeta|$ large ($K = 10, 12$); lower row $|\zeta|$ small with ($K = 12, 14$).

Linearization vs. CGO, $R = 12$, 2048 boundary points

Linearization σ^{Cal}

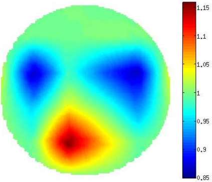


CGO method σ

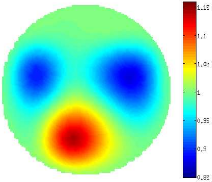


Noise (0.1%), $R = 6$, 968 boundary points

Linearization σ^{Cal}



Exact method σ



Conclusions

- Numerical implementation by F. Delbary
- CGO method works well
- When is CGO method worth the effort?
- Theory for $|\zeta|$ minimal choice
- Explicitly build in prior information?
- Partial data?