Numerical methods for EIT

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Outline

1. The Calderón problem and linearization
2. The CGO-method for reconstruction in 2D
3. The CGO-method for reconstruction in 3D
4. The Calderón problem with partial data

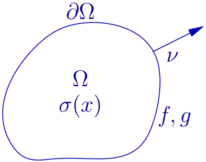
Mathematical model for EIT

Smooth bounded domain $\Omega \subset \mathbb{R}^2$, conductivity coefficient

$$0 < c \leq \sigma \leq C < \infty; \qquad \sigma \equiv 1 \text{ near } \partial\Omega.$$

Voltage potential u in Ω generated by boundary voltage potential f

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = f.$$



Current field: $J = \sigma \nabla u$.
Normal component of current field at $\partial\Omega$:

$$g = \nu \cdot J = \sigma \partial_\nu u|_{\partial\Omega}.$$

Dirichlet to Neumann (voltage to current) map

$$\Lambda_\sigma : f \mapsto g.$$

2. The CGO-method for reconstruction in 2D (aka Dbar method)

The Schrödinger equation

Equivalent problem for Schrödinger equation: $v = \sigma^{-1/2} u$

$$(-\Delta + q)v = 0 \text{ in } \Omega \qquad v|_{\partial\Omega} = f,$$

with

$$q = \Delta\sigma^{1/2}/\sigma^{1/2} \Leftrightarrow (-\Delta + q)\sigma^{1/2} = 0.$$

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Dirichlet to Neumann map:

$$\Lambda_q f = \frac{\partial v}{\partial \nu}|_{\partial\Omega} = \frac{\sigma}{\partial u}\partial\nu|_{\partial\Omega} = \Lambda_\sigma f$$

Identity

$$\langle(\Lambda_\sigma - \Lambda_{\sigma_0})f_1, f_0\rangle = \int_\Omega (q_1 - q_0)v_1\overline{v_0}dx,$$

with

$$(-\Delta + q_j)v_j = 0 \text{ in } \Omega \qquad v_j|_{\partial\Omega} = f_j.$$

For $\sigma_0 = 1,\ q_0 = 0$:

$$\langle(\Lambda_\sigma - \Lambda_1)f_1, f_0\rangle = \int_\Omega q v_1\overline{v_0}dx.$$

Complex Geometrical Optics solutions

$$\langle (\Lambda_\sigma - \Lambda_1) f_1, f_0 \rangle = \int_\Omega q v_1 \overline{v_0} dx.$$

Choose

$$v_1(x,\zeta) \approx e^{ix\cdot\zeta}, \qquad \zeta\cdot\zeta = 0,$$
$$v_0(x) = e^{ix\cdot(\xi+\overline{\zeta})}, \quad (\xi+\zeta)\cdot(\xi+\zeta) = 0.$$

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Complex Geometrical Optics (CGO)solutions $\psi(x,\zeta),\ \zeta\in\mathbb{C}^n,\ \zeta\cdot\zeta = 0$:

$$(-\Delta + q)\psi(x,\zeta) = 0 \text{ in } \mathbb{R}^n,$$
$$\psi(x,\zeta) \approx e^{ix\cdot\zeta} \text{ for large } |x| \text{ or } |\zeta|.$$

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Write $\psi(x,\zeta) = e^{ix\cdot\zeta}(1+\omega(x,\zeta))$; then

$$(-\Delta - 2i\zeta\cdot\nabla + q)\omega = -q.$$

Faddeev's Green's function

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$$g_\zeta(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ix\cdot k}}{|p|^2 + 2p\cdot\zeta}\, dp\ ,\ x \in \mathbb{R}^d \setminus \{0\},$$

$$G_\zeta(x) = e^{ix\cdot\zeta} g_\zeta(x), \qquad \Delta G_\zeta = \delta, \quad G_\zeta \sim e^{ix\cdot\zeta}.$$

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Then ψ, ω must satisfy the Lippmann-Schwinger-Faddeev equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} + \int_\Omega G_\zeta(x-y) q(y) \psi(y,\zeta) dx,$$

$$\omega(x,\zeta) = -g_\zeta * q + \int_\Omega g_\zeta(x-y) q(y) \omega(y,\zeta).$$

1. In 2D this equation is uniquely solvable for all ζ.
2. In 3D this equation is uniquely solvable for $|\zeta|$ large (or small).

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And

$$\|\omega\|_{L^2(\Omega)} \le \frac{C}{|\zeta|}.$$

$\psi|_{\partial\Omega}$ at the boundary

Then $\psi|_{\partial\Omega}$ satisfies for fixed $\zeta \in \mathbb{C}^n$ the boundary integral equation

$$\psi(x,\zeta) + \int_{\partial\Omega} G_\zeta(x-y)(\Lambda_\gamma - \Lambda_1)\psi(y,\zeta)ds = e^{ix\cdot\zeta}, \quad x \in \partial\Omega.$$

Written in terms of layer potentials

$$\psi + S_\zeta(\Lambda_\gamma - \Lambda_1)\psi = e^{ix\cdot\zeta}, \quad x \in \partial\Omega.$$

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Conclusion: from data we can find $\psi|_{\partial\Omega}$.

Solving the equation is severely ill-posed!

The scattering transform

Introduce intermediate object, *non-physical scattering/Fourier transform*

$$\begin{aligned} \mathbf{t}(\xi,\zeta) &= \langle(\Lambda_\sigma - \Lambda_1)\psi(\cdot,\zeta), e^{ix\cdot(x+\bar{\zeta})}\rangle \\ &= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)dx. \end{aligned}$$

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t satisfies the estimate

$$|\hat{q}(\xi) - \mathbf{t}(\xi,\zeta)| = \mathcal{O}(1/|\zeta|).$$

Reconstruction algorithm

$$\Lambda_\gamma \to \mathbf{t}(\xi,\zeta) \to \gamma(x).$$

Second step depends on 2D or 3D.

Reconstruction method in 2D

Parameterize $\zeta = (k, ik),\ k \in \mathbb{C}$ and put $x = x_1 + x_2$.
The Dbar reconstruction algorithm

1. Solve the boundary integral equation (BIE)

$$\psi + S_k(\Lambda_\sigma - \Lambda_1)\psi = e^{ixk}, \quad x \in \partial\Omega,$$

   and compute

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\sigma - \Lambda_1)\psi(\,\cdot\,, k)d\sigma.$$

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2. Solve the Dbar equation

$$\overline{\partial}_k \mu(x,k) = \frac{1}{4\pi\overline{k}}\mathbf{t}(k)e_{-x}(k)\overline{\mu(x,k)}, \quad k \in \mathbb{C},$$

   with $e_{-x}(k) := e^{-i(kx+\overline{k}\bar{x})}$; then $\sigma(x) = \mu(x,0)^2$.

Remarks

- Solving boundary integral equation is exponentially ill-posed.
- Solving dbar-equation is well-posed
- Reconstruction method is exact for $\sigma \in W^{1+\epsilon,p}(\Omega),\ p > 2$.

Numerical details (unit disk)

1. Represent DN-map in Fourier basis $\{e^{in\theta}\}$

$$(\Lambda_\sigma - \Lambda_1)e^{im\theta} = \sum_n c_n e^{in\theta}.$$

   Requires solution of BVP by FEM.
2. Expand all terms in BIE in Fourier basis.
3. After trunctaion we obtain matrix equation

$$Ac = b,$$

   solve i.e. iteratively (GMRES).

Ill posedness is handled in two ways:

- Upper limit on number of basis functions.
- Truncation for $|k| < R$.

Numerical solution of the $\overline{\partial}$-equation

For the scattering / inverse scattering problem solving a $\overline{\partial}$-equation is important. We will now consider the numerical solution of such an equation: We would like to solve the integral equation

$$v(k) = 1 - \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{T(k')}{k - k'} \overline{v(k')} dk_1' dk_2', \qquad k = k_1' + ik_2' \in \mathbb{C}, \tag{1}$$

or

$$v(k) = 1 - g * (T(k)\overline{v(k)}). \tag{2}$$

Assume that T is compactly supported in a bounded domain Ω.

Discretization

Let $S = [-s, s]^2$ be a square such that $\overline{\Omega} \subset S$. Choose $m \in \mathbb{Z}_+,\ M = 2^m,\ h = 2s/M$. Define a grid $\mathcal{G}_m \subset S$ by

$$\mathcal{G}_m = \{jh \,|\, j \in \mathbb{Z}_m^2\},$$
$$\mathbb{Z}_m^2 = \{j = (j_1, j_2) \in \mathbb{Z}^2 \,|\, -2^{m-1} \leq j_l < 2^{m-1}\}.$$

Grid approximation $\phi_h : \mathbb{Z}_m^2 \to \mathbb{C}$ of a function $\phi \in C(\overline{S})$ by

$$\phi_h(j) = \phi(jh), \qquad \text{for } j \in \mathbb{Z}_m^2.$$

Grid approximation of Green's function:

$$g_h(j) = \begin{cases} g(jh), & j \in \mathbb{Z}_m^2, j \neq (0,0), \\ 0, & j = (0,0). \end{cases}$$

Discrete approximation

The discrete convolution operator A_h

$$(A_h\phi_h)(j) = h^2 \sum_{l\in\mathbb{Z}_m^2} g_h(j-l)\phi_h(l), \qquad \text{for } j \in \mathbb{Z}_m^2.$$

Important fact:

$$A_h\phi_h = h^2\,\mathrm{IFFT}\big(\mathrm{FFT}(g_h)\cdot\mathrm{FFT}(\phi_h)\big),$$

i.e. the implementation is fast.
We approximate the integral equation by the discrete eqation

$$[I + A_h(T_h\cdot^-)]w_h = 1. \tag{3}$$

It has a solution for sufficiently large m; solved by GMRES.

Properties

- Linear convergence of algorithm $\mathcal{O}(h)$
- Complexity of algorithm is $\mathcal{O}(M^2 \log(M))$ for each x.
- Multigrid extension of algorithm is possible.
- Speed up possible [Huhtanen and Perämäki, 2010]

Connection to Calderón reconstruction

Near-field scattering transform:

$$\mathbf{t}^{\exp}(\xi,\zeta) = \left\langle (\Lambda_\sigma - \Lambda_1) e^{ix\cdot\zeta}, e^{-ix\cdot(\zeta+\xi)} \right\rangle$$
$$= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x) v^{\exp}(x,\zeta)\,dx,$$

with $(-\Delta + q)v^{\exp} = 0$ in Ω and $v^{\exp}|_{\partial\Omega} = e^{ix\cdot\zeta}$.

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with $(-\Delta + q)v^{\exp} = 0$ in Ω and $v^{\exp}|_{\partial\Omega} = e^{ix\cdot\zeta}$.
Scattering transform:

$$\mathbf{t}(\xi,\zeta) = \left\langle (\Lambda_\sigma - \Lambda_1)\psi, e^{-ix\cdot(\zeta+\xi)} \right\rangle$$
$$= \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x) \psi(x,\zeta)\,dx,$$

where $(-\Delta + q)\psi = 0$ in $\mathbb{R}^n$ and $\psi \sim e^{ix\cdot\zeta}$ for x near ∞.

Regularization of the algorithm

Noise model: $\tilde{\Lambda}_\sigma = \Lambda_\sigma + \mathcal{E}$

Problem: we don't know if $\tilde{\Lambda}_\sigma \in$ range (Λ).

This assumption is often made in stability estimates for the inverse problem resulting in results

$$\|\sigma_1 - \sigma_0\|_X \leq w(\|\Lambda_{\sigma_1} - \Lambda_{\sigma_0}\|_Y),$$
$$w(t) = C|\ln(t)|^{-2/(n+1)}.$$

Related to the notoriously difficult problem of the charaterization of range (Λ).

Non-linear regularization

Definition

A family of continuous mappings $\Gamma_\alpha : Y \to L^\infty(\Omega)$ *parameterized by* $0 < \alpha < \infty$ *is a* regularization strategy *for F if*

$$\lim_{\alpha \to 0} \|\Gamma_\alpha \Lambda_\sigma - \sigma\|_{L^\infty(\Omega)} = 0$$

for each fixed σ*. Further, a regularization strategy with a choice* $\alpha = \alpha(\varepsilon)$ *of regularization parameter as function of noise level is called* admissible *if*

$$\alpha(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

and for any fixed σ *the following holds:*

$$\sup_{\Lambda_\sigma^\varepsilon} \left\{ \|\Gamma_{\alpha(\varepsilon)} \Lambda_\sigma^\varepsilon - \sigma\|_{L^\infty(\Omega)} \, : \, \|\Lambda_\sigma^\varepsilon - \Lambda_\sigma\|_Y \leq \varepsilon \right\} \to 0 \text{ as } \varepsilon \to 0.$$

Concrete strategy

Let $\alpha(\epsilon) = 1/R(\epsilon)$. Define $\Gamma_{\alpha(\varepsilon)}\Lambda_\sigma$ by the steps

1. Solve

$$\tilde{\varphi}(z,k) = e^{izk} - S_k(\tilde{\Lambda}_\sigma - \Lambda_1)\tilde{\varphi}, \quad |k| < R(\epsilon),$$

   and

$$\tilde{\mathbf{t}}(k) = \int_{\partial\Omega} e^{i\overline{zk}}(\tilde{\Lambda}_\sigma - \Lambda_1)\tilde{\varphi}(\cdot,k)d\sigma(z). \quad |k| < R(\epsilon)$$

2. Solve

$$\overline{\partial}_k\tilde{\mu}(x,k) = \frac{1}{4\pi\overline{k}}\tilde{\mathbf{t}}(k)e_{-x}(k)\overline{\tilde{\mu}(x,k)}, \quad k \in \mathbb{C},$$

   and compute $\Gamma_{\alpha(\varepsilon)}\Lambda_\sigma = \tilde{\sigma}(x) = (\tilde{\mu}(x,0))^2$.

Regularization theorem

Theorem
Suppose $\Lambda_\sigma^\varepsilon = \Lambda_\sigma + \mathcal{E}$ with $\|\mathcal{E}\| < \varepsilon$. For $R(\epsilon) = -\frac{1}{10}\log(\varepsilon)$, $\Gamma_{\alpha(\varepsilon)}\Lambda_\sigma$ is an admissible regularization strategy and

$$\|\Gamma_{\alpha(\varepsilon)}\Lambda_\sigma^\varepsilon - \sigma\|_{L^\infty(\Omega)} \leq C(-\log\varepsilon)^{-1/14}.$$

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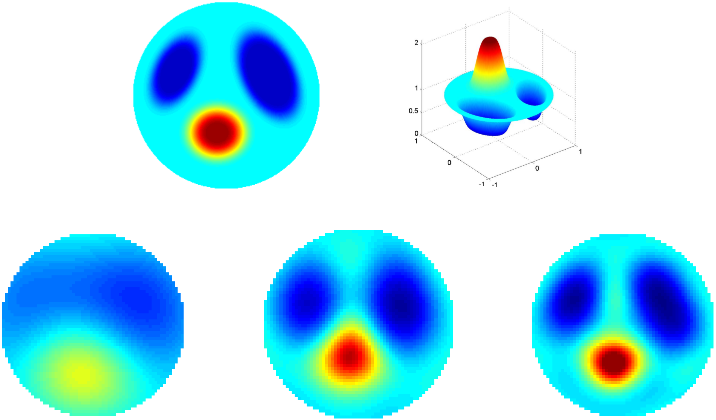
$$\|\Gamma_{\alpha(\varepsilon)}\Lambda_\sigma^\varepsilon - \sigma\|_{L^\infty(\Omega)} \leq C(-\log\varepsilon)^{-1/14}.$$

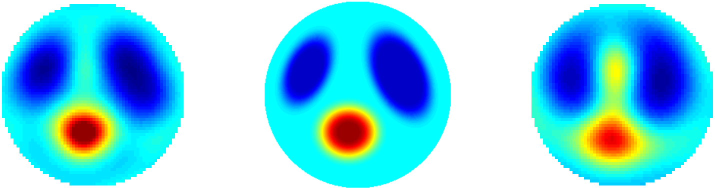
Note, we do not assume $\Lambda_\sigma^\varepsilon$ is in the range of Λ.

Numerical results



Reconstructions with noiselevel $10^{-2}, 10^{-4}$ and 10^{-6}. Error in approximation is $52\%, 14\%$ and 12% respectively.

Comparison to Calderón method



Reconstructions with noiselevel 10^{-6}.
Left: Dbar method; error 12%
Right: Calderon method (linearized); error 23%

Convergence

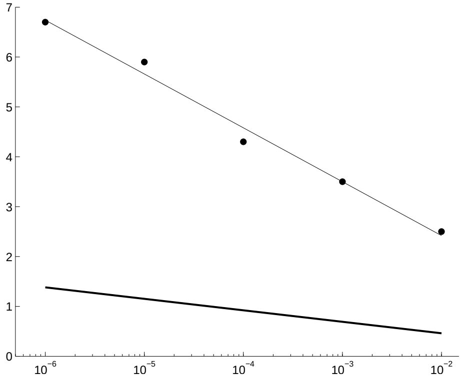
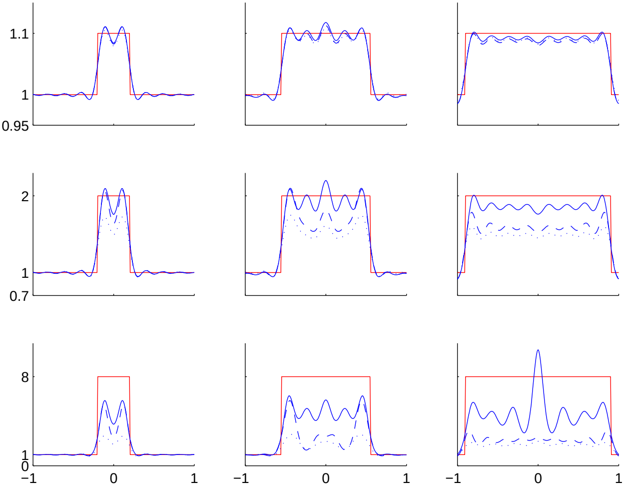


Figure: Truncation against error

Reconstruction of non-smooth conductivities



Conclusion 2D

- Presentation of direct non-linear reconstruction algorithm in 2D.
- Implementation of non-linear method for computing conductivity.
- Rigorous regularization method in 2D.
- Method works reasonable well - is it worth the effort?
- Better understanding of truncation of $\mathbf{t}(k)$ as prior.
- What about 3D?