DTU

Numerical methods for EIT

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Advanced Instructional School of Theoretical and Numerical Aspects of Inverse Problems
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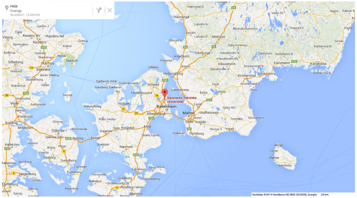
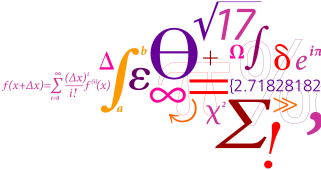
About me...



HDtomo

HIGH - DEFINITION TOMOGRAPHY



Outline

1. The Calderón problem and linearization
2. The CGO-method for reconstruction in 2D
3. The CGO-method for reconstruction in 3D
4. The Calderón problem with partial data

1. The Calderón problem and linearization

Motivation

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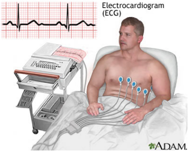
The use of electricity in treatment and monitoring of patients:

- Electrocardiography (EKG)



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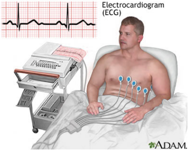

- Electrocardiography (EKG)
- Electro stimulation

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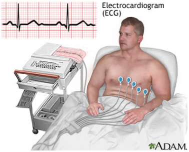

- Electro stimulation

- Defillibrator

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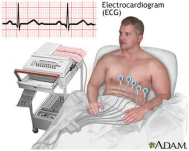

- Electrocardiography (EKG)

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- Electrical Impedance Tomography

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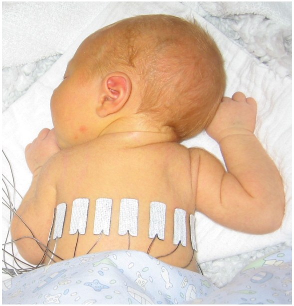
- Barber-Brown (1989):

Tissue	Conductivity (mS/cm)
Blood	6.7
Liver	2.8
Skeletal muscle	8.0 (long.), 0.6 (trans.)
Cardiac muscle	6.3(long.), 2.3 (trans.)
Lung (expiration-inspiration)	1.0 - 0.4
Fat	0.36
Bone	0.06

Measurement setup EIT



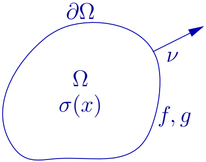
Mathematical model for EIT

Smooth bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, conductivity coefficient

$$0 < c \leq \sigma \leq C < \infty; \qquad \sigma \equiv 1 \text{ near } \partial\Omega.$$

Voltage potential u in Ω generated by boundary voltage potential f

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = f.$$



Current field: $J = \sigma \nabla u$.
Normal component of current field at $\partial\Omega$:

$$g = \nu \cdot J = \sigma \partial_\nu u|_{\partial\Omega}.$$

Dirichlet to Neumann (voltage to current) map

$$\Lambda_\sigma : f \mapsto g.$$

Dirichlet to Neumann map

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$$\begin{aligned}\Lambda_\sigma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega)\\ f &\mapsto g.\end{aligned}$$

Weakly defined for $h \in H^{1/2}(\partial\Omega)$ by

$$\langle \Lambda_\sigma f, h\rangle = \int_{\partial\Omega} (\Lambda_\sigma f)\overline{h}\, ds(x) = \int_\Omega \sigma \nabla u \cdot \overline{\nabla v} dx, \quad v \in H^1(\Omega) : \; v|_{\partial\Omega} = h.$$

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Note:

- Λ_σ bounded operator $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$
- Λ_σ is unbounded, selfadjoint in $L^2(\partial\Omega)$
- $\Lambda_\sigma - \Lambda_1$ is compact in $L^2(\partial\Omega)$

The Calderón problem

Forward problem:

$$\Lambda\colon \sigma \mapsto \Lambda_\sigma$$

From interior conductivity to boundary fields.

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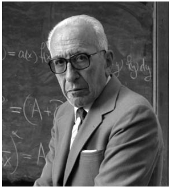

A P Calderón

Inverse problem (Calderón 1980):

- **Uniqueness:** is Λ injective?

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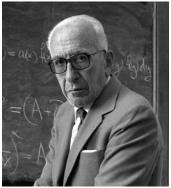

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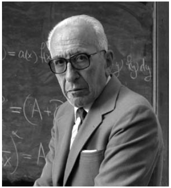

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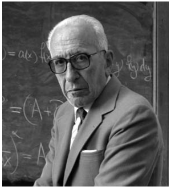

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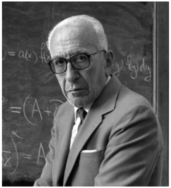

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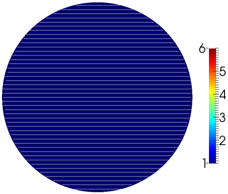
Calderón problem is ill-posed:

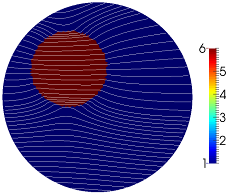
- Large change in conductivity yields small change in data
- Small change in data yields large change in reconstruction

Example with $f(\theta) = \cos(\theta)$

Current flow $J = \sigma \nabla u$:



Homogeneous conductivity



Perturbed conductivity

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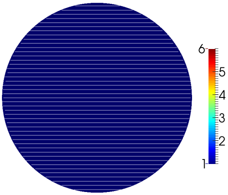

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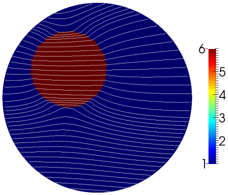
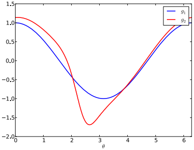
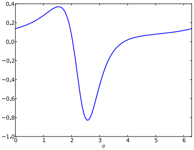


Perturbed conductivity

Boundary normal current $g = \sigma \partial_\nu u$:



Boundary currents



Difference

Example with $f(\theta) = \cos(4\theta)$

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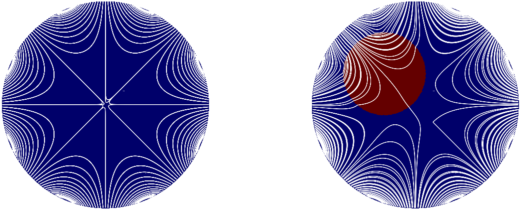

Homogeneous conductivity   Perturbed conductivity

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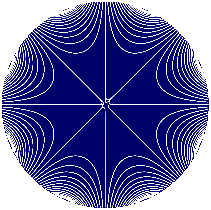
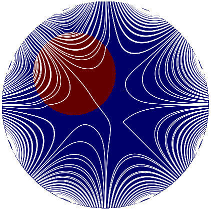


Homogeneous conductivity



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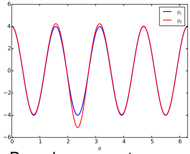

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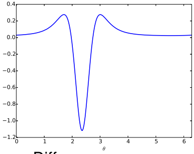

Difference

Short and incomplete history

1980 Calderón: Problem posed, uniqueness for linearized problem, and linear, approximate reconstruction algorithm

3D

1987 Sylvester and Uhlmann: Uniqueness for smooth conductivities. Implicit reconstruction algorithm

1987-88 Novikov, Nachman-Sylvester-Uhlmann, Nachman: Uniqueness for conductivities with 2 derivatives and explicit high frequency reconstruction algorithm. Multidimensional D-bar equation.

1990 Alessandrini: Stability

2003 Brown-Torres, Päivärinta-Panchenko-Uhlmann: Uniqueness for conductivities with $3/2$ derivatives.

2006 Cornean-Knudsen-Siltanen: Low frequency reconstruction algorithm

2010 Bikowski-Knudsen-Mueller: Numerical implementation of simplified reconstruction algorithm

2011-14 Debary - Hansen- Knudsen: Implementation of more accurate numerical reconstruction method

2012 Haberman - Tataru: Uniqueness for Lipschitz conductivities

2D

1996 Nachman: Uniqueness and reconstruction for $W^{2,p}(\Omega)$ conductivities.

1997 Liu: Stabilty for $W^{2,p}(\Omega)$ conductivities

1997 Brown-Torres: Uniqueness for $W^{1,p}(\Omega)$ conductivities

2001 Barceló-Barceló-Ruiz: Stability for $C^{1+\epsilon}$ conductivities

2001 Knudsen-Tamasan: Reconstruction for $C^{1+\epsilon}$ conductivities

2005 Astala-Päivärinta: Uniqueness and reconstruction for $L^{\infty}(\Omega)$

2009 Knudsen-Lassas-Mueller-Siltanen: Regularized $\overline{\partial}$-method

2010 Clop-Faraco-Ruiz: Stability for discontinuous conductivities

+ many more

Reconstruction algorithms

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  Minimize over γ

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  for suitable (semi-)norms X, Y.
- Bayesian inversion:

$$\pi(x|y) = \frac{\pi(y|x)\pi_{\mathrm{pr}}(x)}{\pi(y)}$$

  for $x = \sigma$ and $y = \Lambda_\sigma$.

Linearization

Green's formula yields

$$\langle(\Lambda_{\sigma_1} - \Lambda_{\sigma_0})f_1, f_0\rangle = \int_\Omega (\sigma_1 - \sigma_0)\nabla u_1 \cdot \overline{\nabla u_0} dx,$$

$$\nabla \cdot \sigma_j \nabla u_j = 0 \text{ in } \Omega, \quad u_j|_{\partial\Omega} = f_j, \; j = 0, 1.$$

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Linear approximation (the Fréchet derivative):

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For $\sigma_0 = 1$

$$\langle(\Lambda_{1+\delta\sigma} - \Lambda_1)f_1, f_0\rangle \approx \langle(d\Lambda[1]\delta\sigma)f_1, f_0\rangle = \int_\Omega \delta\sigma \nabla v_1 \cdot \overline{\nabla v_0} dx$$

with $\Delta v_j = 0$, $v_j|_{\partial\Omega} = f_j$.

The linearized problem

Concerns the inversion of the mapping

$$\delta\sigma \mapsto (d\Lambda[1])(\delta\sigma).$$

We want to compute $\delta\sigma$ by knowing:

$$\int_\Omega \delta\sigma \nabla v_1 \cdot \overline{\nabla v_0} dx$$

for all harmonic functions v_0, v_1.
Questions:

- Uniqueness?
- Stable reconstruction?

Comparison

Problem: find σ from knowing

$$K\sigma(x) = \int_\Omega \sigma(y)k(x,y)dy.$$

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1. Calderón problem: unknown diffuse kernel

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2. Linearized Calderón problem: known diffuse kernel

$$k((i,j),y) = \nabla v_i \cdot \overline{\nabla v_j}, \qquad \Delta v = 0.$$

3. Computerized Tomography: known localized kernel

$$k(\theta, r, y) = \delta(y \cdot \theta - r).$$

Exponentially growing harmonics

Fix real vector $\xi \in \mathbb{R}^n$ and complex vector $\zeta \in \mathbb{C}^n$ such that

$$\zeta \cdot \zeta = (\xi + \zeta) \cdot (\xi + \zeta) = 0.$$

Harmonic functions in $\mathbb{R}^n$.

$$v_1(x, \zeta) = e^{ix \cdot \zeta}, \qquad v_0(x, \zeta) = e^{ix \cdot (\xi + \bar{\zeta})}$$

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and

$$\begin{aligned} \langle d\Lambda_1[\delta\sigma] e^{ix\cdot\zeta}, e^{ix\cdot(\xi+\bar{\zeta})} \rangle &= -\frac{1}{2}|\xi|^2 \int_\Omega \delta\sigma e^{-ix\cdot\xi} dx \\ &= -\frac{1}{2}(2\pi)^{n/2}|\xi|^2 \widehat{\delta\sigma}(\xi) \end{aligned}$$

$$\Leftrightarrow \qquad \widehat{\delta\sigma}(\xi) = -\frac{2}{(2\pi)^{n/2}|\xi|^2} \langle d\Lambda_1[\delta\sigma] e^{ix\cdot\zeta}, e^{ix\cdot(\xi+\bar{\zeta})} \rangle.$$

Uniqueness and stable reconstruction

$$\widehat{\delta\sigma}(\xi) = -\frac{2}{(2\pi)^{n/2}|\xi|^2}\langle d\Lambda_1[\delta\sigma]e^{ix\cdot\zeta}, e^{ix\cdot(\xi+\bar{\zeta})}\rangle.$$

- **Uniqueness:** Injectivity of Fourier transform.

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Stabilization: Noise amplified exponential harmonics.
Remedy: Avoid high frequencies by low-pass filtering:

$$\delta\sigma_R(x) = -\frac{2}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\langle d\Lambda_1[\delta\sigma]e^{ix\cdot\zeta}, e^{ix\cdot(\xi+\bar{\zeta})}\rangle}{|\xi|^2}e^{ix\cdot\xi}\chi_R(\xi)\,d\xi.$$

Regularization scheme if R is chosen correctly.

Calderón's reconstruction method

Treat non-linear data

$$\mathbf{t}^{\exp}(\xi,\zeta) := \langle(\Lambda_\sigma - \Lambda_1)e^{ix\cdot\zeta}, e^{ix\cdot(\xi+\overline{\zeta})}\rangle = \int_\Omega \delta\sigma \nabla u \cdot \nabla e^{-ix\cdot(\xi+\zeta)} dx$$

as linear data in previous formula

$$\sigma^{\mathrm{Cal}}(x) = 1 - \frac{2}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\mathbf{t}^{\exp}(\xi,\zeta)}{|\xi|^2} e^{ix\cdot\xi}\chi_R(\xi)\, d\xi.$$

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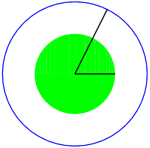
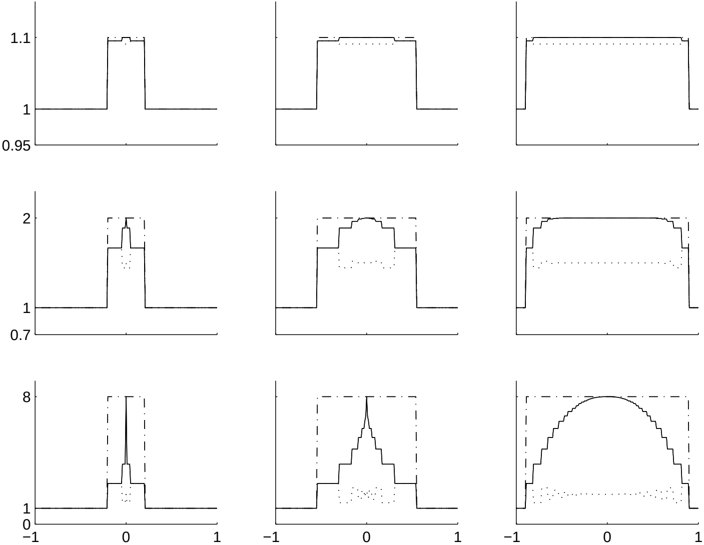
Estimate:

$$\begin{aligned} \|\sigma^{\mathrm{Cal}}(x) - \eta_\gamma * \sigma\|_{L^\infty} &\le \|R(k)\hat{\eta}(k/\gamma)\|_{L^1(\mathbb{R}^2)} \\ &\le C\|\sigma - 1\|_{L^\infty(\Omega)}^{1+\alpha} (\log(\|\sigma - 1\|_{L^\infty(\Omega)}))^2. \end{aligned}$$

Concentric reconstruction (linear)





Non concentric reconstruction (linear)

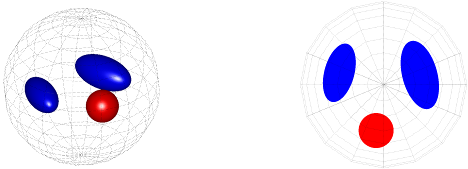
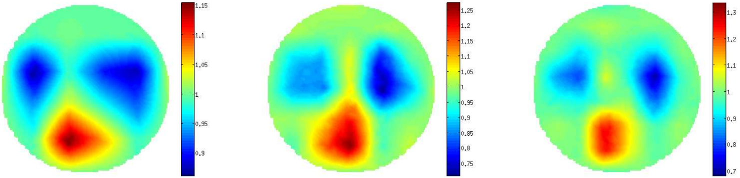


Figure: Increasing Fourier truncation for Calderón's method



Conclusion, linearized reconstruction

- Easy to implement and fast: based on FFT
- Recovers (with no regularization) the support of perturbation $\delta\sigma$ (cf [von Harrach - Seo, 2010] and [Knudsen - Lassas - Mueller - Siltanen, 2007])
- Recovers well low contrast perturbations
- Regularized algorithm recovers smooth approximatrion
- Can rigorous mathematics allow better reconstructions?