Bangalore Lectures on Elasticity and Topological Mechanics

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These lectures deal with properties of mechanical continua and mechanical lattices. They touch on everything from classical elasticity, which is often not treated with sufficient care in physics text books, to the application of topological ideas honed in the quantum world to classical mechanical systems, with stops intermediate stops to investigate the properties of marginally stable elastic media.

This review and the lectures themselves rely heavily on articles (including two review articles) I have written with a number of collaborators. The text presented here is often taken verbatim or nearly so from the published versions of these articles. The relevant articles are listed at the beginning of most sections. The articles themselves have many references, which are not listed in these notes.

The outline of these notes follows below. It is fairly long, and it is not clear that I will be able to cover all topics. If so, I will cut some material to be sure to arrive at a detailed detailed discussion of the new subject of topological mechanics.

Outline of Course

I. CLASSICAL LAGRANGIAN ELASTICITY

References

Ye, F. F., and T. C. Lubensky. "Phase Diagrams of Semisoft Nematic Elastomers." Journal of Physical Chemistry B 113, no. 12 (Mar 2009): 3853-72.

Lubensky, T. C., and F. F. Ye. "Elastic Response and Ward Identities in Stressed Nematic Elastomers." Physical Review E 82, no. 1 (Jul 2010): 011704.

The beginning parts of these articles deal with Langrangian Elasticity, and later study examples, which we probably witll not cover, of its usage.

A. Introduction

Elasticity, reaching back to the early 19ty century and before, is by now an ancient subject. Yet it remains an important discipline of critical importance to the synthesis and of control of materials ranging from airplane wings to living cells. Along with electronic systems, it is undergoing a renaissance at the frontier of science as a result of new understanding of the existence of topological properties of excitation spectra and how they influence a range of physical properties, particularly edge response. This set of lectures will begin with a review of classical nonlinear continuum elasticity, which will cover topics that are not usually found in physics textbooks. It will then consider lattice models and their continuum (i.e., elastic) and linearized limits before turning to marginally stable systems described by the Maxwell-Calladine index theorem and finally topological mechanics.

B. Deformation and Strain

Elastic materials resist changes of state. If they are stretched, compressed, or sheared, they will return to their original shape when external forces causing these deformations are removed. The distorted state has higher energy than the original undistorted one. Crystalline solids, glasses, and rubbers and elastomers are all elastic solids, at least at small enough shape or size distortions. Many solids will undergo irreversible plastic shape changes if deformations are large enough. Here we will consider only elastic media that return do to their original shape.

In the absence of external forces, an elastic material has an equilibrium preferred shape and size. In addition each mass point has a preferred position relative to other mass points. We represent the positions of these mass points by a vector $\mathbf{x}$ of dimension equal to the dimension d of the material. Thus $\mathbf{x}$ is of dimension $d = 2$ for an elastic membrane, and of dimension $d = 3$ for a $3D$ solid. The set of points $\mathbf{x} = (x_1, x_2, ...) \equiv (x, y, ...)$ constitute what we call the *reference space*. When the material is distorted, points $\mathbf{x}$ are mapped into points $\mathbf{R}(\mathbf{x})$ in what we will call the *target space* of dimension D as depicted schematically in Fig. (1). For now, we will restrict our attention to situations in which the dimensions of the reference and target spaces are the same, i.e., $D = d$. We will distinguish the two spaces by using Greek letters $\alpha, \beta, \cdots$ to index refernce space positions ($\mathbf{x} \to x_\alpha$) and roman indices $(i, j, \cdots)$ to index target space positions ($\mathbf{R} \to R_i$). In an undistorted material, that is neither translated nor rotated, $\mathbf{R}(\mathbf{x})$ is identical to $\mathbf{x}$. Deviations from this state are described by the *displacement vector* $\mathbf{u}(\mathbf{x})$ defined through

$$\mathbf{R}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}). \tag{1.1}$$

Under uniform translation through $\mathbf{x}_0$, $\mathbf{u}(\mathbf{x}) = \mathbf{x}_0$. Under uniform rotation, $R_i \to O_{i\alpha} x_\alpha$ where $O_{i\alpha}$ is a rotation matrix and $u_i = (O_{i\alpha} - \delta_{i\alpha}) x_\alpha$, where as usual, the summation convention on repeated indices is understood.

Elasticity describes macroscipic distortions that vary slowly compared to any microsopic scale in the materials of interest. Local deformations can and are thus described by a *deformation matrix*, $\underset{\sim}{\mathbf{\Lambda}}$ with components $\Lambda_{i\alpha}$, that involves only first derivatives with respect to x_α.

$$R_i = \Lambda_{i\alpha} x_\alpha, \tag{1.2}$$

where the summation convention is understood and where

$$\Lambda_{i\alpha} = \frac{\partial R_i}{\partial x_\alpha} \equiv \partial_\alpha R_i = \delta_{i\alpha} + \eta_{i\alpha}; \qquad \eta_{i\alpha} = \partial_\alpha u_i \tag{1.3}$$

where $\eta_{i\alpha} = \partial_\alpha u_i$. Note that there is no requirement that $\Lambda_{i\alpha}$ be symmetric; it describes both rotations and deformations. In addition, the indices i and α transform under different group operations: The right index i, associated with $\mathbf{R}$, transforms under operations in the target space, whereas the left index α, associated with $\mathbf{x}$ transforms under operations in the reference space. Thus, the α part of $\Lambda_{i\alpha}$ is invariant under the symmetry operations of the reference space, whereas the i part transforms with $\mathbf{R}$. If $\Lambda_{i\alpha}$ is spatially uniform as is often

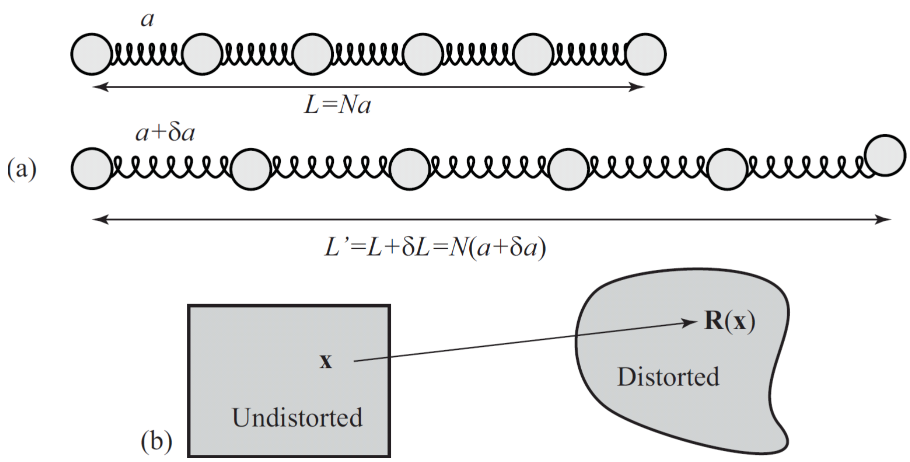

FIG. 1. (a) Stretched and unstrethed springs. (b) Schematic representation of the mapping from an undistorted equilibrium material in the reference space to its distorted form in the target space. reference space to the

the case, $R_i(\mathbf{x}) = \Lambda_{i\alpha} x_\alpha$. A uniform orthogonal (thus anti-symmetric) $\Lambda_{i\alpha}$ corresponds to a pure rotations, whereas a symmetric $\Lambda_{i\alpha}$ corresponds to a deformation. A $\Lambda_{i\alpha}$ that is neither symmetric nor orthogonal describes a deformation followed by a rotation or vice versa. The *polar decomposition theorem* (see below) of matrix algebra says that any square real matrix can be written as a product of an orthogonal rotation matrix times a symmetric deformation matrix (or vice versa).

The Jacobian of the transformation form $\mathbf{x}$ to $\mathbf{R}(\mathbf{x})$ is simply the determinant of the deformation matrix:

$$d^d R = \det \underset{\sim}{\boldsymbol{\Lambda}} d^d x. \tag{1.4}$$

The volume V in the target space of original volume V_0 in the reference space is thus

$$\begin{aligned} V &= \det \underset{\sim}{\boldsymbol{\Lambda}} V_0 \\ \frac{\delta V}{V_0} &= \det(\underset{\sim}{\mathbf{1}} + \underset{\sim}{\eta}) - 1 \approx \operatorname{Tr} \underset{\sim}{\eta}, \end{aligned} \tag{1.5}$$

Special names are given to certain simple deformations. A uniform *compression* or *dilation* [Fig. 2(b)] is described by an isotropic deformation tensor:

$$\underset{\sim}{\boldsymbol{\Lambda}} = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix}. \tag{1.6}$$

Deformations that preserve volume are called *shear*, and two versions of shear deformations [Fig. 2(c)] are given special names; *pure shear* in which

$$\underset{\sim}{\mathbf{\Lambda}} = \begin{pmatrix} \Lambda^{-1/2} & 0 & 0 \\ 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \tag{1.7}$$

and simple shear in which

$$\underset{\sim}{\mathbf{\Lambda}} = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.8}$$

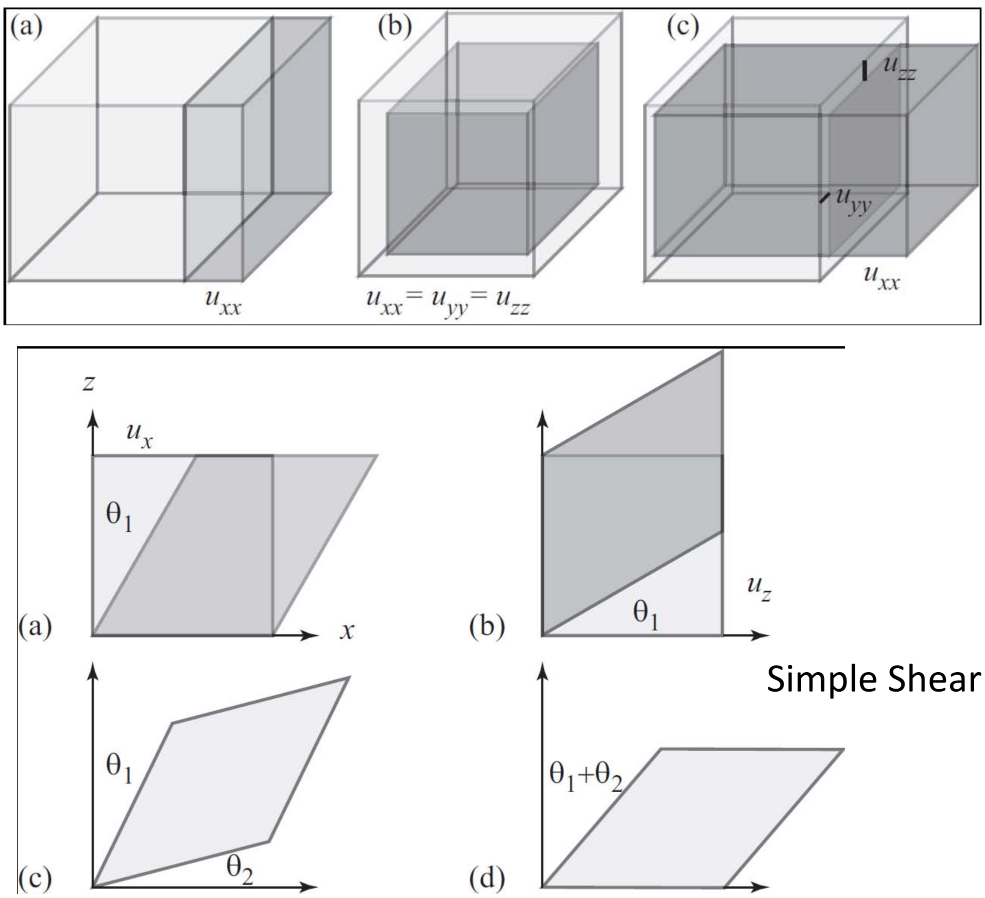

FIG. 2. Elastic deformations:Top (a) Uniaxial stretch, (b) uniform compression, (c) pure shear; Bottom different realizations of simple shear.

The deformations described by $\underset{\sim}{\mathbf{\Lambda}}$ correspond to metric changes. Two nearby points separated by a distance $dx = \sqrt{dx_\alpha dx_\alpha}$ in the reference space are separated by a distance

dR in the target space satisfying

$$(dR)^2 = \Lambda^T_{i\alpha}\Lambda_{i\beta} dx_\alpha dx_\beta \equiv g_{\alpha\beta} dx_\alpha dx_\beta, \tag{1.9}$$

where

$$g_{\alpha\beta} = \partial_\alpha R_i \partial_\beta R_i \qquad \text{or} \qquad \underset{\sim}{\mathbf{g}} = \underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}} \tag{1.10}$$

is the metric tensor (the right Cauchy-Green tensor in the mechanical engineering literature), which is invariant under arbitrary rotations in the target space.

If the reference space is not deformed, then $(dR)^2 = (dx)^2$, and the metric tensor is just the unit tensor. Thus deformations arise when $\underset{\sim}{\mathbf{g}}$ differs from the unit matrix. This leads us to define the nonlinear strain tensor as deviations of $\underset{\sim}{\mathbf{g}}$ from the unit matrix:

$$\begin{aligned} \underset{\sim}{\mathbf{u}} &= \tfrac{1}{2}(\underset{\sim}{\mathbf{g}} - \underset{\sim}{\mathbf{1}}) = \tfrac{1}{2}(\underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}} - \underset{\sim}{\mathbf{1}}) \approx \tfrac{1}{2}(\underset{\sim}{\eta} + \underset{\sim}{\eta}^T) \\ u_{\alpha\beta} &= \tfrac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha u_k \partial_\beta u_k) \approx \tfrac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha). \end{aligned} \tag{1.11}$$

In many cases, the strain is small enough that the nonlinear part of the nonlinear strain tensor can be ignored and the linearized form of Eq. (1.11) can be used. In this case the change in volume is

$$\begin{aligned} \frac{\delta V}{V_0} &= \det(\underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}})^{1/2} - 1 \\ &= \det(\underset{\sim}{\mathbf{1}} + 2\underset{\sim}{\mathbf{u}})^{1/2} - 1 = \exp \frac{1}{2} \mathrm{Tr} \ln(\underset{\sim}{\mathbf{1}} + 2\underset{\sim}{\mathbf{u}}) - 1 \approx \mathrm{Tr}\, \underset{\sim}{\mathbf{u}} \approx \nabla \cdot \mathbf{u}. \end{aligned} \tag{1.12}$$

The strain tensor has $d(d+1)/2$ (3 in $d = 2$ and 6 in $d = 3$) independent degrees of freedom, one of which is $\mathrm{Tr}\underset{\sim}{\mathbf{u}}$, which as we have seen reduces to the fractional volume change in the linearized limit. The other $(d^2 + d - 2)/2$ degrees of freedom correspond to shear strains that do not change volume. The are the independent degrees of freedom in the symmetric and traceless strain tensor

$$\underset{\sim}{\mathbf{u}}^{ST}_{\alpha\beta} = u_{\alpha\beta} - (1/d)\delta_{\alpha\beta} u_{\gamma\gamma}. \tag{1.13}$$

The strain tensor of Eq.(1.11) is a tensor in the reference space. An alternative strain is the Eulerian strain that is a tensor in the target space. It is defines through

$$(dR)^2 - (dx)^2 = 2u^E_{ij} dR_i dR_j \tag{1.14}$$

$$u^E_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial x_\alpha}{\partial R_i}\frac{\partial x_\alpha}{\partial R_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial R_j} + \frac{\partial u_j}{\partial R_i} - \frac{\partial u_\alpha}{\partial R_i}\frac{\partial u_\alpha}{\partial R_j}\right) = \frac{1}{2}\left(\delta_{ij} - (\Lambda^T)^{-1}_{i\alpha}\Lambda^{-1}_{\alpha j}\right) \tag{1.15}$$

This strain is invariant under arbitrary rotations in the reference space, which is OK when the elastic medium is isotropic, but presents problems when the medium is a crystal with discrete point-group symmetries rather than full isotropy. The matrix $(\underset{\sim}{\mathbf{\Lambda}}^T)^{-1}\underset{\sim}{\mathbf{\Lambda}}^{-1}$ is called the *Finger tensor*. Other measures of strain are provided by the *Left Cauchy-Green* tensor $\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{\Lambda}}^T$.

C. Elastic Energy

Energy should be invariant under arbitrary rigid operations of the D-dimensional Euclidean group of the target space vectors and under symmetry operations (discrete or continuous) of the reference space. Allowing for the moment for $d \neq D$, let $\underset{\sim}{\mathbf{U}}$ be the $D \times D$ dimensional matrix describing Euclidean operations in the target space and $\underset{\sim}{\mathbf{V}}$ the $d \times d$ dimensional matrix describing symmetry operations in the reference space. Thus the elastic free energy density $f_{\rm el}$ expressed as a function of $\underset{\sim}{\mathbf{\Lambda}}$ satisfies

$$f_{\rm el}(\underset{\sim}{\mathbf{\Lambda}}) = f_{\rm el}(\underset{\sim}{\mathbf{U}}\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{V}}^{-1}), \tag{1.16}$$

where $\underset{\sim}{\mathbf{V}}$ describes point-group symmetry operations in the reference space. The operations in $\underset{\sim}{\mathbf{U}}$ act on the post-deformed sample without changing the energy, and are usually not of great interest. We can, therefore, express $f$as a function of $\underset{\sim}{\mathbf{u}}$ rather than $\underset{\sim}{\mathbf{\Lambda}}$:

$$f_{\rm el} = f_{\rm el}(\underset{\sim}{\mathbf{u}}) = f_{\rm el}(\underset{\sim}{\mathbf{V}}\underset{\sim}{\mathbf{u}}\underset{\sim}{\mathbf{V}}^{-1}). \tag{1.17}$$

The total elastic energy is, of course,

$$\mathcal{F} = \frac{1}{2}\int d^d x f_{\rm el}(\underset{\sim}{\mathbf{u}}), \tag{1.18}$$

where to harmonic order in the nonlinear strain, the elastic free energy density is

$$f_{\rm el}(\underset{\sim}{\mathbf{u}}) = \frac{1}{2}\int d^d x K_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta}, \tag{1.19}$$

where $K_{\alpha\beta\gamma\delta}$ is the elastic tensor. It is necessarily symmetric under interchange of α and β and of γ and δ because $u_{\alpha\beta}$ is symmetric under interchange of α, and β and, in addition, it is symmetric under interchange of the pairs $\alpha\beta$ and $\gamma\delta$:

$$K_{\alpha\beta\gamma\delta} = K_{\gamma\delta\alpha\beta} = K_{\beta\alpha\gamma\delta} = K_{\alpha\beta\delta\gamma}. \tag{1.20}$$

In isotropic systems, $f_{\rm el}$ is invariant with respect to arbitrary operations described by $\underset{\sim}{\mathbf{V}}$ and, $f_{\rm el}$ must be a function only of combinations of $\underset{\sim}{\mathbf{u}}$ that are scalars with respect to $\underset{\sim}{\mathbf{V}}$. The result is that to harmonic order in $\underset{\sim}{\mathbf{u}}$,

$$f_{\rm el} = \frac{1}{2}\lambda u_{\alpha\alpha}^2 + \mu u_{\alpha\beta}u_{\alpha\beta} \equiv \frac{1}{2}B({\rm Tr}\underset{\sim}{\mathbf{u}})^2 + \mu {\rm Tr}(\underset{\sim}{\mathbf{u}}^{ST})^2 \tag{1.21}$$

where λ and μ are the Lame coefficients, μ is the shear modulus, and $B = \lambda + 2(\mu/d)$ is the bulk modulus and

$$\mu^{ST} = \underset{\sim}{\mathbf{u}} - (1/d){\rm Tr}\,\underset{\sim}{\mathbf{u}} \tag{1.22}$$

is the symmetric-traceless strain that measures shear in the linearized limit. This form corresponds to an elastic constant tensor composed of the only tensor available in an isotropic system, namely the unit matrix $\delta_{\alpha\beta}$, and satisfying the symmetry relations of Eq. (1.20):

$$K_{\alpha\beta\gamma\delta} = \lambda\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{1.23}$$

In uniaxial systems, tensors can be formed from the unit vector n_α parallel to the uniaxial axis:

$$\begin{aligned} K_{\alpha\beta\gamma\delta} =& K_1 n_\alpha n_\beta n_\gamma n_\delta + K_2(n_\alpha n_\beta \delta^T_{\gamma\delta} + n_\gamma n_\delta \delta^T_{\alpha\beta}) + K_3 \delta^T_{\alpha\beta}\delta^T_{\gamma\delta} \\ &+ K_4(\delta^T_{\alpha\gamma}\delta^T_{\beta\delta} + \delta^T_{\alpha\delta}\delta^T_{\beta\gamma}) + K_5(n_\alpha n_\gamma \delta^T_{\beta\delta} + n_\alpha n_\delta \delta^T_{\beta\gamma} + n_\beta n_\delta \delta^T_{\alpha\gamma} + n_\beta n_\gamma \delta^T_{\alpha\delta}), \end{aligned} \tag{1.24}$$

where $n_\alpha = (0,0,1)$ and $\delta^T_{\alpha\beta} = \delta_{\alpha\beta} - n_\alpha n_\beta$. The elastic energy density is then

$$f_{\rm el} = \frac{1}{2}[K_1 u_{zz}^2 + 2K_2 u_{zz}u_{\nu\nu} + K_3 u_{\nu\nu}^2 + 2K_4 u_{\nu\tau}^2 + 4K_5 u_{\nu z}^2], \tag{1.25}$$

where $\mu, \nu = x$ or y.

As we have seen, the strain tensor is a $d \times d$ tensor but because it is symmetric, it has $d(d+1)/2$ rather than d^2 independent components, which we can take to be

$$\underset{\sim}{\mathbf{u}} = (u_{xx}, u_{yy}, u_{xy}) \qquad \text{in 2d} \tag{1.26}$$

$$\underset{\sim}{\mathbf{u}} = (u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, u_{yz}) \qquad \text{in 3d.} \tag{1.27}$$

The ordering of terms is different in the original Voigt notation: $\underset{\sim}{\mathbf{u}} = (u_{xx}, u_{yy}, u_{zz}, u_{yz}, u_{xz}, u_{xy})$. This allows the energy density to be expressed as a quadratic form in the $\underset{\sim}{\mathbf{u}}$

$$f_{\rm el} = \frac{1}{2}\underset{\sim}{\mathbf{u}}^T \underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{u}} \tag{1.28}$$

where $\underset{\sim}{\mathbf{K}}$ is a $d(d-1)/2 \times d(d-1)/2$ dimensional matrix. It appears with coefficients determined by the number of equivalent terms that appear in the free energy of Eq. (1.19). For example, there is only one term in $K_{\alpha\beta\gamma\delta}$ multiplying u_{xx}^2 or u_{yy}^2, whereas there are 2 symmetry equivalent terms multiplying $u_{xx}u_{yy}$ and four terms multiplying $u_{xx}u_{xy}$ and u_{xy}^2. thus in two-dimensions,

$$\underset{\sim}{\mathbf{K}} = \begin{pmatrix} K_{xxxx} & K_{xxyy} & 2K_{xxxy} \\ K_{xxyy} & K_{yyyy} & 2K_{yyxy} \\ 2K_{xxxy} & 2K_{yyxy} & 4K_{xyxy} \end{pmatrix} = \begin{pmatrix} K_1 & K_2 & 2K_5 \\ K_2 & K_3 & 2K_6 \\ 2K_5 & 2K_6 & 4K_4 \end{pmatrix}. \tag{1.29}$$

There are other ways to represent this energy by rescaling the strain and entries in $\underset{\sim}{\mathbf{K}}$. In the *Voigt* notation, the factors of 2 and 4 are replaced by unity but the xy component of the strain is multiplied by 2: $\underset{\sim}{\mathbf{u}} = (u_{xx}, u_{yy}, 2u_{xy})$, and in the Kelvin notation u_{xy} is replaced by $\sqrt{2}u_{xy}$, K_{xxxy} and K_{yyxy} by $\sqrt{2}K_{xxxy}$ and $\sqrt{2}K_{yyxy}$, and K_{xyxy} by $2K_{xyxy}$. Both forms give the same energy as above, and each has certain advantages, but the form defined above is sufficient for our purposes. We will refer to the representation of Eq. (1.29) as the Voigt notation even though strictly speaking it is not that notation.

D. Forces and Stresses

So far we have considered only the nature of geometric distortions of an elastic medium and the energy relative to the reference state generated by these distortions. We now turn to the response of the medium to forces and stresses. We begin with the internal force $\mathsf{f}_i(\mathbf{x})$ exerted on a volume element at $\mathbf{x}$ by its surrounding elastic medium. This force, which exists in the target space, is transmitted to the volume element across its boundaries implying that it must be the gradient of a stress tensor measuring the force per unit area of the reference space.

Stress is defined to be a force per unit area. Since both the direction of surface normal and the direction of the force relative to that normal can vary, stress is described by a second-rank tensor. Physicists usually think of the *Cauchy Stress Tensor*, σ_{ij}^C, which is the force in direction i across a surface oriented along j in normal Euclidean space, which in the case of Lagrangian elasticity is the target space, e.g. σ_{zx}^C is the force per area in the z direction in the target space on a surface with normal along the x direction. The force per unit volume f_i exerted on a small volume element δD by the stresses exerted by surrounding matter on the boundary $\partial\delta D$ of δD is the divergence of the Cauchy stress tensor:

$$\mathsf{f}_i = \frac{\partial \sigma_{ij}^C}{\partial R_j}, \tag{1.30}$$

where R_i is the position in Euclidean space. In an elastic medium, the stress tensor represents forces exerted across boundaries by matter within an elastic medium, and as a result, σ_{ij}^C is zero outside the medium. Internal forces, measured by σ_{ij}^C generate no macroscopic net force F_i or torque τ_i on a sample. The absence of net force follows from the fact σ_{ij}^C is zero outside the medium:

$$\mathsf{F}_i = \int_D d^3R\,\mathsf{f}_i = \int_D d^3R \frac{\partial \sigma_{ij}^C}{\partial R_j} = \int_{\partial D} dS_j \sigma_{ij}^C = 0, \tag{1.31}$$

where the boundary ∂D lies entirely outside the medium in question. The final relation follows because σ_{ij}^C is zero on ∂D. The second constraint arises from the conservation of angular momentum which requires that the total torque exerted by internal forces must vanish: This requires σ_{ij}^C to be symmetric:

$$\begin{aligned}\tau_i &= \int_D d^3R \epsilon_{ijk} R_j f_k = \int_D d^3R \epsilon_{ijk} R_j \frac{\partial \sigma_{kl}^C}{\partial R_l} \\ &= \int_{\partial D} dS_l \epsilon_{ijk} R_j \sigma_{kl}^C - \int_D d^3R \epsilon_{ijk} \sigma_{kj}^C = - \int_D d^3R \epsilon_{ijk} \sigma_{kj}^C = 0,\end{aligned} \tag{1.32}$$

where again the final relation, which sets the anti-symmetric part of σ_{kj}^C to zero, follows because σ_{kl} is zero on the boundary ∂D.

Forces exerted on an infinitesimal volume of mass in an elastic medium by the matter surrounding it are transmitted across the surfaces of that volume. We can take the surfaces to be either those surrounding the volume in the undeformed reference space or in the target space. The latter gives us the Cauchy stress tensor just discussed. The former give the first Piola-Kirchhoff (PK) stress tensor, σ_{ij}^I, and a force density

$$\mathsf{f}_i(\mathbf{x}) = \partial_\alpha \sigma_{i\alpha}^I, \tag{1.33}$$

where the force is what we measure in the target space (i.e., the same one that the derivative of the Cauchy stress tensor measures though at positions $\mathbf{x}$ rather than $\mathbf{R}(\mathbf{x})$), and as before $\partial_\alpha = \partial/\partial x_\alpha$. Unlike σ_{ij}^C, $\sigma_{i\alpha}^I$ does not have to be symmetric, and in general, it is not so.

To see how σ_{ij}^I is related to the elastic free energy, we consider the work done on an elastic medium by internal forces f_i. The change in total (free) energy, δF is equal to minus the work done by internal forces. As before, we consider a domain D' whose boundary $\partial D'$ lies entirely outside the volume D' occupied by the medium. Thus

$$\delta \mathcal{F} = -\int_{D'} d^d x \mathsf{f}_i \delta R_i = -\int_{D'} d^d x \partial_\alpha \sigma^I_{i\alpha} \delta R_i$$
$$= \int d^d x \sigma^I_{i\alpha} \delta \Lambda_{i\alpha}, \tag{1.34}$$

where the surface term on $\partial D'$ vanishes because $\sigma^I_{i\alpha}$ is zero outside matter and $\delta\Lambda_{i\alpha} = \partial_\alpha \delta R_i$. Recall that $\delta R_i = \delta u_i$ so that the force density is

$$\mathsf{f}_i = -\frac{\delta \mathcal{F}}{\delta u_i(\mathbf{x})}. \tag{1.35}$$

Equation (1.34) implies that

$$\sigma^I_{i\alpha} = \frac{\delta \mathcal{F}}{\delta \Lambda_{i\alpha}} = \frac{\partial f_{\text{el}}}{\partial \Lambda_{i\alpha}} = \frac{\partial f_{\text{el}}}{\partial u_{\beta\gamma}} \frac{\partial u_{\beta\gamma}}{\partial \Lambda_{i\alpha}} \equiv \Lambda_{i\beta} \sigma^{II}_{\beta\alpha}, \tag{1.36}$$

where

$$\sigma^{II}_{\alpha\beta} = \frac{\partial f_{\text{el}}}{\partial u_{\alpha\beta}} \tag{1.37}$$

is the second PK stress tensor, which is symmetric (because $u_{\alpha\beta}$ is), invariant under rotations in the target space, and a second-rank tensor with respect to rotations in the reference space. In deriving Eq. (1.36), we used $\partial u_{\beta\gamma}/\partial \Lambda_{i\alpha} = \frac{1}{2}(\delta_{\alpha\gamma}\Lambda_{i\beta} + \Lambda_{i\gamma}\delta_{\alpha\beta})$ and the fact that $\sigma^{II}_{\alpha\beta}$ is symmetric. Inserting Eq. (1.36) into Eq. (1.34) yields

$$\delta \mathcal{F} = \int d^d x \sigma^{II}_{\beta\alpha} \Lambda_{i\beta} \delta \Lambda_{i\alpha} = \int d^d x \sigma^{II}_{\beta\alpha} \delta u_{\alpha\beta} \tag{1.38}$$

which serves as a consistency check on Eq. (1.37).

Finally, the Cauchy stress tensor is obtained by transforming the volume integral over $\mathbf{x}$ in Eq. (1.34) to an integral over $R_i = \Lambda_{i\alpha} x_\alpha$ and by transforming the derivative with respect to x_α in the same equation to one with respect to R_i via $\partial/\partial x_\alpha = (\partial R_l/\partial x_\alpha)\partial/\partial R_l$:

$$\delta \mathcal{F} = -\int d^d R \frac{1}{\det \underset{\sim}{\mathbf{\Lambda}}} \sigma^I_{i\beta} \Lambda_{l\beta} \frac{\partial \delta u_i}{\partial R_l} = \int d^d X \frac{\partial \sigma^C_{ij}}{\partial R_j} \delta u_j, \tag{1.39}$$

implying

$$\sigma^C_{ij} = \frac{1}{\det \underset{\sim}{\mathbf{\Lambda}}} \sigma^I_{i\alpha} \Lambda_{j\alpha} = \frac{1}{\det \underset{\sim}{\mathbf{\Lambda}}} \Lambda_{i\alpha} \sigma^{II}_{\alpha\beta} \Lambda^T_{\beta j}. \tag{1.40}$$

As required, σ^C_{ij} is symmetric.

When the harmonic elastic energy energy is used, the second Piola-Kirchhoff stress tensor is

$$\sigma^{II}_{\alpha\beta} = \frac{\partial f_{\text{el}}}{\partial u_{\alpha\beta}} = \frac{1}{2}\frac{\partial}{\partial u_{\alpha\beta}} K_{\gamma\delta\rho\tau} u_{\gamma\delta} u_{\rho\tau}$$
$$= \frac{1}{2}[K_{\gamma\delta\rho\tau}(u_{\gamma\delta}\delta_{\rho\alpha}\delta_{\tau\beta} + u_{\rho\tau}\delta_{\gamma\alpha}\delta_{\delta\beta})]$$
$$= K_{\alpha\beta\gamma\delta}u_{\gamma\delta}. \tag{1.41}$$

The 2nd-PK stress tensor for isotropic solids follows from Eq. (1.23):

$$\sigma^{II}_{\alpha\beta} = \lambda\delta_{\alpha\beta}u_{\gamma\gamma} + 2\mu u_{\alpha\beta} = B\delta_{\alpha\beta}u_{\gamma\gamma} + 2\mu\left(u_{\alpha\beta} - \tfrac{1}{d}\delta_{\alpha\beta}u_{\gamma\gamma}\right), \tag{1.42}$$

where $B = \lambda + (2\mu/d)$ is the bulk modulus and the seccond expressions divides the contributions of $\sigma^{II}_{\alpha\beta}$ into a part arising from isotropic compression or expansion and a part from shear distortions, at least in the linearized limit. The stress tensor of the lowest-symmetry two-dimensional crystal is

$$\sigma^{II}_{xx} = K_{xx\alpha\beta}u_{\alpha\beta} = K_{xxxx}u_{xx} + K_{xxyy}u_{yy} + (K_{xxxy} + K_{xxyx})u_{xy}$$
$$= K_{xxxx}u_{xx} + K_{xxyy}u_{yy} + 2K_{xxxy}u_{xy} \tag{1.43}$$
$$\sigma^{II}_{yy} = K_{yyxx}u_{xx} + K_{yyyy}u_{yy} + 2K_{yyxy}u_{xy} \tag{1.44}$$
$$\sigma^{II}_{xy} = K_{xyxx}u_{xx} + K_{xyyy}u_{yy} + 2K_{xyxy}u_{xy}. \tag{1.45}$$

In all the cases, the elastic energy can be expressed as

$$f_{\text{el}} = \tfrac{1}{2}\sigma^{II}_{\alpha\beta}u_{\alpha\beta} \tag{1.46}$$

as can be seen from Eq. (1.41). A word of caution about using the energy expressed in terms of the Voigt version of the elastic tensor, Eq. (1.28). Though this is a correct energy, taking derivatives with respect to $u_{\alpha\beta}$ does not yield the stress tensor, as can be see by comparing the expression for $\sigma^{II}_{\alpha\beta}$ that results from differentiating that expression with respect to $u_{\alpha\beta}$ with Eq. (1.41).

We have focused here on theories of elasticity that preserve invariance with respect to arbitrary uniform post-deformation rotations, which requires the use of the nonlinear strain tensor. In many, if not most cases, strain is small enough that the linearized limit of $u_{\alpha\beta}$ provides an excellent description of elastic response, and all three stress tensors are equivalent. In this case, the it is not so useful to distinguish between the reference and target spaces with they both have the same spatial dimension, and we can simply take

$$u_{\alpha\beta} \to \tfrac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{1.47}$$

$$\sigma_{ij} = K_{ijkl} u_{kl}. \tag{1.48}$$

E. The Polar Decomposition Theorem and Its Uses

The deformation tensor $\Lambda_{i\alpha}$ describes both rotations and shape distortions. An explicit division of $\underset{\sim}{\mathbf{\Lambda}}$ into the product of a rotation matrix times a symmetric matrix (the square root of the metric tensor) is accomplished as follows:

$$\underset{\sim}{\mathbf{\Lambda}} = \underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}})^{-1/2}(\underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}})^{1/2} \equiv \underset{\sim}{O}\,\underset{\sim}{g}^{1/2} \tag{1.49}$$

$$\underset{\sim}{O} = \underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}})^{-1/2} = \underset{\sim}{\mathbf{\Lambda}}\,\underset{\sim}{g}^{-1/2}. \tag{1.50}$$

We now limit our attention to the case $d = D$ so that $\underset{\sim}{\mathbf{\Lambda}}$ is a square matrix, which necessarily has an inverse. Thus,

$$\begin{aligned} \underset{\sim}{O}^T \underset{\sim}{O} &= \underset{\sim}{g}^{-1/2} \underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}}\,\underset{\sim}{g}^{-1/2} = \underset{\sim}{g}^{-1/2} \underset{\sim}{g}\,\underset{\sim}{g}^{-1/2} = \underset{\sim}{1} \\ \underset{\sim}{O}\,\underset{\sim}{O}^T &= \underset{\sim}{\mathbf{\Lambda}}\,\underset{\sim}{g}^{-1/2}(\underset{\sim}{\mathbf{\Lambda}}\,g^{-1/2})^T = \underset{\sim}{\mathbf{\Lambda}}\,\underset{\sim}{g}\,\underset{\sim}{\mathbf{\Lambda}}^T = \underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Lambda}}^T \underset{\sim}{\mathbf{\Lambda}})^{-1} \underset{\sim}{\mathbf{\Lambda}}^T = \underset{\sim}{1}, \end{aligned} \tag{1.51}$$

and $\underset{\sim}{O}$ is an orthogonal matrix with determinant equal to either $+1$ (rotation) or -1 (improper rotation). In what follows, we restrict our attention to the former case.

Though $\underset{\sim}{O}$ is a square matrix, it is still useful to retain a notation that distinguishes between the reference and target space, in which the the components of $\underset{\sim}{O}$ and $\underset{\sim}{O}^T$ are $O_{i\alpha}$ and $O^T_{\alpha i}$. In the linearized limit,

$$\begin{aligned} \underset{\sim}{O} = \underset{\sim}{\mathbf{\Lambda}}\,\underset{\sim}{g}^{-1/2} &= (\underset{\sim}{\delta} + \underset{\sim}{\eta})[(\underset{\sim}{\delta}^T + \underset{\sim}{\eta}^T)(\underset{\sim}{\delta} + \underset{\sim}{\eta})]^{-1/2} \\ &\approx \underset{\sim}{\delta} + \underset{\sim}{\eta} - \tfrac{1}{2}(\underset{\sim}{\eta}^T + \underset{\sim}{\eta}) = \underset{\sim}{\delta} + \tfrac{1}{2}(\underset{\sim}{\eta} - \underset{\sim}{\eta}^T) \\ O_{i\alpha} &\approx \delta_{i\alpha} + \tfrac{1}{2}(\partial_\alpha u_i - \partial_i u_\alpha) = \delta_{i\alpha} - \epsilon_{i\alpha k}\Omega_k, \end{aligned} \tag{1.52}$$

where $\Omega_k = \frac{1}{2}\epsilon_{klm}\partial_l u_m$. Thus $\underset{\sim}{O}$ takes any vector A_α expressed in terms of components in the reference space and rotates it to a vector with components A_i in the target space, and

$\underset{\sim}{O}^T$ takes a vector in the target space and converts it to one in the reference space. :

$$A_i = O_{i\alpha} A_\alpha; \qquad A_\alpha = O_{\alpha i} A_i. \tag{1.53}$$

These are useful transformations because they allow us to find minimal expressions for interactions of strain with external electric, magnetic, or other fields that are created in the target space. For example, a rotational-invariant coupling between the electric field E_α and strain expressed in the reference space can be converted to one expressed in terms of the more physical target space electric field E_i:

$$u_{\alpha\beta} E_\alpha E_\beta = E_i O_{i\alpha} u_{\alpha\beta} O^T_{\beta j} E_j \equiv v_{ij} E_i E_j \tag{1.54}$$

$$\begin{aligned} \underset{\sim}{O}\underset{\sim}{u}\underset{\sim}{O}^T &= \tfrac{1}{2}\underset{\sim}{\mathbf{\Lambda}}(\underset{\sim}{\mathbf{\Lambda}}^T\underset{\sim}{\mathbf{\Lambda}})^{-1/2}(\underset{\sim}{\mathbf{\Lambda}}^T\underset{\sim}{\mathbf{\Lambda}} - \underset{\sim}{\delta})(\underset{\sim}{\mathbf{\Lambda}}^T\underset{\sim}{\mathbf{\Lambda}})^{-1/2}\underset{\sim}{\mathbf{\Lambda}}^T = \tfrac{1}{2}(\underset{\sim}{\mathbf{\Lambda}}\underset{\sim}{\mathbf{\Lambda}}^T - \underset{\sim}{\delta}) \\ &= \underset{\sim}{v} \approx \tfrac{1}{2}(\underset{\sim}{\eta} + \underset{\sim}{\eta}^T) \to \tfrac{1}{2}(\partial_i u_j + \partial_j u_i). \end{aligned} \tag{1.55}$$

Note the dependence on the left Cauchy-Green tensor. Thus, to linear order in strain, we do not have to worry about the difference between the target and reference external fields.

II. ELASTIC WAVES AND ELASTIC RESPONSE

Reference:

Chaikin and Lubensky,"Principles of Condensed Matter Physics," Cambridge Press.

This chapter will review normal modes (sound waves) of response of elastic media in the linearized limit in which the strain is simply $(\partial_i u_j + \partial_j u_i)/2$. [We now ignore the difference between the reference and target spaces.] Though the elastic energy is expressed in terms of strains, it is a functional of the the displacement fields, which in the linearized limit can be expressed as a simple bilinear form :

$$\mathcal{F} = \frac{1}{2}\int d^dx d^dx' u_i(\mathbf{x}) K_{ij}(\mathbf{x},\mathbf{x}') u_j(\mathbf{x}') = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} u_i(-\mathbf{q}) K_{ij}(\mathbf{q}) u_j(\mathbf{q}), \tag{2.1}$$

where $u_i(\mathbf{q})$ is the Fourier transform of $u_i(\mathbf{q})$

$$K_{ij}(\mathbf{x},\mathbf{x}') = -K_{ikjl}\partial_k \delta(\mathbf{x}-\mathbf{x}')\partial'_l \tag{2.2}$$

$$K_{ij}(\mathbf{q}) = K_{ikjl} q_k q_l. \tag{2.3}$$

In isotropic media,

$$K_{ij}(\mathbf{q}) = (\lambda + 2\mu) q_i q_j + \mu q^2 \delta^t_{ij}, \tag{2.4}$$

where $\delta^t_{ij} = \delta_{ij} - \hat{q}_i \hat{q}_j$.

A. Bulk Waves

The dynamics of a Lagrangian elastic medium follows from Newton's laws:

$$\rho \ddot{u}_i = \mathsf{f}_i = \partial_j \sigma_{ij} = \partial_k \partial_l K_{ikjl} u_j \tag{2.5}$$

$$\omega^2 u_i(\mathbf{q}, \omega) = D_{ij}(\mathbf{q}) u_j(\mathbf{q}, \omega). \tag{2.6}$$

where ρ is the mass density and $D_{ij}(\mathbf{q}) = K_{ij}(\mathbf{q})/\rho$ is the dynamical matrix. The square of normal mode frequencies are the eigenvalues of the dynamical matrix. In isotropic systems, the dynamical matrix is

$$\rho D_{ij} = (\lambda + 2\mu) q_i q_j + \mu q^2 \delta_{ij}^t \tag{2.7}$$

which is diagonalized by a longitudinal wave with $u_i^L = A^L \hat{q}_i$, where $\hat{q}_i = (q_x, q_y, q_z)/q$, and a transverse waves $u_i^T = \delta_{ij} A_j^T$, which in two dimensions reduces to $u_i^T = A^T \hat{e}_i$, where $e_i = (-q_y, q_x)/q$ is the vector perpendicular to $\hat{q}_i$. The normal modes are the usual longitudinal and transverse sound waves with frequencies

$$\omega_L^2 = c_L^2 q^2; \quad c_L^2 = (\lambda + 2\mu)/\rho = [B + 2(d-1)(\mu/d)]/\rho \tag{2.8}$$

$$\omega^2 = c_T^2 q^2; \qquad c_T^2 = \mu/\rho. \tag{2.9}$$

Note that both c_L and c_T remain positive even when the bulk modulus B is zero or mildly negative, indicting that the system is stable against finite-wavenumber excitations even when it is unstable with respect to uniform compression. Also note that in two dimensions, $c_L = c_T$.

The dynamical matrix of the lowest symmetry (polar) two-dimensional material is

$$D_{ij}(\mathbf{q}) = \frac{1}{\rho} \begin{pmatrix} K_1 q_x^2 + K_4 q_y^2 + 2K_5 q_x q_y & (K_2 + K_4) q_x q_y + K_5 q_x^2 + K_6 q_y^2 \\ (K_2 + K_4) q_x q_y + K_5 q_x^2 + K_6 q_y^2 & K_3 q_y^2 + K_4 q_x^2 + 2K_6 q_x q_y \end{pmatrix} \tag{2.10}$$

B. Surface Rayleigh Waves

Solid materials have surface waves, termed Rayleigh waves, as well bulk waves. Calculating their frequencies is an algebraically somewhat messier task than it is for bulk waves in that they require dealing with boundary conditions at the surface. The "philosophy" of the surface calculations is also different from that of bulk calculations. First the inverse penetrations depths κ are calculate as a function of the surface wavenumber q and the frequency

ω. Then the frequency as a function of wavenumber is determined by the suface boundary conditions. Here we present the calculation of the Rayleigh-wave frequency (and thus speed) for an isotropic solid. We take the surface to be parallel to the x axis, and we first assume that the surface waves penetrate into the bulk with an exponential decay. An exponential decay along y is equivalent to replacing q_y by $i\kappa$. Thus the surface waves that satisfy the bulk equations of motion are simply the waves already calculated with this replacement. As in the bulk case, there are longitudinal and trans verse solutions:

$$u_i^L(x,y) = A_L e^{iqx-\kappa_L y}(q, i\kappa_L) \tag{2.11}$$

$$u_i^T(x,y) = A_T e^{iqx-\kappa_T y}(-i\kappa_T, q), \tag{2.12}$$

where

$$\kappa_L^2 = q^2 - \omega^2/c_L^2 \tag{2.13}$$

$$\kappa_T^2 = q^2 - \omega^2/c_T^2. \tag{2.14}$$

or

$$u_x(x,y) = e^{iqx}\left(qA_L e^{-\kappa_L y} - i\kappa_T A_T e^{-\kappa_T y}\right) \tag{2.15}$$

$$u_y(x,y) = e^{iqx}\left(i\kappa_L A_L e^{-\kappa_k y} + qA_T e^{-\kappa_T y}\right). \tag{2.16}$$

The surface boundary condition is that the stress on the surface vanish:

$$\begin{aligned}\sigma_{yy} &= (\lambda + 2\mu)u_{yy} + \lambda u_{xx} = \rho c_L^2 u_{yy} + \rho(c_L^2 - 2c_T^2)u_{xx} \\ \sigma_{xy} &= 2\mu u_{xy} = 2\rho c_T^2 u_{xy},\end{aligned} \tag{2.17}$$

or

$$\sigma_{yy}(y=0) = \rho e^{iqx}[i(c_L^2(q^2-\kappa_L^2) - 2q^2c_T^2)A_L - 2c_T^2 q\kappa_T A_T] = 0 \tag{2.18}$$

$$\sigma_{xy}(y=0) = \rho c_T^2 e^{iqx}[-2q\kappa_L A_L + i(q^2+\kappa_T^2)A_T] = 0, \tag{2.19}$$

which can be re-expressed as

$$\begin{pmatrix} i[c_L^2(q^2-\kappa_L^2) - 2q^2c_T^2] & -2q\kappa_T c_T^2 \\ -2q\kappa_L & i(q^2+\kappa_T^2)\end{pmatrix}\begin{pmatrix} A_L \\ A_T\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix}. \tag{2.20}$$

The determinant of the preceding matrix must vanish:

$$-(q^2+\kappa_T^2)(c_L^2(q^2-\kappa_L^2) - 2q^2c_T^2) - 4q^2c_T^2\kappa_T\kappa_L \tag{2.21}$$

$$= (1/c_T^2)(2c_T^2q^2 - \omega^2)^2 - 4q^2c_T^2\sqrt{[q^2-(\omega^2/c_L^2)][q^2-(\omega^2/c_T^2)]} = 0. \tag{2.22}$$

Setting $\omega^2 = c_T^2 s^2 q^2$ renders every term in the above expression proportional to q^4 leaving

$$s^6 - 8s^4 + 8s^2[3 - 2(c_T^2/c_L^2)] - 16[1 - (c_T^2/c_L^2)] = 0. \qquad (2.23)$$

Physically acceptable solutions to this equation must satisfy $s^2 > 0$, $\kappa_L^2 > 0$, $\kappa_T^2 > 0$, and importantly $s^2 < 1$ because the frequency of the surface waves must lie below the frequencies of all bulk waves. Only one of the three solutions for s^2 to Eq. (2.23) satisfies these criteria producing a unique solution for the surface mode velocity $c_S = c_T g(c_T^2/c_L^2)$. This solution has some interesting features: When the bulk modulus $B = \lambda + \mu$ is zero and $c_L^2 = c_T^2$. In this limit, the three solutions to Eq. (2.23) are

$$s = 0, \qquad s\sqrt{\ } = 4\left(1 \pm \sqrt{1/2}\right). \qquad (2.24)$$

Only the solution $s = 0$ ($\omega^2 = 0$) satisfies the constraint $s < 1$. In this case, the Rayleigh-wave frequency is zero for all q (Of course, the continuum approximation breaks down at as q increases, but we will find that this result holds for all q_x in lattice models), and the two inverse penetrations are equal q:

$$\kappa_L = \kappa_T = q. \qquad (2.25)$$

In the opposite limit in which either $\mu \to 0$ or $\lambda \to \infty$, $c_S = 0.955 c_T$.

C. Response and Thermal Fluctuations

The force generated by internal stresses is the negative derivative of the elastic energy with respect to $\mathbf{u}_i$. In static equilibrium, the force must equal minus the external force $\mathsf{f}_i^{\text{ext}}(\mathbf{x})$:

$$\mathsf{f}_\mathsf{i}(\mathbf{x}) = -\frac{\delta \mathcal{F}}{\delta u_i(\mathbf{x})} = -\int d^d x K_{ij}(\mathbf{x}, \mathbf{x}')u_j(\mathbf{x}') = \partial_k \partial_l K_{ikjl} u_j = -\mathsf{f}_i^{\text{ext}}(\mathbf{x}). \qquad (2.26)$$

Taking the derivative of this equation with respect to $\mathsf{f}_i^{\text{ext}}(\mathbf{x})$ yield the static displacement response function:

$$\chi_{ij}(\mathbf{x}, \mathbf{x}') = \frac{\delta u_i(\mathbf{x})}{\delta \mathsf{f}_j^{\text{ext}}(\mathbf{x}')} = K_{ij}^{-1}(\mathbf{x}, \mathbf{x}') \qquad (2.27)$$

$$\chi_{ij}(\mathbf{q}) = K_{ij}^{-1}(\mathbf{q}) \xrightarrow{\text{isotropic}} \frac{\hat{q}_i \hat{q}_j}{(\lambda + 2\mu)q^2} + \frac{\delta_{ij} - \hat{q}_i \hat{q}_j}{\mu q^2}. \qquad (2.28)$$

Fluctuations in $u_i(\mathbf{x})$ are the temperature T times the response function:

$$S_{ij}(\mathbf{x}, \mathbf{x}') = \langle u_i(\mathbf{x}) u_j(\mathbf{x}')\rangle - \langle u_i(\mathbf{x})\rangle \langle u_j(\mathbf{x}')\rangle = T\chi_{ij}(\mathbf{x}, \mathbf{x}'), \qquad (2.29)$$

from which we obtain:

$$S_{ij} = T\chi_{ij}(\mathbf{q}) : \qquad \langle u^2(\mathbf{x})\rangle \sim \int \frac{d^dq}{(2\pi)^2}\frac{T}{Kq^2} \xrightarrow{d\leq 2} \infty. \tag{2.30}$$

This divergent fluctuation implies that there is no long-range periodic order in two-dimensional crystals even though the have elastic rigidity as can be seen by taking the average of the mass density expressed in an expansion of in density waves at reciprocal lattice vectors $\mathbf{G}$:

$$\langle\rho(\mathbf{x})\rangle = \sum_{\mathbf{G}} \rho_{\mathbf{G}}\langle e^{i\mathbf{G}\cdot[\mathbf{x}-\mathbf{u}(\mathbf{x})]}\rangle \tag{2.31}$$

$$\langle e^{i\mathbf{G}\cdot[\mathbf{x}-\mathbf{u}(\mathbf{x})]}\rangle \approx e^{-G_i\langle u_i u_j\rangle G_j/2} \xrightarrow{d\leq 2} 0. \tag{2.32}$$

D. Non-affine Response

Reference:

DiDonna, B. A., and T. C. Lubensky. ”Nonaffine Correlations in Random Elastic Media.” Physical Review E 72, no. 6 (Dec 2005): 066619/1-23.

So far our treatment of elasticity applies only to spatially homogeneous continuum systems or to perfectly periodic lattices with a small number of atoms per unit cell. These systems undergo what is called *affine response* in which the distortion of every small unit of volume undergoes the same shape change as the macroscopic sample subjected to uniform stresses at its boundaries. In general lattices, mass points will relax to lower elastic energy in response to strains at boundaries: they exhibit nonaffice response. Figure 3 depicts non-affine response. Imagine mass points whose positions lie on parallel grid. When the surface is sheared, the grid will stretch or rotate in response to the deformation at the boundaries. If points originally on any set of grid lines stay on those lines under deformation, the response is affine, otherwise, if they move off the grid lines, the response is nonaffine.

Here we consider a model that illustrates nonaffine response and its effects on elastic moduli. In this model, the elastic tensor has small spatially random variations, produced for example by variations in local coordination number z, relative to a uniform value:

$$K_{ijkl}(\mathbf{x}) = K_{ijkl} + \delta K_{ijkl}(\mathbf{x}), \tag{2.33}$$

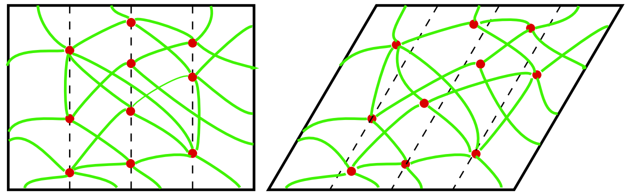

FIG. 3. (a) schematic of an elastic medium with red mass points that lie along a grid of straight lines. (b) The same medium under simple shear. In non-affine respones, the red sites do not stay on the grid of parallel lines.

where the average $\langle \delta K_{ijkl}(\mathbf{x})\rangle$ of δK_{ijkl} is zero, but its variance is not. In addition, we allow for random stresses $\tilde{\sigma}_{ij}(\mathbf{x})$ with zero mean but non-zero variance. We chose the reference state so that the system is in equilibrium at $\mathbf{u}(\mathbf{x}) = 0$. In this case, the force due to random stresses at each point $\mathbf{x}$ must vanish, imposing the constraint that

$$\partial_j \tilde{\sigma}_{ij}(\mathbf{x}) = 0. \tag{2.34}$$

The elastic energy is then

$$\mathcal{F} = \frac{1}{2}\int d^d x [K_{ijkl}(\mathbf{x}) u_{ij} u_{kl} + \tilde{\sigma}_{ij}(\mathbf{x})\partial_i u_k(\mathbf{x})\partial_j u_k(\mathbf{x})], \tag{2.35}$$

where the linear part of the strain tensor following $\tilde{\sigma}(\mathbf{x})$ vanishes because of the constraint of Eq. (2.34).

If the deformation were completely uniform the target space position of point $\mathbf{x}$ would simply be $\mathbf{R}_i(\mathbf{x}) = \Lambda_{ij} x_j$ and the displacement vector would be $u_i(\mathbf{x}) = \gamma_{ij} x_j$ at all $\mathbf{x}$. If there are non-affine displacements then

$$R_i(\mathbf{x}) = \Lambda_{ij} x_j + u'_i(\mathbf{x}) \qquad u_i(\mathbf{x}) = \gamma_{ij} x_j + u'_i(\mathbf{x}) \tag{2.36}$$

with the boundary condition that $u'_i(\mathbf{x}) = 0$ on boundary sites $\mathbf{x}_B$ so that $R_i(\mathbf{x}_B) = \Lambda_{ij} x_j$. The strain expressed in terms of u'_i is thus,

$$u_{ij} = \gamma^S_{ij} + \tfrac{1}{2}(\partial_i u'_j + \partial_j u'_i + \gamma_{pi}\partial_j u'_p + \gamma_{pj}\partial_i u'_p), \tag{2.37}$$

where $\gamma^S_{ij} = (\gamma_{ij} + \gamma_{ji})/2$, and, to linear order in γ_{kl},

$$\begin{aligned}\delta\mathcal{F} &= \frac{1}{2}\int d^d x \{K_{ijkl}\partial_j u'_i \partial_l u'_k + [\delta K_{ijkl} + \delta_{ik}\tilde{\sigma}_{jl}]\partial_j u'_i \partial_l u'_k + 2\delta K_{ijkl}(\mathbf{x})\gamma_{kl}\partial_j u'_i\} \\ &\propto \frac{1}{2}\int d^d x d^d x' u'(\mathbf{x})\chi^{-1}(\mathbf{x},\mathbf{x}')u'(\mathbf{x}') - \int d^d x\, \partial(\delta K(\mathbf{x}))\gamma u(\mathbf{x}),\end{aligned} \tag{2.38}$$

where in the second line, only lowest order terms are included and the $\tilde{\sigma}$ is dropped, the $\propto$ symbol signifies that was follows is "schematic" (i.e., does not keep track of all of the indices, etc.), and $\chi^{-1}(\mathbf{q}) = Kq^2$. Minimizing $\delta\mathcal{F}$ over the free local displacement $u_i'(\mathbf{x})$ in the presence of the driving strain γ_{ij} yields

$$u_i'(\mathbf{x}) = \int d^d x' \chi_{ip}(\mathbf{x},\mathbf{x}')\partial_j' \delta K_{pjkl}(\mathbf{x}')\gamma_{kl} \propto \chi\partial\delta K\gamma \tag{2.39}$$

$$\begin{aligned} u_i'(\mathbf{q}) &= iq_j\gamma_{kl}\chi_{ip}(\mathbf{q})\delta K_{pjkl}(\mathbf{q}), \\ &\propto \chi(\partial\delta K)\gamma \end{aligned} \tag{2.40}$$

where

$$\chi_{ik}^{-1}(\mathbf{x},\mathbf{x}') = -\partial_j K_{ijkl}^T(\mathbf{x})\partial_l \delta(\mathbf{x}-\mathbf{x}') \tag{2.41}$$

$$K_{ijkl}^T = K_{ijkl} + \delta K_{ijkl}(\mathbf{x}) + \delta_{ik}\tilde{\sigma}_{jl}. \tag{2.42}$$

To linear order in the random perturbations, $\delta K_{ijkl}(\mathbf{x})$ and $\tilde{\sigma}_{ij}(\mathbf{x})$ can be dropped in Eq. (2.42). In general, the calculation of even the zeroth order χ_{ij} is complicated. We can, however, see the general structure of the results simply by keeping straight the various powers of q. Figure 4 shows a numerically calculated typical pattern of non-affine displacements in response to a random local elastic tensor.

Inserting Eq. (2.40) into Eq. (2.38) yields a negative negative $\delta\mathcal{F}F$)

$$\begin{aligned} \delta\mathcal{F} &\propto \frac{1}{2}\int (\chi(\partial\delta K)\gamma)\chi^{-1}(\chi(\partial\delta K)\gamma) - 2(\partial\delta K)\gamma(\chi\partial\delta K)\gamma \\ &\propto -\frac{1}{2}\int d^d x d^d x' \gamma^2(\partial'\delta K(\mathbf{x}'))\chi(\mathbf{x},\mathbf{x}')(\partial\delta K(\mathbf{x})). \end{aligned} \tag{2.43}$$

This should be averaged over the random δK. The result is a reduction in the elastic modulus resulting from a purely affine distortion.

We are interested in the statistical properties of $u_i'(\mathbf{x})$. Equation (2.42) is linear in $\delta K_{ijkl}(\mathbf{x})$, whose average is by definition zero, implying that the average of $\mathbf{u}'(\mathbf{x})$ is also zero, but its variance is not. Schematically (i.e., focusing only on q dependence and ignoring tensor all tensor indices), $\delta\mathbf{u}'$ can be expressed as

$$u'(q) \propto q\chi(q)\delta K\gamma \tag{2.44}$$

and defining

$$G_{ij}'(\mathbf{x},\mathbf{x}') = \langle u_i'(\mathbf{x})u_j'(\mathbf{x}')\rangle \tag{2.45}$$

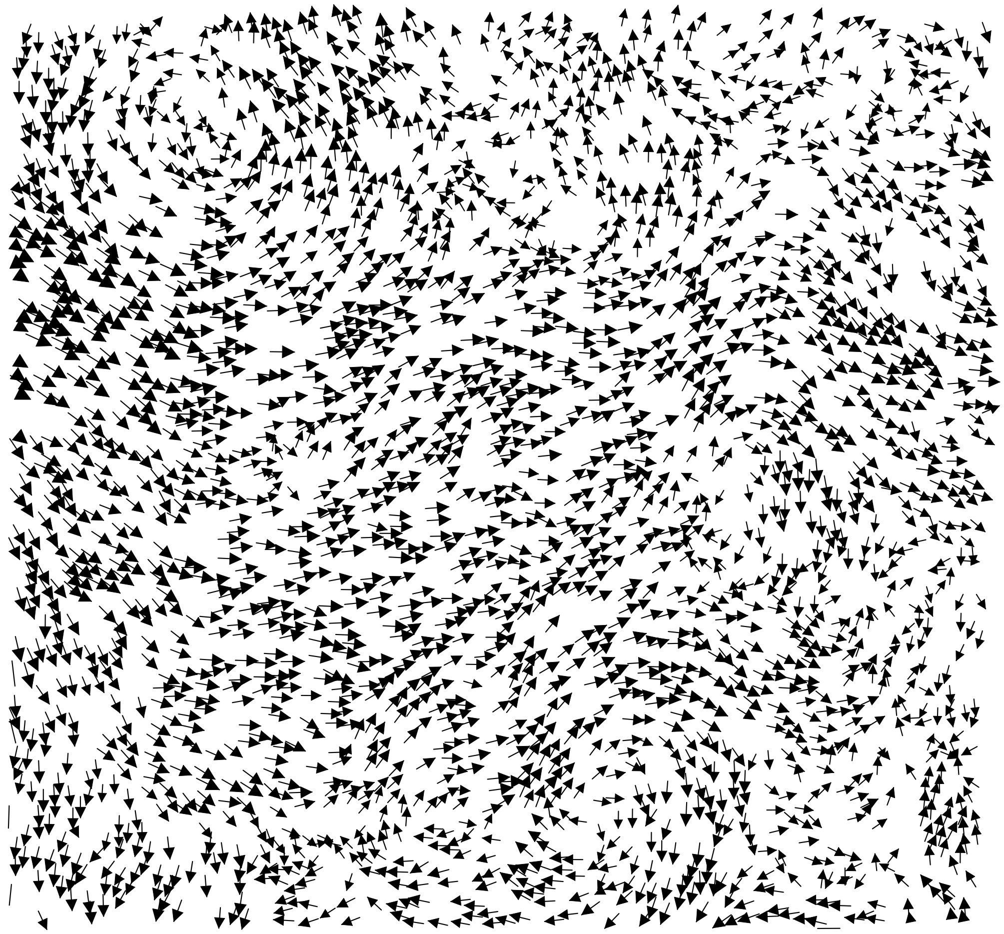

FIG. 4. Pattern of displacements in a medium with a random local elastic tensor.

which is translationally invariant,

$$G'(q) \sim \gamma^2 \chi(q) q \langle \delta K \delta K \rangle q \chi(q) \propto \gamma^2 \frac{\langle \delta K \delta K \rangle}{K^2 q^2}. \tag{2.46}$$

This result which gives diverges results for $G_{ij}(\mathbf{x}, \mathbf{x})$ determines the difference correlation function

$$\mathcal{G}_{ij}(\mathbf{x}) = \langle [u_i'(\mathbf{x}) - u_i'(0)][u_j'(\mathbf{x}) - u_j'(0)] \rangle \tag{2.47}$$

from which we obtain

$$\mathcal{G}(\mathbf{x}) = \mathcal{G}_{ii}(\mathbf{x}) = \langle [\mathbf{u}'(\mathbf{x}) - \mathbf{u}'(0)]^2 \rangle \tag{2.48}$$

$$\sim A \ln(|\mathbf{x}|)/B) \qquad d = 2 \tag{2.49}$$

$$\sim C - D|\mathbf{x}|^{-1} \qquad d = 3. \tag{2.50}$$

Another measure based of non-affinity involves on comparing the angle $\theta \equiv \theta(\mathbf{x}', \mathbf{x})$ that the vector connecting two sites originally at $\mathbf{x}$ and $\mathbf{x}'$ makes with some fixed axis after nonaffine distortion under shear from the angle $\theta_0 \equiv \theta_0(\mathbf{x}') - \theta_0(\mathbf{x})$ that that vector would make if the points were affinely distorted: Angular correlations can also be calculated:

$$\mathcal{G}_\theta(\mathbf{x}' - \mathbf{x}) = \langle [\theta(\mathbf{x}', \mathbf{x}) - \theta_0(\mathbf{x}', \mathbf{x})]^2 \rangle$$
$$\mathcal{G}_\theta(\mathbf{x}) = \epsilon_{ij}\epsilon_{kl}\frac{x_i x_j}{|\mathbf{x}|^4}\mathcal{G}_{kl}(\mathbf{x}), \tag{2.51}$$

where where $\epsilon_{ij} = \epsilon_{zij}$ is the two-dimensional antisymmetric symbol.

III. LATTICE MODELS

A. Notation and Model Energies

We consider model elastic networks in which particles occupy sites on periodic or random lattices in their force-free equilibrium state. Thus, particle ℓ is at lattice position $\mathbf{R}_{\ell 0}$ in equilibrium. When the lattices are distorted, particle ℓ undergoes a displacement $\mathbf{u}_\ell$ to a new position

$$\mathbf{R}_\ell = \mathbf{R}_{\ell 0} + \mathbf{u}_\ell. \tag{3.1}$$

We will refer to the equilibrium lattice, with lattice positions $\mathbf{R}_{\ell 0}$, as the **reference lattice** or **reference space**, and the space into which the lattice is distorted via the displacements $\mathbf{u}_\ell$ as the **target space**. Pairs of particles ℓ and ℓ' are connected by unbreakable central-force springs on the bond $b \equiv <\ell', \ell>$. The coordination number of each particle (or site) is equal to the number of particles (or sites) to which it is connected by bonds. The potential energy, $V_b(R_b)$ (which is rotationally invariant), of the spring on bond b depends only on the magnitude,

$$R_b = |\mathbf{R}_{\ell'} - \mathbf{R}_\ell|, \tag{3.2}$$

of the vector connecting particles ℓ and ℓ'. The total potential energy is thus

$$U_T = \sum_b V_b(R_b) \equiv \frac{1}{2}\sum_{\ell,\ell'} V_{<\ell',\ell>}(|\mathbf{R}_{\ell'} - \mathbf{R}_\ell|). \tag{3.3}$$

We will consider anharmonic potentials

$$V_b = \frac{1}{2}k_b(R_b - R_{bR})^2 + \frac{1}{4}g_b(R_b - R_{bR})^4, \tag{3.4}$$

with both harmonic and quartic components, where R_{bR} is the rest length of bond b. We assume that both k_b and g_b are finite. The harmonic limit is obtained when the quartic coefficient g_b vanishes, in which case, k_b is the harmonic spring constant.

We will only study systems in which there is an equilibrium reference state with particle positions $\{\mathbf{R}_{\ell 0}\}$ in which the force on each site is zero. The length $R_{b0} \equiv |\mathbf{R}_{\ell' 0} - \mathbf{R}_{\ell 0}|$ of each bond b in this configuration does not have to coincide with its rest length R_{bR}. As we shall see in more detail shortly, it is possible to have the total force on every site be zero but still have nonzero forces on each bond.

The potential energy of the lattice can be expanded in terms of the discrete lattice nonlinear strain ,

$$v_b = \frac{1}{2}(R_b^2 - R_{b0}^2) = \mathbf{R}_{b0} \cdot \Delta \mathbf{u}_b + \frac{1}{2}(\Delta \mathbf{u}_b \cdot \Delta \mathbf{u}_b) \tag{3.5}$$

relative to the reference state, where $\Delta \mathbf{u}_b = \mathbf{u}_{\ell'} - \mathbf{u}_\ell$. The discrete strain variable, v_b, is by construction invariant with respect to rigid rotations of the sample, i.e., it is invariant under $R_{\ell i} \rightarrow U_{ij} R_{\ell j}$, where U_{ij} is any ℓ-independent rotation matrix. To second order in v_b in an expansion about a reference lattice with lattice sites $\mathbf{R}_{\ell 0}$, the potential energy is

$$\Delta U_T = \sum_b R_{b0}^{-1} \tilde{F}(b) v_b + \frac{1}{2} \sum_b R_{b0}^{-2} k(b) v_b^2, \tag{3.6}$$

where $\tilde{F}(b) = |\tilde{\mathbf{F}}(b)|$ is the magnitude of the force,

$$\tilde{\mathbf{F}}(b) = -V_b'(R_{b0}) \mathbf{R}_{b0} / R_{b0}, \tag{3.7}$$

acting on bond b and

$$k(b) = V_{b0}''(R_{b0}) - R_{b0}^{-1} V_{b0}'(R_{b0}) \tag{3.8}$$

is the effective spring constant of bond b, which reduces to k_b when $R_{b0} = R_{bR}$. $k(b)$ is never infinite because we assume k_b and g_b are finite.

The equilibrium bond-length R_{b0} for each bond is determined by the condition that the total force at each site ℓ vanish at $\mathbf{u}_\ell = 0$:

$$F_i(\ell) = - \left. \frac{\partial \Delta U_T}{\partial u_{\ell i}} \right|_{\mathbf{u}_\ell = 0} = \sum_{\ell'} \tilde{F}_i(<\ell', \ell>). \tag{3.9}$$

This equilibrium condition only requires that the total force on each site, arising from all of the springs attached to it, be zero . It does not require that the force $\tilde{\mathbf{F}}(b)$ be equal to zero on every bond b.

B. Harmonic Limit

In equilibrium, when Eq. (3.9) is satisfied, the part of v_b linear in $\Delta \mathbf{u}_b$ disappears from ΔU_T. In this case, it is customary to express ΔU_T to harmonic order in $\Delta \mathbf{u}_b$:

$$\Delta U_T^{\text{har}} = \frac{1}{2}\sum_b [V_b'' e_{b0i}e_{b0j} + R_{b0}^{-1}V_b'(\delta_{ij} - e_{b0i}e_{b0j})]\Delta u_{bi}\Delta u_{bj}, \tag{3.10}$$

where $e_{b0i} = R_{b0i}/R_{b0}$ is the unit vector directed along bond b. Thus the harmonic potential on each bond decomposes into a parallel part, proportional to V_b'', directed along the bond and a transverse part, proportional to $R_{b0}^{-1}V_b'$, directed perpendicular to the bond. The transverse part vanishes when the quilibrium force on the bond vanishes.

The harmonic energy ΔU_T^{har} does not preserve the invariance with respect to arbitrary rotations of the full nonlinear strain energy ΔU_T of Eq. (**??**), under which

$$\begin{aligned} R_{b0i} + \Delta u_{bj} = U_{ij}(R_{b0j} + \Delta u_{bj}) &= R_{b0}' + \Delta u_{bi}' \\ \Delta u_{bi} \to \Delta u_{bi}' &= (U_{ij} - \delta_{ij})R_{b0j} + U_{ij}\Delta u_{bj}, \end{aligned} \tag{3.11}$$

where U_{ij} is a rotation matrix. It does, however preserve this invariance up to order θ^2 but not order $\theta^2 \Delta u_b$ and $\theta(\Delta u_b)^2$, where θ is a rotation angle. For small $\boldsymbol{\theta}$,

$$\Delta \mathbf{u}_b' = \Delta \mathbf{u}_b + \boldsymbol{\theta} \times \mathbf{R}_{b0} + O(\theta^2, \theta \Delta u_b), \tag{3.12}$$

and $\mathbf{e}_{b0} \cdot \Delta \mathbf{u}_b' = \mathbf{e}_{b0} \cdot \Delta \mathbf{u}_b + O(\theta^2, \theta \Delta u_b)$. Thus, the part of the harmonic energy arising from the $k(b)$ term in Eq. (3.6) is invariant to the order stated above. The invariance of the force term of Eq. (3.6) is more subtle. Under the above transformation of Eq. (3.12), $(\Delta u_b')^2 = (\Delta u_b)^2 + 2\boldsymbol{\theta} \times \mathbf{R}_b \cdot \Delta \mathbf{u}_b + (\boldsymbol{\theta} \times \mathbf{R}_b)^2 + O(\theta^2 \Delta u_b, \theta(\Delta u_b)^2)$, and it would seem that there are terms of order $\theta \Delta u_b$, and θ^2 in ΔU_T^{har}. These terms vanish, however, upon summation over ℓ and ℓ' because of the equilibrium force condition of Eq. (3.9). Thus, the full ΔU_T^{har} is invariant under rotations up to order θ^2.

C. Continuum Limit

In the continuum limit, when spatial variations are slow on a scale set by the lattice spacing, the equilibrium lattice positions become continuous positions $\mathbf{x}$ in the reference space: $\mathbf{R}_{\ell 0} \to \mathbf{x}$; and the target-space position and displacement vectors become functions of $\mathbf{x}$: $\mathbf{R}_\ell \to \mathbf{R}(\mathbf{x})$ and $\mathbf{u}_\ell \to \mathbf{u}(\mathbf{x})$. In this limit, the lattice strain v_b becomes

$$v_b \approx R^0_{bi} R^0_{bj} u_{ij}(\mathbf{x}), \tag{3.13}$$

where

$$u_{ij}(\mathbf{x}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i \mathbf{u} \cdot \partial_j \mathbf{u}) \tag{3.14}$$

is the full Green-Saint Venant Lagrangian nonlinear strain , which is invariant with respect to rigid rotations in the target space [i.e., with respect to rigid rotations of $\mathbf{R}(\mathbf{x})$]. Sums over lattice sites of the form $\sum_\ell S(\ell)$, for any function $S(\ell)$, can be replaced by integrals $\int d^d x S(\mathbf{x})/v(\mathbf{x})$ where $v(\mathbf{x})$ is the volume of the Voronoi cell centered at position $\mathbf{x} = \mathbf{R}_{\ell 0}$. The continuum energy is then

$$\mathcal{F} = \int d^d x \left[\frac{1}{2} K_{ijkl}(\mathbf{x}) u_{ij}(\mathbf{x}) u_{kl}(\mathbf{x}) + \tilde{\sigma}_{ij}(\mathbf{x}) u_{ij}(\mathbf{x}) \right], \tag{3.15}$$

where

$$\tilde{\sigma}_{ij}(\mathbf{x}) = -\frac{1}{2v(\mathbf{x})} \sum_{\ell'} \tilde{F}_i(b) R_{b0j}|_{b=<\ell',\ell>} \tag{3.16}$$

is a local symmetric stress tensor at $\mathbf{x}$ where the sum over ℓ' is over all bonds with one end at ℓ and

$$K_{ijkl}(\mathbf{x}) = \frac{1}{2v(\mathbf{x})} \sum_{\ell'} k(b) R_{b0}^{-2} R_{b0i} R_{b0j} R_{b0k} R_{b0l}|_{b=<\ell',\ell>} \tag{3.17}$$

is the local elastic-modulus tensor . Because it depends only on the full nonlinear strain $u_{ij}(\mathbf{x})$, the continuum energy $\mathcal{H}$ of Eq. (3.15) is invariant with respect to rigid rotations in the target space. This is a direct result of the fact that we consider only internal forces between particles. The stress tensor $\tilde{\sigma}_{ij}(\mathbf{x})$ is generated by these internal forces, and as a result, it multiplies u_{ij} in $\mathcal{H}$. It is necessarily symmetric, and it transforms like a tensor in the reference space. (It is not, however, the second Piola-Kirchoff tensor , $\sigma^{II}_{ij} = \delta\mathcal{H}/\delta u_{ij}(\mathbf{x}) = K_{ijkl} u_{kl} + \tilde{\sigma}_{ij}$, which also transforms in this way.) External stresses, on the other hand, specify a force direction in the target space and couple to the linear part of the strain.

Since $K_{ijkl}(\mathbf{x})$ in Eq. (3.17) arises from central forces on bonds, it and its average over randomness obey the Cauchy relations , $K_{ijkl}(\mathbf{x}) = K_{ikjl}(\mathbf{x}) = K_{iljk}(\mathbf{x})$, in addition to the more general symmetry relations, $K_{ijkl}(\mathbf{x}) = K_{jikl}(\mathbf{x}) = K_{ijlk}(\mathbf{x}) = K_{klij}(\mathbf{x})$. The Cauchy relations reduce the number of independent elastic moduli in the average modulus $K_{ijkl} = \langle K_{ijkl}(\mathbf{x}) \rangle$ below the maximum number permitted for any given point-group symmetry (for the lowest symmetry, from 21 to 15). In particular, they reduce the number of independent moduli in isotropic and hexagonal systems from two to one, setting the Lamé coefficients λ and μ equal to each other. In our analytical calculations, we will, however, treat λ and μ as independent. The Cauchy limit is easily obtained by setting $\lambda = \mu$. However, as we have seen [Eq. (2.43)]p, random δK leads to corrections to Eq. (3.17) when relaxation relative to the affine limit are allowed. These corrections violate the Cauchy relations.

IV. MARGINALLY COORDINATED LATTICES AND THE MAXWELL-CALLADINE THEOREM

References:

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A. Introduction

So far, these lectures have focused on continuum elastic media or lattices that are mechanically stable. In what follows, we will study lattices that are on the verge of mechanical stability. Such lattices include familiar ones like the square and kagome lattices with nearest neighbor (NN) springs, Penrose tilings and quasicrystals with NN springs, and packed spheres at the jamming transitions.

In a remarkable 1864 paper , James Clerk Maxwell undertook the first systematic study of the mechanical stability of frames consisting of points, which we will refer to as *sites*, with connections, which we will usually refer to as *bonds*, between them as a model for such real-world structures as the Warren Truss (patented in 1848) shown in Fig. 5. He defined a "stiff" frame as one in which "the distance between two points cannot be altered without changing the length of one or more connections". He showed that a stiff frame containing