Holography and Quantum Gravity

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Plan

1. What is holography?

2. How do we find examples of holography?

3. How do we test holography?

What is holography?

A theory of quantum gravity on certain space-time, known as anti-de Sitter (AdS) space-time, is equivalent to a conformally invariant quantum field theory (CFT) in one less dimensions.

What is an AdS space-time?

Consider a flat space-time with n space coordinates $(w^1, \cdots w^n)$ and two time coordinates (y^1, y^2)

In this space describe an (n+1) dimensional subspace by the equation:

$$(w^1)^2 + \cdots (w^n)^2 - (y^1)^2 - (y^2)^2 = -R^2$$

Locally this describes an n+1 dimensional AdS space AdS_{n+1}.

By a suitable coordinate transformation we can use coordinates $x^0, \cdots x^{n-1}, z$ such that the metric is given by

$$R^2 z^{-2} \ \text{diagonal matrix} \, (-1, 1, \cdots 1)$$

$$\text{metric} = R^2 z^{-2} \text{ diagonal matrix } (-1, 1, \cdots 1)$$

$$ds^2 = R^2 z^{-2} \{-(dx^0)^2 + (dx^1)^2 + \cdots (dx^{n-1})^2 + dz^2\}$$

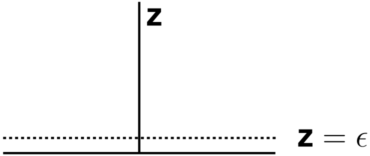
R: size of AdS space

ds: proper distance element

Note that the metric component goes to infinity as z approaches 0.

The distances between points separated by finite difference between the x^μ's become larger and larger in this limit.

$$ds^2 = R^2 z^{-2} \left\{ -(dx^0)^2 + (dx^1)^2 + \cdots (dx^{n-1})^2 + dz^2 \right\}$$



z=0 is called the boundary of the anti de Sitter space-time.

For practical calculations we place the boundary at $z = \epsilon$ where ϵ is some small number.

What is a CFT?

A quantum field theory which, besides being Lorentz invariant, is invariant under scale and special conformal transformations:

$$\mathbf{x}^\mu \to \lambda\,\mathbf{x}^\mu, \quad \mathbf{x}^\mu \to (\mathbf{x}^\mu - \mathbf{b}^\mu \mathbf{x}^2)/(1 - 2\mathbf{b}.\mathbf{x} + \mathbf{b}^2\mathbf{x}^2)$$

Example: Maxwell's theory without sources is conformally invariant in four space-time dimensions.

We shall denote by CFT_n a conformal field theory in n space-time dimensions.

Recall AdS_{n+1} metric:

$$ds^2 = R^2 z^{-2}\left\{-(dx^0)^2 + (dx^1)^2 + \cdots (dx^{n-1})^2 + dz^2\right\}$$

Note that for $z = \epsilon$ the metric in the $x^0, \cdots x^n$ direction is just the Minkowski metric up to an overall factor of $1/\epsilon^2$:

$$ds^2 = R^2 \epsilon^{-2}\left\{-(dx^0)^2 + (dx^1)^2 + \cdots (dx^{n-1})^2\right\}$$

<u>Holography</u>: A theory of quantum gravity in AdS_{n+1} is equivalent to a CFT_n in the n-dimensional Minkowski space at the boundary of the AdS space. Maldacena

Why is holography important?

Often it maps difficult problems in quantum field theory to simple questions in classical gravity in appropriate limit.

More importantly, holography allows us to give a precise definition of quantum gravity in certain space-times by relating it to an ordinary quantum field theory whose rules are quite well understood.

How do we find examples of holography?

Is there a systematic way of deciding which theory of quantum gravity on AdS_{n+1} is dual to which CFT_n?

At present we only know the answer to this in a small class of examples.

(Each ‘class’ contains an infinite number of examples)

From now on we shall restrict our discussion to string theory.

String theory is a quantum theory of gravity in which elementary ‘particles’ are replaced by one dimensional string like objects.

However besides these fundamental strings, string theory contains many other ‘composite’ objects which could have higher or lower number of dimensions.

Example: Type IIB string theory in ten space-time dimensions has an object extending in 3 directions

– called a D3-brane (Dirichlet 3-brane)

(generalization of membrane which extends in 2-directions).

In general, the dynamics of these composite objects is complicated involving infinite number of ‘internal vibrational modes’.

However in the low energy limit the dynamics of a p-brane – a p-dimensional extended object – is described by some quantum field theory in p+1 dimensions.

Examples:

Dynamics of a point-like object (0-brane) is described by quantum mechanics – a 0+1 dimensional quantum field theory.

Dynamics of a string (1-brane) is described by a 1+1 dimensional quantum field theory.

A non-trivial example:

The dynamics of N D3-branes in 9+1 dimensional type IIB string theory is described by a specific 3+1 dimensional quantum field theory known as $\mathcal{N} = 4$ supersymmetric non-abelian gauge theory based on the group U(N).

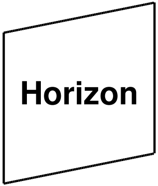
This theory is known to be invariant under conformal transformations.

However there is another way of describing the same system.

Since string theory is a theory of gravity, the D3-branes produce gravitational field (and also other fields).

Thus we can try to describe them as solutions to classical equations of motion of gravity (and other fields which are present in string theory.)

It turns out that the solution corresponds to the zero temperature (extremal) limit of a black 3-brane solution.



Due to gravitational red shift, modes of excitation near the horizon of this black hole appear to be of extremely low energy from the point of view of an observer far away from the branes.

The dynamics of these low energy modes is given by string theory in the ‘near horizon geometry’ – the geometry of the black brane near the horizon.

For D3-branes the near horizon geometry is $AdS_5 \times S^5$ – the direct product of a five dimensional AdS space and a five dimensional sphere.

Thus we have two different descriptions of the low energy dynamics of a system of N D3-branes.

– an $\mathcal{N} = 4$ supersymmetric U(N) gauge theory

– type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ background.

Conjecture: These two theories must be equivalent.

Maldacena

A consistency check: The symmetries of the two theories match.

The first theory has supersymmetry + conformal symmetry

This agrees with the supersymmetry and the geometrical symmetries of the second theory.

Another example

String theory in certain limit reduces to 11 dimensional supergravity.

In this theory there is a classical soliton solution that extends in two space directions.

– a membrane, known as the M2-brane.

We can put several of these membranes on top of each other.

Again this system has two equivalent descriptions.

The low energy dynamics of these membranes is described by some quantum field theory in 2+1 dimensions.

– known to be some kind of supersymmetric Chern-Simons theory (ABJM theory) in 2+1 dimensions.

Aharony, Bergman, Jafferis, Maldacena

On the other hand the near horizon geometry is $AdS_4 \times S^7$.

Conjecture: String theory on $AdS_4 \times S^7$ must be equivalent to the ABJM theory.

A third class of examples

In many string theories we have supersymmetric (extremal) black holes.

A typical situation:

1. Begin with string theory on $K \times R^{3,1}$ where K is some compact space and $R^{3,1}$ is a Minkowski space with 3 space and 1 time direction.

2. This at low energy gives some gravitational theory in 3+1 space-time dimensions.

3. Now in this theory we can find extremal, i.e. zero temperature, black hole solutions.

Near horizon geometries of these black holes take the form

$$AdS_2 \times S^2 \times K$$

S^2: an 2 dimensional sphere, labelled by the usual polar and azimuthal angles.

Thus we expect that the low energy dynamics of these black holes is described by string theory on

$$AdS_2 \times S^2 \times K$$

If we can find an alternative low energy description then we again have an example of AdS/CFT correspondence.

Look for the microscopic system which produces this black hole geometry in the string theory on

$$K \times R^{3,1}$$

Typically these are given by a combination of many different kinds of branes oriented in different ways along the compact space K.

Since many different kinds of branes are involved, the dynamics is usually more complicated, but overall it produces a quantum mechanical system.

The spectrum contains degenerate ground states separated by an energy gap from the excited states.

Thus in the low energy limit we only keep the ground state.

$\Rightarrow$ the low energy description is a quantum mechanical system with a finite dimensional Hilbert space, all with the same energy (which can be taken to be zero by a shift)

This leads to the equivalence between:

1. String theory on

$$AdS_2 \times S^2 \times K$$

and

2. A quantum mechanical system with finite dimensional Hilbert space and zero Hamiltonian.

This can be generalized to cases where the black hole is embedded in n+1 dimensional space-time instead of 3+1 dimensional space-time.

There is also another approach to finding examples of AdS/CFT correspondence.

Make a clever guess.

There have been many recent discoveries of such correspondences involving higher spin theories of gravity.

Sezgin, Sundell; Klebanov, Polyakov; Giombi, Yin; Gaberdiel, Gopakumar; ...

How is an AdS/CFT correspondence tested?

For this we need to know a more precise form of the conjecture.

What quantities in $\mathrm{CFT_n}$ are related to what quantities in quantum gravity on $\mathrm{AdS_{n+1}}$?

One approach to this that we shall follow is based on the euclidean version of this correspondence.

Gubser, Klebanov, Polyakov; Witten

Recall metric of $\mathrm{AdS_{n+1}}$:

$$\mathbf{ds^2} = \mathbf{R^2}\, \mathbf{z^{-2}} \left\{ -(\mathbf{dx^0})^2 + (\mathbf{dx^1})^2 + \cdots (\mathbf{dx^{n-1}})^2 + \mathbf{dz^2} \right\}$$

It is useful to analytically continue the time coordinate $\mathrm{x^0}$ to imaginary values ($\mathrm{x^0} \to \mathrm{ix^0}$) and write the metric of 'euclidean $\mathrm{AdS_{n+1}}$'

$$\mathbf{ds^2} = \mathbf{R^2 z^{-2}} \left\{ (\mathbf{dx^0})^2 + (\mathbf{dx^1})^2 + \cdots (\mathbf{dx^{n-1}})^2 + \mathbf{dz^2} \right\}$$

The boundary at $\mathrm{z} = \epsilon$ is an n-dimensional euclidean space with flat metric:

$$\epsilon^{-2} \left\{ (\mathbf{dx^0})^2 + (\mathbf{dx^1})^2 + \cdots (\mathbf{dx^{n-1}})^2 \right\}$$

Claim:

Correlation functions of operators in the CFT are given by the path integral over all the fields in string theory with appropriate boundary condition in the AdS space.

Different operators correspond to different boundary conditions.

A more precise statement:

Suppose $\mathbf{O_i}(\mathbf{x})$ for i=1,2,... are a complete set of local operators in the CFT.

The observables in CFT are different correlation functions e.g.

$$\langle \mathbf{O_1}(\mathbf{x_1})\mathbf{O_2}(\mathbf{x_2})\ldots\rangle$$

Define

$$\mathbf{Z_{CFT}}[\phi_1, \phi_2, \ldots] =< \mathbf{exp}[\int \mathbf{d^n x}(\phi_1(\mathbf{x})\mathbf{O_1}(\mathbf{x})+\phi_2(\mathbf{x})\mathbf{O_2}(\mathbf{x})+\ldots)] >$$

Then according to AdS/CFT correspondence $\mathbf{Z_{CFT}}$ is given by path integral over all the fields in string theory with boundary condition at $\mathbf{z} = \mathbf{0}$ set by the functions $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \cdots$.

It is some time useful to work in a different coordinate system in which the boundary of AdS space is a sphere instead of flat euclidean space.

$$ds^2 = R^2 \left(dy^2 + \sinh^2 y \, d\Omega^2 \right)$$

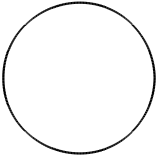
$d\Omega$: distance element on a unit n-dimensional sphere

e.g. for n=2 we have $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ where (θ, ϕ) are the polar and azimuthal angles.

Boundary of AdS is at $y = \infty$.

Regulated boundary at $y = y_0$ describes a sphere of radius $\sinh y_0$.

$$ds^2 = R^2 \left(dy^2 + \sinh^2 y \, d\Omega^2 \right)$$



$y = y_0$

In these coordinates the path integral over fields in string theory gives the correlation function of the CFT on a sphere.

Note: even though for an ordinary field theory correlation functions on a sphere and flat space are quite different, for a conformal field theory they are related by conformal transformation.

This precise relation between CFT and string theory on AdS has different kinds of applications.

1. We can use this to test AdS/CFT correspondence provided we can compute both sides of the relation.

2. If we accept AdS/CFT correspondence to be correct, then we can use this to map a complicated problem in gauge theory to a simple problem in string theory and vice versa.

Our goal in the rest of this talk will be to discuss tests of AdS/CFT.

In order to test AdS/CFT correspondence we need to compute some quantities independently on both sides.

On the CFT side these quantities depend on various parameters of the CFT, e.g. N of U(N) and the Yang-Mills coupling g_{YM} for $\mathcal{N} = 4$ supersymmetric Yang-Mills.

On the AdS side also the quantities depend on various parameters like the size R of AdS and the string theory coupling g_s for type IIB on $AdS_5 \times S^5$.

AdS/CFT correspondence tells us precise relation between these parameters.

In principle we should be able to compare the various quantities in AdS and CFT as functions of these parameters.

However typically quantities which are easy to calculate on one side in certain limit of the parameters are difficult on the other side in the same limit and vice versa.

This makes testing AdS/CFT difficult.

Nevertheless many clever tests of AdS/CFT have been performed by focussing on quantities which can be computed on both sides

– makes use of supersymmetry, large N limit etc.

Most of these tests make use of a limit in which on the string theory side we can use classical description i.e. in the limit of small string coupling.

The reason for this is that while quantizing string theory has been well understood in certain background, including flat space-time, extending this to AdS background is still not well-developed.

Thus while the correspondence between classical string theory on AdS and the dual CFT has been well tested, its extension to quantum string theory is still in its infancy.

One could take the point of view that once we have established the correspondence at the classical level, we can use the conformal field theory as the definition of quantum gravity theory in AdS space.

Nevertheless it is useful to ask if this ‘holographically defined’ quantum gravity obeys the usual rules of local quantum field theory / string theory, at least in some approximation.

Example: In quantum field theory, there is continuous production and annihilation of particle-antiparticle pair in the vacuum.

Can we see this effect in holographically defined quantum gravity theory?

A possible strategy

In the low energy limit string theory contains certain massless particles, including gravitons – the mediator of gravitational force.

We have standard rules for quantizing the low energy modes of massless fields in the AdS background.

So if we can isolate quantum effects which depend only on the low energy modes of massless fields we can compare them with the prediction from the CFT side.

Example: Logarithmic corrections

Various quantities computed in the string theory on AdS may contain terms proportional to log R for large R.

Such terms typically arise only from quantum effects of low energy modes of massless fields and does not require knowledge about massive states or high energy modes.

Thus these can be computed on the string theory side and compared with the result from the CFT side if the latter are calculable.

Example: $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence

Recall that AdS_2 arises in the near horizon geometry of extremal black holes, and the CFT_1 is a quantum mechanical system with a finite dimensional Hilbert space.

The rules of AdS/CFT correspondence tells us that if Ω is the total number of states in the Hilbert space of CFT, then we have

$$\Omega = Z_{AdS}$$

Z_{AdS}: the result of carrying out path integral over all fields in AdS_2 with vanishing boundary condition at infinity.

In the classical limit this correspondence reduces to the famous relation between Bekenstein - Hawking - Wald entropy of black holes and the log of degeneracy of states.

Strominger, Vafa; ...

$$\log \Omega = \log Z_{AdS}$$

The size R of AdS_2 is determined by the charge Q of the black hole.

$$R \propto |Q|$$

If we can compute the log Q terms in log Ω and log R terms in log Z_{AdS}, then we have a way to test AdS/CFT correspondence at the quantum level.

For a class of black holes in string theory we have exact results for Ω and hence log Ω.

Dijkgraaf, Verlinde, Verlinde; Shih, Strominger, Yin; Gaiotto; David, Jatkar, A.S., $\cdots$

For the same system we can also calculate the log R terms in log Z_{AdS} by quantizing massless fields in the AdS background.

One finds perfect agreement.

Results:

S. Banerjee, Gupta, Mandal, A.S.; Ferrara, Marrani; A.S.

The theory	scaling of charges	logarithmic contribution to $\log Z_{AdS}$	result for $\log \Omega$
$\mathcal{N} = 4$ with n_V matter	$Q_i \sim \Lambda, \quad R \sim \Lambda$	0	$\surd$
$\mathcal{N} = 8$	$Q_i \sim \Lambda, R \sim \Lambda$	$-8 \ln \Lambda$	$\surd$
$\mathcal{N} = 2$ with n_V vector and n_H hyper	$Q_i \sim \Lambda, \quad R \sim \Lambda$	$\frac{1}{6}(23 + n_H - n_V) \ln \Lambda$	?
$\mathcal{N} = 6$	$Q_i \sim \Lambda, \quad R \sim \Lambda$	$-4 \ln \Lambda$	?
$\mathcal{N} = 5$	$Q_i \sim \Lambda, \quad R \sim \Lambda$	$-2 \ln \Lambda$	?
$\mathcal{N} = 3$ with n_V matter	$Q_i \sim \Lambda, \quad R \sim \Lambda$	$2 \ln \Lambda$	?
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with n_V vectors	$Q_1, Q_5, n \sim \Lambda$; $J \sim \Lambda^{3/2}, \quad R \sim \Lambda^{1/2}$	$-\frac{1}{4}(n_V - 3) \ln \Lambda$	$\surd$
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with n_V vectors	$Q_1, Q_5, n \sim \Lambda$; $J = 0, \quad R \sim \Lambda^{1/2}$	$-\frac{1}{4}(n_V + 3) \ln \Lambda$	$\surd$

A similar analysis can be carried out for $\text{AdS}_4/\text{CFT}_3$ correspondence.

In this case we have a relation

$$Z_{\text{AdS}} = Z_{\text{CFT}}$$

Z_{CFT}: partition function of the CFT on a sphere with all the ϕ_i's set to zero.

On the AdS side this corresponds to path integral with all fields set to zero at the boundary.

$$\log Z_{AdS} = \log Z_{CFT}$$

Recall that here the CFT is described by the low energy dynamics of N membranes given by ABJM model.

The size R of AdS is proportional to $N^{1/6}$.

Z_{CFT} has been exactly calculated as a function of N.

Drukker, Marino, Putrov

Thus if we can calculate log R terms in $\log Z_{AdS}$, then we can compare it with the log N term in $\log Z_{CFT}$ to test AdS/CFT at the quantum level.

Result: $-\frac{3}{2}\log R$ on both sides!

More tests are in progress.

Bhattacharyya, Grassi, Marino, A.S.

While the above analysis tests the quantum nature of the massless particles in AdS, it does not address the following question:

Is the holographically defined quantum gravity theory a quantum string theory?

If so, the vacuum should have continuous creation and annihilation of pairs of heavy string states.

Can we see the effect of this in the dual CFT?

In the AdS_2/CFT_1 example, this can be seen in certain terms in the expression for $\log \Omega$ as a function of the charges.

Cardoso, de Wit, Kappeli, Mohaupt; David, Jatkar, A.S.

A term in the expression for log Ω and log Z_{AdS} in a specific example:

$$\mathbf{12}\ln\tau_{\mathbf{2}}+\mathbf{24}\ln\eta(\tau_{\mathbf{1}}+\mathbf{i}\tau_{\mathbf{2}})+\mathbf{24}\ln\eta(-\tau_{\mathbf{1}}+\mathbf{i}\tau_{\mathbf{2}})$$

η**: Dedekind function**

τ_1, τ_2 **are functions of ratios of charges.**

In log Ω this comes from the exact result.

In Z_{AdS} this comes from the effect of creation and annihilation of massive string states in the vacuum.

This (and other similar) agreements cannot be accidental, and indicates that AdS/CFT correspondence really holds at the quantum level.

Summary

AdS/CFT correspondence has given us a new way of defining quantum theory of gravity in Anti-de-Sitter space-times.

We have also seen that whenever old ways of dealing with quantum gravity / string theory are valid, the results of the CFT agree perfectly with the old ways.

Perhaps this will show us a way to modify our old ways of describing quantum gravity so as to make this into an exact description, valid in general background.

Or perhaps there is even a completely new description of both quantum gravity and quantum field theories which demystifies many of the mysteries in either description.