Eigenfunction Expansions of Ultradifferentiable Functions and Ultradistributions

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Outline

1. Compact Lie groups
   - Motivation
   - Fourier analysis on Compact Lie groups
   - Result
     - Gevrey Classes
     - Ultradistributions
2. Compact Manifold
   - Aim
   - Fourier Analysis on Compact Manifold
   - Results
     - Ultradifferentiable functions
     - Ultradistributions

On $\mathbb{R}^n$, $n \geq 2$,

Heat Operator:

$$L = \frac{\partial}{\partial x_n} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

The fundamental Solution:

$$E(x) = \begin{cases} (4\pi x_n)^{\frac{(1-n)}{2}} \exp[-\frac{(x_1^2+\ldots+x_{n-1}^2)}{4x_n}], & x_n > 0 \\ 0 & x_n \leq 0 \end{cases}$$

$E \in C^\infty(\mathbb{R}^n \setminus 0)$, but is not analytic for $x_n = 0$.

But for any fixed compact subset $K \subset \mathbb{R}^n$, $0 \notin K$,

$$|\partial^\alpha E(x)| \leq C^{|\alpha|+1}(\alpha!)^2, \quad x \in K.$$

$$\Downarrow$$

Generalizing this the Gevrey spaces on $\mathbb{R}^n$ were defined.

L. Rodino, "Linear partial differential operators in Gevrey spaces." *World Scientific Publishing Co., Inc., River Edge, NJ*, 1993.

Let G be a compact group of dimension n.

$\widehat{G}$: set of equivalence classes of continuous irreducible unitary representations of G.

Irreducible Unitary representation:
A strongly continuous mapping $\xi : G \to \mathcal{U}(H_\xi)$,

- $\xi(x)^* = \xi(x)^{-1}$ (unitary)
- $\xi(xy) = \xi(x)\xi(y)$ (preserves group structure)
- $\xi \neq \xi_1 \oplus \xi_2$ for some unitary representations ξ_1, ξ_2(irreducible)

$\dim(\xi) = \dim(\mathcal{H}_\xi) = d_\xi$

$\xi_{ij}(x) = \langle e_i, \xi(x)e_j \rangle$, where $\{e_j\}_{j=1}^{d_\xi}$ is an orthonormal basis of $\mathcal{H}_\xi$

$\xi = (\xi_{ij})_{i,j=1}^{d_\xi} \in \mathbb{C}^{d_\xi \times d_\xi}$

- **Fourier coefficient:** $\hat{f}(\xi)$ of $f \in C^{\infty}(G)$ at $\xi \in \hat{G}$ is

$$\hat{f}(\xi) = \int_G f(x)\xi(x)^* dx$$

Note that $\hat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ is a matrix!

- **Plancheral Identity**:

$$||f||_{L^2(G)} = \left(\sum_{[\xi]\in\hat{G}} d_\xi ||\hat{f}(\xi)||_{HS}^2 \right)^{1/2} = ||\hat{f}||_{l^2(\hat{G})}$$

where $||B||_{HS} = \sqrt{\text{Tr}(BB^*)}$

- **Fourier inversion formula/ Fourier series:**

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \sum_{\xi\in\hat{G}} d_\xi \text{Tr}\left(\hat{f}(\xi)\xi(x)\right).$$

- For each $[\xi] \in \hat{G}$,

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda_\xi^2 \xi_{ij}(x), \quad 1 \leq i,j \leq d_\xi$$

$\{\sqrt{d_\xi}\xi_{ij}\}$ is an orthonormal basis of $L^2(G)$.

Denote $|\xi| := \lambda_\xi \geq 0$

- $\langle \xi \rangle = \left(1 + \lambda_{[\xi]}^2\right)^{1/2}$ are the eigenvalues of elliptic first order pseudo-differential operators $(I - \mathcal{L}_G)^{1/2}$.

Gevrey Functions

Let $0 < s < \infty$.

Definition

The Gevrey-Romieu class, $\gamma_s(G)$ (Gevrey-Beurling class, $\gamma_{(s)}(G)$) is the space of functions $\phi \in C^\infty(G)$ for which there exists (for every) constants $A > 0$ and $C > 0$ (there exists $C_A > 0$) such that and for all multi-indices α, we have

$$||\partial^\alpha \phi||_{L^\infty} = \sup_{x \in G} |\partial^\alpha \phi(x)| \leq C A^{|\alpha|} (\alpha!)^s.$$

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$$||\partial^\alpha \phi||_{L^\infty} = \sup_{x \in G} |\partial^\alpha \phi(x)| \leq C_A A^{|\alpha|} (\alpha!)^s.$$

Gevrey Functions

Theorem

(R) *We have* $\phi \in \gamma_s(G)$ *if and only if there exists* $B > 0$ *and* $K > 0$ *such that*

$$||\widehat{\phi}(\xi)||_{\mathrm{HS}} \leq Ke^{-B\langle\xi\rangle^{1/s}}$$

holds for all $[\xi] \in \widehat{G}$.

(B) *We have* $\phi \in \gamma_{(s)}(G)$ *if and only if for every* $B > 0$ *there exists* $K_B > 0$ *such that*

$$||\widehat{\phi}(\xi)||_{\mathrm{HS}} \leq K_B e^{-B\langle\xi\rangle^{1/s}}$$

holds for all $[\xi] \in \widehat{G}$.

Gevrey Ultradistributions

$\gamma'_s(G)$ (or $\gamma'_{(s)}(G)$): the space of continuous linear functionals on $\gamma_s(G)$ (or $\gamma_{(s)}(G)$) is called the space of ultradistributions.

Theorem

Let $1 \leq s < \infty$.
We have $v \in \gamma'_s(G)$ (or $\gamma'_{(s)}(G)$) if and only if for every $B > 0$ there exists $K_B > 0$ (there exists $B > 0$ and $K > 0$) such that

$$||\widehat{v}(\xi)||_{\mathrm{HS}} \leq K_B e^{B\langle\xi\rangle^{1/s}}$$

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$$||\widehat{v}(\xi)||_{\mathrm{HS}} \leq K_B e^{B\langle\xi\rangle^{1/s}}$$

$$\left(||\widehat{v}(\xi)||_{\mathrm{HS}} \leq K e^{B\langle\xi\rangle^{1/s}}\right), \ \forall\ [\xi] \in \hat{G}.$$

$\widehat{v}(\xi) = v_\xi := \langle v, \xi^* \rangle = v(\xi^*)$.

A. Dasgupta and M. Ruzhansky, “Gevrey functions and ultradistributions on compact Lie grups and homogenous spaces.” *Bull. Sci. Math.*, 138(6): 756-782, 2014.

Applications

Consider the Cauchy problem for the wave equation

$$\partial_t^2 u - a(t)\mathcal{L}_G u = 0.$$

$$\Downarrow$$

Global Characterisations of Gevrey space + Energy inequality $\Longrightarrow$ well posedness result.

C. Garetto and M. Ruzhansky, "Wave equation for sums of squares on compact Lie groups." *J. Differ. Equ.*, 258(12): 4324-4347, 2015.

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Analytic functions

X be a compact analytic manifold.
E : an analytic, elliptic, positive differential operator of order ν.
$\{\lambda_j\}$ and $\{\phi_j\}$ be respectively eigenvalues and eigenfunctions of E.

A smooth function ,f,

- $f = \sum_j f_j \phi_j$ is analytic $\iff$ there is a constant $C > 0$ such that for all $k \geq 0$
$$\sum_j \lambda_j^{2k} |f_j|^2 \leq ((\nu k)!)^2 \, C^{2k+2}.$$
- $f = \sum_j f_j \phi_j$ is analytic $\iff$ the sequence $\{A^{\lambda_j^{1/\nu}} f_j\}$ is bounded for some $A > 1$.

R. T. Seeley. "Eigenfunction expansions of analytic functions". *Proc. Amer. Math. Soc.*, 21: 734-738, 1969.

Let X be a compact C^∞ manifold of dimension n.

$E \in \Psi^{\nu}_{+e}(X)$:a positive elliptic pseudo-differential operators of an integer order ν.

The eigenvalues of E form a sequence $\{\lambda_j\}$ and assume that $0 < \lambda_1 < \lambda_2 < ....$

H_j : The corresponding eigenspace for the eigen value λ_j.
$d_j := \dim H_j$, $H_0 := \ker E$, $\lambda_0 := 0$, $d_0 := \dim H_0$.
$\{e_j^k\}, \ 1 \leq k \leq d_j$, the orthonormal basis of H_j, so $H_j = \text{span}\{e_j^k\}_{k=1}^{d_j}$.
$L^2(X) = \bigoplus_{j=0}^{\infty} H_j$.

It can shown that,
$\sum_{j=1}^{\infty} d_j(1+\lambda_j)^{-q} < \infty$ if and only if $q > \frac{n}{\nu}$.

Fourier Coefficients:

For $f \in L^2(X), \widehat{f}(j,k) := \left(f, e_j^k\right)_{L^2}$
and

$$\widehat{f}(j) = \begin{pmatrix} \widehat{f}(j,1) \\ \vdots \\ \\ \vdots \\ \widehat{f}(j,d_j) \end{pmatrix} \in \mathbb{C}^{d_j}.$$

Plancheral formula :

$$||f||_{L^2(X)} = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j,k)|^2 = \sum_{j=0}^{\infty} ||\widehat{f}(j)||^2_{\text{HS}}.$$

Let $\{M_k\}$ be a sequence of positive numbers such that

(M.1) $M_0 = 1$

(M.2) $M_{k+1} \leq AH^k M_k$, for some $A > 0$, $H \geq 1$ and $k = 0, 1, 2, \ldots$

(M.3) $M_{2k} \leq AH^{2k} M_k^2$

$\Updownarrow$

$(M.3')$ $M_k \leq AH^k \min_{0 \leq q \leq k} M_q M_{k-q}$, $k = 0, 1, 2, \ldots$

(M.4) $\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty$

Ultradifferentiable functions on $\mathbb{R}^n$

Definition

The space of functions $\psi \in C^\infty(\mathbb{R}^n)$ such that for every compact $K \subset \mathbb{R}^n$ there exist $h > 0$ and a constant C such that

$$\sup_{x \in K} |\partial^\alpha \psi(x)| \leq C h^{|\alpha|} M_{|\alpha|}.$$

H. Komatsu, " Ultradistributions. I. Structure theorems and a characterization." *J. Fac. Sci. Univ. Tokyo. Sect. IA. Math.*, 20: 25-105, 1973.

Ultradifferentiable functions on X

Komatsu Type Ultradifferentiable Functions:

$\{M_k\}$ satisfies $(M.1)$, $(M.2)$, $(M.3)$ and
$k! \leq C_l l^k M_k, \forall k \in \mathbb{N}_0,$ for some $l,\ C_l > 0.$ (Romieu Class),
for all $l > 0$ there is $C_l > 0$ (Beurling Class).

Definition

The Romieu class, $\Gamma_{\{M_k\}}(X)$ (Beurling class, $\Gamma_{(M_k)}(X)$) is the space of C^∞ functions ϕ on X such that there exist (for every) $h > 0$ and $C > 0$ (there exists $C_h > 0$) such that we have

$$||E^k\phi||_{L^2(X)} \leq Ch^{\nu k}M_{\nu k},\quad k = 0, 1, 2, ...(Romieu),\quad \nu \in \mathbb{N}.$$

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$$||E^k\phi||_{L^2(X)} \leq Ch^{\nu k}M_{\nu k}, \quad k = 0, 1, 2, ...(Romieu), \quad \nu \in \mathbb{N}.$$

$$||E^k\phi||_{L^2(X)} \leq C_h h^{\nu k}M_{\nu k}, \quad k = 0, 1, 2, ...(Beurling), \quad \nu \in \mathbb{N}.$$

Results

We have proved the following properties:

1. $\Gamma_{\{M_k\}}(X)$ is independent of the choice of an operator $E \in \Psi^{\nu}_{e}(X)$.

2. $\phi \in \Gamma_{\{M_k\}}(X)$ if and only if there exist $h > 0$ and $C > 0$ such that
$$||\partial^{\alpha}\phi||_{L^2(X)} \leq Ch^{|\alpha|}M_{|\alpha|},$$
where $\partial^{\alpha} = \partial^{\alpha_1}_{j_1} ... \partial^{\alpha_k}_{j_k}$, with $1 \leq j_1, .., j_k \leq N$, $\sum_{j=1}^{N} \partial^2_j$ is elliptic and $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_k$.

3. Assume X and E are analytic. Then the class $\Gamma_{\{M_k\}}(X)$ is preserved by analytic change of variables, and is well defined in X. Moreover, in every local coordinate chart, it consists of functions locally belonging to the class $\Gamma_{\{M_k\}}(\mathbb{R}^n)$.

Gevrey Spaces

If $M_k = (k!)^s$ for $s \geq 1$ then $\gamma^s(X) = \Gamma_{\{(k!)^s\}}(X)$, that is functions which belong to the Gevrey clases $\gamma^s(\mathbb{R}^n)$ in local coordinate charts, which means,
$\phi \in \gamma^s(\mathbb{R}^n) \Rightarrow \exists A > 0, C > 0$ such that $\forall\alpha$,

$$|\partial^\alpha \phi(x)| \leq CA^{|\alpha|}(\alpha!)^s.$$

For $s = 1$ it will be the space of analytic functions.

Associated function:

$M(r) := \sup_{k\in\mathbb{N}} \log \frac{r^{\nu k}}{M_{\nu k}}$, $r > 0$, and set $M(0) = 0$

For Gevrey Class, $M(r) \approx r^{1/s}$, that is, $\frac{s}{4\nu e} r^{1/s} \leq M(r) \leq sr^{1/s}$.

Theorem

$\phi \in \Gamma_{\{M_k\}}(X)$ *if and only if there exist constants* $C > 0$, $L > 0$ *such that*

$$||\widehat{\phi}(l)||_{HS} \leq C \exp\{-M(L\lambda_l^{1/\nu})\}, \quad \textit{for all } l \geq 1,$$

where $M(r) = \sup_{k \in \mathbb{N}} \log \frac{r^{\nu k}}{M_{\nu k}}$.

Theorem

$\phi \in \Gamma_{\{M_k\}}(X)$ if and only if there exist constants $C > 0, L > 0$ such that

$$||\widehat{\phi}(l)||_{HS} \leq C \exp\{-M(L\lambda_l^{1/\nu})\}, \quad \textit{for all } l \geq 1,$$

where $M(r) = \sup_{k \in \mathbb{N}} \log \frac{r^{\nu k}}{M_{\nu k}}$.

Corollary

Let X and E be analytic and let $s \geq 1$. Then $\phi \in \gamma^s(X)$ if and only if there exists constants $C > 0, L > 0$ such that

$$||\widehat{\phi}(l)||_{HS} \leq C \exp\left(-L\lambda_l^{1/s\nu}\right), \quad \textit{for all } l \geq 1.$$

In particular, for $s = 1$, we recover the characterisation of analytic functions by Seeley.

Ultradistributions

Definition

The space $\Gamma'_{\{M_k\}}(X)$ is the set of all linear forms u on $\Gamma_{\{M_k\}}(X)$ such that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|u(\phi)| \leq C_\epsilon \sup_\alpha \epsilon^{|\alpha|} M_{\nu|\alpha|}^{-1} \sup_{x \in X} |E^{|\alpha|}\phi(x)|$$

holds for all $\phi \in \Gamma_{\{M_k\}}(X)$.

Theorem

$u \in \Gamma'_{\{M_k\}}(x)$ if and only if for every $L > 0$ there exists $K = K_L > 0$ such that

$$||\widehat{u}(l)||_{HS} \leq K \exp\left(M(L\lambda_l^{1/\nu})\right)$$

holds for all $l \in \mathbb{N}$.

Here the Fourier coefficients of $u \in \Gamma'_{\{M_k\}}(x)$ are defined by $\widehat{u}(e_l^k) := u(\overline{e_l^k})$ and $\widehat{u}(l) := \widehat{u}(e_l) := \left[u(\overline{e_l^k})\right]_{k=1}^{d_l}$.

α-duals

Definition

The α-dual of the space $\Gamma_{\{M_k\}}(X)$ of ultradifferentiable functions, denoted by $[\Gamma_{\{M_k\}}(X)]^{\wedge}$, is defined as

$$\left\{ v = (v_l)_{l\in\mathbb{N}_0} : \sum_{l=0}^{\infty}\sum_{j=1}^{d_l} |(v_l)_j||\widehat{\phi}(l,j)| < \infty, v_l \in \mathbb{C}^{d_l}, \text{ for all } \phi \in \Gamma_{\{M_k\}}(X) \right\}.$$

G. Köthe, “Topological vector spaces. I.” Band 159. Springer-Verlag New York Inc., New York, 1969.

Theorem

The following statements are equivalent:

1. $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$;
2. *for every* $L > 0$ *we have*

$$\sum_{l=1}^{\infty} \exp\left(-M(L\lambda_l^{1/\nu})\right) ||v_l||_{HS} < \infty;$$

3. *for every* $L > 0$ *there exists* $K = K_L > 0$ *such that*

$$||v_l||_{HS} \leq K \exp\left(M(L\lambda_l^{1/\nu})\right), \quad \forall l \in \mathbb{N}.$$

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3. *for every* $L > 0$ *there exists* $K = K_L > 0$ *such that*

$$||v_l||_{HS} \leq K \exp\left(M(L\lambda_l^{1/\nu})\right), \quad \forall l \in \mathbb{N}.$$

Theorem

$v \in \Gamma'_{\{M_k\}}(x)$ *if and only if* $v \in [\Gamma_{\{M_k\}}(X)]^\wedge$.

Beurling Class

Theorem

$\phi \in \Gamma_{(M_k)}(X)$ if and only if for every $L > 0$ there exists $C_L > 0$ such that

$$||\widehat{\phi}(l)||_{HS} \leq C_L \exp\{-M(L\lambda_l^{1/\nu})\}, \quad \text{for all } l \geq 1.$$

For the dual space and for the α-dual, the following statements are equivalent:

1. $v \in \Gamma_{(M_k)}(X)'$
2. $v \in [\Gamma_{(M_k)}(X)]^\wedge$
3. *there exists $L > 0$ such that we have*

$$\sum_{l=0}^{\infty} \exp\left(-M(L\lambda_l^{1/\nu})\right) ||v_l||_{HS} < \infty$$

Continued..

Theorem

4. *there exists* $L > 0$ *and* $K > 0$ *such that*

$$||v_l||_{HS} \leq K \exp\left(M(L\lambda_l^{1/\nu})\right)$$

holds for all $l \in \mathbb{N}_0$.

A. Dasgupta and M. Ruzhansky, "Eigenfunction expansions of ultradifferentiable functions and ultradistributions." *Trans. Amer. Math. Soc.*, 368(12): 8481-8498, 2016.

A. Dasgupta and M. Ruzhansky, "Eigenfunction expansions of ultradifferentiable functions and ultradistributions II- Tensor Representations." *Trans. Amer. Math. Soc.*, accepted.

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3. M.Ruzhansky, V.Turunen, "Pseudo-Differential Operators and Symmetries", Birkhäuser, Basel, 2010.
4. Y. Taguchi, "Fourier coefficients of periodic functions of Gevrey classes and ultradistributions", *Yoko-hama Math.J.* 35(1987) 51-60.

Thank you!