Nematic Liquid Crystals in Lipschitz domains

Anupam Pal Choudhury
Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Linz, Austria

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In this talk, we shall discuss the simplified Ericksen–Leslie model for nematic liquid crystals in three dimensional bounded Lipschitz domains. Applying a semilinear approach, we shall discuss the proof of local and global well-posedness (assuming a smallness condition on the initial data) in critical spaces for initial data in L^3_σ for the fluid and $W^{1,3}$ for the director field. The analysis of such models, so far, has been restricted to domains with smooth boundaries.

Based on a joint work with Amru Hussein and Patrick Tolksdorf

Reference:
Nematic Liquid Crystals in Lipschitz domains, arXiv:1709.06384 .

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Outline of the talk

- Introduction
- The case of Neumann boundary data
- The case of Dirichlet boundary data
- Concluding remarks and Future directions

Anupam Pal Choudhury
Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria

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- Such materials possess mechanical and symmetric properties of both the solid and liquid states at the same time.

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Crystals- Groups of molecules are regularly stacked. Molecular order is present, that is, certain pattern repeats.
Isotropic liquids- Properties independent of directions in which they are measured. Molecular patterns are not present.
Liquid crystals- Order exists at least in one-direction in space combined with anisotropy.

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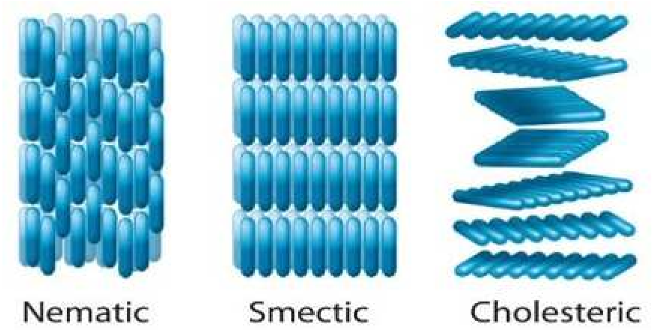
Three phases of liquid crystals:

- Nematic phase: No positional order present but a long-range translational order. Thus, it differs from isotropic liquids in which molecules are arbitrarily oriented. Preferred direction varies throughout the medium.
- Smectic phase: Set of two-dimensional liquid layers on top of each other. Preferred direction within each layer but the preference varies over layers.
- Cholesteric phase: Two-dimensionally ordered systems in three dimensions following a helical structure. As in the smectic phase, no positional ordering within layers but a preferred direction in each layer.

In this talk, we shall focus on a simplified model for nematic liquid crystals.

Anupam Pal Choudhury
Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria



Ref: http://www.qsstudy.com/chemistry/liquid-crystals

Mathematical Preliminaries

Anupam Pal Choudhury
Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria

Definition (Lipschitz boundary)

The boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^n$ is said to be *Lipschitz* if for each point $x \in \partial\Omega$, there exist $r > 0$ and a *Lipschitz continuous mapping* $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes if necessary, we have

$$\Omega \cap Q(x, r) := \{y | \gamma(y_1, \cdots, y_{n-1}) < y_n\} \cap Q(x, r),$$

where

$$Q(x, r) := \{y || y_i - x_i| < r, i = 1, \cdots, n\}.$$

In other words, near each point $x \in \partial\Omega$, the boundary is the graph of a Lipschitz continuous function.

The isothermal simplified Ericksen-Leslie model

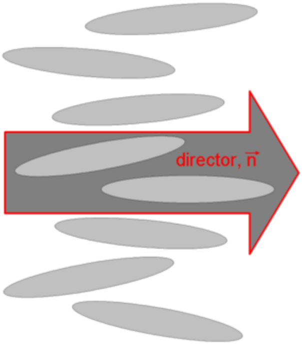
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The isothermal simplified Ericksen-Leslie model is given by

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla \pi &= -\lambda \operatorname{div}([\nabla d]^\top \nabla d) \text{ in } (0, T) \times \Omega, \\ \partial_t d + (u \cdot \nabla)d &= \gamma(\Delta d + |\nabla d|^2 d) \text{ in } (0, T) \times \Omega, \\ \operatorname{div} u &= 0 \text{ in } (0, T) \times \Omega, \\ (u, d)\Big|_{t=0} &= (a, b) \text{ in } \Omega. \end{aligned}$$

- $u : (0, \infty) \times \Omega \to \mathbb{R}^n$ denotes the velocity field, $\pi : (0, \infty) \times \Omega \to \mathbb{R}$ denotes the pressure.
- $d : (0, \infty) \times \Omega \to \mathbb{R}^n$ denotes the molecular orientation of the liquid crystal at the macroscopic level (we shall also refer to this as the director field). This physical interpretation of d further imposes the condition

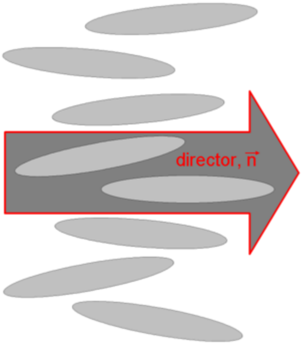
$$|d| = 1 \text{ in } (0, T) \times \Omega.$$

- Without loss of generality, we shall also assume that $\nu = \gamma = \lambda = 1$.



Ref: Wikipedia

- The notation n corresponds to d in the pde.
- Ericksen-Leslie theory for nematic liquid crystals was developed in 1960's.



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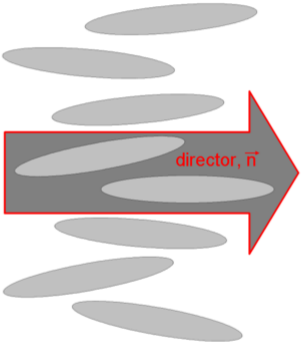
- Earliest mathematical analysis of nematic liquid models can be dated back to Lin-Liu (1995).



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Boundary conditions:

$u = 0$ on $(0, T) \times \partial\Omega$,

$\partial_\nu d = 0$ on $(0, T) \times \partial\Omega$ (*Neumann boundary condition*) *or*

$d = \tilde{d}$ on $(0, T) \times \partial\Omega$ (*Dirichlet boundary condition*).

In case of Dirichlet boundary conditions, we shall also assume that

$$|\tilde{d}(t, x)| = 1 \text{ on } (0, T) \times \partial\Omega.$$

Two types of approach: Fluid-type approach (couple equation for d with Navier-stokes), Geometric approach (fluid equation coupled with theory of harmonic maps on spheres).

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- In the case of smooth domains and with Neumann boundary conditions, a complete dynamic theory is known by analyzing the system as a quasilinear parabolic evolution equation: work of *Hieber-Nesensohn-Pruss-Schade*(2016).
- A semilinear approach to study the system with Neumann boundary data was followed in Li-Wang (2012) but their proof contained some crucial errors.
- It was observed in Choudhury-Hieber-Hussein (unpublished note) that the quasilinear approach, as in the Neumann boundary data case, can be adapted when the Dirichlet boundary data for the director field is a constant vector e.

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The case of Neumann boundary data

Applying the Helmholtz projection $\mathbb{P}$, one considers the space

$$X = L^q_\sigma(\Omega) \times L^q(\Omega)^3,$$

and obtains the equivalent equation without pressure.

Then omitting the condition $|d| = 1$ one obtains

$$\begin{cases} \partial_t u + Au &= -\mathbb{P}u \cdot \nabla u - \mathbb{P}\mathrm{div}\left[(\nabla d)^T (\nabla d)\right], \quad \text{in } \Omega \times (0,T), \\ \partial_t d + Bd &= -u \cdot \nabla d - |\nabla d|^2 d, \quad \text{in } \Omega \times (0,T), \end{cases} \tag{1}$$

with initial conditions $u(0) = a$ and $d(0) = b$.

The operators A, B are defined as

$$A := -\mathbb{P}\Delta, \;\; B := -\Delta_{N,q}.$$

- Define for any $b \in L^1(\Omega)$ the average and the complementary mean-value free part

$$\overline{b} := \frac{1}{|\Omega|}\int_\Omega b d\omega \qquad \text{and} \qquad b_s := b - \overline{b}, \tag{2}$$

where the subscript in b_s refers to stable, since for the Cauchy problem defined by the Neumann Laplacian it corresponds to the exponentially stable part.

- Note that (2) define bounded projections in all L^p spaces, $p \in [1, \infty]$,

$$P_c d = \overline{d} \qquad \text{and} \qquad P_s d = d_s.$$

Now, using (2) one defines the variables

$$x = \overline{d} - \overline{b} \qquad \text{and} \qquad y = d_s,$$

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where $x(0) = 0$ and $y(0) = b_s$.

We, therefore, obtain the following reformulation of (1)

$$\begin{cases} \partial_t u + Au &= -\mathbb{P}u\cdot\nabla u - \mathbb{P}\mathrm{div}\,\left[(\nabla y)^T(\nabla y)\right], \quad \text{in } \Omega\times(0,T),\\ \partial_t y + By &= -u\cdot\nabla y - P_s|\nabla y|^2(x+y+\bar{b}), \quad \text{in } \Omega\times(0,T),\\ \partial_t x &= -P_c|\nabla y|^2(x+y+\bar{b}), \quad \text{in } \Omega\times(0,T) \end{cases} \tag{3}$$

which defines a system in the space

$$L^q_\sigma(\Omega)\times L^q_0(\Omega)^3\times\mathbb{R}^3.$$

Mild formulation

The non-linear terms are comprised using the representation $u \cdot \nabla u = \operatorname{div} u \otimes u$ for $\operatorname{div} u = 0$ by the notation

$$F_u(u, \nabla y) = -\mathbb{P}\operatorname{div} \left(u \otimes u + (\nabla y)^T (\nabla y) \right),$$

$$F_y(u, \nabla y, y, x, \overline{b}) = -u \cdot \nabla y - P_s |\nabla y|^2 (x + y + \overline{b}),$$

$$F_x(\nabla y, y, x, \overline{b}) = -P_c |\nabla y|^2 (x + y + \overline{b}).$$

Starting with the mild formulation of the problem one can define now the iteration scheme as follows.

$$u_0 := e^{-tA} a, \ \ u_{j+1} := u_0 + \int_0^t e^{-(t-s)A} F_u(u_j(s), \nabla y_j(s)) ds,$$

$$y_0 := e^{-tB} b_s,$$

$$y_{j+1} := y_0 + \int_0^t e^{-(t-s)B} F_y(u_j(s), \nabla y_j(s), y_j(s), x_j(s), \overline{b}) ds,$$

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$$y_{j+1} := y_0 + \int_0^t e^{-(t-s)B} F_y(u_j(s), \nabla y_j(s), y_j(s), x_j(s), \overline{b})ds,$$
$$x_0 = 0, \;\; x_{j+1} := \int_0^t F_x(\nabla y_j(s), y_j(s), x_j(s), \overline{b})ds.$$

$L^p - L^q$ estimates

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists $\varepsilon > 0$ such that,

(a) *there exists $\omega > 0$ and a constant $C > 0$ such that for $\frac{3}{2} - \varepsilon < p \leq q < 3 + \varepsilon$ and $t > 0$,*

$$\|e^{-tA} f\|_{L^q_\sigma(\Omega)} \leq C e^{-\omega t} t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p_\sigma(\Omega)}, \; f \in L^p_\sigma(\Omega),$$

$$\|e^{-tA} \mathbb{P}\mathrm{div}\, F\|_{L^q_\sigma(\Omega)} \leq C e^{-\omega t} t^{-\frac{1}{2} - \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|F\|_{L^p(\Omega)^{3\times 3}}, \; F \in L^p(\Omega)^{3\times 3},$$

(b) *there exists $\omega > 0$ and a constant $C > 0$ such that for all $1 < p \leq q \leq \infty$ with $p < \infty$ and $t > 0$,*

$$\|e^{-tB} f\|_{L^q(\Omega)^3} \leq C e^{-\omega t} t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\Omega)^3}, \quad f \in L^p_0(\Omega)^3.$$

Moreover, for all $\frac{3}{2} - \varepsilon < p \leq q < 3 + \varepsilon$ and for $t > 0$,

$$\|\nabla e^{-tB} f\|_{L^q(\Omega)^{3\times 3}} \leq C e^{-\omega t} t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|\nabla f\|_{L^p(\Omega)^{3\times 3}}, \; f \in L^p_0 \cap W^{1,p},$$

$$\|\nabla e^{-tB} f\|_{L^q(\Omega)^{3\times 3}} \leq C e^{-\omega t} t^{-\frac{1}{2} - \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\Omega)^3}, \; f \in L^p_0(\Omega)^3.$$

For $0 < T \leq \infty$ and $3 \leq p < q$, the class of solutions considered is defined using

$$S_q^u(T) := \left\{ u \in C((0,T); L^q_\sigma(\Omega)) \mid \sup_{0<s<T} e^{\frac{\omega s}{2}} s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(s)\|_{L^q_\sigma(\Omega)} < \infty \right\},$$

$$S_q^d(T) := \left\{ d \in C((0,T); W^{1,q}(\Omega)^3) \mid \sup_{0<s<T} e^{\frac{\omega s}{2}} s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\nabla d(s)\|_{L^q(\Omega)^{3\times 3}} < \infty \right\},$$

where $\omega > 0$ is the minimum of the corresponding constants appearing earlier.

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, then there exists $\varepsilon > 0$ such that given initial conditions $a \in L^p_\sigma(\Omega)$ and $b \in W^{1,p}(\Omega)^3 \cap L^\infty(\Omega)^3$ where $3 \leq p < 3 + \varepsilon$, the following hold true for $q \in (p, 3+\varepsilon)$.

(a) *There exists $T > 0$ depending on the initial data such that equation (1) has a local mild solution (u, d) satisfying*

$$u \in S_q^u(T) \cap BC([0,T); L^p_\sigma(\Omega)), \ \overline{d} \in BC([0,T); \mathbb{R}^3),$$
$$d_s \in S_q^d(T) \cap BC([0,T); W^{1,p}(\Omega)^3) \cap BC([0,T); L^\infty(\Omega)^3),$$

where in the limit $s \to 0+$, one has

$$\|u(s) - a\|_{L^p_\sigma(\Omega)} \to 0, \|d(s) - b\|_{L^\infty(\Omega)^3} \to 0, \|\nabla[d(s) - b]\|_{L^p(\Omega)^{3\times 3}} \to 0.$$

(b) In the limit $s \to 0+$, the solutions satisfy

$$s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|u(s)\|_{L^q_\sigma(\Omega)} \to 0 \quad \text{and} \quad s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\nabla d(s)\|_{L^q(\Omega)^{3\times 3}} \to 0.$$

(c) If a and ∇b are sufficiently small, then the solution exists globally in the class

$$u \in S^u_q(\infty) \cap BC([0,\infty); L^p_\sigma(\Omega)),$$
$$d_s \in S^d_q(\infty) \cap BC([0,\infty); W^{1,p}(\Omega)^3) \cap BC([0,\infty); L^\infty(\Omega)^3),$$
$$\overline{d} \in BC([0,\infty); \mathbb{R}^3).$$

(d) The solution is unique in the class given in (a) provided $p > 3$, and in the case $p = 3$, it is unique in the subset of this class satisfying in addition the limit conditions (b).

(e) Equation (1) subject to Neumann boundary conditions preserves the condition $|d| = 1$ if $|d(0)| = |b| = 1$.

The smallness condition in the previous theorem can be made precise in the sense that there exists a constant $C > 0$ depending only on p, q, and Ω such that if

$$\max\{\kappa, \kappa^2\}(1 + \|b\|_{L^\infty(\Omega)^3}) < C, \text{ where } \kappa := \|a\|_{L^p_\sigma(\Omega)} + \|\nabla b\|_{L^p(\Omega)^{3\times 3}},$$

then the solution exists globally.

Theorem

For every $s \in (1, 2)$, the solution has the following additional regularity properties

$$u \in W^{1,s}(0, T; W_\sigma^{-1,\frac{p}{2}}(\Omega)) \cap L^s(0, T; W_{0,\sigma}^{1,\frac{p}{2}}(\Omega)),$$

$$d', B_{\frac{p}{2}} d \in L^s(0, T; L^{\frac{p}{2}}(\Omega)^3).$$

- For $0 < T \leq \infty$, let us define the quantities

$$k_j^u(T) := \sup_{0<s<T} e^{\omega s/2} s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u_j(s)\|_{L^q_\sigma(\Omega)},$$

$$k_j^{\nabla y}(T) := \sup_{0<s<T} e^{\omega s/2} s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\nabla y_j(s)\|_{L^q(\Omega)^3},$$

$$k_j^y(T) := \sup_{0<s<T} \|y_j(s)\|_{L^\infty(\Omega)^3}, \ k_j^x(T) := \sup_{0<s<T} \|x_j(s)\|_{\mathbb{R}^3}.$$

- Let us introduce the notations

$$k_j^q := k_j^u + k_j^{\nabla y}, \ k_j^\infty := k_j^y + k_j^x.$$

- Next, let us denote

$$W_j(t) := u_{j+1}(t) - u_j(t), \ Z_j(t) := \nabla y_{j+1}(t) - \nabla y_j(t),$$
$$Y_j(t) := y_{j+1}(t) - y_j(t), \ X_j(t) := x_{j+1}(t) - x_j(t),$$

and the corresponding quantities

$$\delta_j^u(T) := \sup_{0<s<T} e^{\omega s/2} s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|W_j(s)\|_{L^q_\sigma(\Omega)},$$

$$\delta_j^{\nabla y}(T) := \sup_{0<s<T} e^{\omega s/2} s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|Z_j(s)\|_{L^q(\Omega)^3},$$

$$\delta_j^y(T) := \sup_{0<s<T} \|Y_j(s)\|_{L^\infty(\Omega)^3}, \ \delta_j^x(T) := \sup_{0<s<T} \|X_j(s)\|_{\mathbb{R}^3}.$$

Now, one estimates for $3 \le p < q < 3+\epsilon$,

$$
\begin{aligned}
k^u_{j+1}(T) &\le k^u_0(T) + CC_{1,T}[k^u_j(T)^2 + k^{\nabla y}_j(T)^2],\\
k^{\nabla y}_{j+1}(T) &\le k^{\nabla y}_0(T)\\
&\quad + CC_{1,T}\Big[k^u_j(T)k^{\nabla y}_j(T) + k^{\nabla y}_j(T)^2(k^x_j(T) + k^y_j(T) + \|\bar{b}\|)\Big],\\
k^y_{j+1}(T) &\le k^y_0(T)\\
&\quad + CC_{2,T}\Big[k^u_j(T)k^{\nabla y}_j(T) + k^{\nabla y}_j(T)^2(k^x_j(T) + k^y_j(T) + \|\bar{b}\|)\Big]\\
k^x_{j+1}(T) &\le CC_{3,T}k^{\nabla y}_j(T)^2(k^x_j(T) + k^y_j(T) + \|\bar{b}\|).
\end{aligned}
$$

- Let $\tilde{C}_T := \max\{C_{1,T}, C_{2,T}, C_{3,T}\}$. We note that $\lim_{T\to 0}\tilde{C}_T = 0$ and if $T = +\infty$, $\tilde{C}_T$ is a constant independent of T.

Now, one estimates for $3 \leq p < q < 3 + \epsilon$,

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k^{x}_{j+1}(T) &\leq CC_{3,T}k^{\nabla y}_{j}(T)^2(k^{x}_{j}(T) + k^{y}_{j}(T) + \|\bar{b}\|).
\end{aligned}
$$

- Let $\tilde{C}_T := \max\{C_{1,T}, C_{2,T}, C_{3,T}\}$. We note that $\lim_{T\to 0}\tilde{C}_T = 0$ and if $T = +\infty$, $\tilde{C}_T$ is a constant independent of T.

For $3 \leq p < q < 3+\varepsilon$,

$$\begin{aligned}
\|u_{j+1}\|_{L^q_\sigma(\Omega)} &\leq \|u_0\|_{L^q_\sigma(\Omega)} + \Big\| \int_0^t e^{-(t-s)A}\mathbb{P}\mathrm{div}\,(u_j \otimes u_j) + e^{-(t-s)A}\mathbb{P}\mathrm{div}\,([\nabla y_j]^\top \nabla y_j)\,\mathrm{d}s \Big\|_{L^q_\sigma(\Omega)} \\
&\leq \|u_0\|_{L^q_\sigma(\Omega)} \\
&\quad + C\int_0^t e^{-(t-s)\omega}(t-s)^{-\frac{1}{2}-\frac{3}{2q}} \big(\|u_j \otimes u_j\|_{L^{q/2}_\sigma(\Omega)} + \|[\nabla y_j]^\top \nabla y_j\|_{L^{q/2}(\Omega)^{3\times 3}}\big)\,\mathrm{d}s \\
&\leq \|u_0\|_{L^q_\sigma(\Omega)} + C\int_0^t e^{-t\omega}(t-s)^{-\frac{1}{2}-\frac{3}{2q}} s^{-3(\frac{1}{p}-\frac{1}{q})} \\
&\qquad\qquad \Big\{ \Big(e^{\frac{s\omega}{2}} s^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|u_j\|_{L^q_\sigma(\Omega)}\Big)^2 + \Big(e^{\frac{s\omega}{2}} s^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|\nabla y_j\|_{L^q(\Omega)^{3\times 3}}\Big)^2 \Big\}\,\mathrm{d}s \\
&\leq \|u_0\|_{L^q_\sigma(\Omega)} + C\Big(e^{-t\omega}\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}} s^{-3(\frac{1}{p}-\frac{1}{q})}\,\mathrm{d}s\Big)[k_j^u(T)^2 + k_j^{\nabla y}(T)^2]
\end{aligned}$$

which implies, multiplying by the factor $e^{\frac{\omega t}{2}} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}$ and taking $\sup_{0<t<T}$, that

$$k_{j+1}^u(T) \leq k_0^u(T) + C\Big(\sup_{0<t<T} e^{-\frac{\omega t}{2}} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}} s^{-3(\frac{1}{p}-\frac{1}{q})}\,\mathrm{d}s\Big)[k_j^u(T)^2 + k_j^{\nabla y}(T)^2]. \tag{4}$$

Since $3 \leq p < q < 3+\varepsilon$, it follows that $\frac{1}{2} - \frac{3}{2p} \geq 0,\ 1 - 3(\frac{1}{p} - \frac{1}{q}) > 0,\ \frac{1}{2} - \frac{3}{2q} > 0$, and hence

$$\begin{aligned}
\sup_{0<t<T} e^{-\frac{\omega t}{2}} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} &\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}} s^{-3(\frac{1}{p}-\frac{1}{q})}\,\mathrm{d}s \\
&= \Big(\sup_{0<t<T} e^{-\frac{\omega t}{2}} t^{\frac{1}{2}-\frac{3}{2p}}\Big) B\big(1 - 3(\tfrac{1}{p} - \tfrac{1}{q}),\ \tfrac{1}{2} - \tfrac{3}{2q}\big),
\end{aligned}$$

where $B(x,y)$ denotes the beta function for $x, y > 0$. Therefore, setting $C_{1,T} := \sup_{0<t<T} e^{-\frac{\omega t}{2}} t^{\frac{1}{2}-\frac{3}{2p}}$, equation (4) turns into

$$k_{j+1}^u(T) \leq k_0^u(T) + CC_{1,T}[k_j^u(T)^2 + k_j^{\nabla y}(T)^2]. \tag{5}$$

Similarly for the differences we can write

$$
\begin{aligned}
\delta_j^u(T) &\le C\tilde{C}_T\Big[\delta_{j-1}^u(T)(k_j^q(T)+k_{j-1}^q(T)) + \delta_{j-1}^{\nabla y}(T)(k_j^q(T)+k_{j-1}^q(T))\Big],\\
\delta_j^{\nabla y}(T) &\le C\tilde{C}_T\Big[\delta_{j-1}^u(T)k_j^q(T) + k_{j-1}^q(T)\delta_{j-1}^{\nabla y}(T)\\
&\quad + (\delta_{j-1}^{\nabla y}(T)k_j^q(T) + k_{j-1}^q(T)\delta_{j-1}^{\nabla y}(T))(k_j^\infty(T)+k_j^\infty(T)+\|\overline{b}\|)\\
&\quad + k_{j-1}^q(T)^2(\delta_{j-1}^x(T)+\delta_{j-1}^y(T))\Big],\\
\delta_j^y(T) &\le C\tilde{C}_T\Big[\delta_{j-1}^u(T)k_j^q(T) + k_{j-1}^q(T)\delta_{j-1}^{\nabla y}(T)\\
&\quad + (\delta_{j-1}^{\nabla y}(T)k_j^q(T) + k_{j-1}^q(T)\delta_{j-1}^{\nabla y}(T))(k_j^\infty(T)+k_j^\infty(T)+\|\overline{b}\|)\\
&\quad + k_{j-1}^q(T)^2(\delta_{j-1}^x(T)+\delta_{j-1}^y(T))\Big],\\
\delta_j^x(T) &\le C\tilde{C}_T\Big[(\delta_{j-1}^{\nabla y}(T)k_j^q(T) + k_{j-1}^q(T)\delta_{j-1}^{\nabla y}(T))(k_j^\infty(T)+k_j^\infty(T)\\
&\qquad + \|\overline{b}\|) + k_{j-1}^q(T)^2(\delta_{j-1}^x(T)+\delta_{j-1}^y(T))\Big].
\end{aligned}
$$

- Let us denote $\delta_j(T) := \delta_j^u(T) + \delta_j^{\nabla y}(T) + \delta_j^y(T) + \delta_j^x(T)$.
- Then if
$$144 C \tilde{C}_T K (1 + \|b\|_{L^\infty(\Omega)^3}) < 1, \qquad (6)$$
where $K = \max\{k_0^q(T), k_0^q(T)^2\}$, we have a contraction in terms of δ_j, which gives us the existence.

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We give a short summary of the conditions that provide the validity of (6).

(a) In the case $T = +\infty$, if $\|a\|_{L^p_\sigma(\Omega)} + \|\nabla b\|_{L^p(\Omega)^{3\times 3}}$ is small enough, the validity of (6) can be inferred. This implies eventually the global existence under a suitable smallness condition on the initial data a and ∇b.

(b) If $p > 3$, then (6) follows from the fact that $\lim_{T\to 0} \tilde{C}_T = 0$. This will imply local existence of solutions without any smallness assumptions on the initial data.

(c) If $p = 3$, then (6) follows for small times T using a similar argument to that of $p > 3$. Again this will imply local existence of solutions without any smallness assumption on the initial data.

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- The above estimates are not enough because we cannot take $q = p$. Therefore, using the estimates we need to extend our consideration to L^s with time weight $\sigma' = \frac{3}{2}(\frac{1}{p} - \frac{1}{s})$ where

$$3 \leq \max\{p, \frac{q}{2}\} \leq s \leq q < 3 + \epsilon.$$

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Retrieving the condition $|d| = 1$

- Let u and d be mild solutions to our problem such that the initial conditions satisfy $a \in \mathrm{L}^p_\sigma(\Omega)$ and $b \in W^{1,p}(\Omega; \mathbb{C}^3)$ with $|b| = 1$ in Ω for some $3 \le p < 3 + \varepsilon$.
- Assume further, that for every $p < q < 3 + \varepsilon$

$$
\begin{aligned}
t \mapsto t^{\frac{3}{2}(\frac{1}{p} - \frac{1}{q})} u(t) &\in \mathrm{BC}([0, T); \mathrm{L}^q_\sigma(\Omega)), \\
t \mapsto t^{\frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \nabla d(t) &\in \mathrm{BC}([0, T); \mathrm{L}^q_\sigma(\Omega)), \\
d &\in \mathrm{BC}([0, T); \mathrm{L}^\infty(\Omega; \mathbb{C}^3)).
\end{aligned}
$$

- Then for all $\vartheta \in \mathrm{W}^{1,(p/2)'}(\Omega; \mathbb{C}^3)$,

$$
\begin{aligned}
\int_\Omega d_t(t) \cdot \overline{\vartheta} \, \mathrm{d}x + \int_\Omega \nabla d(t) \cdot \overline{\nabla \vartheta} \, \mathrm{d}x = - \int_\Omega (u(t) \cdot \nabla) d(t) \cdot \overline{\vartheta} \, \mathrm{d}x \\
+ \int_\Omega |\nabla d(t)|^2 d(t) \cdot \overline{\vartheta} \, \mathrm{d}x.
\end{aligned}
\tag{7}
$$

Retrieving the condition $|d| = 1$

- Define
$$\varphi := |d|^2 - 1.$$
- Note that by assumption $\varphi(t) \in \mathrm{L}^\infty(\Omega)$ for every $t \in (0, T)$ and that
$$\partial_k \varphi(t) = 2\overline{d(t)} \cdot \partial_k d(t) \in \mathrm{L}^p(\Omega), \varphi_t(t) = 2\overline{d(t)} \cdot d_t(t) \in \mathrm{L}^{p/2}(\Omega). \tag{8}$$
- It follows from (8) that for almost every $t \in (0, T)$ we have $\varphi(t) \in \mathrm{W}^{1,p}(\Omega) \subset \mathrm{W}^{1,(p/2)'}(\Omega)$.
- Therefore, φ itself is an admissible test function and we can deduce
$$\int_\Omega \varphi_t \varphi \, \mathrm{d}x + \int_\Omega |\nabla \varphi|^2 \, \mathrm{d}x = - \int_\Omega u \cdot \nabla \varphi \varphi \, \mathrm{d}x + 2 \int_\Omega |\nabla d|^2 \varphi^2 \, \mathrm{d}x. \tag{9}$$

Retrieving the condition $|d| = 1$

- Then for any $t < T$ we find that

$$\frac{1}{2}\int_\Omega \phi(t)^2\,dx + \int_0^t\int_\Omega |\nabla\phi(s)|^2\,dx\,ds = 2\int_0^t\int_\Omega |\nabla d(s)|^2\phi(s)^2\,dx\,ds.$$

- We shall now show that Gronwall's inequality can be applied and we can infer that $\phi = 0$.
- For $0 < \tilde{t} < t$ to be chosen appropriately, we write

$$\underbrace{\int_0^t\int_\Omega |\nabla d(s)|^2\phi(s)^2\,dx\,ds}_{A} = \underbrace{\int_0^{\tilde{t}}\int_\Omega |\nabla d(s)|^2\phi(s)^2\,dx\,ds}_{I} + \underbrace{\int_{\tilde{t}}^t\int_\Omega |\nabla d(s)|^2\phi(s)^2\,dx\,ds}_{II}. \tag{10}$$

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Retrieving the condition $|d| = 1$

We estimate the integral II in the following way.

$$\begin{aligned}
\int_{\tilde t}^{t}\int_\Omega |\nabla d(s)|^2 \phi(s)^2 \, dx \, ds &\le \int_{\tilde t}^{t} \| |\nabla d(s)|^2 \|_{L^{\frac{q}{2}}} \|\phi(s)^2\|_{L^{(\frac{q}{2})'}} \, ds \\
&= \int_{\tilde t}^{t} \|\nabla d(s)\|_{L^q}^2 \|\phi(s)\|_{L^{2\cdot(\frac{q}{2})'}}^2 \, ds \\
&\le C \int_{\tilde t}^{t} s^{-3(\frac{1}{p}-\frac{1}{q})} \|\phi(s)\|_{L^{\frac{2q}{q-2}}}^2 \, ds \\
&\overset{\text{interpolation}}{\le} C \int_{\tilde t}^{t} s^{-3(\frac{1}{p}-\frac{1}{q})} \|\phi(s)\|_{L^2}^{2(1-\alpha)} \|\phi(s)\|_{L^6}^{2\alpha} \, ds \\
&\le C \int_{\tilde t}^{t} s^{-3(\frac{1}{p}-\frac{1}{q})} \|\phi(s)\|_{L^2}^{2(1-\alpha)} \|\phi(s)\|_{H^1}^{2\alpha} \, ds \\
&\overset{Young's}{\le} \int_{\tilde t}^{t} \frac{[Cs^{-3(\frac{1}{p}-\frac{1}{q})}\|\phi(s)\|_{L^2}^{2(1-\alpha)}]^{\frac{1}{1-\alpha}}}{\frac{1}{1-\alpha}} + \frac{\|\phi(s)\|_{H^1}^{\frac{2\alpha}{\alpha}}}{\frac{1}{\alpha}} \\
&\le \int_{\tilde t}^{t} \left[(1-\alpha) C^{\frac{1}{1-\alpha}} \tilde t^{-\frac{3}{1-\alpha}(\frac{1}{p}-\frac{1}{q})} + \alpha\right] \|\phi(s)\|_{L^2}^2 \, ds \\
&\quad + \alpha \int_{\tilde t}^{t} \|\nabla \phi(s)\|_{L^2}^2 \, ds.
\end{aligned} \tag{11}$$

Retrieving the condition $|d| = 1$

To estimate the integral I we choose an approximating sequence $\{b_j\}$ (of smooth functions) to b in $W^{1,3}$ and proceed as follows. We write

$$\begin{aligned}\int_0^{\tilde{t}} \int_\Omega |\nabla d(s)|^2 \phi(s)^2 \, dx \, ds \leq\ & C \int_0^{\tilde{t}} \int_\Omega |\nabla d(s) - \nabla b|^2 \phi(s)^2 \, dx \, ds \\ & + C \int_0^{\tilde{t}} \int_\Omega |\nabla b - \nabla b_j|^2 \phi(s)^2 \, dx \, ds \\ & + C \int_0^{\tilde{t}} \int_\Omega |\nabla b_j|^2 \phi(s)^2 \, dx \, ds.\end{aligned}$$

Now choosing j such that $\|\nabla(b - b_j)\|_{L^3} \leq \sqrt{\epsilon}$, we can write

$$\begin{aligned} & C \int_0^{\tilde{t}} \int_\Omega |\nabla b - \nabla b_j|^2 \phi(s)^2 \, dx \, ds \\ & \leq C \int_0^{\tilde{t}} \|\nabla(b - b_j)\|_{L^3}^2 \|\phi(s)\|_{L^6}^2 \, ds \qquad (12) \\ & \leq C\epsilon \int_0^{\tilde{t}} \|\phi(s)\|_{L^2}^2 \, ds + C\epsilon \int_0^{\tilde{t}} \|\nabla \phi(s)\|_{L^2}^2 \, ds.\end{aligned}$$

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Similarly if $\tilde{t}$ is chosen small enough, we have

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Again

$$\begin{aligned} C\int_0^{\tilde{t}}\int_\Omega |\nabla b_j|^2 \phi(s)^2 \, dx \, ds &\leq C\int_0^{\tilde{t}} \|\nabla b_j\|_{L^q}^2 \|\phi(s)\|_{\frac{2q}{q-2}}^2 \, ds \\ &\leq C\int_0^{\tilde{t}} \|\phi(s)\|_{L^2}^{2(1-\alpha)} \|\phi(s)\|_{H^1}^{2\alpha} \, ds \\ &\leq \int_0^{\tilde{t}} (1-\alpha) C^{\frac{1}{1-\alpha}} \|\phi(s)\|_{L^2}^2 \, ds + \alpha \int_0^{\tilde{t}} \|\phi(s)\|_{H^1}^2 \, ds \\ &\leq \int_0^{\tilde{t}} \left[(1-\alpha) C^{\frac{1}{1-\alpha}} + \alpha\right] \|\phi(s)\|_{L^2}^2 \, ds \\ &\quad + \alpha \int_0^{\tilde{t}} \|\nabla\phi(s)\|_{L^2}^2 \, ds. \end{aligned} \quad (14)$$

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Retrieving the condition $|d| = 1$

Combining (12)-(14) with (11) we can write

$$\begin{aligned} A &\le I + II \\ &\le \int_0^t \underbrace{\Big[(1-\alpha)C^{\frac{1}{1-\alpha}}\,\tilde{t}^{-\frac{3}{1-\alpha}(\frac{1}{p}-\frac{1}{q})} + \alpha + (1-\alpha)C^{\frac{1}{1-\alpha}} + C\epsilon\Big]}_{\text{constant function and therefore continuous}} \|\phi(s)\|_{L^2}^2 \; ds \\ &\qquad + (C\epsilon + \alpha)\int_0^t \|\nabla\phi(s)\|_{L^2}^2 \; ds. \end{aligned} \tag{15}$$

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Further regularity

- Let $\Phi : [L^{p'}_\sigma(\Omega)]^* \to L^p_\sigma(\Omega)$ denote the canonical isomorphism between $[L^{p'}_\sigma(\Omega)]^*$ and $L^p_\sigma(\Omega)$, and recall the duality pairing

$$(\Phi^{-1}u)(v) = \langle \Phi^{-1}u, v\rangle_{[L^{p'}_\sigma]^*, L^{p'}_\sigma} = \langle u, v\rangle_{L^p_\sigma, L^{p'}_\sigma} = \int_\Omega u \cdot \overline{v}\, \mathrm{d}x.$$

  We regard Φ^{-1} also as the canonical inclusion of $L^p_\sigma(\Omega)$ into $W^{-1,p}_\sigma(\Omega)$ by

$$\langle \Phi^{-1}u, v\rangle_{W^{-1,p}_\sigma, W^{1,p'}_{0,\sigma}} = \langle u, v\rangle_{L^p_\sigma, L^{p'}_\sigma}, \quad u \in L^p_\sigma(\Omega),\ v \in W^{1,p'}_{0,\sigma}(\Omega).$$

  In this sense, we define the weak Stokes operator $\mathcal{A}_p$ in $W^{-1,p}_\sigma(\Omega)$ by $\mathrm{dom}(\mathcal{A}_p) := \Phi^{-1}W^{1,p}_{0,\sigma}(\Omega)$ and

$$\begin{aligned} &\mathcal{A}_p : \mathrm{dom}(\mathcal{A}_p) \subset W^{-1,p}_\sigma(\Omega) \to W^{-1,p}_\sigma(\Omega), \\ &w \mapsto \Big[v \mapsto \int_\Omega \nabla\Phi w \cdot \overline{\nabla v}\, \mathrm{d}x\Big]. \end{aligned} \tag{16}$$

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- The proof of regularity follows by interpreting the integral equations as linear equations with a known right-hand side and using the maximal regularity properties of the weak Stokes and Neumann operators.

The case of Dirichlet boundary data

- In the case of Lipschitz domains, the global existence and uniqueness (under suitable smallness assumptions) can be studied using the semilinear approach.
- Let us denote $\delta = d - e$. Then we can rewrite the system as

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi = -\mathrm{div}([\nabla\delta]^\top \nabla\delta) \text{ in } (0,T)\times\Omega, \\ \partial_t \delta + (u\cdot\nabla)\delta = \Delta\delta + |\nabla\delta|^2\delta + |\nabla\delta|^2 e \text{ in } (0,T)\times\Omega, \\ \mathrm{div}\ u = 0 \text{ in } (0,T)\times\Omega, \\ (u,\delta) = (0,0) \text{ on } (0,T)\times\partial\Omega, \\ (u,\delta)\Big|_{t=0} = (a,\tilde{b}) \text{ in } \Omega, \end{aligned} \tag{17}$$

where $\tilde{b} = b - e$.
- We would like to note that the above system (17) in (u, δ) has homogeneous Dirichlet boundary conditions.

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Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, then there exists $\varepsilon > 0$ such that given initial conditions $a \in L^p_\sigma(\Omega)$ and $b \in W^{1,p}(\Omega)^3 \cap L^\infty(\Omega)^3$ with $b = e$ on $\partial\Omega$ for some $e \in \mathbb{S}^2$ where $3 \leq p < 3 + \varepsilon$, the following hold true for $q \in (p, 3+\varepsilon)$.

(a) *There exists $T > 0$ depending on the initial data such that equation (17) with Dirichlet boundary conditions has a local mild solution (u, δ) satisfying*

$$u \in S^u_q(T) \cap BC([0,T]; L^p_\sigma(\Omega)),$$
$$\delta \in S^d_q(T) \cap BC([0,T]; W^{1,p}_0(\Omega)^3) \cap BC([0,T]; L^\infty(\Omega)^3),$$

where in the limit $s \to 0+$, one has

$$\|u(s) - a\|_{L^p_\sigma(\Omega)} \to 0, \|\delta(s) - \tilde{b}\|_{L^\infty(\Omega)^3} \to 0, \|\nabla[\delta(s) - \tilde{b}]\|_{L^p(\Omega)^{3\times 3}} \to 0.$$

(b) *In the limit $s \to 0+$, the solutions satisfy*

$$s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|u(s)\|_{L^q_\sigma(\Omega)} \to 0 \quad \text{and} \quad s^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\nabla\delta(s)\|_{L^q(\Omega)^{3\times 3}} \to 0.$$

(c) If a and ∇b are sufficiently small, then the solution exists globally in the class

$$u \in S_q^u(\infty) \cap BC([0,\infty); L_\sigma^p(\Omega)),$$
$$\delta \in S_q^d(\infty) \cap BC([0,\infty); W_0^{1,p}(\Omega)^3) \cap BC([0,\infty); L^\infty(\Omega)^3).$$

(d) The solution is unique in the class given in (a) provided $p > 3$, and in the case $p = 3$, it is unique in the subset of this class satisfying in addition the limit conditions (b).

(e) The condition $|d| = 1$ is preserved if $|d(0)| = |b| = 1$.

Theorem

For every $s \in (1,2)$, the solution has the following additional regularity properties

$$u \in W^{1,s}(0,T; W_\sigma^{-1,\frac{p}{2}}(\Omega)) \cap L^s(0,T; W_{0,\sigma}^{1,\frac{p}{2}}(\Omega)),$$
$$\delta \in W^{1,s}(0,T; L^{\frac{p}{2}}(\Omega)^3) \cap L^s(0,T; \mathrm{dom}(B_{\frac{p}{2}})).$$

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Existence:

- In this case, we do not need to split the director field as the homogeneous Dirichlet boundary conditions ensure exponential decay.
- The proof then follows, as in the previous case, by considering the integral formulations for u, δ and $\nabla\delta$.

Retrieving the unit modulus condition for the director field:

- Once we have proved the existence of δ, we can go back to the original variable d which satisfies the same regularity.
- We can then proceed as in the proof in the previous case. The only point to notice is that we can use $\phi := |d|^2 - 1$ as a test function and this vanishes on the boundary.
- The discussion on regularity remains the same.

Concluding remarks and Future directions

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- In the case of a smooth domain Ω our approach yields similar results as has been obtained by Hieber et al. using quasilinear techniques.

  More concretely, their approach requires initial data in Besov spaces

$$a \in B^{2\mu-2/p}_{qp}(\Omega)^3 \cap L^p_\sigma(\Omega),$$
$$b \in B^{2\mu-2/p}_{qp}(\Omega)^3, \quad \frac{2}{p} + \frac{3}{q} < 1, \frac{1}{2} + \frac{1}{p} + \frac{3}{2q} < \mu \leq 1,$$

  using the fact that the embedding $B^{2\mu-2/p}_{qp}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ holds. These initial data are much more regular than the ones assumed by us.

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- Note that there are also other versions of the simplified Ericksen–Leslie model.

  For instance, some authors drop the assumption $|d| = 1$ and replace the dynamical equation for the director field d by

  $$\partial_t d - \Delta d + (u \cdot \nabla) d = -\gamma f(d), \quad \gamma > 0,$$

  for a bounded vector valued penalty function f.

  The method we presented here can be adapted for this setting as well.

Future directions

The complete *Ericksen-Leslie* model

$$\partial_t u + (u \cdot \nabla) u = Div\ \sigma \text{ in } J \times \Omega,$$
$$div\ u = 0 \text{ in } J \times \Omega,$$
$$d \times \Big(g + Div\big(\frac{\partial W^{OF}(d, \nabla d)}{\partial(\nabla d)}\big) - \frac{\partial W^{OF}(d, \nabla d)}{\partial d} \Big) = 0 \text{ in } J \times \Omega,$$
$$|d| = 1 \text{ in } J \times \Omega,$$
$$(u, d)\Big|_{t=o} = (u_0, d_0) \text{ in } \Omega.$$

- u velocity, σ stress, d director field
- The Oseen-Frank density

$$W^{OF} = \frac{1}{2}\Big[k_1 (div\ d)^2 + k_2 |d \times (\nabla \times d)|^2 + k_3 |d \cdot (\nabla \times d)|^2 + (k_2 + k_4)(tr(\nabla d)^2 - (div\ d)^2) \Big]$$

where $k_1, \cdots, k_4 \subset \mathbb{R}$ are elasticity coefficients.

Anupam Pal Choudhury
Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria

- The stress is defined as

$$\sigma = -\pi Id - \left[\frac{\partial W^{OF}(d, \nabla d)}{\partial \nabla d}\right]^T \nabla d + \sigma_L,$$

where

$$\sigma_L := \alpha_1 [dd^T : D] dd^T + \alpha_2 dN^T + \alpha_3 Nd^T + \alpha_4 D + \alpha_5 d[Dd]^T + \alpha_6 [Dd] d^T,$$

$D = \frac{1}{2}([\nabla u]^T + \nabla u)$, $V = \frac{1}{2}([\nabla u]^T - \nabla u)$ and
$N = \partial_t d + (u \cdot \nabla) d - Vd$.

- In the case of the simplified model, Giga's iteration technique seems to work fine if we want to use a semilinear approach.
- The major question is if the semilinear approach might be suitable for more complex models even in smooth domains.
- How do we deal with them in the Lipschitz domains?

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Thank You