# Properties of Spectrum for Anderson type random operator

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# Anderson type operator

- The model was introduced to study spin wave propagation over doped semi-conductors.
- Anderson developed it to show that random medium causes the eigenfunctions to decay exponentially. This phenomenon is now known as Anderson localization.
- The continuous version of the Anderson Hamiltonian is given by

$$H^\omega = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \sum_{n \in \mathbb{Z}^d} \omega_n \chi_{n+[0,1)^d}$$

  on $L^2(\mathbb{R}^d)$.

There are basically three type of questions

- Spectrum of the operator (including behavior of density of states).
- Local structure of spectrum (behavior of gaps between eigenvalue).
- Multiplicity of spectrum.

# Anderson type operator

## Anderson tight binding model/Random Schrödinger operator

- For ergodic case, the spectrum is almost surely constant and there are regularity results for density of states.
- Recently there is some results on local spectral statistics in region of Anderson localization, which in turn provides answer for multiplicity.

## Anderson type operator

- There are many intermediate models like dimer/polymer models which falls between Anderson tight binding model and random Schrödinger operator.
- For the ergodic case, many results involving density of state can be extended from theory of Anderson tight binding model to these cases.
- The questions of spectral statistics and multiplicity are being studied.

# Multiplicity bounds of singular spectrum for certain Random operators

# Multiplicity bounds of spectrum

## Family of operator

The family of random operator $A^\omega$ is of the form

$$A^\omega = A + \sum_n \omega_n C_n,$$

where $A$ is essentially self adjoint operator and $\{C_n\}_n$ are finite rank non-negative operators. Here $\{\omega_n\}_n$ are independent real random variables with absolutely continuous distribution.

## Example

On $\ell^2(\mathbb{Z})$ consider the operator $\Delta + V^\omega$, where $\Delta$ is Laplacian and

$$(V^\omega u)(n) = \omega_{\lfloor \frac{n}{N} \rfloor} u(n) \qquad \forall n \in \mathbb{Z}.$$

This example can be generalized to $\mathbb{Z}^d$ case.

# Multiplicity bounds of spectrum

## Main results of [AM, D. R. Dolai, arXiv:1709.01774][1]

Under the assumption that $A^\omega$ is almost surely essentially self adjoint operator.
The multiplicity of singular spectrum is bounded above by maximum multiplicity of eigenvalues of $\sqrt{C_n}(A^\omega - z)^{-1}\sqrt{C_n}$ for almost all $(\omega, z)$.

As an expression, it is given by

$$\sup_n \operatorname*{ess\text{-}sup}_{z\in\mathbb{C}\setminus\mathbb{R}} Mult_n^\omega(z),$$

where

$$Mult_n^\omega(z) := \text{Maximum multiplicity of roots of polynomial } \det(\sqrt{C_n}(A^\omega - z)^{-1}\sqrt{C_n} - xI) \text{ in } x.$$

It can be shown that $Mult_n^\omega(z)$ is constant for almost all $\omega$.

[1] Multiplicity theorem of singular Spectrum for general Anderson type Hamiltonian

# Multiplicity bounds of spectrum

## Main results of [AM, M. Krishna, arXiv:1803.06895][2]

- When $C_n$ are finite rank projection satisfying $\sum_n C_n = I$ and $\{\omega_n\}_n$ are independent real random variables following an absolutely continuous distribution with full support.
- Let $J$ be an interval in the region where Simon-Wolff criteria is satisfied, such that the multiplicity of eigenvalues of $A^\omega$ in $J$ is bounded below by $K$ almost surely.

Then, the multiplicity of spectrum, in the region where Simon-Wolff criteria is satisfied, is bounded below by $K$.

## Simon-Wolff criteria

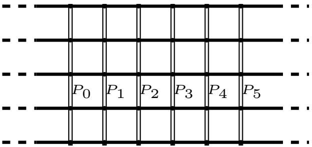
Is the region where only pure-point spectrum exists

$$\bigcap_n \left\{ E \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \left\| (A^\omega - E - \iota\epsilon)^{-1} C_n \right\| < \infty \quad a.s. \right\}$$

[2] Global multiplicity bounds and Spectral Statistics Random Operators

# Example

On $\ell^2(\mathbb{Z} \times \{1, \cdots, 5\})$ consider the operator
$$H^\omega = \tilde{\Delta} + \sum_n \omega_n P_n,$$
where
$$(\tilde{\Delta} u)(x,n) = \left\lceil \frac{n}{3} \right\rceil (u(x+1,n) + u(x-1,n)),$$




and $\omega_n$ are i.i.d real random variables with absolutely continuous distribution. The projection is given by $(P_n u)(x,m) = \begin{cases} u(x,m) & x = n, \\ 0 & o.w \end{cases}$.

Observe that $H^\omega$ restricted to $\ell^2(\mathbb{Z} \times \{1\})$, $\ell^2(\mathbb{Z} \times \{2\})$ and $\ell^2(\mathbb{Z} \times \{3\})$ are unitarily equivalent. Similarly $\ell^2(\mathbb{Z} \times \{4\})$ and $\ell^2(\mathbb{Z} \times \{5\})$ are unitarily equivalent.

- First result will imply that the maximum multiplicity is bounded by three.
- And second result will imply that lower bound of multiplicity is two.

# Technical details (Assuming $C_n$ are projections)

- Setting $\mathscr{H}_n^\omega = \overline{\{f(A^\omega)\phi : f \in C_c(\mathbb{R}), \phi \in C_n\mathscr{H}\}}$ we have
$$(A^\omega, \mathscr{H}_n^\omega) \cong (M_{Id}, L^2(\mathbb{R}, \Sigma_n^\omega, \mathbb{C}^{rank(C_n)})),$$
where $\Sigma_n^\omega$ is matrix valued measure which satisfies
$$\int \frac{1}{x-z}\Sigma^\omega(dx) = C_n(A^\omega - z)^{-1}C_n.$$
- Writing $d\Sigma_n^\omega(E) = W_n^\omega(E)d\sigma_n^\omega(E)$ where $\sigma_n^\omega = tr(\Sigma_n^\omega)$, Poltoratskii's theorem gives
$$W_n^\omega(E) = \lim_{\epsilon\downarrow 0} \frac{1}{tr(C_n(A^\omega - E - \iota\epsilon)^{-1}C_n)} C_n(A^\omega - E - \iota\epsilon)^{-1}C_n,$$
for almost all $E$ w.r.t $\sigma_{n,sing}^\omega$.
- For $E$ in support of $\sigma_{n,sing}^\omega$, we have $tr(C_n(A^\omega - E - \iota\epsilon)^{-1}C_n) \to \infty$ as $\epsilon$ goes to zero.
- If $\mathscr{H}_n^\omega \cap \mathscr{H}_m^\omega = \{0\}$, then $\sigma_{n,sing}^\omega$ and $\sigma_{m,sing}^\omega$ are singular.

# Technical details (Assuming $C_n$ are projections)

To establish multiplicity of $W_n^{\omega}(E)$ for almost all $E$, it is enough to study the case $A_\lambda = A + \lambda C_n$. This is done as:

- Let $G_\lambda(z) = C_n(A_\lambda - z)^{-1}C_n$, then using resolvent equation
$$G_\lambda(z) = G_0(z)(I + \lambda G_0(z))^{-1} = (G_0(z)^{-1} + \lambda I)^{-1}$$
- A consequence of Poltoratskii's theorem is
$$supp(\sigma_{\lambda,sing}) = \{E : \det(I + \lambda G_0(E + \iota 0)) = 0\}$$
- So multiplicity of $G_0(E + \iota 0)$ for almost all $E$ provides the required answer.
- Viewing $\det(I + \lambda G_0(z))$ as polynomial of $\lambda$, we need to find the multiplicity roots as function of $z$. This can be re-stated as a polynomial being non-zero.

Anderson type operators on Cayley like graphs

# Anderson type operators on Cayley like graphs

## Definition [AM, P. A. Narayanan arXiv:1808.05820][4]

Given a finitely generated group $G$ with generators $g_1, \cdots, g_n$ and a sequence of vertices $v_{\pm 1}, v_{\pm 2}, \cdots, v_{\pm n}$ from a finite undirected graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, define the infinite graph $\mathcal{H}_G = (\mathcal{V}_G, \mathcal{E}_G)$ by

- $\mathcal{V}_G = \mathcal{V} \times G = \{(v, g) : v \in \mathcal{V}, g \in G\}$,
- The edge set $\mathcal{E}_G)$ is union of
  $\{\{(v, g), (w, g)\} : \{v, w\} \in \mathcal{E}, g \in G\}$
  and
  $\{\{(v_{-i}, g), (v_i, gg_i)\} : g \in G, 1 \leq i \leq n\}$.

[4] On multiplicity of spectrum for Anderson type operators with higher rank perturbations

# Anderson type operators on Cayley like graphs

On $\ell^2(\mathcal{V}_G)$ let $\Delta$ denote the adjacency operator. Define the Anderson operator as $H^\omega = \Delta + \sum_{g\in G} \omega_g P_g$, where $P_g$ is the projection onto $\ell^2(\mathcal{V} \times \{g\})$ and $\{\omega_g\}_g$ are i.i.d random variables.

Viewing $\omega_g$ as projection from $(\Omega, \mathcal{B}, \mathbb{P}) = (\mathbb{R}^G, \otimes_G \mathcal{B}_\mathbb{R}, \otimes_G \mu)$ to $\mathbb{R}$, for any $h \in G$ define $\theta_h : \Omega \to \Omega$ by $(\theta_h \omega)_g = \omega_{gh}$. Then observe that

$$H^{\theta_h \omega} = U_h H^\omega U_h^*,$$

where for any automorphism $\phi$ of $\mathcal{H}_G$ define the unitary operator $U_\phi$ on $\ell^2(\mathcal{V}_G)$ by

$$(U_\phi u)((v,g)) = u(\phi(v,g)),$$

so the operator $H^\omega$ is ergodic.

# Anderson type operators on Cayley like graphs

## special Unitaries preserving $H^\omega$

$$Aut_{And}(\mathcal{H}_G) = \{\phi \in Aut(\mathcal{H}_G) : H^\omega = U_\phi H^\omega U_\phi^* \ a.s.\}$$

## Theorem

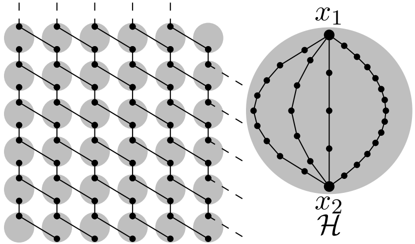
The group homomorphism

$$\Theta : \prod_{g \in G} Aut(\mathcal{H}|\{v_{\pm i}\}_{i=1}^n) \to Aut_{And}(\mathcal{H}_G)$$

defined by $\Theta((\phi_g)_g)((v,h)) = (\phi_h(v), h)$, for $(v,h) \in \mathcal{V}_G$, is ismorphism.
Here
$Aut(\mathcal{H}|\{v_{\pm i}\}_{i=1}^n) := \{\phi \in Aut(\mathcal{H}) : \phi(v_{\pm i}) = v_{\pm i} \ 1 \leq i \leq n\}$.
So if $Aut(\mathcal{H}|\{v_{\pm i}\}_{i=1}^n)$ is trivial, then $Aut_{And}(\mathcal{H}_G)$ is also trivial.

# Anderson type operators on Cayley like graphs

Observe that the zero eigenvalue for Laplacian over $\mathcal{H}$ has non-trivial multiplicity. But by construction of the graph $Aut(\mathcal{H}|\{x_1, x_2\}) = \{1\}$.




Another important observation is that, there is multiple eigenvectors corresponding to zero eigenvalue of Laplacian which is zero at $x_1$ and $x_2$.

If we construct $\mathcal{H}_{\mathbb{Z}^2}$ for the group $\mathbb{Z}^2$ and define the Anderson operator described as above, then the random operator $H^\omega$ has non-trivial multiplicity. By construction of the graph, $Aut_{And}(\mathcal{H}_{\mathbb{Z}^2})$ is trivial, so automorphisms of underlying space is not contributing to any multiplicity of the operator.

## Local Eigenvalue Statistics and regularity of DOS

# Local Eigenvalue Statistics and regularity of DOS

## Model in [AM, D.R. Dolai, arXiv:1506.07132][3]

On the Hilbert space $\ell^2(\mathbb{Z}^d)$ consider the operator

$$(H^\omega u)(n) = \sum_{\|n-m\|_1=1} (u(n) - u(m)) + (1 + \|n\|_2^\alpha)\omega_n u(n) \quad n \in \mathbb{Z}^d,$$

where $\{\omega_n\}_n$ are i.i.d random variables following uniform distribution in $[0,1]$.

It was shown by Gordon-Jakšić-Mochanov-Simon that

- $\sigma_{ess}(H^\omega) = [a_k, \infty)$.
- For $E \in (a_j, a_{j-1})$ for $j \leq k$ the limit

  $$N_j(E) = \lim_{L\to\infty} \frac{1}{L^{d-j\alpha}} \#\{E_n \in \sigma(H_L^\omega) : E_n < E\}$$

  exists and is non-zero almost surely, where $H_L^\omega$ is restriction of $H^\omega$ in $\{-L, \cdots, L\}^d$.

[3] Spectral statistics of random Schrödinger operator with growing potential

# Smoothness property of $N_1$

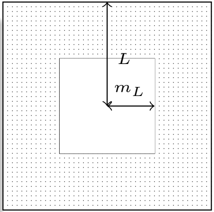
## Result [AM, D.R. Dolai, arXiv:1506.07132]

For $E > 2d$ we have $N_1(E) = C_{d,\alpha}(E - 2d)$.

For $\Lambda_{L,m_L} := \{-L, \cdots, L\}^d \setminus \{-m_L, \cdots, m_L\}^d$ define $H^\omega_{L,m_L}$. The function

$G_L(z) = \frac{1}{L^d}\mathbb{E}\left[Tr\left((H^\omega_{L,m_L} - z)^{-1}\right)\right]$

has analytic continuation for any compact subset of $\{z : Rez > 2d^2\}$ for large enough $L$.




## Random walk expansion

$$\langle \delta_0, (H^\omega - z)^{-1}\delta_0\rangle = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma_L^k(0,0)} \prod_{i=0}^{k} \frac{1}{(1 + \|\gamma_i\|_2^{\alpha})\omega_{\gamma_i} + 2d - z}$$

where $\Gamma_L^k(0,0)$ is set of path starting and ending at $0$ of length $k$.

# Local Statistics for unfolded eigenvalues

## Result [AM, D.R. Dolai, arXiv:1506.07132]

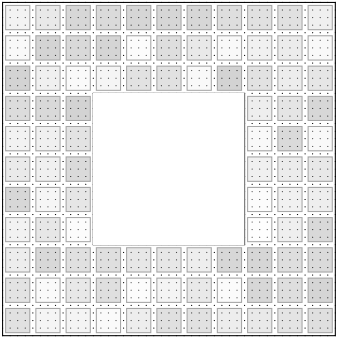
The point process $\Lambda_L^{\omega,t}(f) = Tr(f(L^{d-j\alpha}(N_j(H_L^\omega) - t)))$ for $f \in C_c(\mathbb{R})$ converges to Poisson point process.

The set $\Lambda_{L,m_L}$ can be divided into $\{\Lambda_{l_L}(p)\}_p$, such that for most eigenvalues of $H^\omega_{L,m_L}$ there is a unique $p$ where some eigenvalue $H^\omega_{\Lambda_{l_L}(p)}$ is exponentially close.

So we can re-define the point process with respect to smaller boxes.

$\Lambda^{\omega,t}_{L,p}(f) = Tr(f(L^{d-j\alpha}(N_j(H^\omega_{\Lambda_{l_L}(p)}) - t)))$.

These point processes are independent.



Rest of the work is to show that the point process $\sum_p \Lambda^{\omega,t}_{L,p}$ converges to Poisson point process.

Current problems and future directions

# Current Problems (in progress)

## Regularity properties of density of state

- **Joint work with Prof. Manjunath**: To show that the density of states measure for random Toeplitz/Hankel matrix is absolutely continuous. In case of Toeplitz matrix, the density of states measure is $C^\infty$.
- **Joint work with D.R. Dolai and Prof. M. Krishna**: To show that the density of state measure is $C^\infty$ in the region of dynamical localization, random Schrödinger operator on $L^2(\mathbb{R}^d)$.

The fundamental difference between above work is, there is no well defined understanding of behavior of eigenvectors for the case of Toeplitz/Hankel matrix, but for the other work we are explicitly focusing on the region where the eigenvectors are exponentially localized.

# Future direction

Consider the sequence of random matrices $\{A^\omega_{\alpha,L}\}_{L\in\mathbb{N}}$ given by $A^\omega_{\alpha,L} = \Delta_L + \frac{1}{L^\alpha}V^\omega$, where $\Delta_L$ is Laplacian and $V^\omega$ is multiplication by i.i.d random variables, defined on $\mathbb{C}^L$.
Observe that $A^\omega_{\alpha,L}$ converges strongly to discrete Laplacian on $\ell^2(\mathbb{N})$ for $\alpha > 0$. For $E_0 \in (-2,2)$, consider the point process
$\Lambda^\omega_{\alpha,E_0,L}(f) = Tr(f(L(A^\omega_{\alpha,L} - E_0)))$ for $f \in C_c(\mathbb{R})$.
For $\alpha > \frac{1}{2}$, it can be shown that $\Lambda^\omega_{\alpha,E_0,L}$ converges to clock process, in sense of distribution, as $L \to \infty$.

The main goal is to show that for $\alpha < \frac{1}{2}$, the point process $\Lambda^\omega_{\alpha,E_0,L}$ converges to Poisson point process.
This would imply that local eigenvalue statistics is not a good indicator of the nature of spectrum locally.

# References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