National Research Tomsk State University

Tetrahedral hyperbolic 3-manifolds and links

Andrei Vesnin

Knots Through Web, August 24, 2020 (Online),
ICTS of the Tata Institute of Fundamental Research

Tomsk State University, Russia

Outline

1. Hyperbolic 3-space and 3-manifolds
2. Complexity of cusped 3-manifolds
3. Tetrahedral manifolds
4. A census of tetrahedral manifolds
5. Tetrahedral links in S^3
6. Volumes of tetrahedral manifolds and clasical arithmetic functions

Hyperbolic 3-space and 3-manifolds

Hyperbolic 3-space

Consider a model of the hyperbolic (Lobachevsky) space in the upper half-space $\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$ with

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

- Geodesics = Euclidean half-lines and hemicirclies orthogonal to the plane $\{t = 0\}$.
- Plaines = Euclidean half-planes and hemispheres orthogonal to the plane $\{t = 0\}$.
- Volume: $d\mathrm{vol} = \dfrac{dx\,dy\,dt}{t^3}$.

Poincare extension

Absolute, i.e. the boundary of a hyperbolic 3-space $\partial\mathbb{H}^3 = \overline{\mathbb{C}}$.

A point $(x, y, t) \in \mathbb{H}^3$ can be presented as a quaternion

$$x + y \cdot i + t \cdot j + 0 \cdot k = z + tj.$$

For a linear-fractional transformation

$$z \mapsto \frac{az + b}{cz + d}, \qquad \text{где} \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}$$

of the Riemann sphere $\overline{\mathbb{C}}$, we extend its action to $\mathbb{H}^3$ as a product of quaternions:

$$z + tj \mapsto (a(z + tj) + b)(c(z + tj) + d)^{-1}.$$

The group of orientation-preserving isometries of the space $\mathbb{H}^3$:

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}.$$

Hyperbolic manifolds and hyperbolic knots

By a hyperbolic structure on an n-manifold M, we mean a Riemannian metric on M of constant sectional curvature -1. A structure is complete if the induced metric is complete.

The Mostow–Prasad rigidity theorem. If an orientable n-manifold with $n \geq 3$ admits a complete hyperbolic structure of finite volume, then this structure is unique.

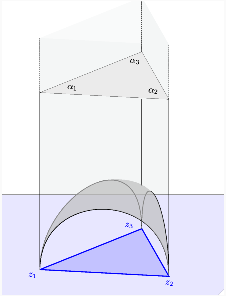
A knot or link K in $\mathcal{S}^3$ is hyperbolic if its complement $\mathcal{S}^3 - K$ admits a complete hyperbolic structure of finite volume.

Theorem [Thurston, 1982]. A prime knot in $\mathcal{S}^3$ is hyperbolic if and only if it is neither a torus knot nor a satellite knot.

Ideal tetrahedra in $\mathbb{H}^3$

Theorem [Andreev, 1970]. An acute-angled tetrahedron in $\mathbb{H}^3$ is uniquely determined by its dihedral angles.

A tetrahedron T is called ideal if all its vertices belongs to $\partial\mathbb{H}^3$.



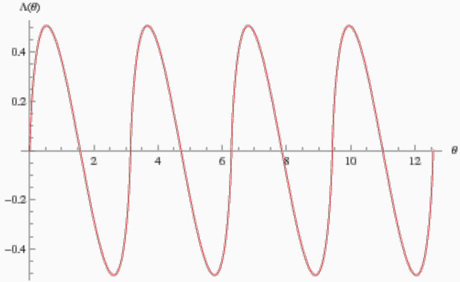
Lemma. Opposite dihedral angles are equal and $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.

Below we denote an ideal tetrahedron by $T(\alpha_1, \alpha_2, \alpha_3)$.

Volumes of ideal tetrahedra

Volumes of 3-dimensional hyperbolic polyhedra and manifolds can be found in terms of the Lobachevsky function

$$\Lambda(\theta) = -\int_0^{\theta} \ln \mid 2 \sin x \mid dx.$$

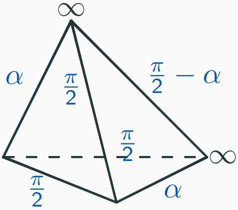


Lobachevsky function

Theorem [Kubert identity]. For any positive integer m the Lobachevsky function satisfies the following relation:

$$\Lambda(m\theta) = m \sum_{k=0}^{m-1} \Lambda\left(\theta + \frac{k\pi}{m}\right).$$

Geometrical meaning of the Lobachevsky function:



Volume of this tetrahedra is equal to $\frac{1}{2}\Lambda(\alpha)$.

Regular ideal tetrahedron

Theorem [Milnor]. Let $T(\alpha, \beta, \gamma)$ be an ideal tetrahedron in $\mathbb{H}^3$. Then

$$\mathrm{vol}(T(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Corollary. A regular ideal tetrahedron $T(\pi/3, \pi/3, \pi/3)$ has the maximum volume among all (finite volume) tetrahedra in $\mathbb{H}^3$. The volume is equal to

$$v_3 = \mathrm{vol}(T(\pi/3, \pi/3, \pi/3)) = 3\Lambda(\pi/3) =$$

$$1.0149416064096536250212025542745202859416893075302997920\ldots$$

Cusped hyperbolic 3-manifolds

Let M be a connected hyperbolic 3-manifold obtained by gluing together a finite set $\mathcal{P}$ of pairwise disjoint ideal (with all vertices are at the absolute) tetrahedra from $\mathbb{H}^3$.

Let $\mathcal{S}$ be the set of all faces of tetrahedra from $\mathcal{P}$. Assume that the gluing is realized by a pairing Θ along faces $\mathcal{S}$ by isometries of $\mathbb{H}^3$.

The pairing Θ extends to a pairing of ideal vertices of tetrahedra from $\mathcal{P}$ that splits all ideal vertices in classes of equivalent.

For an ideal vertex v denote by $[v]$ it equivalence class.

A class of equivalent ideal vertices is called a cusp of M.

Cusped hyperbolic 3-manifolds

Let v be an ideal vertex of a tetrahedra P_v from $\mathcal{P}$. Choose a horoshere Σ_v at v which intersects only those faces of $\mathcal{S}$ which are incident to v. A link of a vertex v is a set $L(v) = P_v \cap \Sigma_v$.

Since $L(v)$ is a compact Euclidean polygon in Σ_v, the pairing Θ induces a gluing of polygons $\{L(u) : u \in [v]\}$ along sides by similarities.

Denote the resulting surface by $L[v]$. The surface $L[v]$ is said to be a link of the cusp $[v]$ of M.

If every isometry from Θ is orientation-preserving, then $L[v]$ is a torus. Otherwise, it can be is a Klein bottle.

Hyperbolic 3-manifold M is complete if and only if for each of its cusps $[v]$ the link $L[v]$ is complete.

Complexity of cusped 3-manifolds

Complexity of cusped hyperbolic 3-manifolds

We say that complexity $c(M)$ of a cusped hyperbolic 3-manifold M is equal to k if M admits an ideal triangulation with k tetrahedra and there is no an ideal triangulation with less number of tetrahedra.

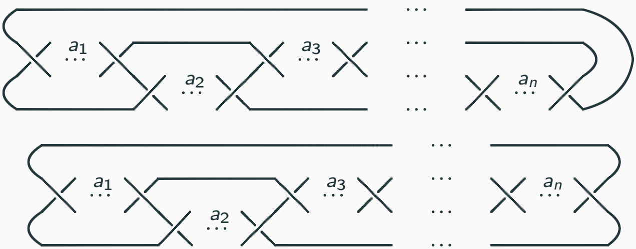
If $c(M) = k$, then $\text{vol}(M) \leqslant v_3 k$, hence $c(M) \geqslant \frac{\text{vol}(M)}{v_3}$.

There is only a finite number of manifolds of a given complexity.

Problem. Classify hyperbolic 3-manifolds (and hyperbolic knots) according to their complexity.

Example: 2-bridge knots and links

$$p/q = [a_1, a_2, \ldots, a_{n-1}, a_n] = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}.$$



The Conway normal form of a two-bridge link $K(p/q)$.
Here a_j denotes a number of half-twists.

Example: complements of 2-bridge knots and links

Proposition [Petronio – V., 2009]. Let $K(p,q)$ be a hyperbolic two-bridge link with $p/q = [a_1, a_2, \ldots, a_n]$, $n \geq 2$, $a_i > 0$ and $a_1, a_n > 1$. Then

$$2n - 2 \leqslant c(S^3 \setminus K(p/q)).$$

Theorem [Ishikawa – Nemoto, 2015]. With above conditions we have:

$$c(S^3 \setminus K(p,q)) \leqslant \sum_{i=1}^{n} a_i + 2(n-3) - \sharp\{a_i = 1\}.$$

Corollary. If $p/q = [a_1, a_2, \ldots, a_n] = [2, 1, \ldots, 1, 2]$, $n \geq 2$, then $c(S^3 \setminus K(p,q)) = 2n - 2$.

Tetrahedral manifolds

Tetrahedral manifolds

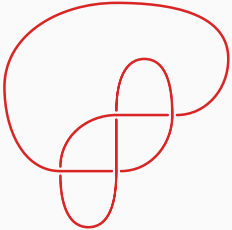
We call a cusped hyperbolic 3-manifold tetrahedral if it can be decomposed into regular ideal tetrahedra.

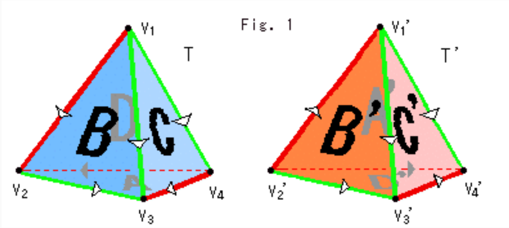
Let M be a tetrahedral manifold which can be decomposed into k regular ideal tetrahedra. Since ideal regular tetrahedron has maximal volume, we have $c(M) = k$.

For $k = 1$ there is a unique tetrahedral manifold is the Gieseking manifold (1912) that is non-orientable.

For $k = 2$ one of orientable tetrahedral manifolds is the figure-eight knot complement.

The figure-eight knot complement

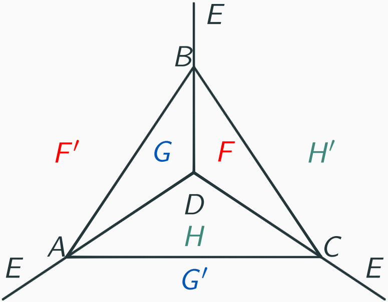




The figure-eight knot and triangulation of its complement into 2 regular ideal tetrahedra.

The figure-eight knot and gluing of ideal tetrahedra

Let $P = ABCD \cup ABCE$, where $ABCD$ and $ABCE$ are regular ideal tetrahedra.



The pairing of faces by isometries:

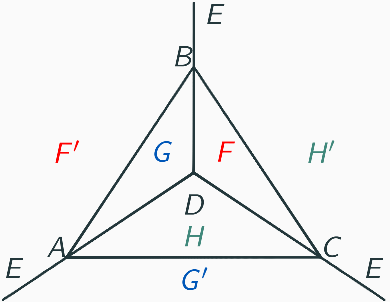
$f\ :\ F = BCD \to F' = BEA,$

$g\ :\ G = ADB \to G' = ACE,$

$h\ :\ H = CDA \to H' = ECB.$

Denote $\Theta = \langle f, g, h \rangle$

Equivalence classes of edges



Edges of P split in classes of equivalent under the group action:

$$BA \xrightarrow{g} EA \xrightarrow{f^{-1}} CD \xrightarrow{h} EC \xrightarrow{g^{-1}} BD \xrightarrow{f} BA,$$

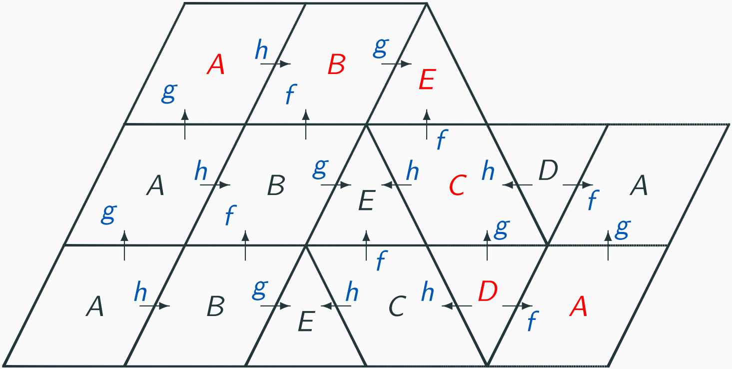
where $g\,f^{-1}\,h\,g^{-1}\,f = 1$; and

$$AD \xrightarrow{g} AC \xrightarrow{h} BE \xrightarrow{f^{-1}} BC \xrightarrow{h^{-1}} AD,$$

where $g\,h\,f^{-1}\,h^{-1} = 1$.

Equivalence classes of ideal vertices

All ideal vertices of P are equivalent to one $[v] := [A] = [B] = [C] = [D] = [E]$ under the action of the group $\Theta = \langle f, g, h \rangle$.



An Euclidean plane $\mathbb{E}^2$ as the universal cover of the link $L[v]$.

A census of tetrahedral manifolds

Orientable tetrahedral manifolds of complexity $\leqslant 8$

[P. Callahan, M. Hildebrand, J. Weeks]: listed all hyperbolic 3-manifolds which can be glued from $\leqslant 7$ ideal (not necessary regular) tetrahedra (4,815 manifolds).

[M. Thistlethwaite]: listed all hyperbolic 3-manifolds which can be glued from 8 ideal (not necessary regular) tetrahedra (12,846 manifolds).

[Fominykh – Tarkaev – V.]: independent generation of orientable tetrahedral manifolds of complexity at most 8. Recognition: by homology and Turaev – Viro quantum invariants of 3-manifolds.

Theorem. There are only 29 orientable tetrahedral manifolds of complexity at most 8. Among them 17 have 1 cusp and 12 have 2 cusps.

Orientable tetrahedral manifolds of complexity 9, 10

Theorem [Fominykh – Tarkaev – V., 2014]. There is unique orientable tetrahedral manifold of complexity 9. This manifold has 1 cusp.

Theorem [Fominykh – Tarkaev – V., 2014]. Let $N(10, k)$ be a number of orientable tetrahedral manifolds of complexity 10 with k cusps. Then $k = 1, 2, 3, 4, 5$ and

1. $11 \leq N(10, 1) \leq 15$;
2. $15 \leq N(10, 2) \leq 20$;
3. $9 \leq N(10, 3) \leq 15$;
4. $N(10, 4) = 3$;
5. $N(10, 5) = 1$.

Bad news: not all constructed manifolds can be recognized by the first homology group and Turaev – Viro invariants.

Computer software to study 3-manifolds

- SnapPy is a computer program for studding the topology and geometry of 3-manifolds, with focus on hyperbolic structures. Written by Marc Culler, Nathan Dunfield, and Mathias Goerner using the SnapPea kernel written by Jeff Weeks.
- Regina is a software package for 3-manifold and 4-manifold topologist, with a focus on triangulations, normal surfaces and angle structures. The primary developers of Regina are Benjamin Burton, Ryan Budney, and Willian Pettersson.
- Recognizer is a computer program to study 3-manifolds given by spines, with a focus on complexity of 3-manifolds. Written by Sergey Matveev and Vladimir Tarkaev.

The main theorem

Theorem [FGGTV, 2016] The number of combinatorial tetrahedral tessellations and tetrahedral manifolds up to 25 tetrahedra for orientable manifolds and up to 21 tetrahedra for non-orientable manifolds are listed in tables below.

EXPERIMENTAL MATHEMATICS
2016, VOL. 25, NO. 4, 466–481
http://dx.doi.org/10.1080/10586458.2015.1114436

Taylor & Francis
Taylor & Francis Group

A Census of Tetrahedral Hyperbolic Manifolds

Evgeny Fominykh[a,b], Stavros Garoufalidis[c], Matthias Goerner[d], Vladimir Tarkaev[a], and Andrei Vesnin[e]

[a]Laboratory of Quantum Topology, Chelyabinsk State University, Chelyabinsk, Russia; [b]Institute of Mathematics and Mechanics, Ekaterinburg, Russia; [c]School of Mathematics, Georgia Institute of Technology, Atlanta, GA, USA; [d]Pixar Animation Studios, Emeryville, CA, USA; [e]Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia

ABSTRACT
We call a cusped hyperbolic 3-manifold *tetrahedral* if it can be decomposed into regular ideal tetrahedra. Following an earlier publication by three of the authors, we give a census of all tetrahedral manifolds and all of their combinatorial tetrahedral tessellations with at most 25 (orientable case) and 21 (non-orientable case) tetrahedra. Our isometry classification uses certified canonical cell decompositions (based on work by Dunfield, Hoffman, and Licata) and isomorphism signatures (an improvement

KEYWORDS
hyperbolic 3-manifolds; regular ideal tetrahedron; census; tetrahedral manifolds; Bianchi orbifolds

arXiv:1502.00383 contains 62 source and data files for computer programs SnapPy and Regina.

Table 1. Number of triangulations in the census.

Tetrahedra	Combinatorial tet. tessellations		Tetrahedral manifolds		Homology links
	Orientable	Non-or.	Orientable	Non-or.	
1	0	1	0	1	0
2	2	2	2	1	1
3	0	1	0	1	0
4	4	4	4	2	2
5	2	12	2	8	0
6	7	14	7	10	0
7	1	1	1	1	0
8	14	10	13	6	5
9	1	6	1	6	0
10	57	286	47	197	12
11	0	17	0	17	0
12	50	117	47	80	7
13	3	8	3	8	0
14	58	134	58	113	25
15	91	975	81	822	0
16	102	175	96	142	32
17	8	52	8	52	0
18	213	1118	199	810	66
19	25	326	25	326	0
20	1886	26,320	1684	22,340	209
21	31	251	31	251	0
22	390	–	381	–	148
23	58	–	58	–	0
24	1544	–	1465	–	378
25	7563	–	7367	–	0

Computations

The most computationally hard part was the combinatorial enumeration of tetrahedral triangulations.

The case of orientable triangulations up to 25 tetrahedra and the cases of non-orientable triangulations up to 21 tetrahedra needed about 6 weeks each. We used the server processor Xeon E5-2630, 2.3 Ghz.

All orientable tetrahedral triangulations with $n \leq 7$ tet.

n	Signatures	Name	n	Signatures	Name
2	cPcbbbdxm	$otet02_{0000}$	6	gLLPQccdfeefqjsqqjj	$otet06_{0000}$
2	cPcbbbiht	$otet02_{0001}$	6	gLLPQccdfeffqjsqqsj	$otet06_{0001}$
4	eLMkbbdddemdxi	$otet04_{0000}$	6	gLLPQceefeffpupuupa	$otet06_{0002}$
4	eLMkbcddddedde	$otet04_{0001}$	6	gLMzQbcdefffhxqqxha	$otet06_{0003}$
4	eLMkbcdddhxqdu	$otet04_{0002}$	6	gLMzQbcdefffhxqqxxq	$otet06_{0004}$
4	eLMkbcdddhxqlm	$otet04_{0003}$	6	gLvQQadfedefjqqasjj	$otet06_{0005}$
5	fLLQcbcedeeloxset	$otet05_{0000}$	6	gLvQQbefeeffedimipt	$otet06_{0006}$
5	fLLQcbdeedemnamjp	$otet05_{0001}$	7	hLvAQkadfdgggfjxqnjnbw	$otet07_{0000}$

For any triangulation $\mathcal{T}$ with n tetrahedra we find isomorphism signature that is the lexicographically smallest $24n$–bit sequence presenting the triangulation. We use 64-version of the dehydration presentation of the sequence taking in account the correspondence between integers $0, 1, \ldots, 63$ and characters:

integer	0	$\cdots$	25	26	$\cdots$	51	52	$\cdots$	61	62	63
character	a	$\cdots$	z	A	$\cdots$	Z	0	$\cdots$	9	+	-

Commensurability and arithmeticity of tetrahedral manifolds

Two manifolds (or orbifolds) are commensurable if they have a common finite cover.

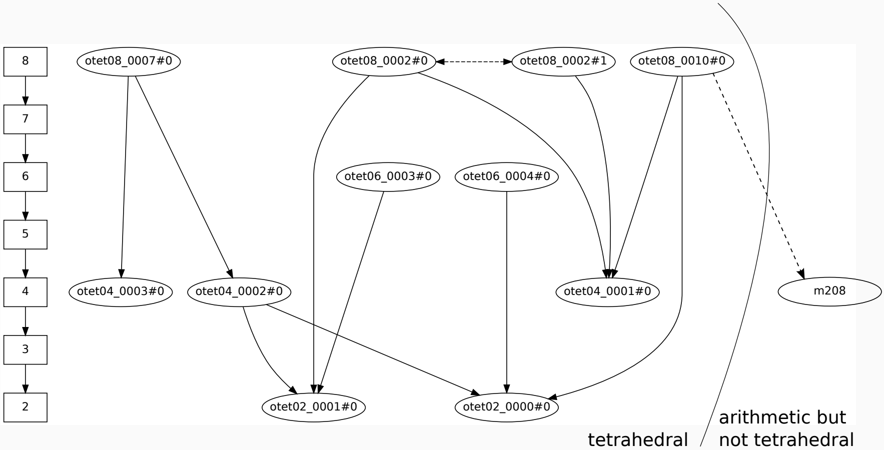
Lemma [due to C. Maclachlan and A. Reid]. For a cusped hyperbolic manifold M, the following are equivalent

- M is commensurable with the figure-eight knot complement.
- M is arithmetic with invariant trace field $\mathbb{Q}(\sqrt{-3})$.
- The invariant trace field of M is $\mathbb{Q}(\sqrt{-3})$ and M has integer traces.

Lemma. All tetrahedral manifolds are arithmetic with invariant trace field $\mathbb{Q}(\sqrt{-3})$ and commensurable to each other.

Remark. The converse to the Lemma doesn't hold. There are (at least 8) arithmetic manifolds commensurable to the figure-eight knot complement which are not tetrahedral.

Covering diagram



Tetrahedral links in S^3

Homological links

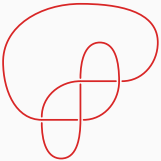
We say that M is a homology link complement if $H_1(M, \mathbb{Z}) = \mathbb{Z}^c$, where c is the number of cusps. This condition is equivalent to the fact that M is a complement of a link in an integer homology sphere.

Homological links up to 25 tet.

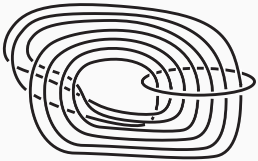
Tetrahedra	homology links
2	1 (K4a1)
4	2 (Berge link, L6a2)
8	5 (4 of them are L10n46, L14n38547, L14n24613, $\mathrm{otet8}_{0005}$)
10	12 (11 of them are L8a20, L10n88, L11n354, L8a21, L10n101, L12n2201, L10n113, L12n1739, $\mathrm{otet10}_{0007}$, $\mathrm{otet10}_{0003}$, $\mathrm{otet10}_{0025}$)
12	7
14	25
16	32
18	66
20	209
22	148
24	378

Conjecture. Every tetrahedral link complement has an even number of tetrahedra.

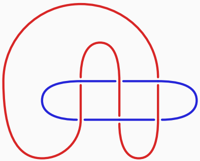
Tetrahedral links with 2, 4, and 8 tetrahedra



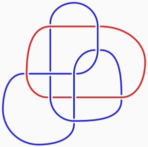
$\mathrm{otet02}_{0001}(K4a1)$



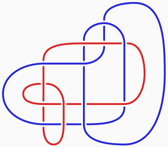
$\mathrm{otet04}_{0000}$



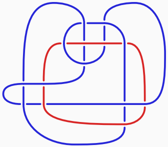
$\mathrm{otet04}_{0001}(L6a2)$



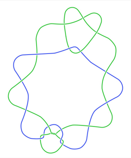
$\mathrm{otet08}_{0002}(L10n46)$



$\mathrm{otet08}_{0009}(L14n38547)$

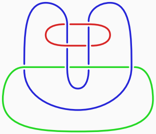


$\mathrm{otet08}_{0001}(L14n24613)$

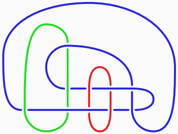


$\mathrm{otet08}_{0005}$

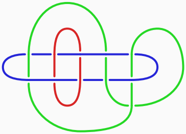
Tetrahedral links with 10 tetrahedra



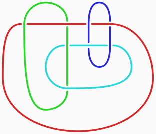
$\mathrm{otet}10_{0006}(L8a20)$



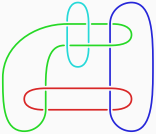
$\mathrm{otet}10_{0042}(L10n88)$



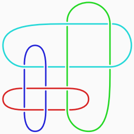
$\mathrm{otet}10_{0008}(L11n354)$



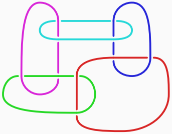
$\mathrm{otet}10_{0011}(L8a21)$



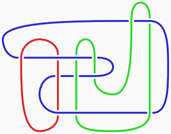
$\mathrm{otet}10_{0014}(L10n101)$



$\mathrm{otet}10_{0028}(L12n2201)$

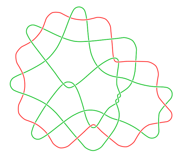


$\mathrm{otet}10_{0027}(L10n113)$

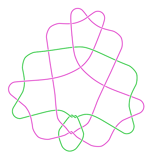


$\mathrm{otet}10_{0043}(L12n$

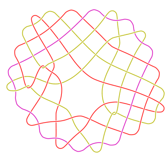
Tetrahedral links with 10 and 12 tetrahedra



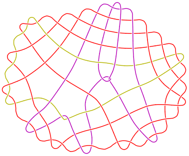
otet10_{0007}



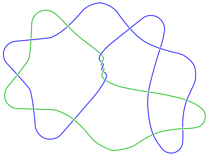
otet10_{0003}



otet10_{0025}

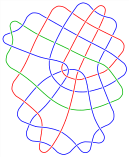


otet12_{0001}

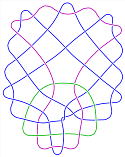


otet12_{0005}

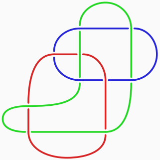
Tetrahedral links with 12 tetrahedra



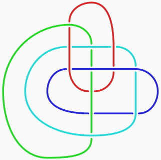
otet12_{0006}



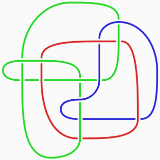
otet12_{0010}



otet12_{0007}($L10a157$)



otet12_{0009}($L12n2208$)



otet12_{0018}($L13n9382$)

Number of cusps and number of tetrahedra

C. Adams, W. Sherman [1991]: Denote by $\sigma(k)$ a minimum number of ideal tetrahedra which are necessary to construct an orientable hyperbolic 3-manifolds with k cusps.

It is known that $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 6$, $\sigma(4) = 8$, $\sigma(5) = 10$.

For $k \geqslant 2$ there is an upper bound $\sigma(k) \leqslant 4k - 4$.

Conjecture. $\sigma(6) = 16$, $\sigma(7) = 20$, $\sigma(9) = 30$.

Denote by $\sigma^r(k)$ the minimum number of regular ideal tetrahedra which are necessary to construct an orientable hyperbolic 3-manifolds with k cusps.

We have got: $\sigma^r(1) = 2$, $\sigma^r(2) = 4$, $\sigma^r(3) = 10$, $\sigma^r(4) = 10$, $\sigma^r(5) = 10$, $\sigma^r(6) = 16$, $\sigma^r(7) = 20$.

Volumes of tetrahedral manifolds and clasical arithmetic functions

Fundamental group of a knot complement

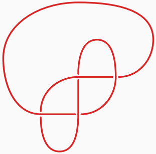
Since any tetrahedral manifold M is commensurable to the figure-eight knot complement, we have

$$\mathrm{vol}(M) = \frac{c(M)}{2}\,\mathrm{vol}(S^3 \setminus \mathcal{F}).$$

The group $\Gamma = \pi_1(S^3 \setminus \mathcal{F}) = \langle a, b \mid ab^{-1}aba^{-1} = b^{-1}aba^{-1}b \rangle$. has a faithful presentation in $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}^+\mathbb{H}^3$:

$$\theta(a) \mapsto \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \qquad \theta(b) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where $\omega = -1/2 + \sqrt{-3}/2$. Hence $\Gamma \subset \mathrm{PSL}(2, \mathbb{Q}(\sqrt{-3}))$.



Arithmetic groups

Let $F = \mathbb{Q}(\sqrt{-d})$ be the imaginary quadratic number field, $d \geqslant 1$, and $\mathcal{O}_d$ be the ring of integers of F.

$\mathrm{PSL}(2, \mathcal{O}_d)$ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, known as Bianchi group. Thus, $\mathbb{H}^3/\mathrm{PSL}(2, \mathcal{O}_d)$ is a hyperbolic 3-orbifold.

Let G be a torsion-free subgroup of $\mathrm{PSL}(2, \mathcal{O}_d)$. Then $\mathbb{H}^3/G$ is an orientable hyperbolic 3-manifold.

Theorem [Riley]. The group $\Gamma = \pi_1(S^3 \setminus \mathcal{F})$, where $\mathcal{F}$ is the figure-eight knot, is a subgroup of $\mathrm{PSL}(2, \mathcal{O}_3)$ of index 12.

Humbert volume formula

Dedekind zeta-function: for $\mathrm{Re}(s) > 1$,

$$\zeta_F(s) = \sum_{\mathcal{I} \subseteq \mathcal{O}_F} \frac{1}{(N(\mathcal{I}))^s},$$

where $\mathcal{O}_F$ is the ring of integers, the sum is taken over all nonzero prime ideals, and $N(\mathcal{I}) = [\mathcal{O}_F : \mathcal{I}]$ is the norm.

Theorem [Humbert, 1919].

$$\mathrm{vol}(\mathbb{H}^3/\mathrm{PSL}(2, \mathcal{O}_d)) = \frac{D_d \sqrt{D_d}}{4\pi^2} \zeta_{\mathbb{Q}(\sqrt{-d})}(2),$$

where $D_d = d$ for $d \equiv 3 \bmod 4$, and $D_d = 4d$ otherwise.

L-function

If $d = 3$ then $D_3 = 3$, and

$$\text{vol}(\mathbb{H}^3/\mathrm{PSL}(2, \mathcal{O}_3)) = \frac{3\sqrt{3}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-3})}(2).$$

Dirichlet L-function is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi(n)$ is a Dirichlet character. Consider a character χ_{-3} by modulo 3:

$$\chi_{-3}(n) = \begin{cases} 0, & n \equiv 0 \mod 3, \\ 1, & n \equiv 1 \mod 3, \\ -1, & n \equiv 2 \mod 3. \end{cases}$$

Volume and L-function

Thus,

$$L(s, \chi_{-3}) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \dots .$$

Recall that

$$\chi_{\mathbb{Q}(\sqrt{-3})}(s) = \zeta(s) L(s, \chi_{-3}).$$

Since $\zeta(2) = \pi^2/6$ we get

$$\mathrm{vol}(\mathbb{H}^3/\mathrm{PSL}(2, \mathcal{O}_3)) = \frac{3\sqrt{3}}{4\pi^2} \cdot \frac{\pi^2}{6} \cdot L(2, \chi_{-3}),$$

whence

$$\mathrm{vol}(S^3 \setminus \mathcal{F}) = 12 \cdot \frac{3\sqrt{3}}{4\pi^2} \cdot \frac{\pi^2}{6} \cdot L(2, \chi_{-3}) = \frac{3\sqrt{3}}{2} L(2, \chi_{-3}).$$

Mahler measure

For polynomial $P \in \mathbb{C}[x_1, x_2, \ldots, x_k]$ define Mahler measure:

$$M(P) = \exp\left\{\int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_k})| dt_1 \cdots dt_k\right\}$$

and logarithmic Mahler measure: $m(P) = \log M(P)$.

Assume $P(x) = a \prod_{i=1}^{n}(x - \alpha_k)$, then

$$M(P) = |a| \cdot \prod_{k=1}^{n} \max\{1, |\alpha_k|\}.$$

For example, $M(ax + b) = \max\{|a|, |b|\}$.

Lehmer conjecture

[Lehmer conjecture, 1933]: There exists $\lambda > 1$ such that: if $P \in \mathbb{Z}[x]$, monic, then

$$M(P) = 1 \qquad \text{or} \qquad M(P) \geq \lambda.$$

The best known $\lambda = 1.72623\ldots$ is the root of the Lehmer polynomial

$$P(x) = x^{10} - x^9 + x^7 - x^6 + x^5 - x^4 + x^3 - x + 1.$$

It is interesting that this polynomial is the Alexander polynomial of the pretzel $(-2, 3, 7)$-knot.

Mahler measure and L-function

Let $P \in \mathbb{C}[x,y]$. It is known that

$$M(1+x+y) = M(\max\{1, |1+x|\})$$
$$M(1+x+y-xy) = M(\max\{|1-x|, |1+x|\})$$

[Smyth, 1981]:

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_{-3}).$$

Thus, we obtained

$$\mathrm{vol}(S^3 \setminus \mathcal{F}) = 2\pi \cdot m(1+x+y).$$

Logarithmic Mahler measure and hyperbolic volume

Let $f_{a,b,c}(x,y) = a + bx + cy$. Consider an Euclidean triangle with side lengths a, b, c. Its angles are α, β, γ such that $\alpha + \beta + \gamma = \pi$. Let $T(\alpha, \beta, \gamma)$ be an ideal hyperbolic tetrahedron with such dihedral angles.

Theorem. [Maillot, 2000] Let $a, b, c \in \mathbb{C}$ be such that there exists an Euclidean triangle with lengths $|a|$, $|b|$, $|c|$, and let α, β, γ be its angles. Denote $z = |\frac{a}{b}|e^{i\gamma}$. Then

$$\pi m(a + bx + cy) = \mathrm{vol}(T_z) + \alpha \log|a| + \beta \log|b| + \gamma \log|c|.$$

If a triangle with such side lengths doesn't exist, then

$$m(a + bx + cy) = \log \max\{|a|, |b|, |c|\}.$$

Here T_z is an ideal tetrahedron with dihedral angles

$$\arg z, \quad \arg \frac{1}{1-z}, \quad \arg(1 - \frac{1}{z}).$$

References and used computer programs.

[1] Fominykh E., Garoufalidis S., Goerner M., Tarkaev V., Vesnin A. A census of tetrahedral hyperbolic manifolds. // Experimental Mathematics, 2016, 25(4), 466–481.

All files with programs and computation results are available at:

[2] supplementary files to arXiv:1502.00383

[3] http://unhyperbolic.org/tetrahedralCensus/

Used computer programs:

[4] SnapPy, a computer program for studying the topology of 3-manifolds. http://snappy.computop.org

[5] Regina, Software for 3-manifold topology and normal surface theory. http://regina.sourceforge.net

[6] HIKMOT, a Python module for verified computations for hyperbolic 3-manifolds. http://www.oishi.info.waseda.ac.jp/takayasu/hikmot

Thank you!