New Polynomial Invariants of Virtual Knots

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The talk is based on the following papers:

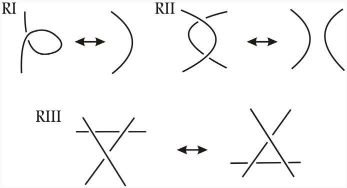
- K. Kaur, M. Prabhakar, A. Vesnin, Two-variable polynomial invariants of virtual knots arising from flat virtual knot invariants, Journal of Knot Theory and Its Ramifications, 2018, 27(13). Preprint version is available on arXiv:1803.05191.
- M. Ivanov, A. Vesnin, F-polynomials of tabulated virtual knots, Journal of Knot Theory and Its Ramifications, 2020. Preprint version is available on arXiv:1906.01976.

Content

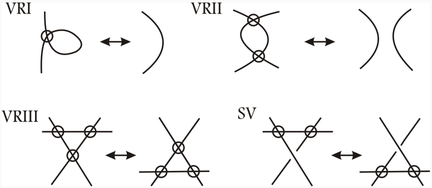
Basic definitions

Virtual knot diagrams

Virtual knots were introduced by Kauffman as a generalization of classical knots and presented by virtual knot diagrams having classical crossings as well as virtual crossings.



(a) Classical Reidemeister moves.



(b) Virtual Reidemeister moves.

Figure 1: Reidemeister moves.

Classical crossing sign

Let D be an oriented virtual knot diagram and $C(D)$ be the set of classical crossings in D. By an arc we mean an edge between two consecutive classical crossings along the orientation.

The sign of classical crossing $c \in C(D)$, denoted by $\text{sgn}(c)$, is defined as in Fig. 2.

$\text{sgn}(c) = +1$ $\quad\quad$ $\text{sgn}(c) = -1$

Figure 2: Crossing signs.

Labelling of arcs

Now assign an integer label to each arc in D in such a way that the labeling around each crossing point of D follows the rule as shown in Fig. 3.

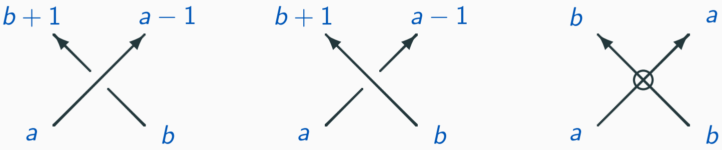


Figure 3: Labeling around crossing.

Such a labelling is also called Cheng coloring.

Kauffman proved that Cheng coloring always exists for an oriented virtual knot diagram.

Index value

Cheng and Gao assigned an index value $\mathrm{Ind}(c)$ to each classical crossing c of a virtual knot diagram as a product:

$$\mathrm{Ind}(c) = \mathrm{sgn}(c)(a - b - 1)$$

with a and b be labels as presented in Fig. 3.

The Kauffman's affine index polynomial is defined by the formula

$$P_D(t) = \sum_{c \in C(D)} \mathrm{sgn}(c)(t^{\mathrm{Ind}(c)} - 1).$$

n-dwrithe

Satoh and Taniguchi introduced for each $n \in \mathbb{Z} \setminus \{0\}$ the n-th writhe $J_n(D)$ of an oriented virtual link diagram D. It is defined as the number of positive sign crossings minus number of negative sign crossings of D with index value n.

Remark, that $J_n(D)$ is the coefficient of t^n in the Kauffman affine index polynomial.

Def. Let $n \in \mathbb{N}$ and D be an oriented virtual knot diagram. The n-th dwrithe of D, denoted by $\nabla J_n(D)$, is defined as

$$\nabla J_n(D) = J_n(D) - J_{-n}(D).$$

Properties of the n-dwrithe

Remark 1. The n-th dwrithe $\nabla J_n(D)$ is a virtual knot invariant, since n-th writhe $J_n(D)$ is an oriented virtual knot invariant.

Remark 2. The n-th dwrithe $\nabla J_n(D) = 0$ for any classical knot diagram.

Remark 3. Let D be a virtual knot diagram. Consider set of all affine index values of crossing points:

$$S(D) = \{|\mathrm{Ind}(c)| \, : \, c \in C(D)\} \subset \mathbb{N}\}.$$

Then $\nabla J_n(D) = 0$ for any $n \in \mathbb{N} \setminus S(D)$.

Flat virtual knots

A flat virtual knot diagram is a virtual knot diagram obtained by forgetting the over/under-information of every classical crossing.

A flat virtual knot is an equivalence class of flat virtual knot diagrams by flat Reidemeister moves which are Reidemeister moves without the over/under information.

We will say that a virtual knot invariant is a flat virtual knot invariant, if it is independent of crossing change operation.

Lemma 1. For any $n \in \mathbb{N}$, the n-th dwrithe $\nabla J_n(D)$ is a flat virtual knot invariant.

Reversing orientation and mirror image

Let D^- be the reverse of D, obtained from D by reversing the orientation and let D^* be the mirror image of D, obtained by switching all the classical crossings in D.

Lemma 2. If D is an oriented virtual knot diagram, then

$$\nabla J_n(D^*) = \nabla J_n(D)$$

and

$$\nabla J_n(D^-) = -\nabla J_n(D).$$

L-polynomials of a virtual knot diagram

Two types of smoothing

Let c be a classical crossing of an oriented virtual knot diagram D. There are two possibility to smooth in c.

One is to smooth along the orientation of arcs shown in Fig. 4.

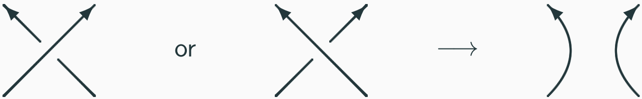


Figure 4: Smoothing along orientation.

Another is smoothing against the orientation of arcs shown in Fig. 5.

Figure 5: Smoothing against orientation.

Definition of L-polynomials

Let us denote by D_c the oriented diagram obtained from D by smoothing at c against orientation of arcs. The orientation of D_c is induced by the orientation of smoothing. Since D is a virtual knot diagram, D_c is also a virtual knot diagram.

Def. Let D be an oriented virtual knot diagram and n be any positive integer. Then n-th L-polynomial of D at n is defined as

$$L_D^n(t,\ell) = \sum_{c \in C(D)} \operatorname{sgn}(c)\left(t^{\operatorname{Ind}(c)} \ell^{|\nabla J_n(D_c)|} - \ell^{|\nabla J_n(D)|}\right).$$

Theorem 1. Let D be a diagram of an oriented virtual knot K. Then for any $n \in \mathbb{N}$, the polynomial $L_D^n(t,\ell)$ is an invariant of K.

To prove we consider the Reidemeister moves

Proof of Theorem 1. Move RI

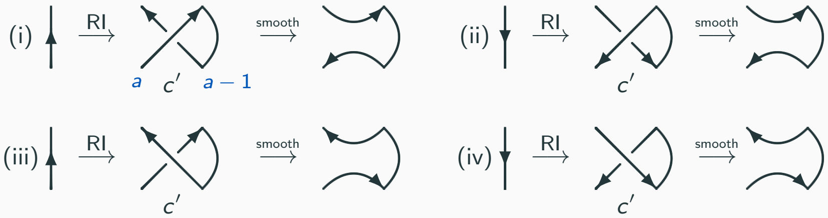


Figure 6: RI-move and smoothing against orientation.

Therefore,

$$\begin{aligned}
L^n_{D'}(t,\ell) &= \sum_{c\in C(D')} \operatorname{sgn}(c)\left(t^{\operatorname{Ind}(c)}\ell^{|\nabla J_n(D'_c)|} - \ell^{|\nabla J_n(D')|}\right)\\
&= \sum_{c\in C(D)} \operatorname{sgn}(c)\left(t^{\operatorname{Ind}(c)}\ell^{|\nabla J_n(D_c)|} - \ell^{|\nabla J_n(D)|}\right)\\
&\qquad + \operatorname{sgn}(c')\left(t^{\operatorname{Ind}(c')}\ell^{|\nabla J_n(D'_{c'})|} - \ell^{|\nabla J_n(D')|}\right)\\
&= L^n_D(t,\ell) + \operatorname{sgn}(c')\left(t^0\ell^{|\nabla J_n(D)|} - \ell^{|\nabla J_n(D)|}\right) = L^n_D(t,\ell).
\end{aligned}$$

Proof of Theorem 1. Move RII

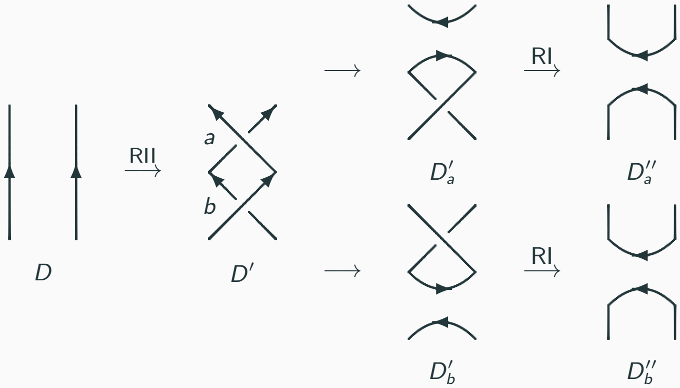


Figure 7: RII-move and smoothing against orientation. Case (1).

Proof of Theorem 1. Move RII

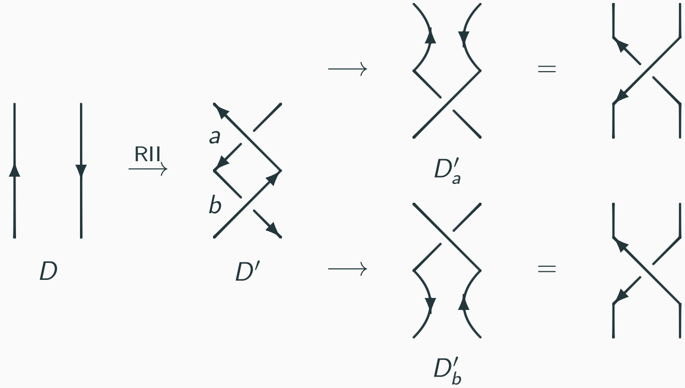


Figure 8: RII-move and smoothing against orientation. Case (2).

In both cases $\mathrm{sgn}(b) = \mathrm{sgn}(a)$ and $\mathrm{Ind}(b) = \mathrm{Ind}(a)$.

Therefore, in both cases we have

$$\begin{aligned} L^n_{D'}(t,\ell) &= L^n_D(t,\ell) + \mathrm{sgn}(a)\left(t^{\mathrm{Ind}(a)}\ell^{|\nabla J_n(D'_a)|} - \ell^{|\nabla J_n(D')|}\right) \\ &\quad + \mathrm{sgn}(b)\left(t^{\mathrm{Ind}(b)}\ell^{|\nabla J_n(D'_b)|} - \ell^{|\nabla J_n(D')|}\right) \\ &= L^n_D(t,\ell) + \mathrm{sgn}(a)\left(t^{\mathrm{Ind}(a)}\ell^{|\nabla J_n(D'_a)|} - \ell^{|\nabla J_n(D')|}\right) \\ &\quad - \mathrm{sgn}(a)\left(t^{\mathrm{Ind}(a)}\ell^{|\nabla J_n(D'_a)|} - \ell^{|\nabla J_n(D')|}\right) = L^n_D(t,\ell). \end{aligned}$$

Proof of Theorem 1. Move RIII

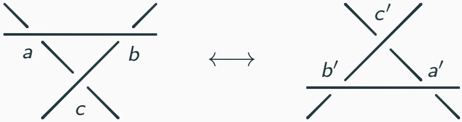


Figure 9: RIII–move.

Proof of Theorem 1. Move RIII

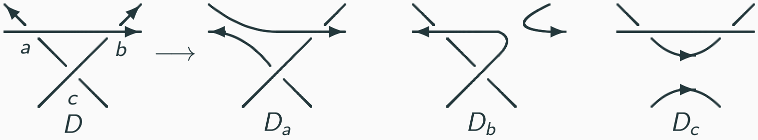


Figure 10: RIII move and orientation reversing smoothing. Case (1).

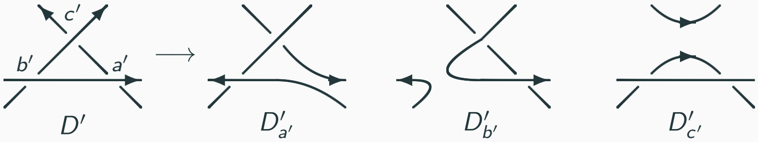


Figure 11: RIII–move and orientation revising smoothing. Case (2).

Proof of Theorem 1. Move SV

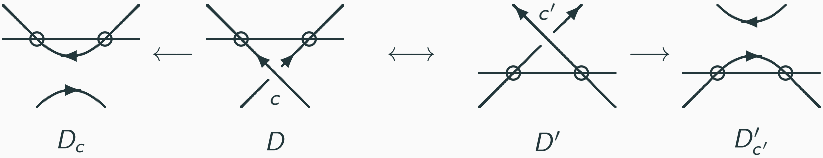


Figure 12: SV-move. Case (1).

Here $\operatorname{sgn}(c') = \operatorname{sgn}(c)$, $\operatorname{Ind}(c') = \operatorname{Ind}(c)$ and $\nabla J_n(D') = \nabla J_n(D)$. Moreover, diagrams $D'_{c'}$ and D_c are equivalent under VRII-moves. Thus, $\nabla J_n(D'_{c'}) = \nabla J_n(D_c)$.

Proof of Theorem 1. Move SV

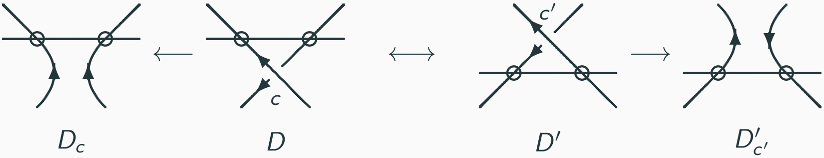


Figure 13: SV-move. Case (2).

We see that $\operatorname{sgn}(c') = \operatorname{sgn}(c)$ and $\operatorname{Ind}(c') = \operatorname{Ind}(c)$. Hence $\nabla J_n(D') = \nabla J_n(D)$. Moreover, diagrams $D'_{c'}$ and D_c are equivalent. Thus, $\nabla J_n(D'_{c'}) = \nabla J_n(D_c)$.

Thus, in Case (1) as well as in Case (2) we get

$$\operatorname{sgn}(c')\left(t^{\operatorname{Ind}(c')}\ell^{|\nabla J_n(D'_{c'})|}-\ell^{|\nabla J_n(D')|}\right)= \\ \operatorname{sgn}(c)\left(t^{\operatorname{Ind}(c)}\ell^{|\nabla J_n(D_c)|}-\ell^{|\nabla J_n(D)|}\right).$$

Therefore, $L^n_{D'}(t,\ell)=L^n_D(t,\ell)$.

Consideration of all cases for moves RI, RII, RIII, and SV give that $L^n_D(t,\ell)$ is a virtual knot invariant.

Example 1(a)

Consider an oriented virtual knot diagram D presented in Fig. 14.

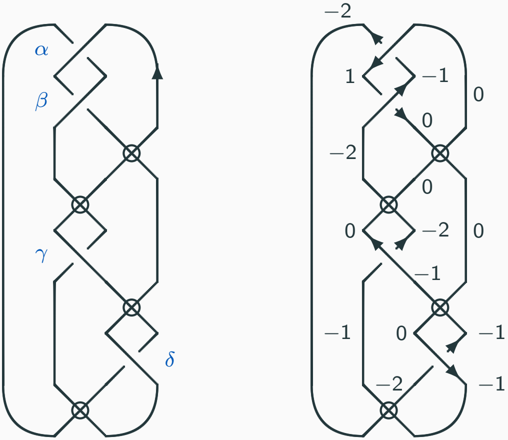


Figure 14: Oriented virtual diagram D and its labelling.

Example 1(b)

Then $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta) = \mathrm{sgn}(\gamma) = -1$ and $\mathrm{sgn}(\delta) = 1$
and index values are

$$\mathrm{Ind}(\alpha) = 2 = (-1)(-1-0-1),\ \mathrm{Ind}(\beta) = -2,\ \mathrm{Ind}(\gamma) = 1,\ \mathrm{Ind}(\delta) = 1$$

by the formula

$$\mathrm{Ind}(c) = \mathrm{sgn}(c)(a - b - 1).$$

Therefore, only writhe numbers $J_1(D)$, $J_2(D)$, and $J_{-2}(D)$ can be non-zero.

We see, that $J_1(D) = \mathrm{sgn}(\gamma) + \mathrm{sgn}(\delta) = 0$, $J_2(D) = \mathrm{sgn}(\alpha) = -1$, and $J_{-2}(D) = \mathrm{sgn}(\beta) = -1$. Then $\nabla J_1(D) = J_1(D) - J_{-1}(D) = 0$ and $\nabla J_2(D) = J_2(D) - J_{-2}(D) = 0$. For any $n \geq 3$ we have $\nabla J_n(D) = 0$.

Example 1(c)

We consider against orientation smoothings at classical crossing points α, β, γ, and δ of D. Results are D_α, D_β, D_γ, and D_δ.

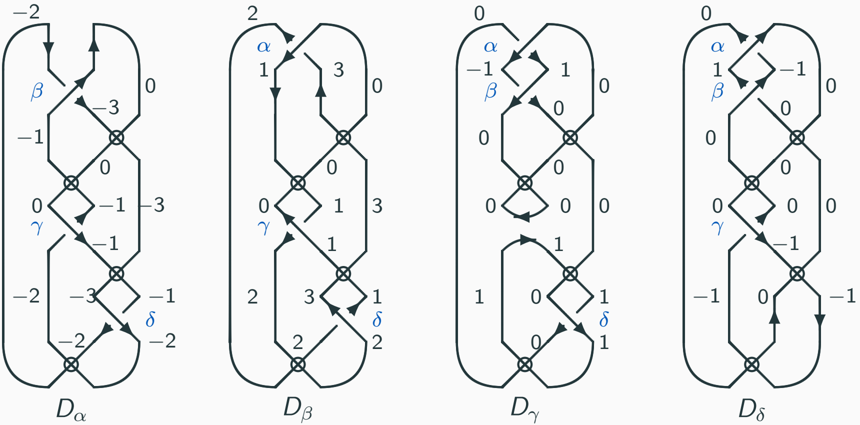


Figure 15: Oriented virtual knot diagrams D_α, D_β, D_γ, and D_δ.

Example 1(d)

Table 1: Values of sign, Ind, and dwrithe for above diagrams.

	Sign	index value	dwrihe
D_α	$\text{sgn}(\beta) = -1$	$\text{Ind}(\beta) = 2$	$\nabla J_1(D_\alpha) = \text{sgn}(\gamma) - \text{sgn}(\delta) = 2$
	$\text{sgn}(\gamma) = 1$	$\text{Ind}(\gamma) = 1$	$\nabla J_2(D_\alpha) = \text{sgn}(\beta) = -1$
	$\text{sgn}(\delta) = -1$	$\text{Ind}(\delta) = -1$	
D_β	$\text{sgn}(\alpha) = -1$	$\text{Ind}(\alpha) = -2$	$\nabla J_1(D_\beta) = \text{sgn}(\delta) - \text{sgn}(\gamma) = -2$
	$\text{sgn}(\gamma) = 1$	$\text{Ind}(\gamma) = -1$	$\nabla J_2(D_\beta) = -\text{sgn}(\alpha) = 1$
	$\text{sgn}(\delta) = -1$	$\text{Ind}(\delta) = 1$	
D_γ	$\text{sgn}(\alpha) = 1$	$\text{Ind}(\alpha) = -1$	$\nabla J_1(D_\gamma) = \text{sgn}(\beta) - \text{sgn}(\alpha) = 0$
	$\text{sgn}(\beta) = 1$	$\text{Ind}(\beta) = 1$	$\nabla J_2(D_\gamma) = 0$
	$\text{sgn}(\delta) = -1$	$\text{Ind}(\delta) = 0$	
D_δ	$\text{sgn}(\alpha) = 1$	$\text{Ind}(\alpha) = 1$	$\nabla J_1(D_\delta) = \text{sgn}(\alpha) - \text{sgn}(\beta) = 0$
	$\text{sgn}(\beta) = 1$	$\text{Ind}(\beta) = -1$	$\nabla J_2(D_\delta) = 0$
	$\text{sgn}(\gamma) = 1$	$\text{Ind}(\gamma) = 0$	

Example 1(e)

Basing on these calculations we obtain L-polynomials for diagram D.

For $n = 1$ and $n = 2$ we get

$$L_D^1(t, \ell) = 2 - t^2\ell^2 - t^{-2}\ell^2,$$

$$L_D^2(t, \ell) = 2 - t^2\ell - t^{-2}\ell.$$

For all $n \geq 3$ L-polynomials coincide with the affine polynomial:

$$L_D^n(t, \ell) = P_D(t) = 2 - t^2 - t^{-2}.$$

L-polynomials and Kauffman affine index polynomial (a)

Let us denote by $|C(D)|$ the cardinality of the set of classical crossings of D. Then by Kauffman's result for any arc of D the absolute value of its label is at most $|C(D)|$. Remark that $|C(D_c)| = |C(D)| - 1$ for any $c \in C(D)$. Hence, for any crossing c' in D or D_c the inequality $|\mathrm{Ind}(c')| \leq 2|C(D)| + 1$ holds.

Thus for any $n > 2|C(D)| + 1$ we have $\nabla J_n(D_c) = \nabla J_n(D) = 0$ and $L_D^n(t, \ell) = P_D(t)$.

Remark 4. The L-polynomials and the Kauffman affine index polynomial coincide on classical knots.

Remark 5. For $\ell = 1$ we get $L_K^n(t, 1) = P_K(t)$. Thus the affine index polynomial becomes a special case of the L-polynomial.

L-polynomials and Kauffman affine index polynomial (b)

Example 2. Consider virtual knot K and its mirror image K^* for which Kauffman affine index polynomial is trivial.

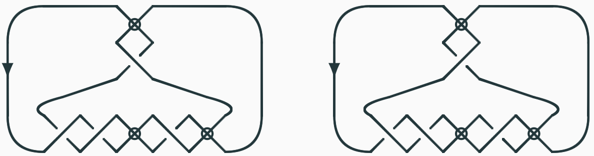


Figure 16: Oriented virtual knots K (on left) and K^* (on right).

$$L_K^1(t,\ell) = t^{-1}\ell^2 + t\ell^2 - t^{-1} - t, \quad L_K^2(t,\ell) = t^{-1}\ell + t\ell - t^{-1} - t,$$

and $L_{K^*}^1(t,\ell) = -L_K^1(t,\ell)$, $L_{K^*}^2(t,\ell) = -L_K^2(t,\ell)$. Therefore, knots K and K^* are both non-trivial and $K^* \neq K$.

Reversing orientation and mirror image

The following result describes behaviour of L-polynomial under reflection of a diagram and under changing its orientation.

Lemma 3. Let D be an oriented virtual knot diagram. Denote by D^* its mirror image and by D^- its reverse. Then for any $n \in \mathbb{N}$ we have

$$L^n_{D^*}(t,\ell) = -L^n_D(t^{-1},\ell) \qquad \text{and} \qquad L^n_{D^-}(t,\ell) = L^n_D(t^{-1},\ell).$$

Application 1: Cosmetic crossing change conjecture

The case of classical knots

A crossing in a knot diagram is said to be nugatory if it can be removed by twisting part of the knot, see Fig. 17. An example of a nugatory crossing is one that can be undone with an RI-move. Obviously, under applying a crossing change operation at a nugatory crossing we will get a diagram equivalent to the original diagram.

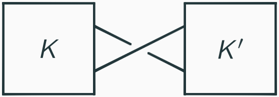


Figure 17: Nugatory crossing.

A Kirby's list problem on classical knots

Def. A crossing change in knot diagram D is said to be cosmetic if the new diagram, say D', is equivalent to D.

A crossing change on nugatory crossing is called trivial cosmetic crossing change. The following question from the Kirby's list of problems is still open.

Problem. Do non-trivial cosmetic crossing changes exit?

This question is often referred to as the cosmetic crossing change conjecture or the nugatory crossing conjecture. It has been answered in the negative for many classes of classical knots.

A problem on virtual knots

For virtual knots similar question also has been answered in the negative for a wide class of knot.

Folwaczny and Kauffman proved that a crossing c in D with $\text{Ind}(c) \neq 0$ is not cosmetic.

The following result gives one more condition on a crossing to say that the crossing is not a cosmetic.

Theorem 2. Let D be a virtual knot diagram and c be a crossing. If $\text{Ind}(c) \neq 0$ or there exists n such that $\nabla J_n(D_c) \neq \pm\nabla J_n(D)$, then c is not a cosmetic crossing.

F-polynomials of a virtual knot diagram

Definition of F-polynomials

We observed that in some cases L-polynomials fail to distinguish given two virtual knots (see Example 3 below). To resolve this problem we modify L-polynomials as follows.

Def. Let D be an oriented virtual knot diagram and n be a positive integer. Then n-th F-polynomial of D is defined as

$$F_D^n(t,\ell) = \sum_{c\in C(D)} \mathrm{sgn}(c) t^{\mathrm{Ind}(c)} \ell^{\nabla J_n(D_c)} - \sum_{c\in T_n(D)} \mathrm{sgn}(c)\ell^{\nabla J_n(D_c)} - \sum_{c\notin T_n(D)} \mathrm{sgn}(c)\ell^{\nabla J_n(D)},$$

where $T_n(D) = \{c \in C(D) \mid \nabla J_n(D_c) = \pm \nabla J_n(D)\}$.

F-polynomials are invariants of virtual knots

Theorem 3. For any $n \in \mathbb{N}$, the polynomial $F_K^n(t, \ell)$ is a virtual knot invariant.

Let us compare L-polynomials and F-polynomials.

Lemma 4. If two oriented virtual knots are distinguished by L-polynomials, then they are distinguished by F-polynomials too. The converse property doesn't hold.

We will give an example of two knots which demonstrate that the converse property doesn't hold.

Example 3

Consider oriented virtual knots K and K' depicted in Fig. 18.

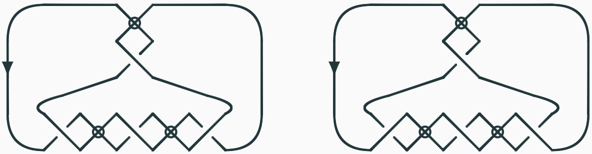


Figure 18: Oriented virtual knots K (on left) and K' (on right).

Virtual knot K	Virtual knot K'
L-polynomials	
$L^1_K(t,\ell) = L^1_{K'}(t,\ell) = t^{-1}\ell^2 - t\ell^2 - t - t^{-3} + 2\ell^3$	
$L^2_K(t,\ell) = L^2_{K'}(t,\ell) = t^{-1}\ell - t\ell - t - t^{-3} + 2$	
$L^3_K(t,\ell) = L^3_{K'}(t,\ell) = t^{-1} - 2t - t^{-3} + 2\ell$	
F-polynomials	
$F^1_K(t,\ell) = t^{-1}\ell^{-2} - t\ell^{-2} - t - t^{-3} + 2\ell^{-3}$	$F^1_{K'}(t,\ell) = t^{-1}\ell^2 - t\ell^2 - t - t^{-3} + 2\ell^{-3}$
$F^2_K(t,\ell) = t^{-1}\ell - t\ell - t - t^{-3} + 2$	$F^2_{K'}(t,\ell) = t^{-1}\ell^{-1} - t\ell^{-1} - t - t^{-3} + 2$
$F^3_K(t,\ell) = F^3_{K'}(t,\ell) = t^{-1} - 2t - t^{-3} + 2\ell$	

Application 2: Mutation by positive involution

Conway mutation

One of useful local transformation of a knot, producing new knot, is a mutation introduced by Conway. The mutation is positive if the orientation of the arcs of the tangle doesn't change under mutation. A positive involution is a positive mutation as in Fig. 19.

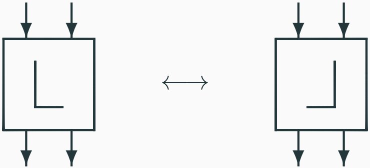


Figure 19: Positive involution mutation.

Conway mutation and F-polynomials

It is known that the Kauffman affine index polynomial fails to identify mutation by positive involution.

Lemma 5. F-polynomials distinguish an infinite number of pairs of virtual knots and their positive reflection mutants.

Example 4

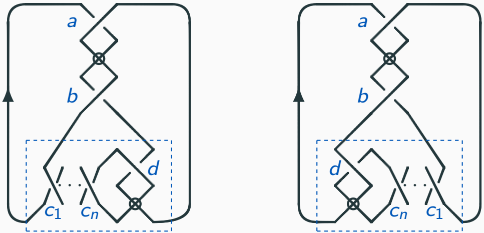


Figure 20: Virtual knot K_n (left) and its mutant MK_n (right).

for n odd:
$F^1_{K_n}(t,\ell) = t^2 + 1 + t^{-1}\ell^2 - \frac{n+1}{2}t - \frac{n-1}{2}t^{-1} - (2-n)\ell^{-2} - \ell^2$ $F^2_{K_n}(t,\ell) = t^2 + 1 + t^{-1}\ell^{-1} - \frac{n+1}{2}t - \frac{n-1}{2}t^{-1} - (2-n)\ell - \ell^{-1}$
$F^1_{MK_n}(t,\ell) = t^2 + 1 + t^{-1}\ell^{-2} - \frac{n+1}{2}t - \frac{n-1}{2}t^{-1} - (3-n)\ell^{-2}$ $F^2_{MK_n}(t,\ell) = t^2 + 1 + t^{-1}\ell - \frac{n+1}{2}t - \frac{n-1}{2}t^{-1} - (3-n)\ell$
for n even:
$F^1_{K_n}(t,\ell) = n - \frac{n}{2}(t^{-1}\ell^2 + t\ell^{-2})$ $F^2_{K_n}(t,\ell) = n - \frac{n}{2}(t^{-1}\ell^{-1} + t\ell)$
$F^1_{MK_n}(t,\ell) = n - \frac{n}{2}(t^{-1}\ell^{-2} + t\ell^2)$ $F^2_{MK_n}(t,\ell) = n - \frac{n}{2}(t^{-1}\ell + t\ell^{-1})$

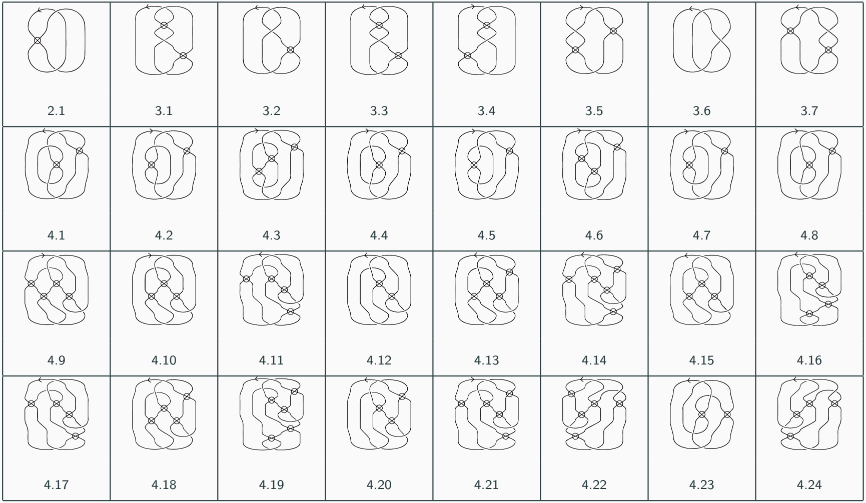
Tabulation of F-polynomials

Enumerating of virtual knots

Virtual knots up to 4 classical crossings were enumerated by Bar-Natan and Green (http://www.math.toronto.edu/drorbn).

8 non-trivial virtual knots up to 3 crossings.

108 virtual knots with 4 crossings.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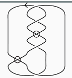
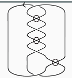
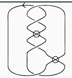
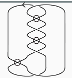
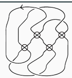
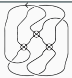
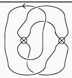
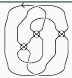
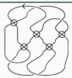
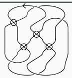
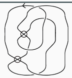
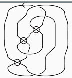
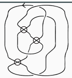
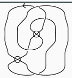
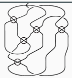
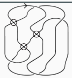
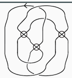
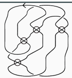
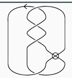
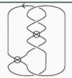
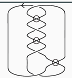
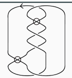
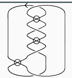
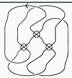
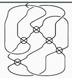
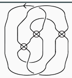
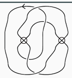
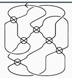
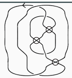
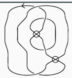
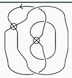
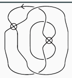
 4.25	 4.26	 4.27	 4.28	 4.29	 4.30	 4.31	 4.32
 4.33	 4.34	 4.35	 4.36	 4.37	 4.38	 4.39	 4.40
 4.41	 4.42	 4.43	 4.44	 4.45	 4.46	 4.47	 4.48
 4.49	 4.50	 4.51	 4.52	 4.53	 4.54	 4.55	 4.56

Table of virtual knots (c)

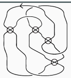
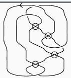
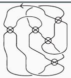
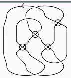
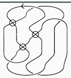
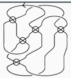
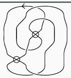
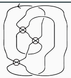
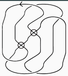
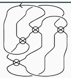
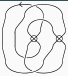
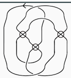
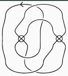
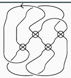
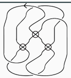
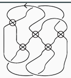
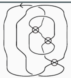
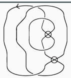
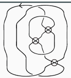
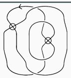
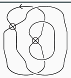
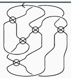
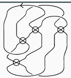
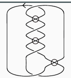
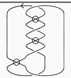
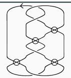
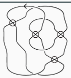
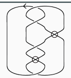
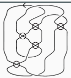
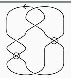
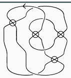
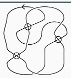
 4.57	 4.58	 4.59	 4.60	 4.61	 4.62	 4.63	 4.64
 4.65	 4.66	 4.67	 4.68	 4.69	 4.70	 4.71	 4.72
 4.73	 4.74	 4.75	 4.76	 4.77	 4.78	 4.79	 4.80
 4.81	 4.82	 4.83	 4.84	 4.85	 4.86	 4.87	 4.88

Table of virtual knots (d)

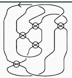
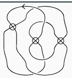
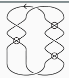
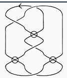
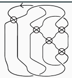
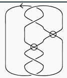
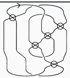
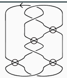
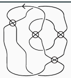
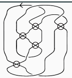
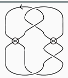
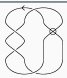
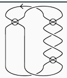
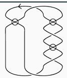
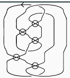
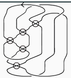
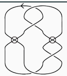
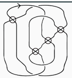
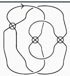
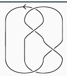
 4.89	 4.90	 4.91	 4.92	 4.93	 4.94	 4.95	 4.96
 4.97	 4.98	 4.99	 4.100	 4.101	 4.102.	 4.103	 4.104
 4.105	 4.106	 4.107	 4.108				

Table of F-polynomials, part 1

name of the virtual knot	n	$F^n(t,\ell)$-polynomials
2.1, **3.2**, **4.4**, **4.5**, **4.30**, **4.40**, **4.54**, **4.61**, **4.69**, **4.74**, **4.94**	1	$-t^{-1}+2-t$
3.1	1	$-t^{-2}+t^{-1}+\ell^{-2}-t$
	2	$-t^{-2}+t^{-1}+\ell-t$
	3	$-t^{-2}+t^{-1}+1-t$
3.3	1	$-t^{-2}+3\ell^{-2}-2t$
	2	$-t^{-2}+3\ell-2t$
	3	$-t^{-2}+3-2t$
3.4	1	$\ell^{-2}-2t+t^2$
	2	$\ell-2t+t^2$
	3	$1-2t+t^2$
3.5, **3.7**, **4.65**, **4.85**, **4.86**. **4.106**	1	$-t^{-2}+2-t^2$
3.6, **4.2**, **4.6**, **4.8**, **4.12**, **4.41**, **4.55**, **4.56**, **4.58**, **4.59**, **4.68**, **4.71**, **4.72**, **4.75**, **4.76**, **4.77**, **4.90**, **4.98**, **4.99**, **4.105**, **4.107**. **4.108**	1	0
4.1, **4.3**, **4.7**, **4.43**, **4.53**, **4.73**, **4.100**	1	$-2t^{-1}+4-2t$
4.9, **4.27**, **4.33**, **4.44**	1	$-t^{-1}\ell^2+2-t\ell^{-2}$
	2	$-t^{-1}\ell^{-1}+2-t\ell$
	3	$-t^{-1}+2-t$
4.10	1	$-t^{-1}-\ell^{-2}-1+3\ell^2+t\ell^{-2}-t^2$
	2	$-t^{-1}+3\ell^{-1}-1-\ell+tl-t^2$
	3	$-t^{-1}+1+t-t^2$
4.11	1	$-t^{-2}\ell^2+2\ell^{-2}+\ell^2-t\ell^{-2}-t$
	2	$-t^{-2}\ell^{-1}+\ell^{-1}+2\ell-t-t\ell$
	3	$-t^{-2}+3-2t$

Table of F-polynomials, part 2

4.13	1	$-t^{-1}\ell^{-2} + t^{-1}\ell^{2}$
	2	$t^{-1}\ell^{-1} - t^{-1}\ell$
	3	0
4.14	1	$t^{-2}\ell^{2} - t^{-1} - t + t^{2}\ell^{-2}$
	2	$t^{-2}\ell^{-1} - t^{-1} - t + t^{2}\ell$
	3	$t^{-2} - t^{-1} - t + t^{2}$
4.15	1	$-t^{-2} + 3\ell^{-2} - 1 + \ell^{2} - t - t\ell^{2}$
	2	$-t^{-2} + \ell^{-1} - 1 + 3\ell - t\ell^{-1} - t$
	3	$-t^{-2} + 3 - 2t$
4.16	1	$-\ell^{-2} + 2 - \ell^{2}$
	2	$-\ell^{-1} + 2 - \ell$
	3	0
4.17, **4.57**	1	$-t^{-2}\ell^{2} + t^{-1} + \ell^{2} - t\ell^{-2}$
	2	$-t^{-2}\ell^{-1} + t^{-1} + \ell^{-1} - t\ell$
	3	$-t^{-2} + t^{-1} + 1 - t$
4.18, **4.60**	1	$-t^{-1}\ell^{-2} + 2 - t\ell^{-2}$
	2	$-t^{-1}\ell + 2 - t\ell$
	3	$-t^{-1} + 2 - t$
4.19	1	$-t^{-1} + \ell^{2} + t\ell^{-2} - t^{2}\ell^{-2}$
	2	$-t^{-1} + \ell^{-1} + t\ell - t^{2}\ell$
	3	$-t^{-1} + 1 + t - t^{2}$
4.20	1	$t^{-2} - t^{-1} - t^{-1}\ell^{2} + 1$
	2	$t^{-2} - t^{-1}\ell^{-1} - t^{-1} + 1$
	3	$t^{-2} - 2t^{-1} + 1$
4.21	1	$t^{-2}\ell^{-2} - 2 + t^{2}\ell^{2}$
	2	$t^{-2}\ell - 2 + t^{2}\ell^{-1}$
	3	$t^{-2} - 2 + t^{2}$

Table of F-polynomials, part 3

4.22	1	$-t^{-1}+1+t^2-t^3\ell^{-2}$
	2	$-t^{-1}+1+t^2-t^3\ell$
	3	$-t^{-1}+1+t^2-t^3$
4.23	1	$-t^{-1}-1+2\ell^2+t\ell^2-t^2$
	2	$-t^{-1}+2\ell^{-1}-1+t\ell^{-1}-t^2$
	3	$-t^{-1}+1+t-t^2$
4.24	1	$t^{-2}-2\ell^{-1}+1-t+t^3\ell^{-2}$
	2	$t^{-2}-\ell^{-1}+1-\ell-t+t^3\ell$
	3	$t^{-2}+1-2\ell-t+t^3$
	4	$t^{-2}-1-t+t^3$
4.25	1	$-t^{-1}-t^{-1}\ell^2+4-t\ell^{-2}-t$
	2	$-t^{-1}\ell^{-1}-t^{-1}+4-t-t\ell$
	3	$-2t^{-1}+4-2t$
4.26	1	$-t^{-3}+t^{-1}+t^{-1}\ell^2-t\ell^2$
	2	$-t^{-3}+t^{-1}\ell^{-1}+t^{-1}-t\ell^{-1}$
	3	$-t^{-3}+2t^{-1}-t$
4.28	1	$t^{-3}-t^{-1}\ell^{-2}-2\ell^3+t\ell^{-2}+t$
	2	$t^{-3}-t^{-1}\ell-2+t+t\ell$
	3	$t^{-3}-t^{-1}-2l^{-1}+2t$
	4	$t^{-3}-t^{-1}-2+2t$
4.29	1	$-t^{-1}\ell^{-2}-t^{-1}+\ell^{-2}+2\ell^2-t^2\ell^2$
	2	$-t^{-1}-t^{-1}\ell+2l^{-1}+l-t^2\ell^{-1}$
	3	$-2t^{-1}+3-t^2$
4.31, **4.51**	1	$t\ell^{-2}-t\ell^2$
	2	$-t\ell^{-1}+t\ell$
	3	0

Table of F-polynomials, part 4

4.32	1	$-t^{-2}+t^{-1}+1-t\ell^{-2}$
	2	$-t^{-2}+t^{-1}+1-t\ell$
	3	$-t^{-2}+t^{-1}+1-t$
4.34	1	$\ell^{-2}-t-t\ell^{2}+t^{2}\ell^{2}$
	2	$\ell-t\ell^{-1}-t+t^{2}\ell^{-1}$
	3	$1-2t+t^{2}$
4.35	1	$-t^{-1}+1+t\ell^{2}-t^{2}$
	2	$-t^{-1}+1+t\ell^{-1}-t^{2}$
	3	$-t^{-1}+1+t-t^{2}$
4.36	1	$t^{-2}-2+t^{2}$
4.37	1	$-t^{-2}-t^{-1}+4-t-t^{2}$
4.38	1	$\ell^{-2}-1+\ell^{2}-t-t\ell^{2}+t^{2}$
	2	$\ell^{-1}-1+\ell-t\ell^{-1}-t+t^{2}$
	3	$1-2t+t^{2}$
4.39	1	$-t^{3}\ell^{-2}-t^{-1}-1+t^{2}+2\ell$
	2	$-t^{-1}-1+t^{2}+2\ell-t^{3}\ell$
	3	$2\ell^{-1}-t^{-1}-1+t^{2}-t^{3}$
	4	$-t^{-1}+1+t^{2}-t^{3}$
4.42	1	$1-t-t^{2}+t^{3}\ell^{-2}$
	2	$\ell^{-1}+1-\ell-t-t^{2}+t^{3}\ell$
	3	$1-t-t^{2}+t^{3}$
4.45	1	$-t^{-3}+t^{-1}+2\ell^{-3}-t\ell^{-2}-t\ell^{2}$
	2	$-t^{-3}+t^{-1}+2-t\ell^{-1}-t\ell$
	3	$-t^{-3}+t^{-1}+2\ell-2t$
	4	$-t^{-3}+t^{-1}+2-2t$

Table of F-polynomials, part 5

4.46	1	$-t^{-1}+t^{-1}\ell^2+t\ell^{-2}-t$
	2	$t^{-1}\ell^{-1}-t^{-1}-t+t\ell$
	3	0
4.47	1	$t^{-3}-t^{-1}\ell^{-2}-t^{-1}\ell^2+t$
	2	$t^{-3}-t^{-1}\ell^{-1}-t^{-1}\ell+t$
	3	$t^{-3}-2t^{-1}+t$
4.48, **4.82**	1	$-t^{-2}\ell^{-2}-t^{-1}+4-t-t^2\ell^2$
	2	$-t^{-2}\ell-t^{-1}+4-t-t^2\ell^{-1}$
	3	$-t^{-2}-t^{-1}+4-t-t^2$
4.49	1	$t^{-2}\ell^{-2}-t^{-1}\ell^{-2}-t^{-1}+\ell^2$
	2	$t^{-2}\ell-t^{-1}-t^{-1}\ell+\ell^{-1}$
	3	$t^{-2}-2t^{-1}+1$
4.50	1	$-t^{-2}+t^{-1}+2\ell^{-2}-1-t\ell^{-2}$
	2	$-t^{-2}+t^{-1}-1+2\ell-t\ell$
	3	$-t^{-2}+t^{-1}+1-t$
4.52	1	$-t^{-1}\ell^2+\ell^{-2}+\ell^2-t\ell^{-2}$
	2	$-t^{-1}\ell^{-1}+\ell^{-1}+\ell-t\ell$
	3	$-t^{-1}+2-t$
4.62	1	$-t^{-2}-t^{-1}+1+2\ell-t^3\ell^{-2}$
	2	$-t^{-2}-t^{-1}+1+2\ell-t^3\ell$
	3	$-t^{-2}-t^{-1}+2\ell^{-1}+1-t^3$
	4	$-t^{-2}-t^{-1}+3-t^3$
4.63	1	$-t^{-2}+\ell^{-2}+1+\ell^2-t-t\ell^2$
	2	$-t^{-2}+\ell^{-1}+1+\ell-t\ell^{-1}-t$
	3	$-t^{-2}+3-2t$

Table of F-polynomials, part 5

4.64	1	$t^{-2} - t^{-1} - t + t^2$
4.66	1	$t^{-2} - 1 - t + t^3\ell^{-2}$
	2	$t^{-2} + \ell^{-1} - 1 - \ell - t + t^3\ell$
	3	$t^{-2} - 1 - t + t^3$
4.67	1	$t^{-2} - t^{-1} - 1 + t\ell^2$
	2	$t^{-2} - t^{-1} - 1 + t\ell^{-1}$
	3	$t^{-2} - t^{-1} - 1 + t$
4.70	1	$-t^{-1} + \ell^2 + t\ell^2 - t^2\ell^2$
	2	$-t^{-1} + \ell^{-1} + t\ell^{-1} - t^2\ell^{-1}$
	3	$-t^{-1} + 1 + t - t^2$
4.78	1	$-t^{-2} - t^{-1} - 1 + 4\ell - t^3\ell^{-2}$
	2	$-t^{-2} - t^{-1} - 1 + 4\ell - t^3\ell$
	3	$-t^{-2} - t^{-1} + 4\ell^{-1} - 1 - t^3$
	4	$-t^{-2} - t^{-1} + 3 - t^3$
4.79	1	$2\ell^{-1} - 1 - t - t^2 + t^3\ell^{-2}$
	2	$3\ell^{-1} - 1 - \ell - t - t^2 + t^3\ell$
	3	$-1 + 2\ell - t - t^2 + t^3$
	4	$1 - t - t^2 + t^3$
4.80	1	$-t^{-3} + 4\ell^{-3} - t\ell^{-2} - t - t\ell^2$
	2	$-t^{-3} + 4 - t\ell^{-1} - t - t\ell$
	3	$-t^{-3} + 4\ell - 3t$
	4	$-t^{-3} + 4 - 3t$
4.81	1	$t^{-3} - t^{-1}\ell^{-2} - t^{-1} - t^{-1}\ell^2 + 2\ell^3$
	2	$t^{-3} - t^{-1}\ell^{-1} - t^{-1} - t^{-1}\ell + 2$
	3	$t^{-3} - 3t^{-1} + 2\ell^{-1}$
	4	$t^{-3} - 3t^{-1} + 2$

Table of F-polynomials, part 6

4.83	1	$-t^{-3}+t^{-2}\ell^{-2}-t^{-1}+2\ell-t^{2}\ell^{-2}$
	2	$-t^{-3}+t^{-2}\ell-t^{-1}+2\ell^{-2}-t^{2}\ell$
	3	$-t^{-3}+t^{-2}-t^{-1}+2\ell-t^{2}$
	4	$-t^{-3}+t^{-2}-t^{-1}+2-t^{2}$
4.84	1	$-t^{-2}\ell^{-2}+t^{-1}+t-t^{2}\ell^{2}$
	2	$-t^{-2}l+t^{-1}+t-t^{2}\ell^{-1}$
	3	$-t^{-2}+t^{-1}+t-t^{2}$
4.87	1	$-t^{-3}-t^{-1}+4\ell-t^{2}\ell^{-2}-t^{2}\ell^{2}$
	2	$-t^{-3}-t^{-1}+4\ell^{-2}-t^{2}\ell^{-1}-t^{2}\ell$
	3	$-t^{-3}-t^{-1}+4\ell-2t^{2}$
	4	$-t^{-3}-t^{-1}+4-2t^{2}$
4.88	1	$t^{-3}-t^{-2}\ell^{-2}-t^{-2}\ell^{2}+t^{-1}$
	2	$t^{-3}-t^{-2}\ell^{-1}-t^{-2}\ell+t^{-1}$
	3	$t^{-3}-2t^{-2}+t^{-1}$
4.89	1	$-2t^{-2}+4-2t^{2}$
4.91	1	$-t^{-3}-t^{-1}+4-t-t^{3}$
4.92, **4.95**, **4.101**	1	$-t^{-3}+2-t^{3}$
4.93	1	$t^{-3}-t^{-2}\ell^{-2}-t^{-2}\ell^{2}+2\ell^{-1}-t$
	2	$t^{-3}-t^{-2}\ell^{-1}-t^{-2}\ell+2\ell^{2}-t$
	3	$t^{-3}-2t^{-2}+2\ell^{-1}-t$
	4	$t^{-3}-2t^{-2}+2-t$
4.96	1	$-t^{-2}\ell^{-2}+2-t^{2}\ell^{2}$
	2	$-t^{-2}\ell+2-t^{2}\ell^{-1}$
	3	$-t^{-2}+2-t^{2}$
4.97	1	$t^{-2}\ell^{-2}-t^{-1}-t^{2}\ell^{-2}+t^{3}$
	2	$t^{-2}\ell-t^{-1}-t^{2}\ell+t^{3}$
	3	$t^{-2}-t^{-1}-t^{2}+t^{3}$

Table of F-polynomials, part 7

4.102	1	$t^{-3} - t^{-1} - t + t^3$
4.103	1	$-t^{-2}\ell^{-2} - t^{-2}\ell^2 + t^{-1} + 2\ell^{-1} - t^3$
	2	$-t^{-2}\ell^{-1} - t^{-2}\ell + t^{-1} + 2\ell^2 - t^3$
	3	$-2t^{-2} + t^{-1} + 2\ell^{-1} - t^3$
	4	$-2t^{-2} + t^{-1} + 2 - t^3$
4.104	1	$t^{-3} - 2 + t^3$

Thank you!