Anderson localization on random regular graphs:
Toy-model of many body-localization

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Talk mainly based on:

K.S. Tikhonov, ADM, and M.A. Skvortsov, Phys Rev B 94, 220203 (2016)

K.S. Tikhonov and ADM, Phys Rev B 99, 024202 (2019)

K.S. Tikhonov and ADM, Phys Rev B 99, 214202 (2019)

K. Tikhonov (Moscow, Karlsruhe $\longrightarrow$ Paris)

M. Skvortsov (Moscow)

Briefly touched in the talk:

Finite Cayley tree, difference to RRG:

K.S. Tikhonov and ADM, Phys. Rev B 94, 184203 (2016)

M. Sonner, K.S. Tikhonov, and ADM, Phys. Rev. B 96, 214204 (2017)

MBL transition with long-range interaction:

K.S. Tikhonov and ADM, Phys. Rev. B 97, 214205 (2018)

MPS-TDVP study of the MBL transition:

E.V.H. Doggen, F. Schindler, K.S. Tikhonov, ADM, T. Neupert, D.G. Polyakov, I.V. Gornyi, Phys. Rev. B 98, 174202 (2018)

Ergodicity and MBL in excited states of many-body systems

Spatially extended systems with short-range interaction

Gornyi, Mirlin, Polyakov, PRL 95, 206603 (2005)
Basko, Aleiner, Altshuler, Ann Phys 321, 1126 (2006)
Oganesyan, Huse, PRB 75, 155111 (2007)

Quantum dots

Altshuler, Gefen, Kamenev, Levitov, PRL 78, 2803 (1997)
Mirlin, Fyodorov, PRB 56, 13393 (1997)
Jacquod, Shepelyansky, PRL 79, 1837 (1997)

Spatially extended systems with power-law interaction

Burin, arXiv:cond-mat/0611387; PRB 91, 094202 (2015)
Yao, Laumann, Gopalakrishnan, Knap, Müller, Demler, Lukin,
PRL 113, 243002 (2014)
Gutman, Protopopov, Burin, Gornyi, Santos, Mirlin, PRB 93, 245427 (2016)

and many further papers

$\longrightarrow$ **Revival of interest to localization on tree-like graphs**

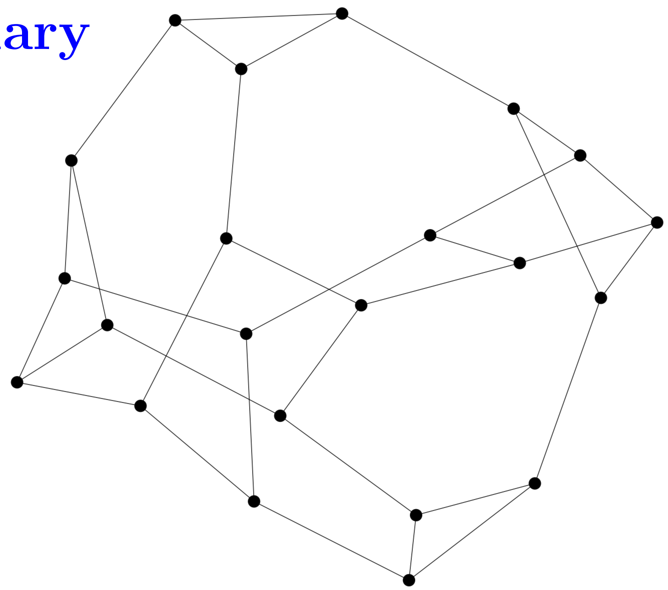
Properties of MBL transition, loc. and deloc. phases, critical regime – ?

One of important questions: Is the delocalized phase ergodic ?

Anderson localization on random regular graphs (RRG)

Random regular graph – random graph with constant connectivity

Locally tree-like (as Bethe lattice) but without boundary



Typical size of loops $\sim \ln N$

$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(c_i^+ c_j + c_j^+ c_i \right) + \sum_{i=1} \varepsilon_i c_i^+ c_i$$

$\varepsilon_i \longrightarrow$ disorder W

Relation to the MBL problem:

Hilbert space size $N \sim m^L$ where L is "linear size"

Sites $\longleftrightarrow$ many-body basis states, links $\longleftrightarrow$ interaction matrix elements

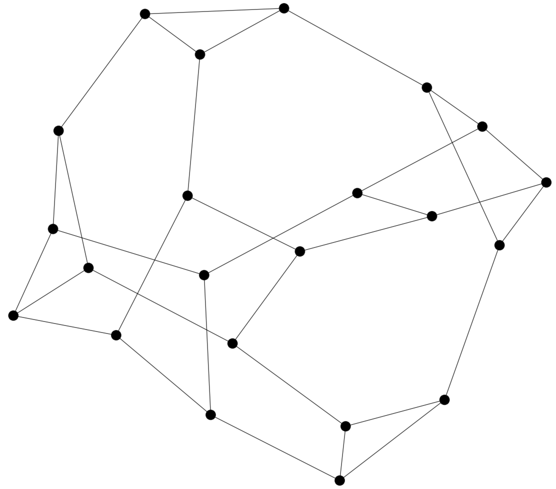
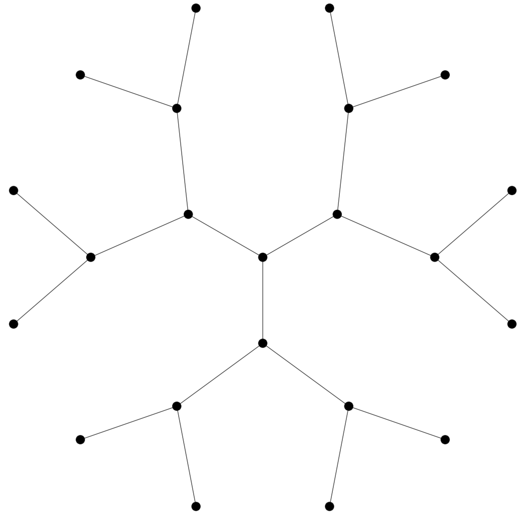
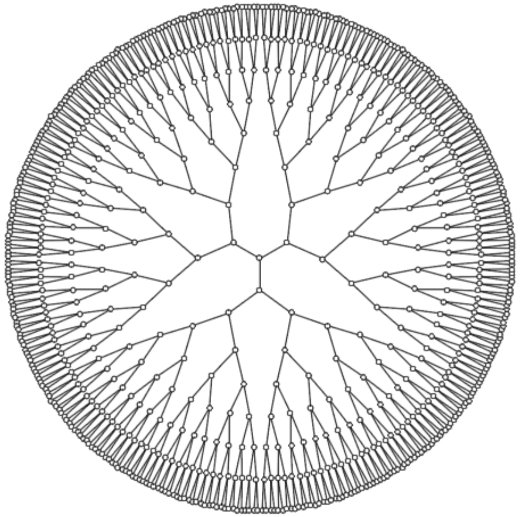
ADM, Fyodorov '91 Supersymmetry theory of Anderson transition in sparse random matrix model ($\sim$ RRG with fluctuating connectivity)

Delocalized phase ($W < W_c$): "ergodicity":

- Wigner-Dyson level statistics
- Wave function statistics: Inverse participation ratio (IPR) $P_2 = \langle \sum_i |\psi(i)^4| \rangle$

$$P_2 \simeq N_\xi(W)/N \ , \quad \ln N_\xi \propto (W_c - W)^{-1/2} \ , \quad N \gg N_\xi$$

RRG vs finite Bethe lattice vs infinite Bethe lattice

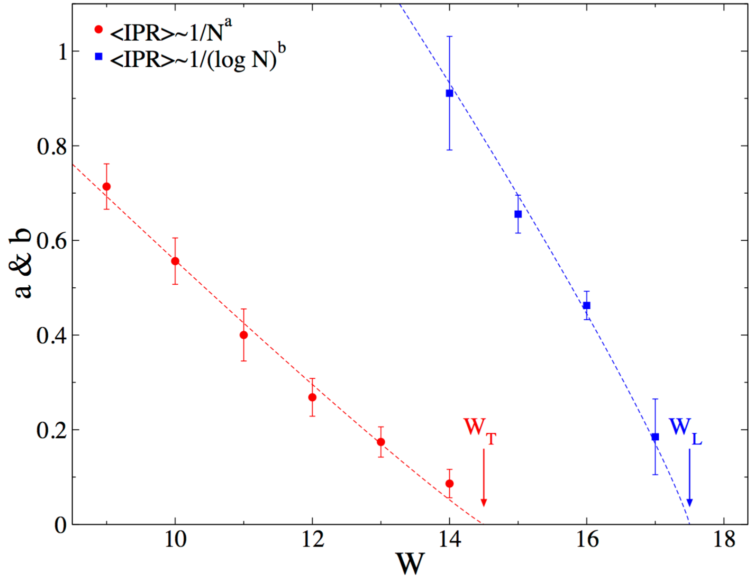
RRG: finite N, one can study properties of individuals eigenstates,
e.g. IPR $P_2 = \langle \sum_i |\psi_n(i)|^4 \rangle \longrightarrow$ this talk

finite BL: finite N, one can study properties of individuals eigenstates,
but they differ crucially from RRG !
Multifractality that depends on W and on position on the tree
Tikhonov and ADM, Phys Rev B 94, 184203 (2016);
Sonner, Tikhonov, and ADM, Phys. Rev. B 96, 214204 (2017)
not considered in this talk

infinite BL: $N = \infty$, one can study statistics of Green functions (e.g. LDOS)
at finite frequency (imaginary or real)

Anderson localization on RRG: Previous numerics

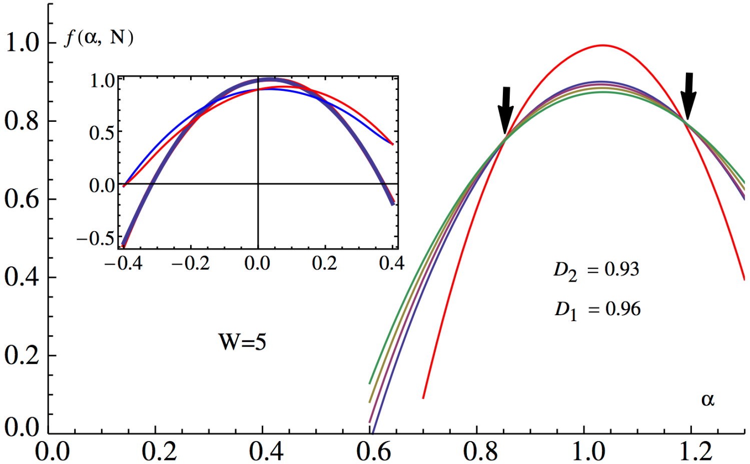
Biroli, Ribeiro-Teixeira, Tarzia, arXiv:1211.7334

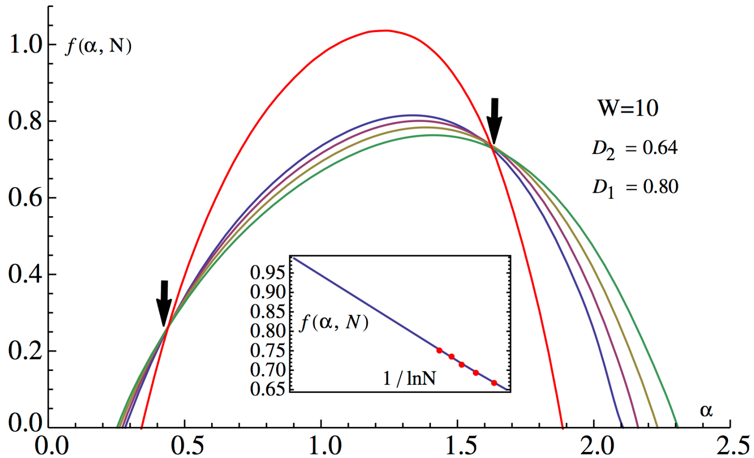


apparent fractality of IPR

$\longrightarrow$ non-ergodictiy of delocalized phase ?!

De Luca, Altshuler, Kravtsov, Scardicchio, Phys Rev Lett '14





"We conclude that the nonergodicity and multifractality persist in the entire region of delocalized states $0 < W < W_c$"

Approaches to Anderson model on RRG

- **Direct numerics: Exact diagonalization**
- **Field theory, Large N $\longrightarrow$ saddle point**

  $\longrightarrow$ self-consistency equation

  - **Analytical solution**
  - **Numerical solution via pool method (population dynamics)**

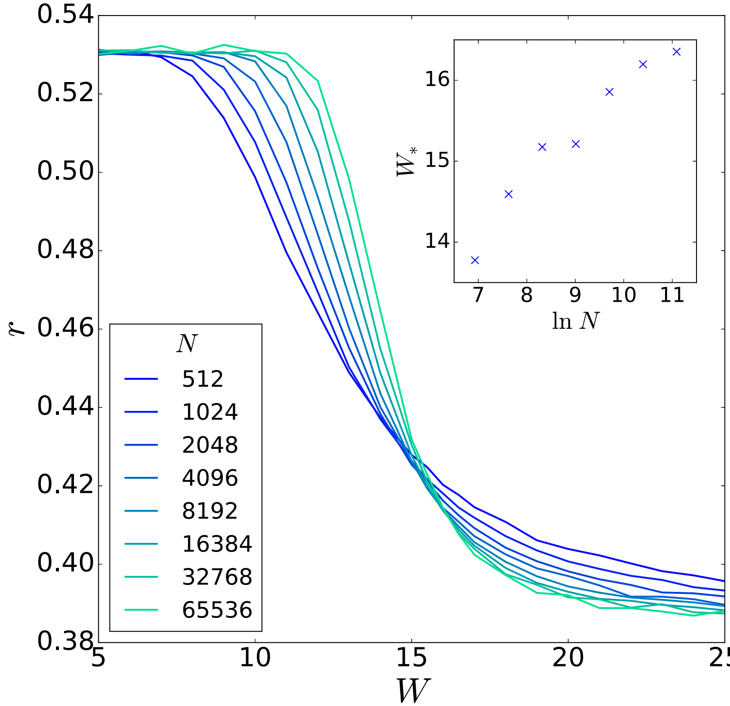
Anderson localization and ergodicity on RRG

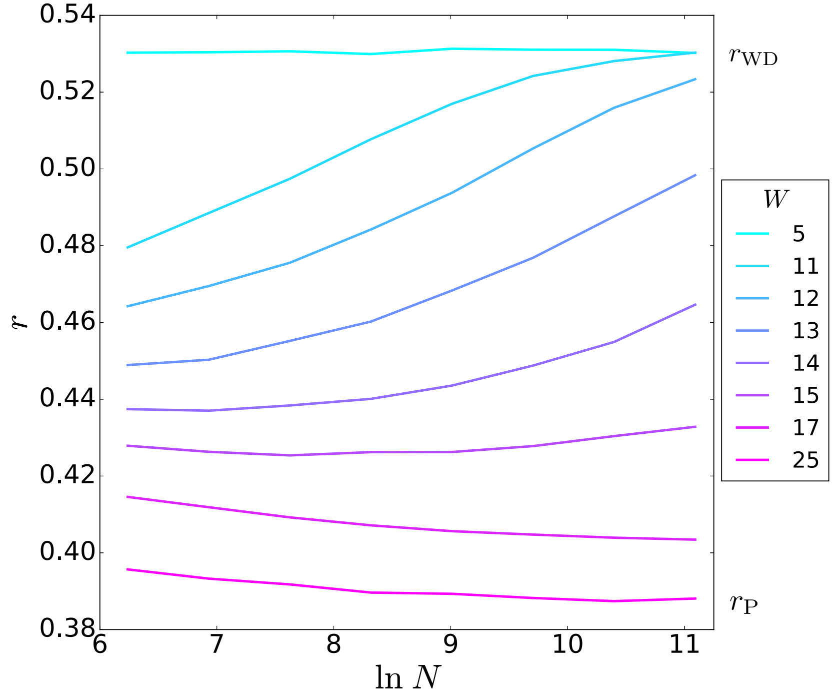
K.S. Tikhonov, ADM, M.A. Skvortsov, PRB 94, 220203(R) (2016)

maximal size $N = 65\,536$; for $W = 11$: $N = 262\,144$

Level statistics:

mean adjacent gap ratio r





Crossing point W_* drifts towards stronger disorder:

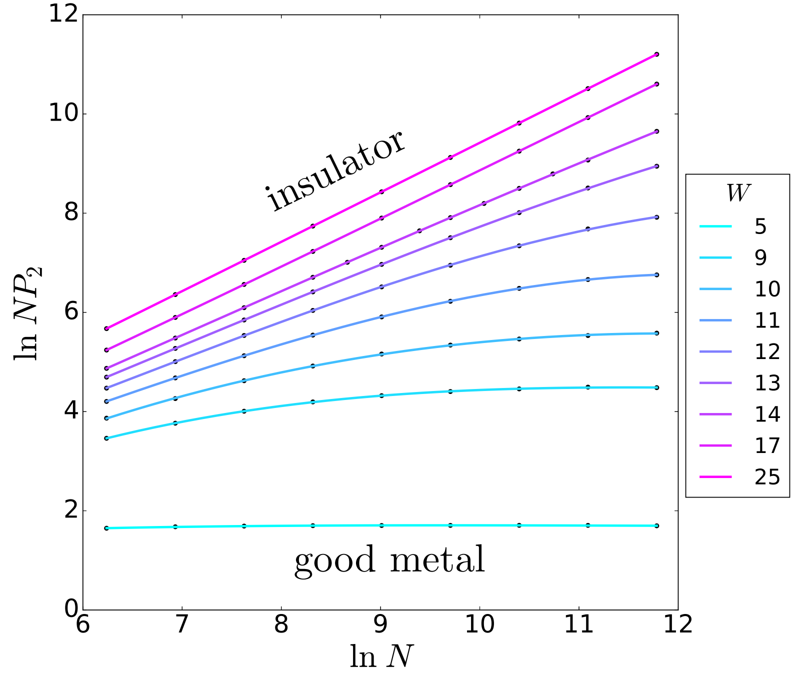
$W_* \simeq 14$ $(N = 512) \longrightarrow W_* \simeq 16$ $(N = 65\,536)$

Equivalently: for given W non-monotonic dependence $r(N)$

Explanation: critical point on tree-like structures (or at $d \to \infty$) has quasi-localized character (Poisson statistics, IPR $\propto N^0$)

Eigenfunction statistics

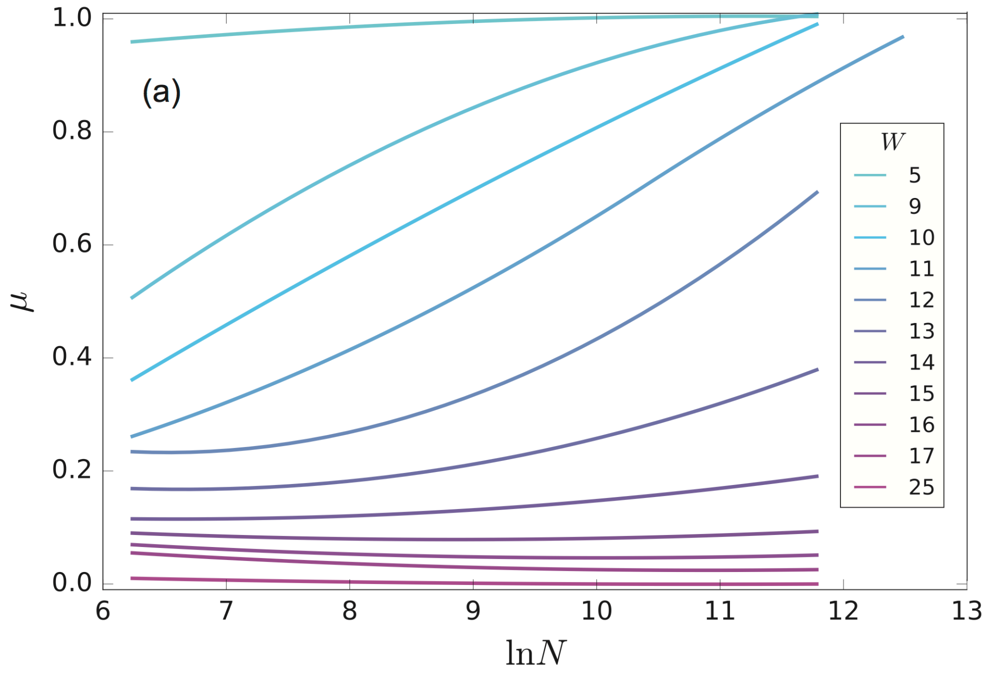
IPR $P_2(W, N)$

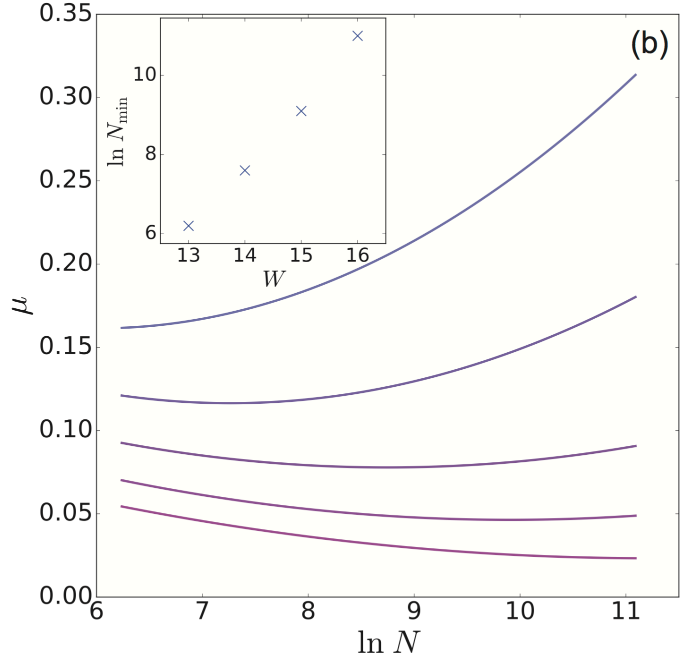


"flowing fractal exponent"

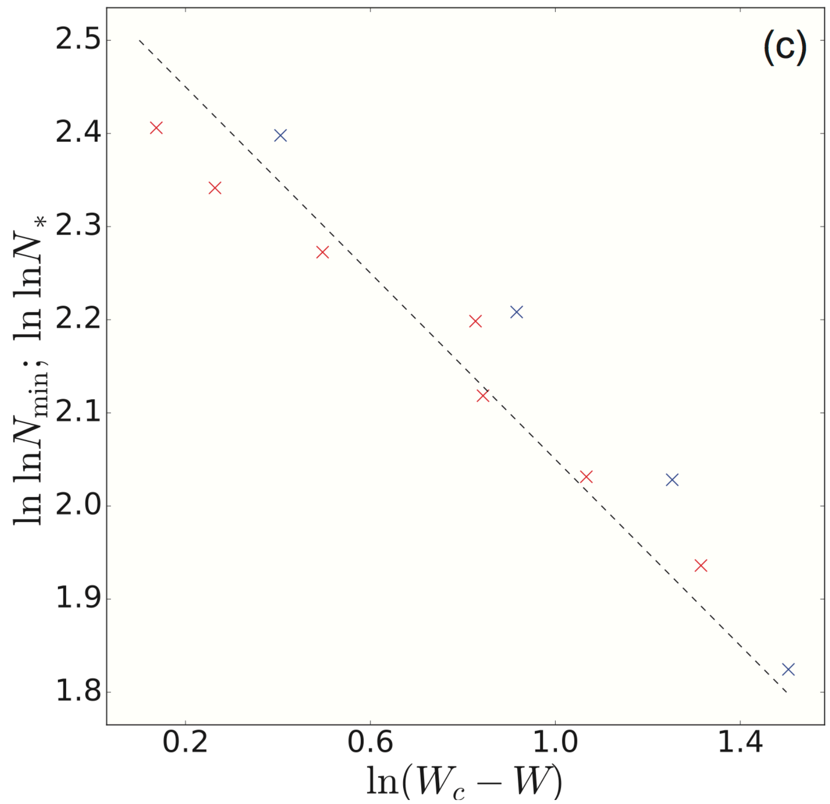
$$\mu(W, N) = -\partial \ln P_2(W, N)/\partial \ln N:$$

non-monotonic N-dependence





Correlation length



× level statistics

× eigenfunction statistics

$$\xi(W) \propto (W_c - W)^{-\nu_d}$$

correlation length

$$N_\xi(W) \sim m^{\xi(W)}$$

correlation volume

data roughly consistent with $\nu_d = \frac{1}{2}$

as expected from the critical behavior of IPR (analytics)

$$P_2 \simeq N_\xi(W)/N\ , \qquad \ln N_\xi \propto (W_c - W)^{-1/2}\ , \qquad N \gg N_\xi$$

But: exact diagonalization not sufficient for accurate numerical determination of critical behavior!

$\longrightarrow$ Field-theoretical approach

RRG: Field-theoretical approach

$\langle \mathcal{O} \rangle = \int \prod_k [d\Phi_k] e^{-\mathcal{L}(\Phi)} U_{\mathcal{O}}(\Phi)$ $\qquad$ $\Phi_{i,s} = (S^{(1)}_{i,s}, S^{(2)}_{i,s}, \chi_{i,s}, \chi^*_{i,s})$ – supervector

Doubling $\Phi_i = (\Phi_{i,1}, \; \Phi_{i,2})$ for retarded (R) and advanced (A) Green functions

$$e^{-\mathcal{L}(\Phi)} = \int \prod_i d\epsilon_i \gamma(\epsilon_i) e^{\frac{i}{2}\Phi_i^\dagger \hat{\Lambda}(E-\epsilon_i)\Phi_i + \frac{i\omega}{4}\Phi_i^\dagger \Phi_i} \prod_{\langle i,j \rangle} e^{-i\Phi_i^\dagger \Phi_j} \qquad \Lambda = \mathrm{diag}(1,-1)_{RA}$$

RRG, connectivity $p = m+1$, distributions of energies $\gamma(\epsilon)$ and hoppings $h(t)$

$$\langle Z \rangle = \int \prod_i d\Phi_i \frac{dx_i}{2\pi} e^{ipx_i} \exp\left\{ \sum_i \left[\frac{i}{2} \Phi_i^\dagger \hat{\Lambda}(E - J_i \hat{K})\Phi_i + \frac{i}{2}\left(\frac{\omega}{2} + i\eta\right) \Phi_i^\dagger \Phi_i \right.\right.$$
$$\left.\left. + \ln \tilde{\gamma}(\frac{1}{2}\Phi_i^\dagger \hat{\Lambda}\Phi_i) \right] + \frac{p}{2N} \sum_{i \neq j} \left[e^{-i(x_i + x_j)} \tilde{h}(\Phi_i^\dagger \hat{\Lambda} \Phi_j) - 1 \right] \right\}$$

Functional generalization of Hubbard-Stratonovich transformation

$\longrightarrow$ integral over functions $g(\Phi)$: $\qquad$ $\langle \mathcal{O} \rangle = \int Dg \, U_{\mathcal{O}}(g) e^{-N\mathcal{L}(g)}$

$$\mathcal{L}(g) = \frac{m+1}{2} \int d\Psi d\Psi' g(\Psi) C(\Psi, \Psi') g(\Psi') - \ln \int d\Psi \, F_g^{(m+1)}(\Psi)$$

$$F_g^{(s)}(\Psi) = \exp\left\{ \frac{i}{2} E \Psi^\dagger \hat{\Lambda} \Psi + \frac{i}{2}\left(\frac{\omega}{2} + i\eta\right) \Psi^\dagger \Psi \right\} \tilde{\gamma}(\frac{1}{2}\Psi^\dagger \hat{\Lambda} \Psi) g^s(\Psi)$$

Field theory for RRG model: Saddle-point treatment

$\langle \mathcal{O} \rangle = \int Dg\, U_{\mathcal{O}}(g) e^{-N\mathcal{L}(g)}$ Large $N \longrightarrow$ saddle-point treatment

IPR $P_2 = \dfrac{1}{\pi\nu} \lim_{\eta\to 0} \eta \, \langle G_R(j,j) G_A(j,j) \rangle$ $\quad G_{R,A}(j,j) = \langle j | (E - \mathcal{H} \pm i\eta)^{-1} | j \rangle$

$\langle G_R(j,j) G_A(j,j) \rangle = \int Dg\, U(g) e^{-N\mathcal{L}(g)}$

$U(g) = \int [d\Psi] \, \frac{1}{16} \left(\Psi_1^\dagger \hat{K} \Psi_1 \right) \left(\Psi_2^\dagger \hat{K} \Psi_2 \right) F_g^{(m+1)}(\Psi)$

$g_0(\Psi) = \int d\Phi \, \tilde{h}(\Phi^\dagger \hat{\Lambda} \Psi) F_{g_0}^{(m)}(\Phi)$ saddle-point equation

identical to the self-consistency equation for infinite Bethe lattice (BL) !

ADM, Fyodorov 1991

Symmetry $\longrightarrow g_0(\Psi) = g_0(x,y); \quad x = \Psi^\dagger \Psi, \quad y = \Psi^\dagger \hat{\Lambda} \Psi$

Laplace (x) - Fourier (y) transf.: $g_0(x,y) \longleftrightarrow$ distribution of Im G and Re G

self-consistency equation in the form of Abou-Chacra, Thouless, Anderson 1973

Field theory for RRG model: Inverse Participation Ratio

- $W \geq W_c$ localized phase and critical point:

single saddle-point $g_0(\Phi) = g_0(x,y)$, characteristic $x \sim \eta^{-1}$

$$\longrightarrow \quad U(g_0) = \frac{C}{\eta}, \quad C \sim 1 \quad \longrightarrow \quad P_2 = \frac{C}{\pi\nu} \sim 1$$

- $W < W_c$ delocalized phase: spontaneous symmetry breaking

manifold of saddle points

$$g_0(\Psi) \longrightarrow g_{0T}(\Psi) = g_0(\hat{T}\Psi) = g_0(\Psi^\dagger \bar{\hat{T}}\hat{T}\Psi,\ \Psi^\dagger\hat{\Lambda}\Psi) \qquad \bar{\hat{T}}\hat{\Lambda}\hat{T} = \hat{\Lambda}$$

$$\langle G_R(j,j)G_A(j,j)\rangle = \int Dg e^{-N\mathcal{L}(g)}U(g) = \int d\mu(\hat{T})\, U(g_{0T})\, e^{-\frac{\pi}{2}N\eta\nu \mathrm{Str}\left[\bar{\hat{T}}\hat{T}\right]}$$

$$P_2 = \frac{1}{\pi\nu}\lim_{\eta\to 0}\eta\,\langle G_R(j,j)G_A(j,j)\rangle = \frac{12}{N}\frac{g_{0,xx}^{(m+1)}}{\pi^2\nu^2} = \frac{3}{N}\frac{\langle\nu^2\rangle_{\mathrm{BL}}}{\nu^2} \qquad N \gg N_\xi$$

Near the transition: $\langle\nu^2\rangle_{\mathrm{BL}}/\nu^2 = N_\xi \gg 1$ — correlation volume $\quad P_2 = 3\frac{N_\xi}{N}$

Exact relations between RRG and infinite BL problems !

Generalized to correlation functions at arbitrary distance r
and of different eigenstates (energy separation ω)

Wave function correlations: Single wave function

RRG: $\alpha(r) = \langle|\psi_k^2(i)\psi_k^2(j)|\rangle$ $\quad r$ – distance between i and j

large $N \longrightarrow$ expressed in terms of infinite Bethe lattice correlation functions:

$$K_1(r) = \langle G_R(i,i)G_A(j,j)\rangle_{\mathrm{BL}} = \langle \tfrac{1}{16}(\Psi_{i,1}^\dagger \hat{K}\Psi_{i,1})(\Psi_{j,2}^\dagger \hat{K}\Psi_{j,2})\rangle_{\mathrm{BL}}$$

$$K_2(r) = \langle G_R(i,j)G_A(j,i)\rangle_{\mathrm{BL}} = \langle \tfrac{1}{16}(\Psi_{j,1}^\dagger \hat{K}\Psi_{i,1})(\Psi_{i,2}^\dagger \hat{K}\Psi_{j,2})\rangle_{\mathrm{BL}}$$

- **Localized phase:**

$$\alpha(r) = \frac{1}{\pi\nu N}\lim_{\eta\to 0}\eta K_1(r,\eta) \sim \frac{1}{N} m^{-r} e^{-r/\zeta} r^{-3/2}$$

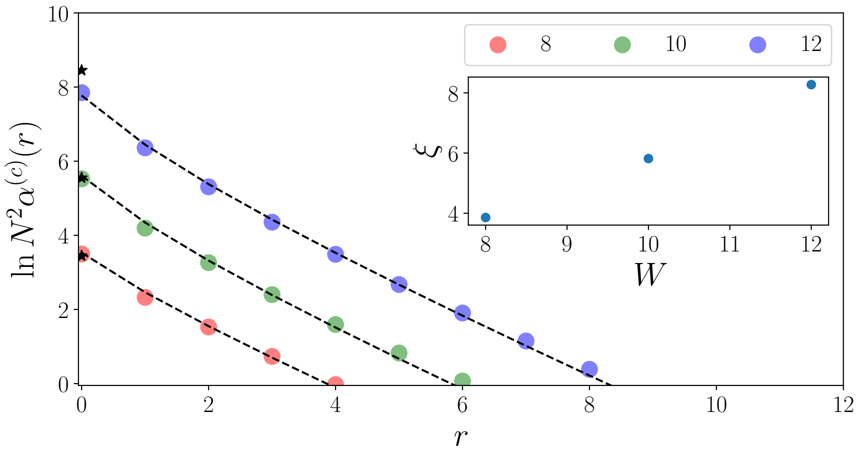
ζ – localization length

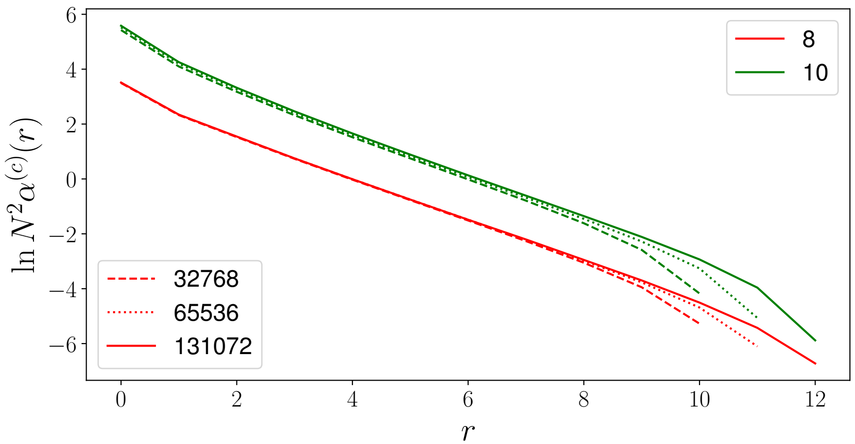
- **Critical point:** $\zeta = \infty \longrightarrow \alpha(r) \sim \dfrac{1}{N}\dfrac{m^{-r}}{r^{3/2}}$

- **Delocalized phase,** $N \gg N_\xi$:

$$\alpha(r) = \frac{1}{2\pi^2 N^2}[K_1(r) + 2K_2(r)] \sim \frac{N_\xi}{N^2}\frac{m^{-r}}{r^{3/2}}$$

$$(r < \xi)$$





Wave function correlations: Different wave functions

RRG: $\beta(r,\omega) = \langle|\psi_k^2(i)\psi_l^2(j)|\rangle \qquad \omega = \epsilon_k - \epsilon_l \qquad r = \text{distance}(i,j)$

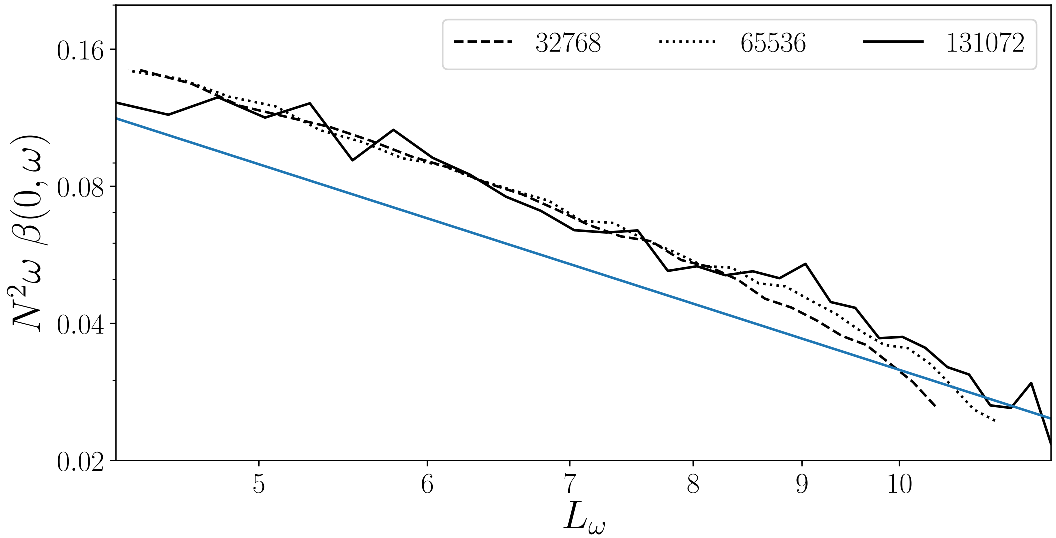
$$\beta(r,\omega) = \frac{1}{2\pi^2 N^2}\,\mathrm{Re}\,K_1(r,\omega)$$ consider first $r = 0$

- **Critical point**

$$K_1(r=0,\omega=2i\eta) \simeq \frac{c_1^{(K)}}{\eta} + \frac{c_2^{(K)}}{\eta\ln^\mu 1/\eta}$$

$$\longrightarrow \quad \beta(0,\omega) \sim \frac{1}{N^2\omega\ln^{\mu+1} 1/\omega}$$

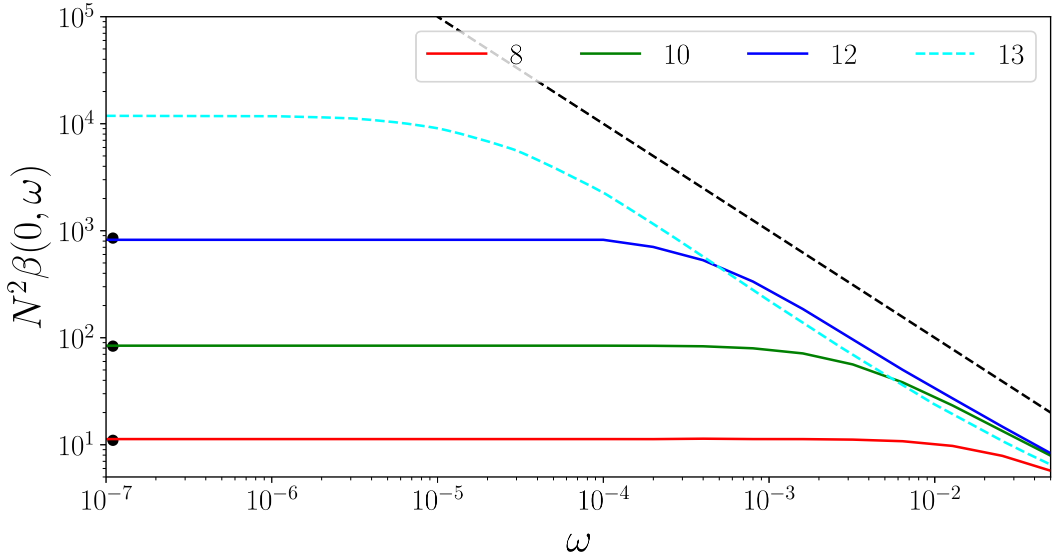
$\mu = 1/2$ from ED
and numerical solution of SC equation



- **Delocalized phase**

$$\beta(0,\omega) \sim \begin{cases} N_\xi/N^2, & \omega < \omega_\xi \\ \dfrac{1}{N^2\omega\ln^{\mu+1} 1/\omega}, & \omega > \omega_\xi \end{cases}$$

$\omega_\xi \sim N_\xi^{-1}$ (with log correction)



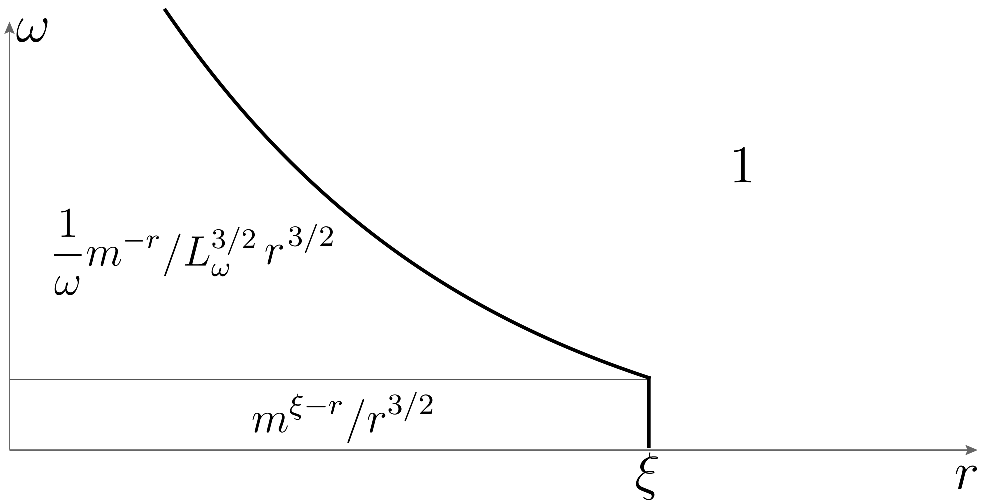
Wave function correlations: $r - \omega$ plane

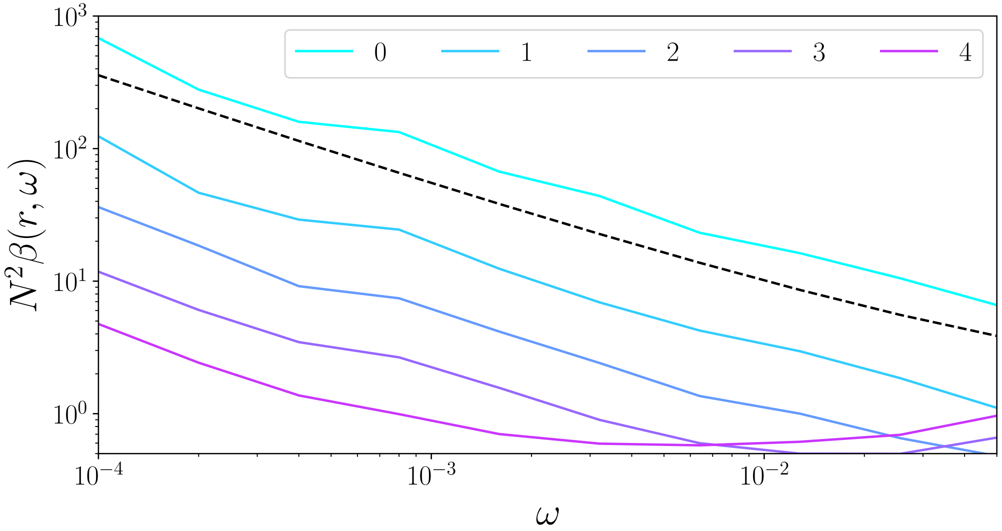
$$\beta(r,\omega) = \langle|\psi_k^2(i)\psi_l^2(j)|\rangle \qquad \omega = \epsilon_k - \epsilon_l \qquad r = \text{distance}(i,j)$$

$$\beta(r,\omega) = \frac{1}{2\pi^2 N^2}\,\text{Re}\,K_1(r,\omega)$$ consider $W < W_c$

$$\beta(r,\omega) \sim \begin{cases} \dfrac{m^{\xi-r}}{N^2 r^{3/2}}, & r < \xi < L_\omega \quad \text{“metallic” regime} \\ \dfrac{m^{-r}}{N^2 \omega L_\omega^{3/2} r^{3/2}}, & r < L_\omega < \xi \quad \text{critical regime} \end{cases}$$

characteristic length scales: $\xi \sim (W_c - W)^{-1/2}$ $\qquad L_\omega = \log_m(1/\omega)$

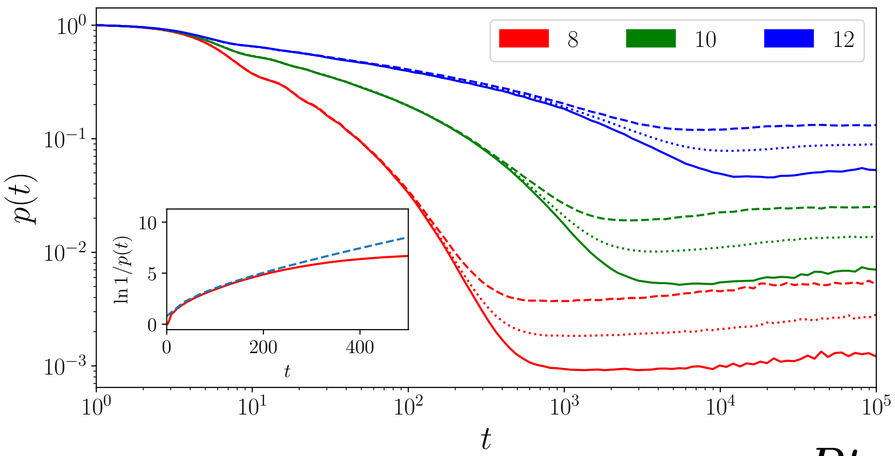


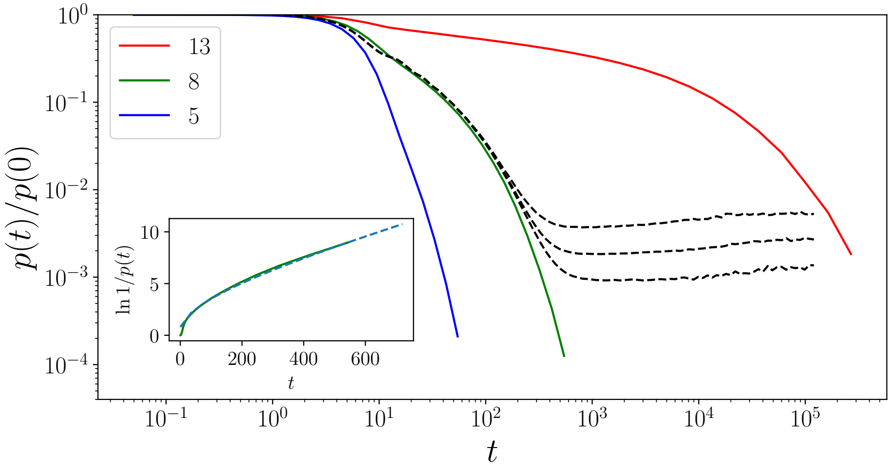


Further dynamical observables: return probability, spectral statistics

Return probability

Eigenstate correlations $\alpha(0)$ and $\beta(0,\omega)$ $\longrightarrow$ return probability $p(t)$

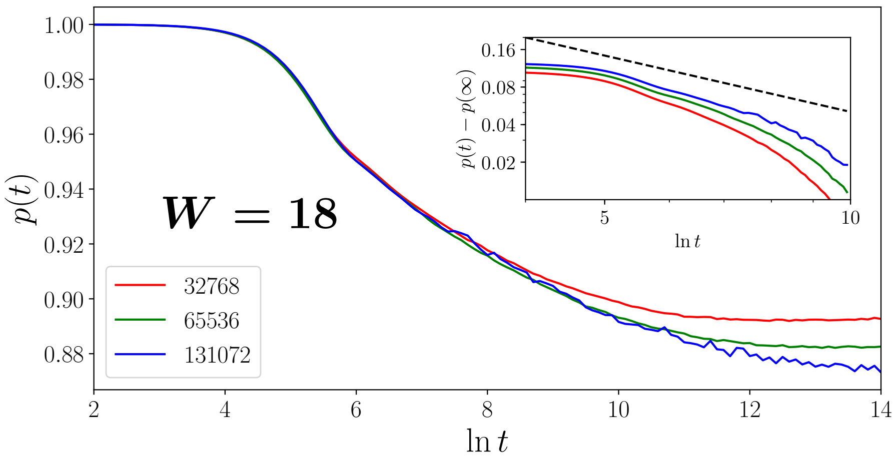




Delocalized phase: $p(t) \sim \dfrac{e^{-Dt}}{(Dt)^{3/2}}$ **"diffusive" (exponential)** $D \sim \dfrac{\ln^3 N_\xi}{N_\xi}$

Saturation: $p(t) = p_\infty = N_\xi / N$ **at** $t > t_{\rm Th}$ $\qquad t_{\rm Th} = D^{-1} \log_m N$

Agreement of ED with numerical solution of saddle-point equation



Critical regime:

$$p(t) = p_\infty + \frac{c}{\ln^{1/2} t}$$

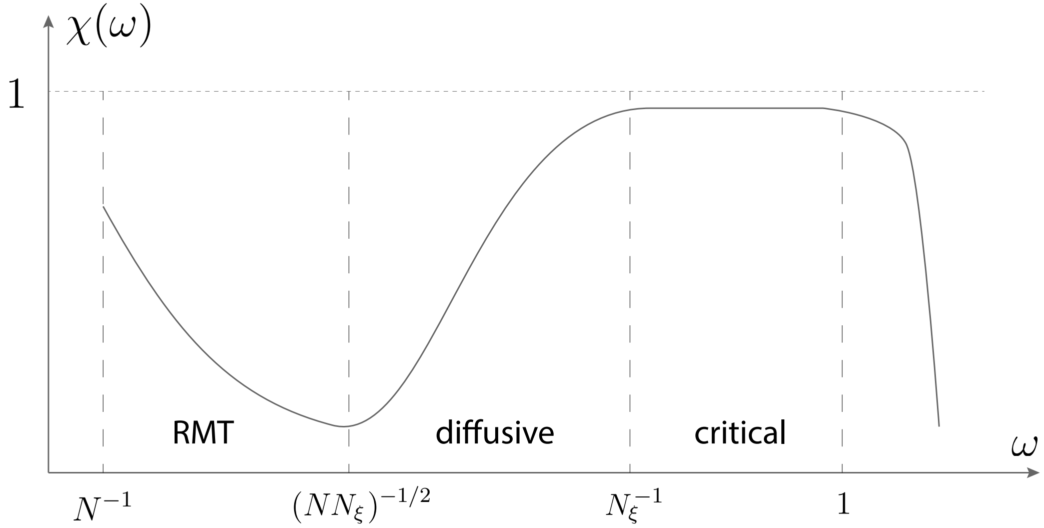
$p_\infty \sim 1$ **as in the localized phase.**

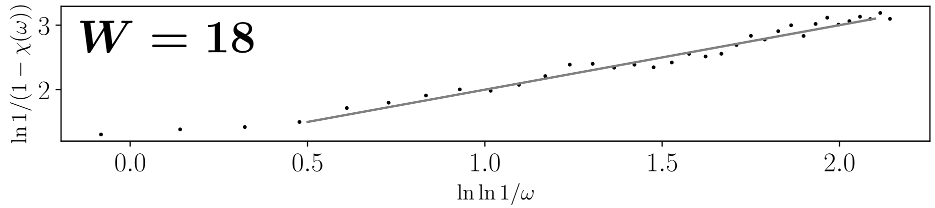
Logarithmically slow dynamics

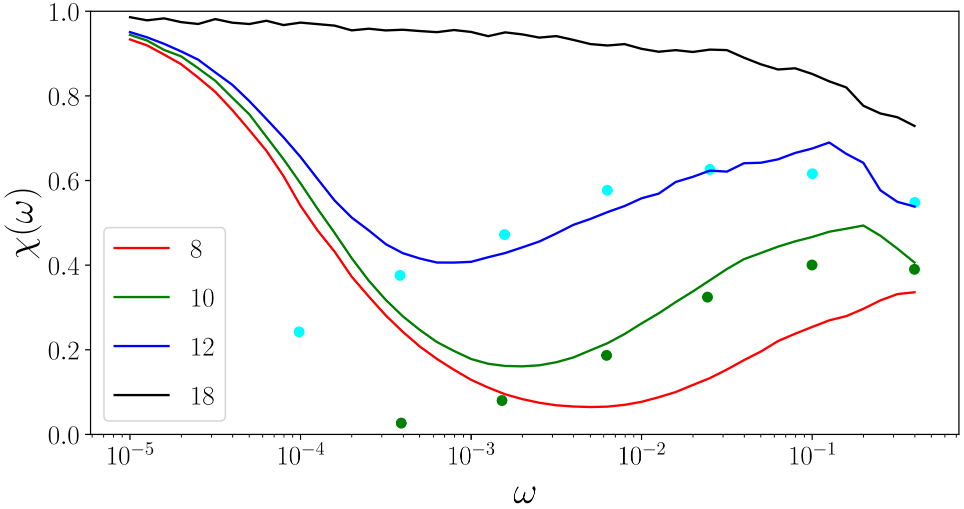
Spectral statistics

Level correlation function $R(\omega)$, **level number variance** $\Sigma_2(\omega) = \mathrm{var}[I(\omega)]$

$\chi(\omega) = \Sigma_2(\omega)/\langle I(\omega)\rangle$ **normalized level number variance**



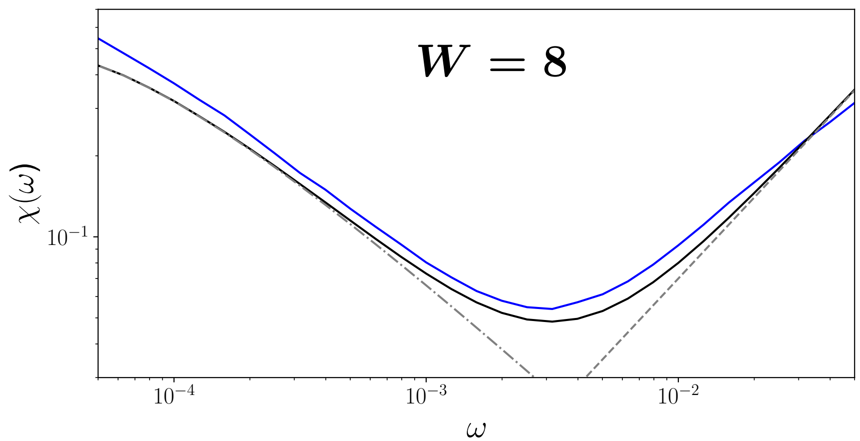




Delocalized phase:

$$R(\omega) = R_{\mathrm{GOE}}(\omega) + R_{\mathrm{diff}}(\omega)$$

$$\longrightarrow \quad \chi(\omega) \simeq \frac{2}{\pi^2}\frac{\Delta}{\omega}\ln\frac{2\pi\omega}{\Delta} + N_\xi\,\omega, \quad \omega < N_\xi^{-1}$$



Critical regime: $$\chi(\omega) = 1 - \frac{c}{\ln(1/\omega)}$$

logarithmically slow dependence

Critical behavior

Correlation volume N_ξ $\longrightarrow$ correlation length ξ

Critical behavior: $\xi \sim (W_c - W)^{-\nu_{\rm del}}$ critical index $\nu_{\rm del} =?$

Self-consistency equation $\longrightarrow$ $m\lambda_\beta = 1$

λ_β – largest eigenvalue of certain integral operator

$\lambda_\beta(W) \simeq \frac{1}{2} - c_1\,(W - W_c) + c_2\left(\beta - \frac{1}{2}\right)^2$, has minimum at $\beta = 1/2$

Localized phase, $W > W_c$: β real

Critical point, $W = W_c$: $m\lambda_{1/2} = 1$ Abou-Chacra et al, 1973

Delocalized phase, $W < W_c$: spontaneous symmetry breaking

β becomes complex: $\beta = \frac{1}{2} \pm i\sigma\,,$ $\sigma \simeq \sqrt{\frac{c_1}{c_2}}(W_c - W)^{1/2}$

Correlation length $\ln N_\xi \simeq \dfrac{\pi}{\sigma}$ $\longrightarrow$ critical index $\boxed{\nu_{\rm del} = 1/2}$

$m = 2 \longrightarrow c_1 \simeq 1.59,\quad c_2 \simeq 0.0154 \longrightarrow \ln N_\xi \simeq 31.9\,(W_c - W)^{-1/2}$

ADM, Fyodorov, 1991, Tikhonov, ADM, 2019

Critical behavior

Numerical verification of $\nu_{\rm del} = 1/2$ **?**

Kravtsov, Altshuler, Ioffe, Ann Phys 2018 **found** $\nu_{\rm del} \approx 1$. **Contradiction?**

We want an accurate determination of W_c **and** $\nu_{\rm del}$

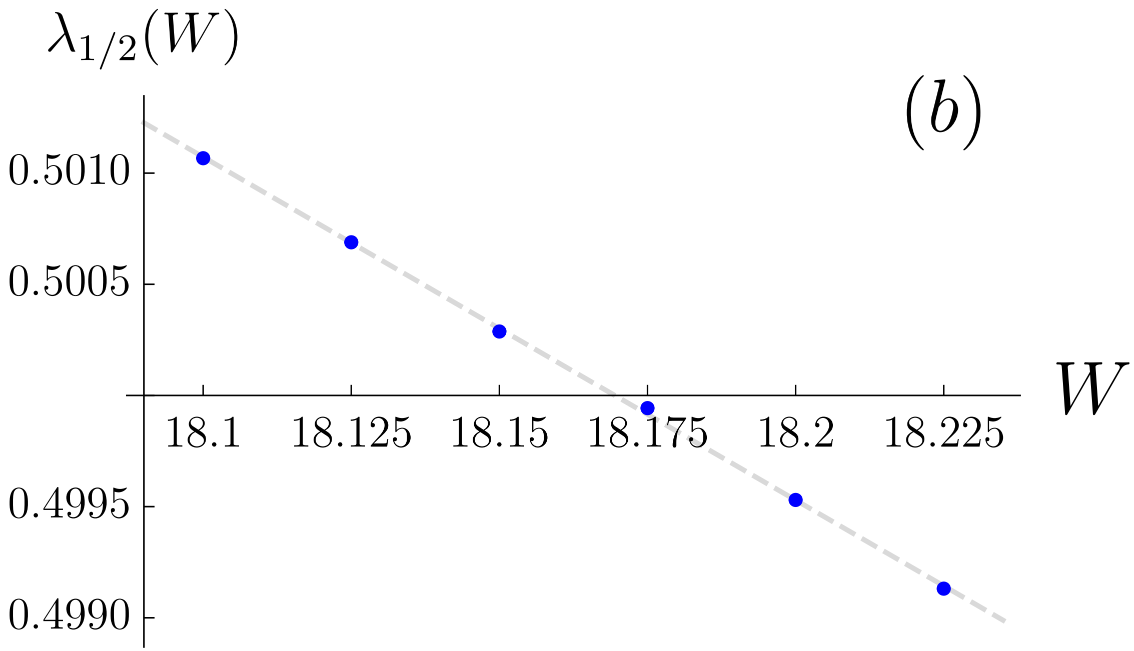
Exact diagonalization for RRG: system sizes not sufficient for this purpose

To approach much closer to the critical point, we use field theory and solve numerically the self-consistency equation

First step: **accurate determination of** W_c **from the equation** $m\lambda_{1/2} = 1$

$m = 2$

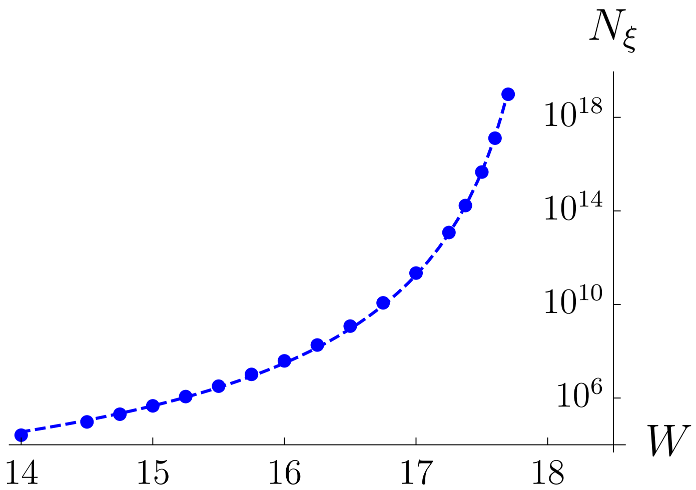
$$\boxed{W_c = 18.17 \pm 0.01}$$

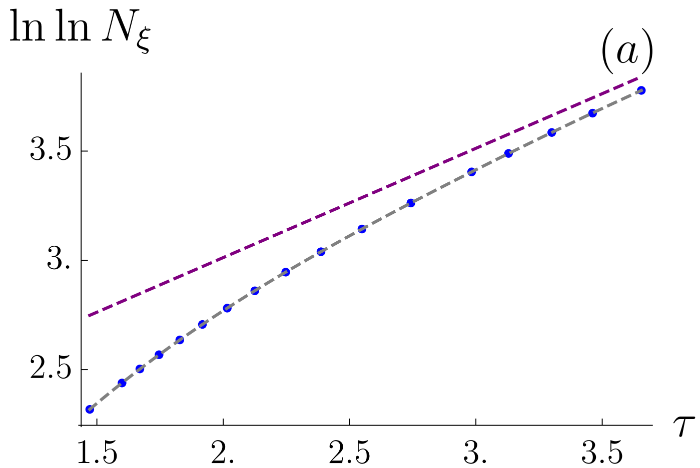


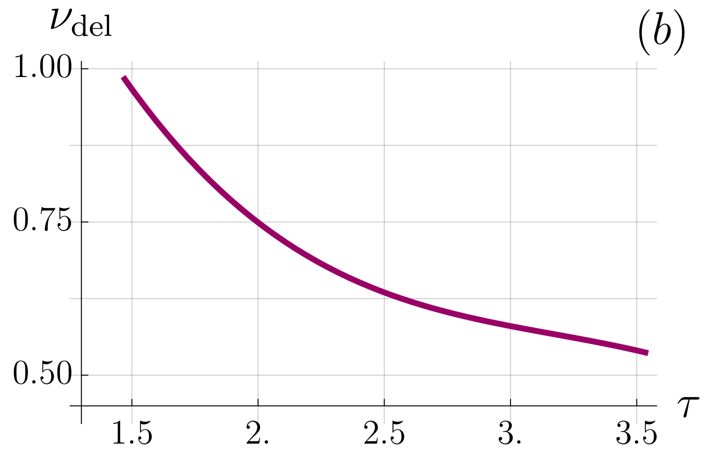
Critical behavior: Numerical confirmation of $\nu_{\rm del} = 1/2$

Solve self-consistency equation by pool method (population dynamics) and thus determine N_ξ

$$\ln N_\xi \sim (W_c - W)^{-\nu_{\rm del}} \longrightarrow \frac{\partial \ln \ln N_\xi}{\partial \ln \tau} = \nu_{\rm del} \qquad \tau = -\ln(1 - W/W_c)$$







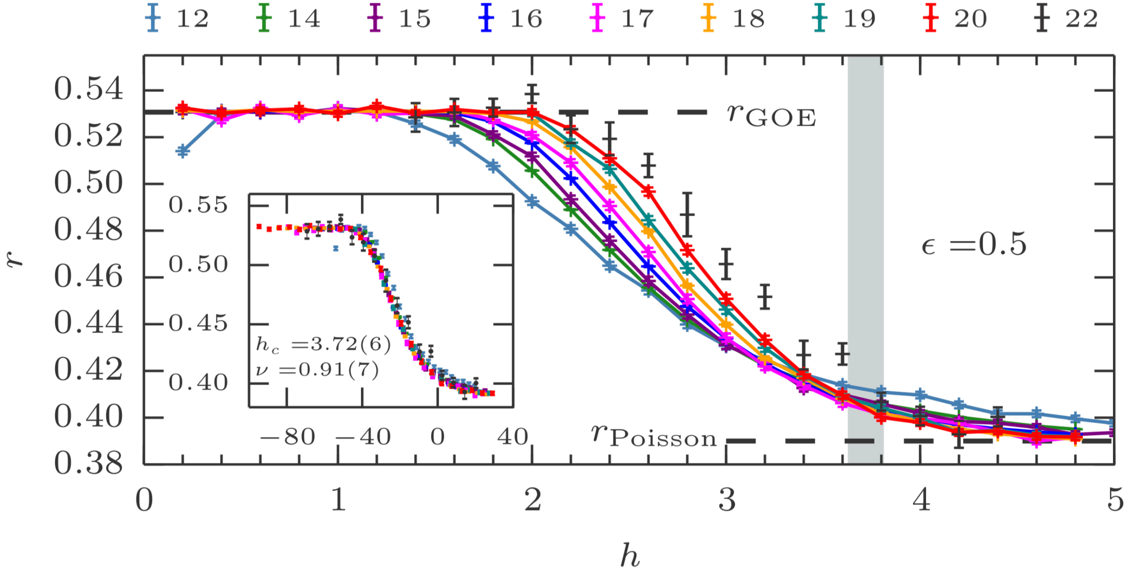
$m = 2 \longrightarrow$ **asymptotics** $\ln N_\xi = 31.9\,(W_c - W)^{-1/2}$

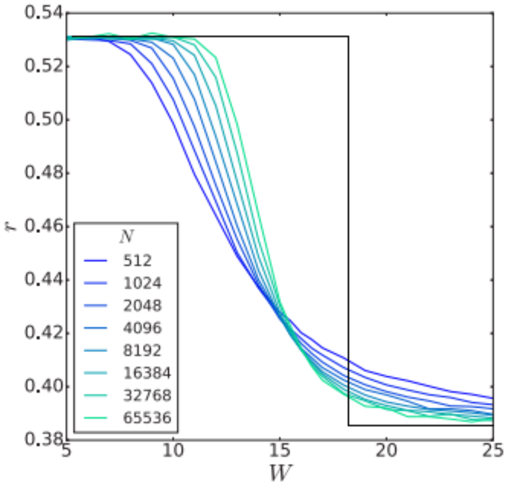
$$\boxed{\nu_{\rm del} = 1/2}$$

MBL: Analogies to RRG and lessons from RRG

MBL with short-range interaction: XXZ spin chain in random field

Luitz, Laflorencie, Alet, PRB (2015)   Striking similarities to RRG

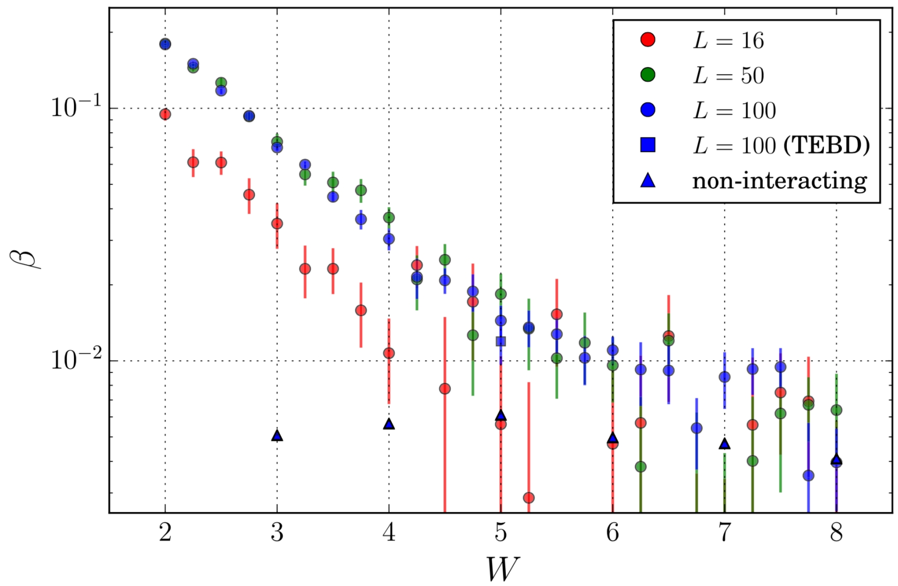




RRG

Tikhonov, ADM, Skvortsov 2016

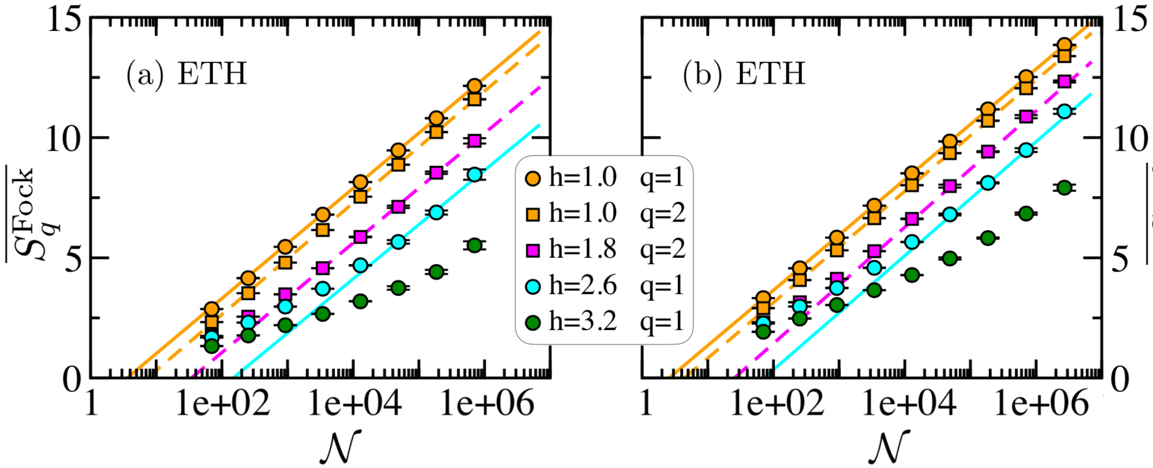
- strong drift of crossing point: strong finite-size effects,
  actual transition
  at considerably stronger disorder,
  as also implied by MPS-TDVP study
  Doggen et al, PRB 98, 174202 (2018)
  $W_c \simeq 5.5 - 6$ rather than $3.7 - 3.8$

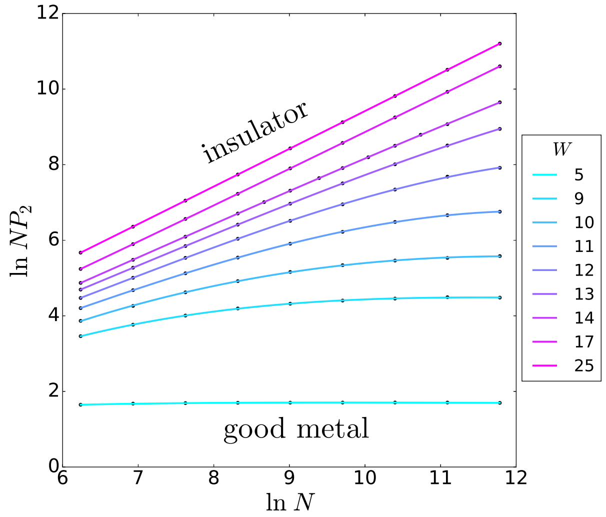


- critical point similar to localized phase

MBL: Analogies to RRG and lessons from RRG (cont'd)

- ergodicity of the delocalized phase achieved for Hilbert space size $N \gg N_\xi$





RRG

Tikhonov, ADM, Skvortsov 2016

Macé, Alet, Laflorencie, PRL 2019

- asymmetry of critical behavior:

$\nu_{\rm del} \simeq 0.45$ and $\nu_{\rm loc} \simeq 0.76$

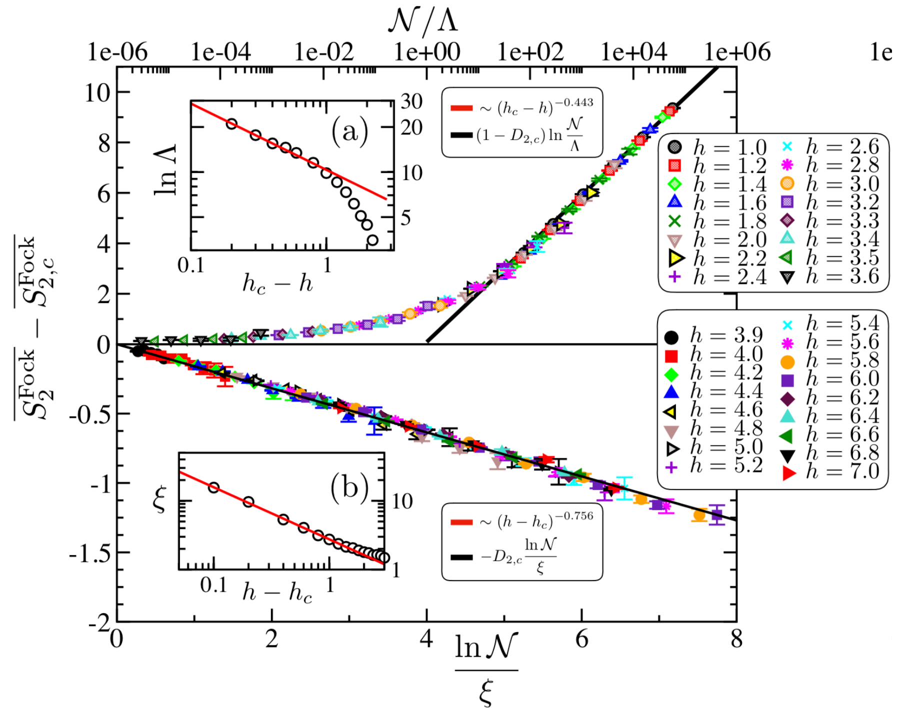
to be compared to

$\nu_{\rm del} = 1/2$ and $\nu_{\rm loc} = 1$ (RRG)

Numerically found exponents for MBL are close to those for RRG and strongly violate Harris criterion

$\longrightarrow$ MBL systems too small to exhibit asymptotic critical behavior

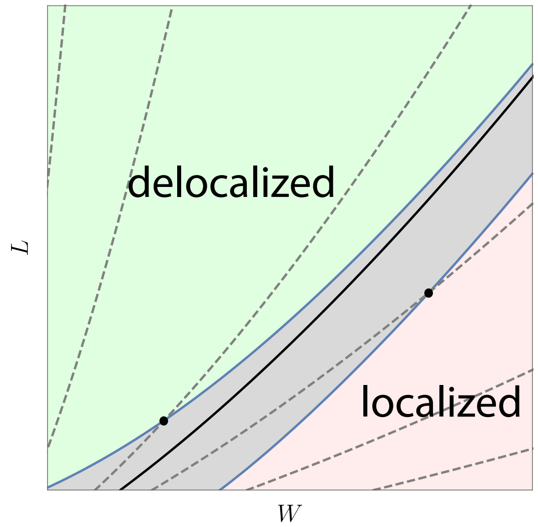
Intermediate, RRG-like fixed point – ?



MBL with long-range interaction and RRG

Random spin chain with $1/r^\alpha$ interaction, $d < \alpha < 2d$

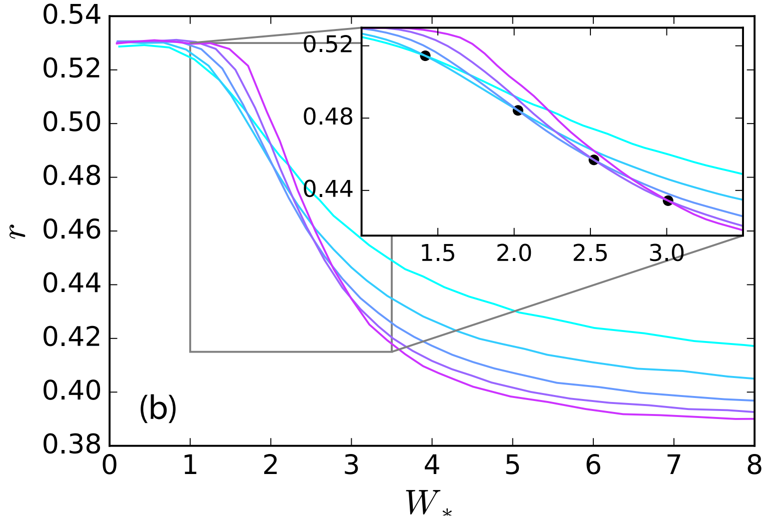
Mapping to RRG $\longrightarrow$ $\boxed{W_c \sim L^{2d-\alpha} \ln L}$

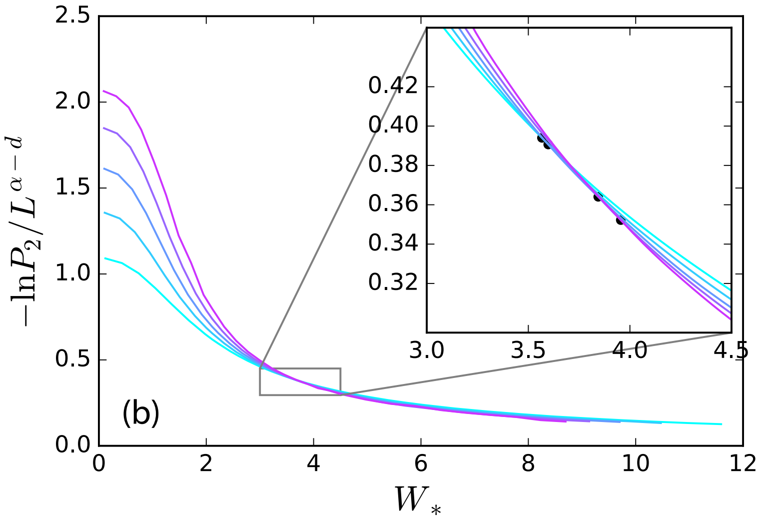


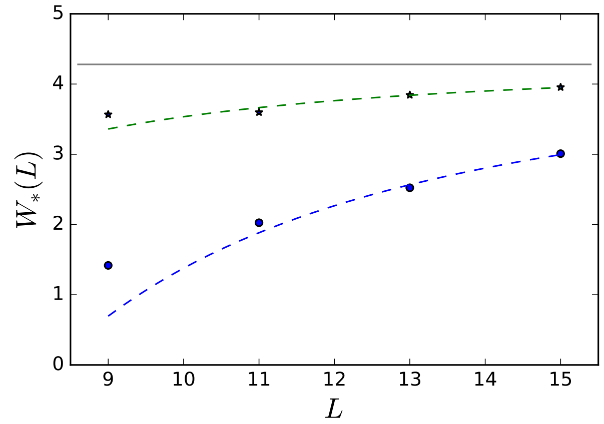
Agreement with exact diagonalization

$d = 1\,, \qquad \alpha = 3/2$

- **Scaling of transition point**
- **Delocalized side: Ergodicity**
- **Critical point $\longrightarrow$ drift towards larger $W_* = W/L^{1/2} \ln L$**







extrapolation to $W_{*c} \simeq 4.3$

Summary

- **Localization transition on RRG. Approaches: (i) exact diagonalization, (ii) analytics, (iii) analytics + population dynamics. Full agreement.**

- **Ergodicity of the delocalized phase $W < W_c$, achieved for $N \gg N_\xi(W)$ with $\ln N_\xi \propto (W_c - W)^{-1/2}$**

- **Critical regime (of nearly localized character) for $N \ll N_\xi(W)$**
  $\longrightarrow$ peculiar crossover from criticality to ergodicity

- **Detailed understanding of eigenfunction fluctuations and correlations, and level statistics.**

- **RRG as a very intricate $d = \infty$ limit of Anderson localization in d dimensions**

- **Index $\nu_{\rm del} = 1/2$ confirmed numerically. Large corrections to scaling. Accurate evaluation of $W_c = 18.17 \pm 0.01$ (for $m = 2$) and of N_ξ up to 10^{19}**

- **RRG as a toy-model of MBL. Quantitative connections to long-range MBL. Strong qualitative analogies with short-range MBL.**
  Lessons for MBL: Strong finite-size effects, ED numerics deceptive in a broad (critical) regime on the delocalized side of the transition