# Hyperbolic Knot Theory

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KNOTS THROUGH WEB (ONLINE)

ICTS INTERNATIONAL CENTRE for THEORETICAL SCIENCES
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

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Talk 2

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Plan of Talks:
Talk 1 : Hyperbolic geometry, ideal tetrahedra, hyperbolic 3-manifolds
Talk 2: Hyperbolic knots, ideal triangulations, Thurstons gluing equations, Example, SnapPy

**Further reading: https://www.math.csi.cuny.edu/abhijit/hypknots-reading.html**

**In yesterday's talk :**

Hyp 3-mfds

$M$ be fin vol., complete, orient hyp 3-mfd

$M$ hyp :

1. $M = \mathbb{H}^3/\Gamma$ $\quad \Gamma < PSL(2,\mathbb{C})$ disc & torsion free
2. $p: \mathbb{H}^3 \to M$ covering map + local isometry
3. $\rho_0 : \pi_1 M \to PSL(2,\mathbb{C})$ discrete + faithful

① Structure of hyp. 3-mfds

Theorem (Mostow-Prasad Rigidity) Let $M_1$ and $M_2$ be hyperbolic 3-manifolds. If $f : M_1 \to M_2$ is a homotopy equivalence then $f$ is homotopic to an isometry $h : M_1 \to M_2$.

Corollary Isomorphic fundamental groups imply isometric manifold i.e. Hyperbolic structure on a 3-manifold is unique.

Corollary Geometric invariants, (e.g. hyperbolic volume), are topological invariants !

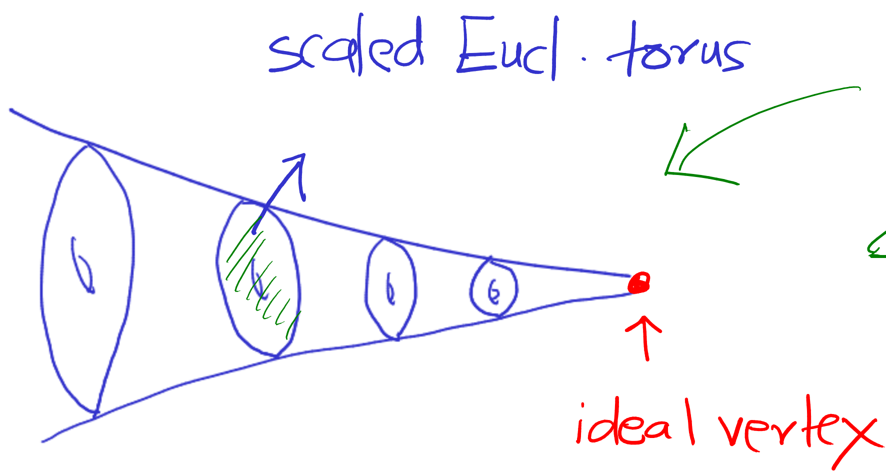
true for dim $n \geq 3$

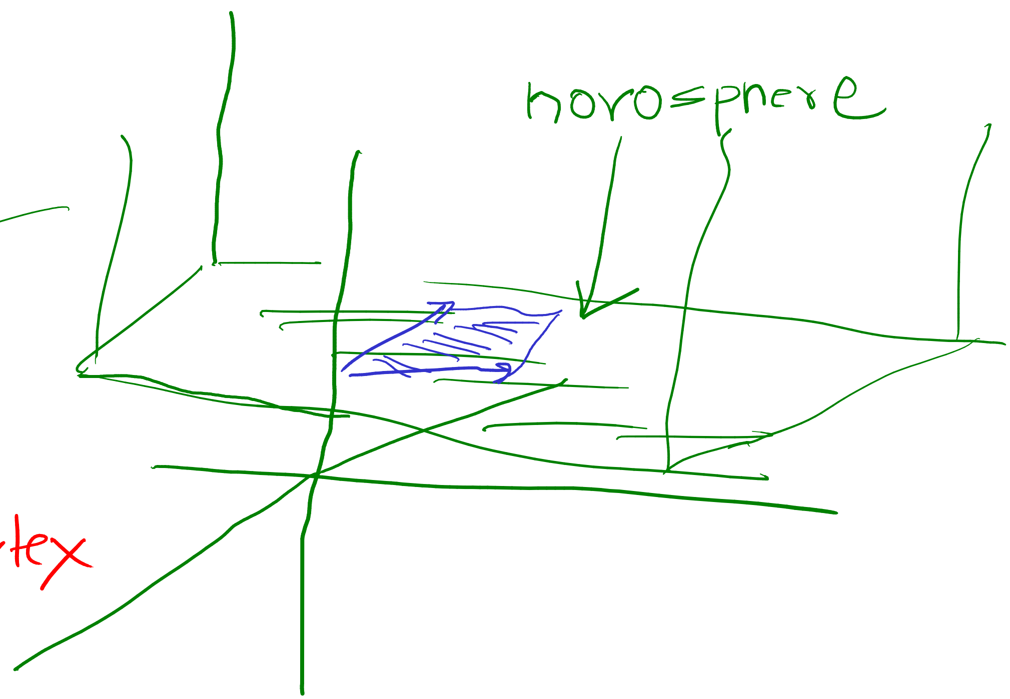
not true in dim $= 2$ & other geometries

Margulis's thick-thin decomposition of non-compact hyperbolic 3-manifold implies thin parts are cusps i.e. $T^2 \times [0, \infty)$ with metric $ds^2_{(x,y,t)} = e^{-2t}(dx^2 + dy^2) + dt^2$.

A cross section tori of the cusps are quotients of horozpheres by $\mathbb{Z}^2$.

Corollary A non-compact hyperbolic 3-manifold is homeomorphic to the interior of a compact 3-manifold with torus boundary.

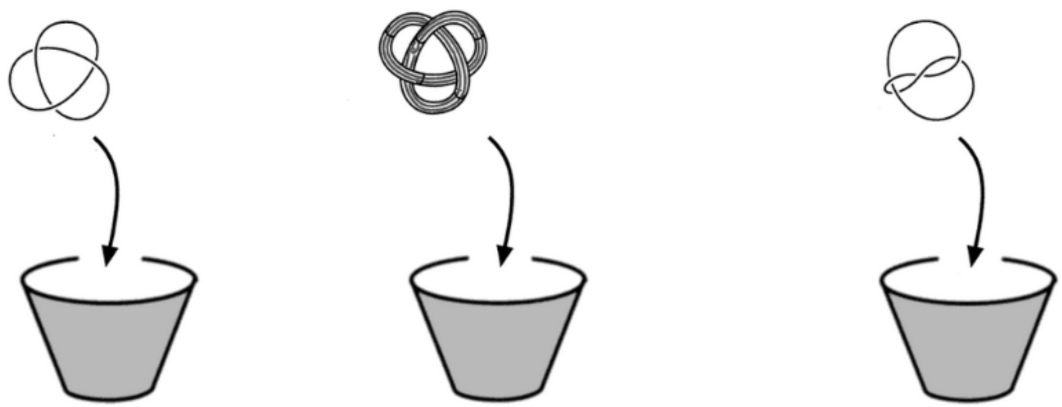


# ② Hyperbolic Knots & links

A knot $K$ in $S^3$ is hyperbolic if $S^3 - K$ is a hyperbolic 3-manifold.

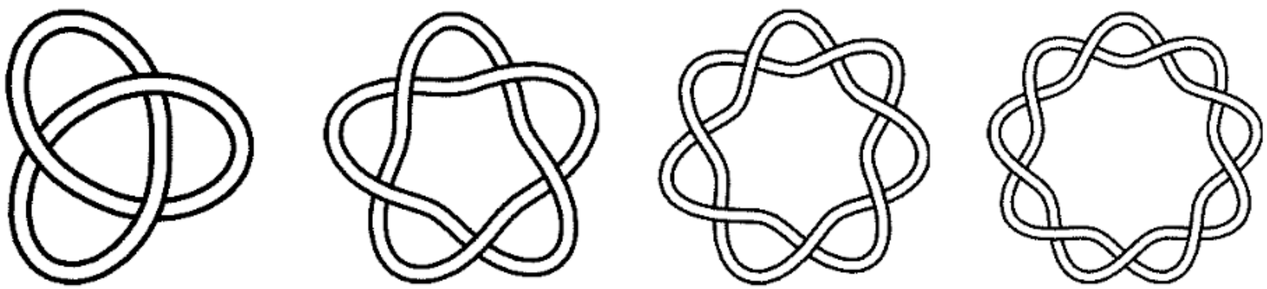
Theorem (Thurston) Every knot in $S^3$ is either a torus knot, a satellite knot or a hyperbolic knot.



Torus knots Satellite knots Hyperbolic knots

Theorem (Menasco) Most prime alternating knot are hyperbolic i.e. if $K$ has a connected prime alternating knot diagram, except the standard $(2, q)$-torus knot diagram, then $K$ is hyperbolic.

$(2, q)$–torus knot diagrams:



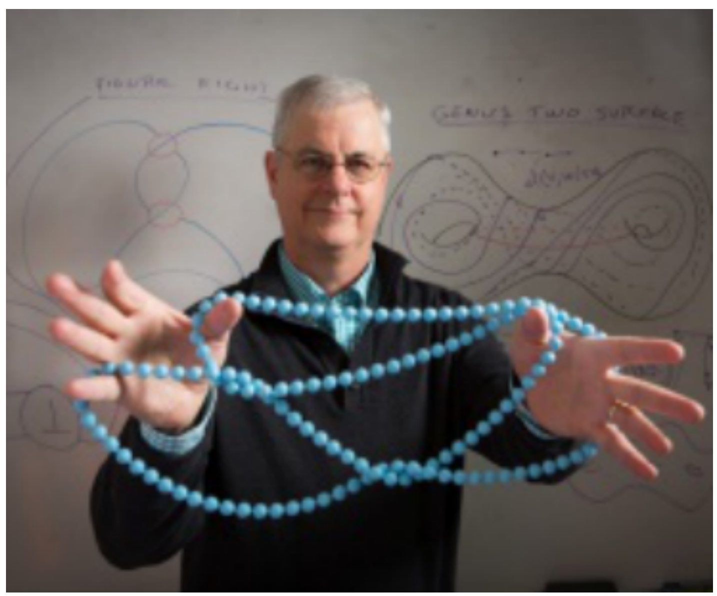
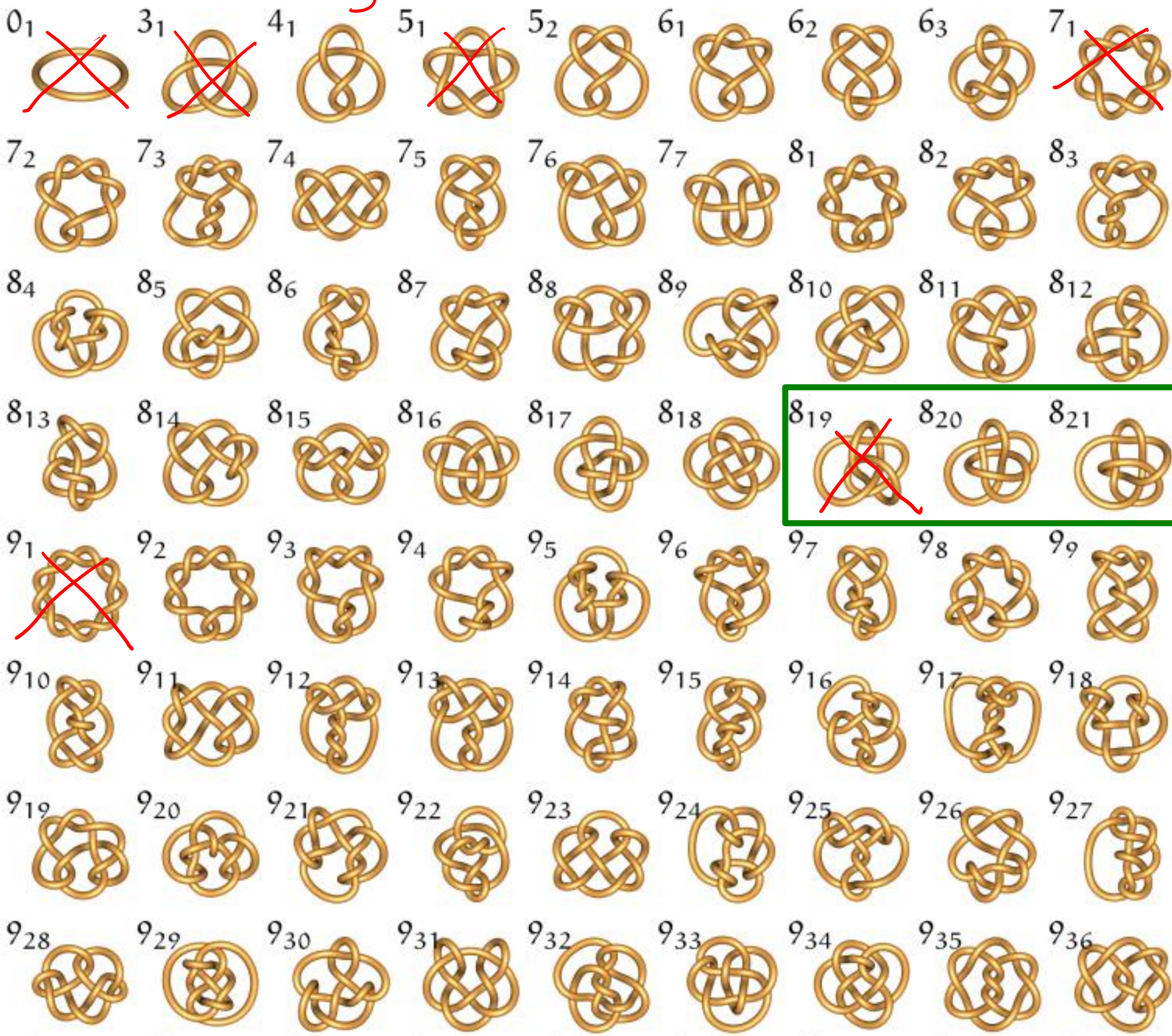


fig-8



$0_1$ $3_1$ $4_1$ $5_1$ $5_2$ $6_1$ $6_2$ $6_3$ $7_1$
$7_2$ $7_3$ $7_4$ $7_5$ $7_6$ $7_7$ $8_1$ $8_2$ $8_3$
$8_4$ $8_5$ $8_6$ $8_7$ $8_8$ $8_9$ $8_{10}$ $8_{11}$ $8_{12}$
$8_{13}$ $8_{14}$ $8_{15}$ $8_{16}$ $8_{17}$ $8_{18}$ $8_{19}$ $8_{20}$ $8_{21}$
$9_1$ $9_2$ $9_3$ $9_4$ $9_5$ $9_6$ $9_7$ $9_8$ $9_9$
$9_{10}$ $9_{11}$ $9_{12}$ $9_{13}$ $9_{14}$ $9_{15}$ $9_{16}$ $9_{17}$ $9_{18}$
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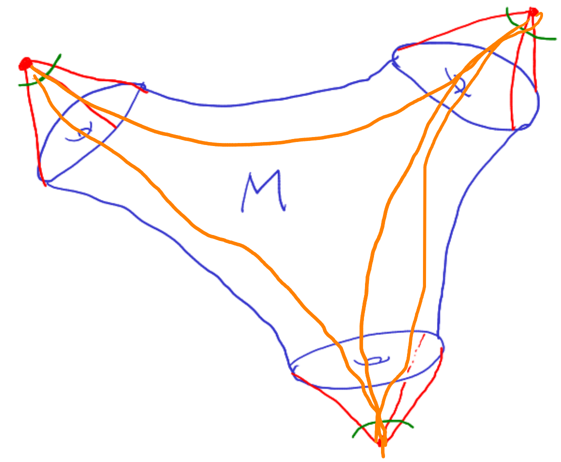
non-alt

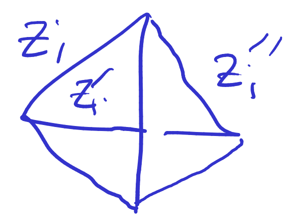
# ③ Ideal triangulations & Gluing eqns

- $M$ = compact oriented 3-manifolds with $\partial M = \sqcup T^2 \neq \phi$
- $\widehat{M} = M$ with each component of $\partial M$ coned to a point
- $\widehat{M}-$ cone points $= M - \partial M$ = int(M)
- $\mathcal{T} = \Delta$-complex structure of $\widehat{M}$ such that $\widehat{M} = \Delta_1 \cup \ldots \Delta_n$ and vertices of $\mathcal{T}$ are cone points of $\widehat{M}$.
- $\mathcal{T}$ is called ideal triangulation of $M$, and $\mathcal{T}-$vertices $= M - \partial M$ = int(M)




Proposition In $\mathcal{T}$, the number of edges is equal to the number of tetrahedra.

Idea: Give a hyp str. on int(M) by making each tet in $\mathcal{T}$ into a hyp. ideal tet.




Let $\Delta_i = \Delta(z_i)$

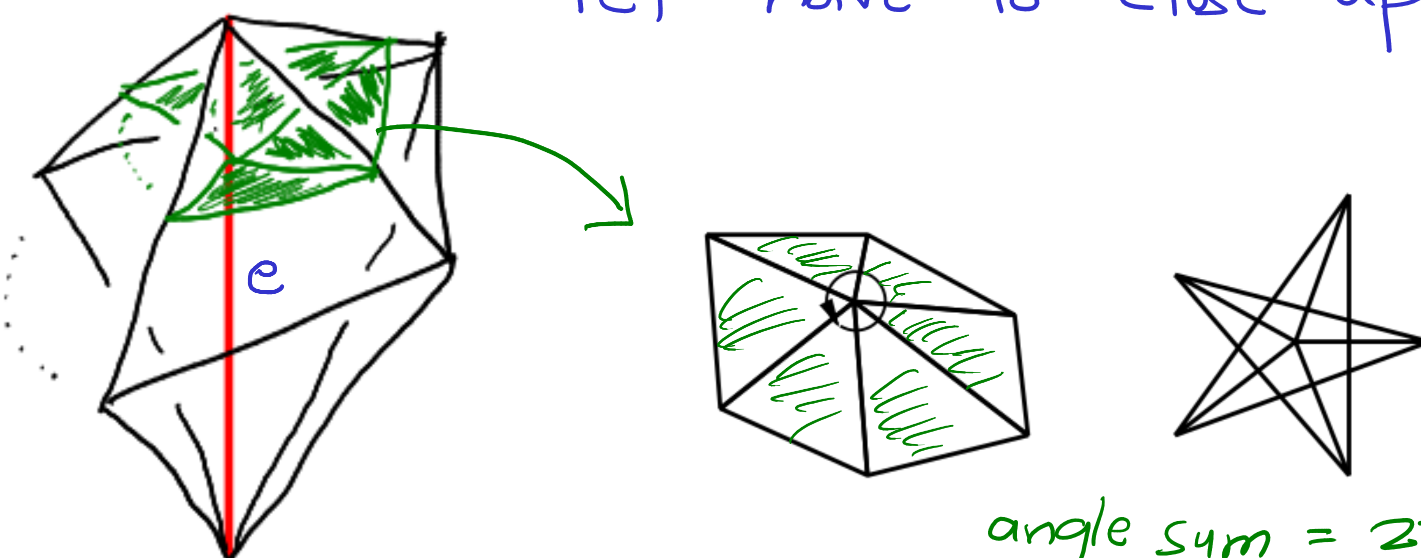
Try to glue tet using isometries –
① faces
② edges
③ "vertices"

① <u>Faces</u>: all ideal triangles in $\mathbb{H}^2$ are isometries

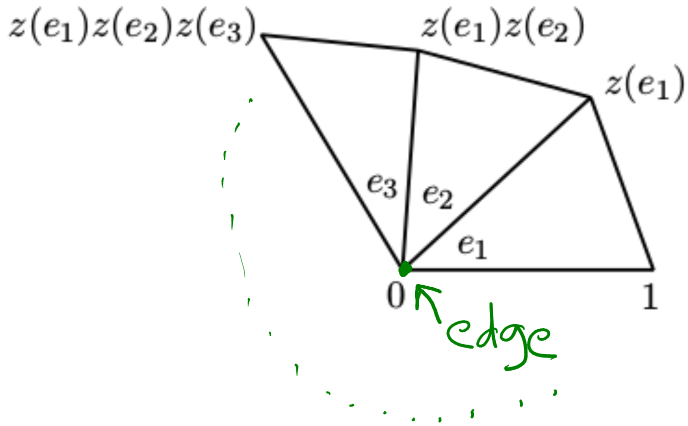
$\Rightarrow$ gluing along faces is ok

② <u>Edges</u>:

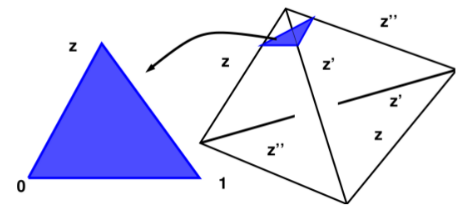
tet have to close up around edges




angle sum $= 2\pi$




Eqn in tet edge parameters for every edge




$$z, z' = \frac{1}{1-z}, z'' = \frac{z-1}{z} = 1 - \frac{1}{z}$$

Edge gluing equations:

$$\prod_{i=1}^{n} z_i^{a_{ij}} z_i'^{a'_{ij}} z_i''^{a''_{ij}} = \pm 1 \quad j = 1, \ldots, n$$

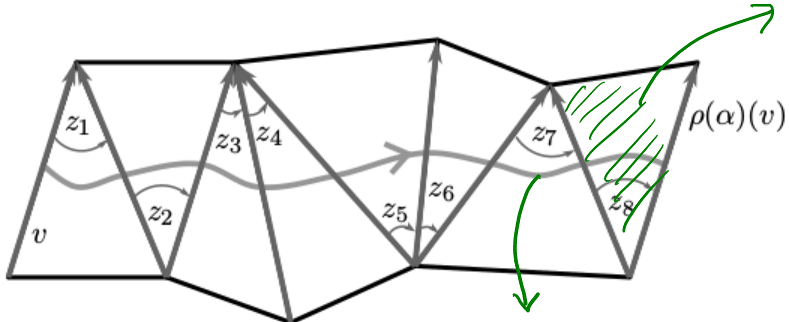
and angle sum equals $2\pi$ at every edge $e_j$.

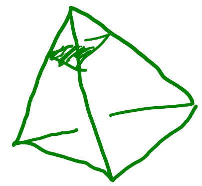
$\Rightarrow$ hyp str extends across edges

However this may not be complete!

③ Vertices: The torus formed by link triangles of vertices of tet get a similarity str.

For completeness this similarity str. has to be Euclidean ⟿ Holonomy is a translation



meridian or torus ⟿ holonomy described in tet parameters

Completeness equations:

$$\prod_{i=1}^{n} z_i^{b_{ik}} z_i'^{\,b'_{ik}} z_i''^{\,b''_{ik}} = \pm 1 \quad k = 1, \ldots, h$$

Thurston's Gluing Eqns

for meridian at every cusp $C_k$ (and similar one for longitude)

Thm (Thurston) Let $\mathcal{T}$ be an ideal triang of $M$ with $n$ tet & $h$ bound tori. If a point $(z_1^0, \dots, z_n^0)$ with $\mathrm{Im}(z_i^0) > 0$ $\forall i = 1, \dots n$ satisfies the gluing eqns (edge + comp.) then $M$ admits a complete hyp str.

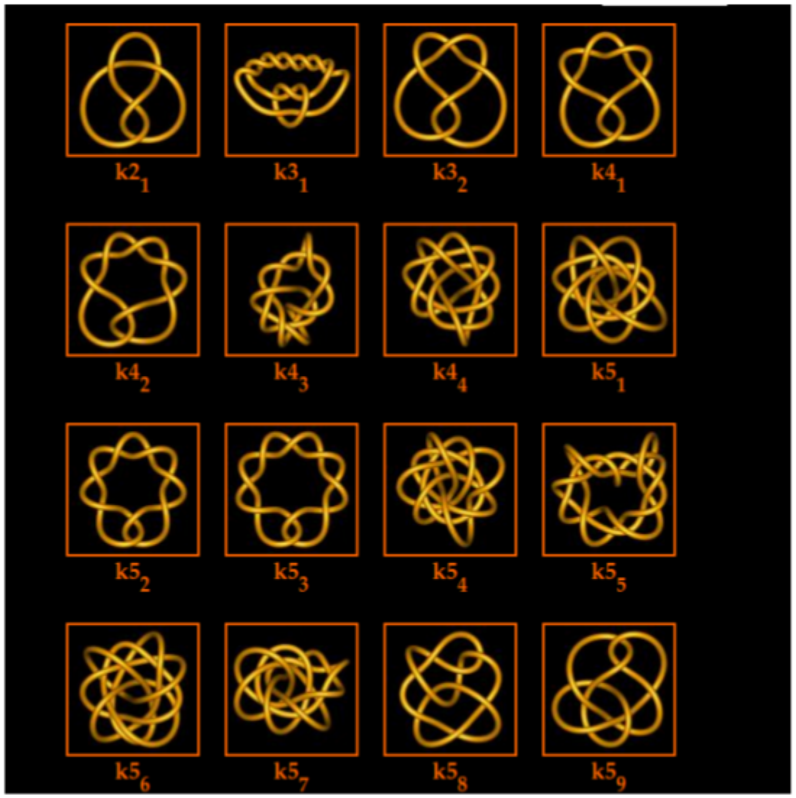
Cor: $\mathrm{vol}(M) = \sum_{i=1}^{n} D(z_i^0)$

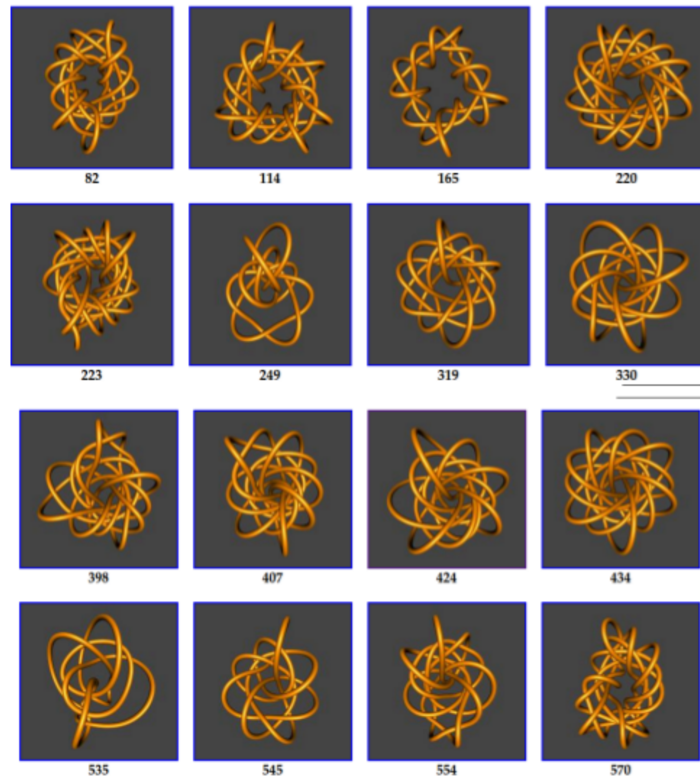
Prop: If $\mathcal{T}$ has the property that every edge is 6-valent then the point $(e^{\pi i/3}, \dots, e^{\pi i/3})$ gives comp. hyp str.

$\Rightarrow$ each tet is reg. ideal tet.

Geometric complexity on knots = min #tet

# Hyperbolic knot census



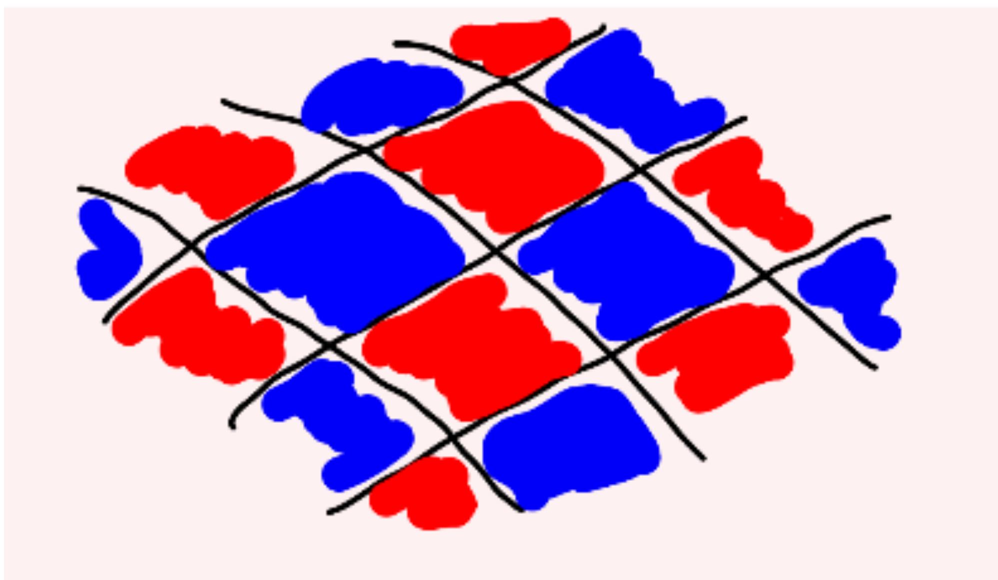



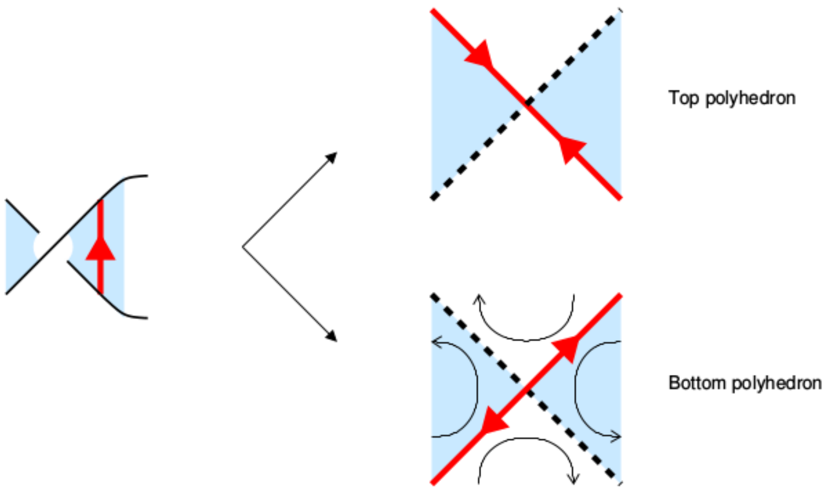

Callahan-Dean-Weeks (1999) – $\leq 6$ tetrahedra – 72 knots
C-Kofman-Patterson (2004) – 7 tetrahedra – 129 knots
C-Kofman-Mullen (2013) – 8 tetrahedra – 301 knots

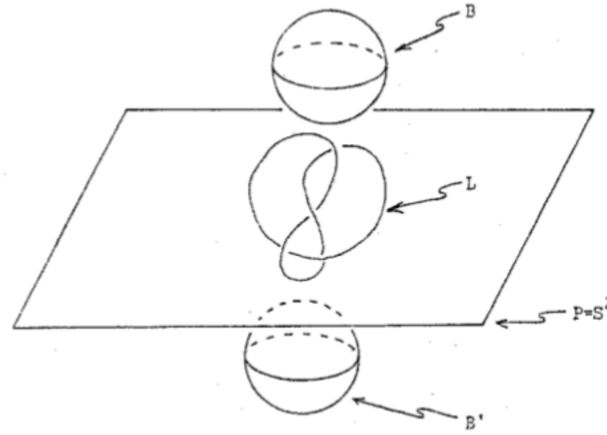
# ④ Checkerboard polyhedra & Examples

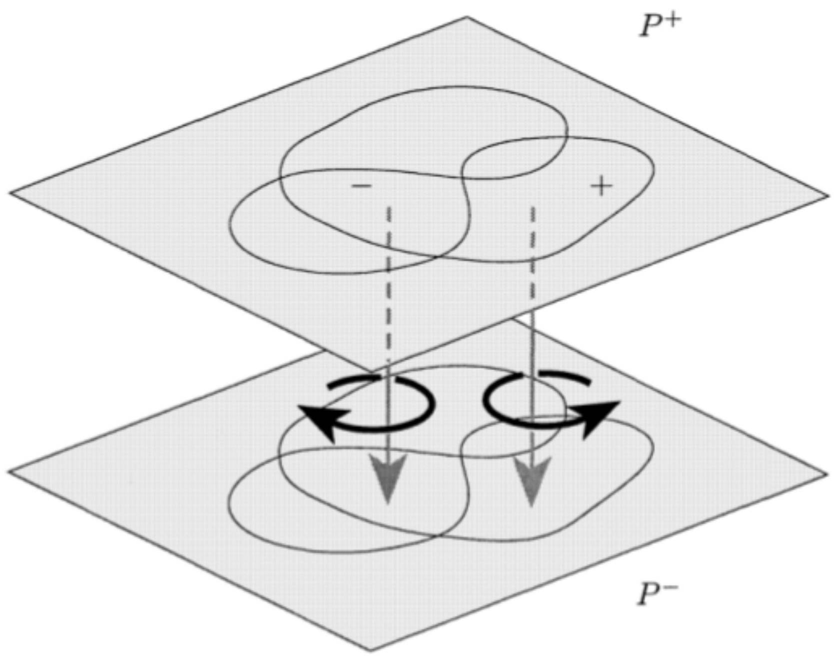
Let $K$ be a knot with reduced, prime, alternating diagram, and associated checkerboard surfaces $B$ and $R$.

Cutting $S^3 - K$ along both $B$ and $R$ simultaneously decomposes it into two identical (topological) ideal polyhedra.

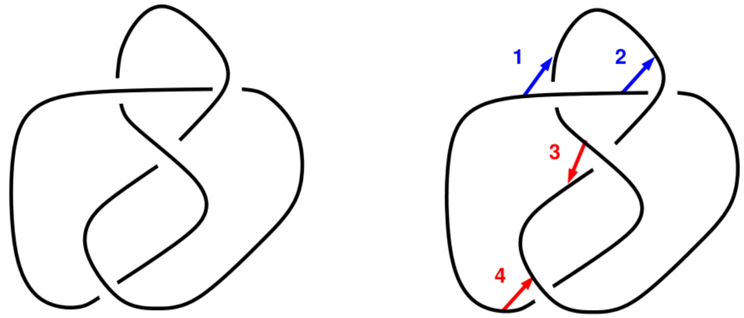



Top polyhedron
Bottom polyhedron


B
L
P=S^2
B'

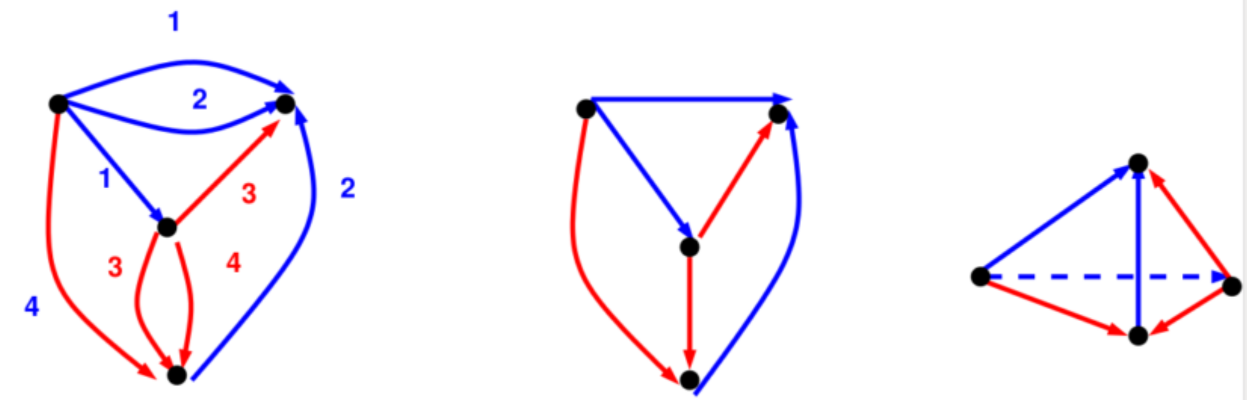

$P^+$
$-$
$+$
$P^-$

Example: Fig-8 knot

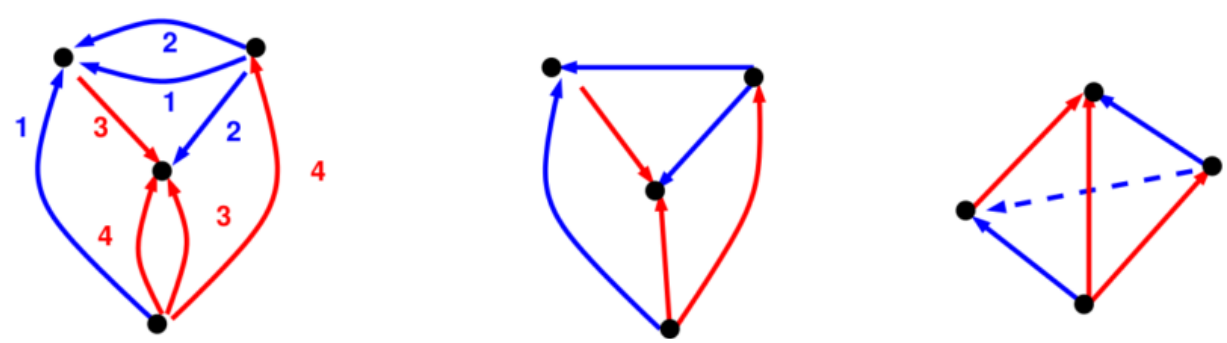


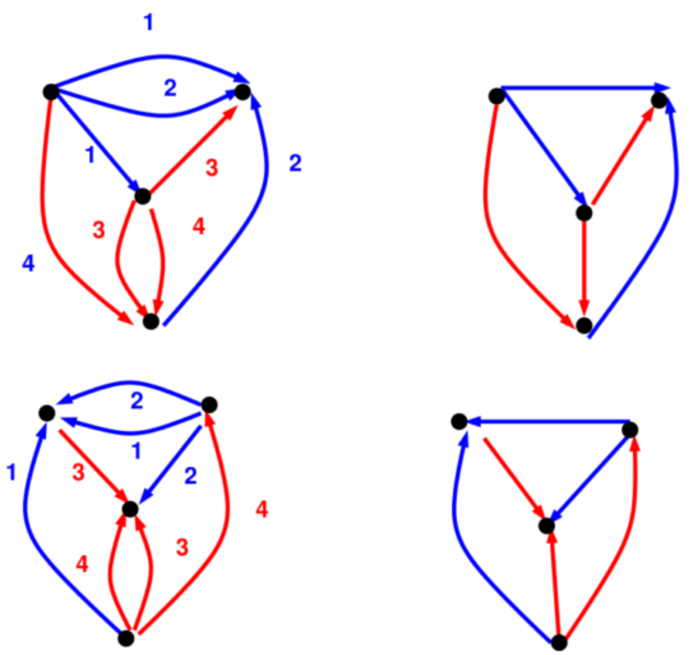

for geometry collapse bigons

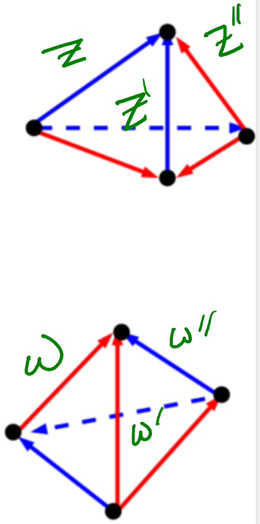
top poly $P^+$




bottom poly $P^-$



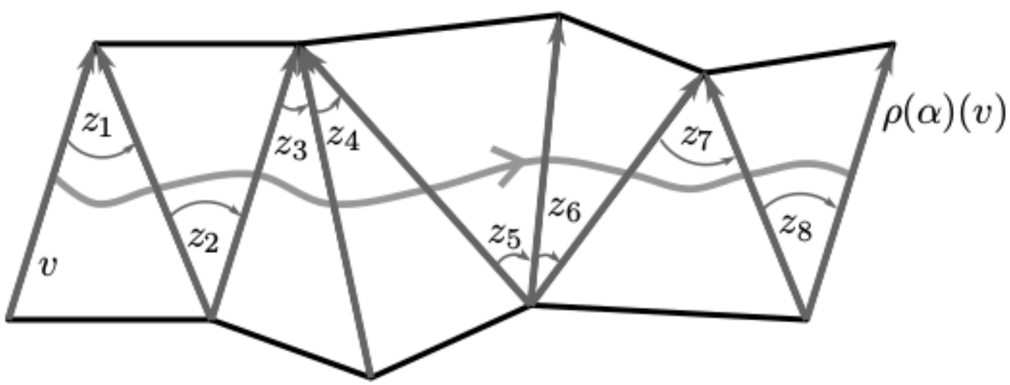



Blue edge: $z \times z' \times z' \times \omega' \times \omega'' \times \omega'' = 1$

$$H(\alpha) = \left(\frac{1}{1-z}\cdot\frac{z-1}{z}\cdot\frac{1-w}{1}\cdot\frac{w}{w-1}\right)^2 = \left(\frac{w}{z}\right)^2 = 1$$

$\Rightarrow \omega = z$




$z' = z$

$\frac{1}{1-z} = z$

$z^2 - z + 1 = 0$

$z = \frac{1+i\sqrt{3}}{2} = e^{\pi i/3}$

$\Rightarrow$ both tet are regular

$\Rightarrow$ vol(fig-8) $= 2\, v_{tet}$

# Computations

math.uic.edu/t3m/SnapPy/

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SnapPy 2.8 documentation »

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# SnapPy

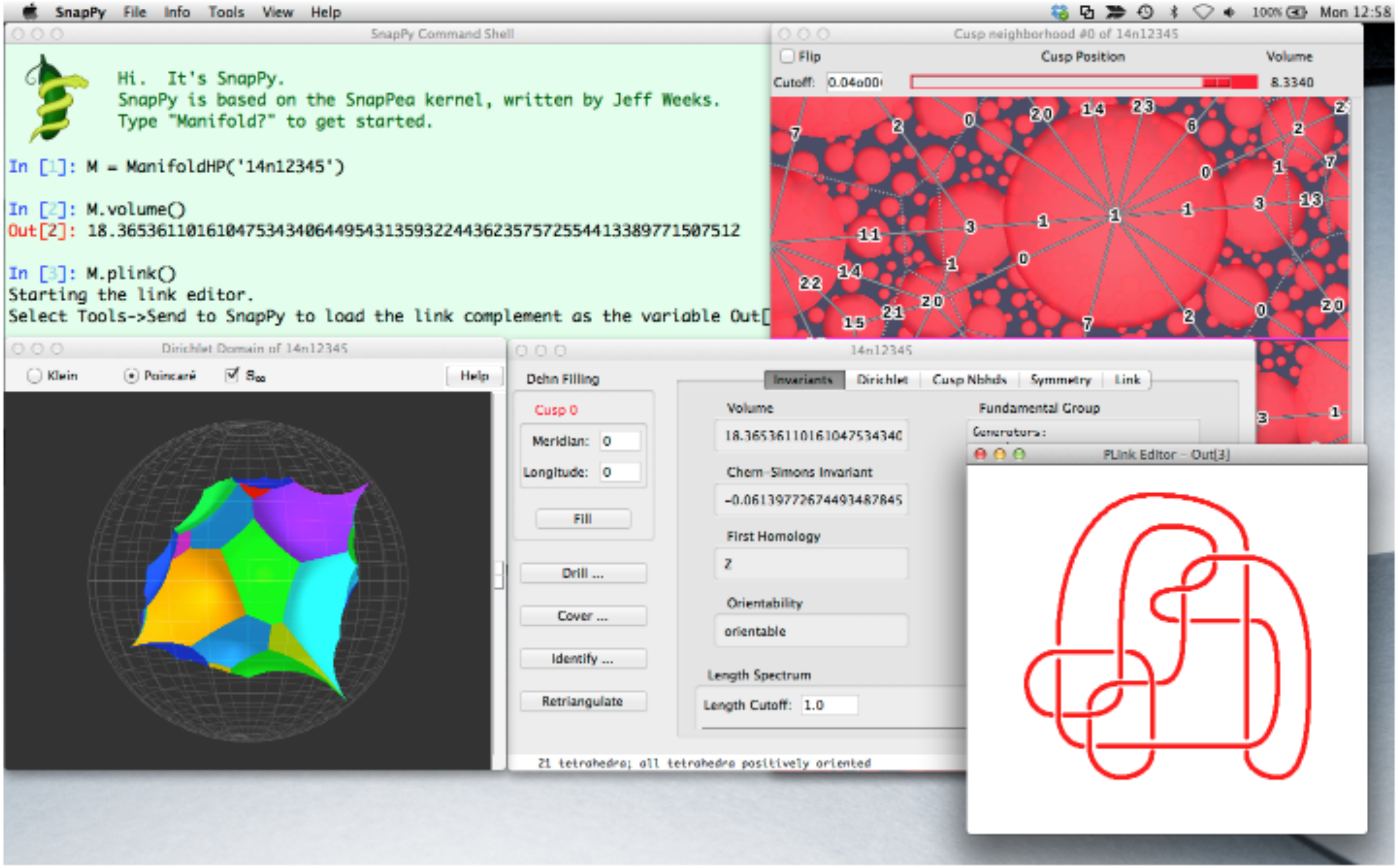
## What is SnapPy?

SnapPy is a program for studying the topology and geometry of 3-manifolds, with a focus on hyperbolic structures. It runs on Mac OS X, Linux, and Windows, and combines a link editor and 3D-graphics for Dirichlet domains and cusp neighborhoods with a powerful command-line interface based on the Python programming language. You can see it in action, learn how to install it, and watch the tutorial.

## News

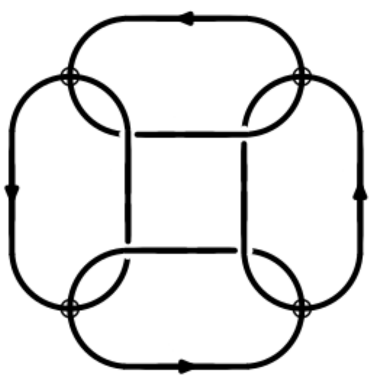
- Version 2.8 (June 2020): New features include:

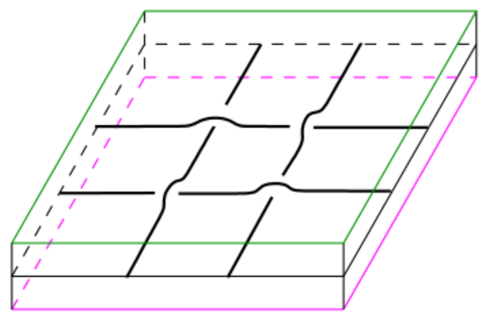
# Screenshot of SnapPy

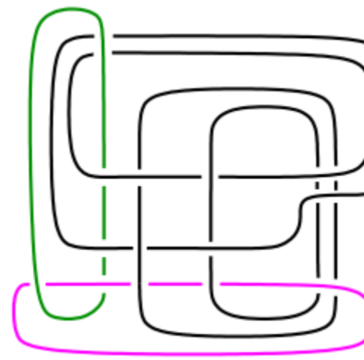


# ⑤ Hyperbolic Geometry of virtual Knots & links

virtual link $\longleftrightarrow$ link in $S \times I$ thickened surface







$L \subset T^2 \times I$

$T^2 \times I - L = S^3 - (L \cup H)$

$V_{oct}$ = vol of reg oct

$\mathrm{vol}(T^2 \times I - L) = 4 V_{oct}$

Colin Adams + REU students, Purcell-Howie

C-Kofman-Purcell - genus 1 case

Theorem(SMALL, 2017): The complement of a prime fully alternating link in S x I is hyperbolic. (S orientable and genus $\geq$ 1).

(A link in S x I is *fully alternating* if it is alternating and all faces of projection surface are disks.)

surface S

genus 1 $\longrightarrow$ link in $S^3$ by Hopf link addition

genus $\geq 2$ $\longrightarrow$ tot geodesic struct on $S \times I - L$