Hyperbolic Knot Theory

KNOTS THROUGH WEB (ONLINE)

INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

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Plan of Talks:
Talk 1 : Hyperbolic geometry, ideal tetrahedra, hyperbolic 3-manifolds
Talk 2: Hyperbolic knots, ideal triangulations, Example, SnapPy

Further reading: https://www.math.csi.cuny.edu/abhijit/hypknots-reading.html

1 Geometries

2-dim - $\mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2$ → hyperbolic geometry

In geometry, the **parallel postulate**, also called **Euclid's fifth postulate** because it is the fifth postulate in Euclid's *Elements*, is a distinctive axiom in Euclidean geometry. It states that, in two-dimensional geometry:

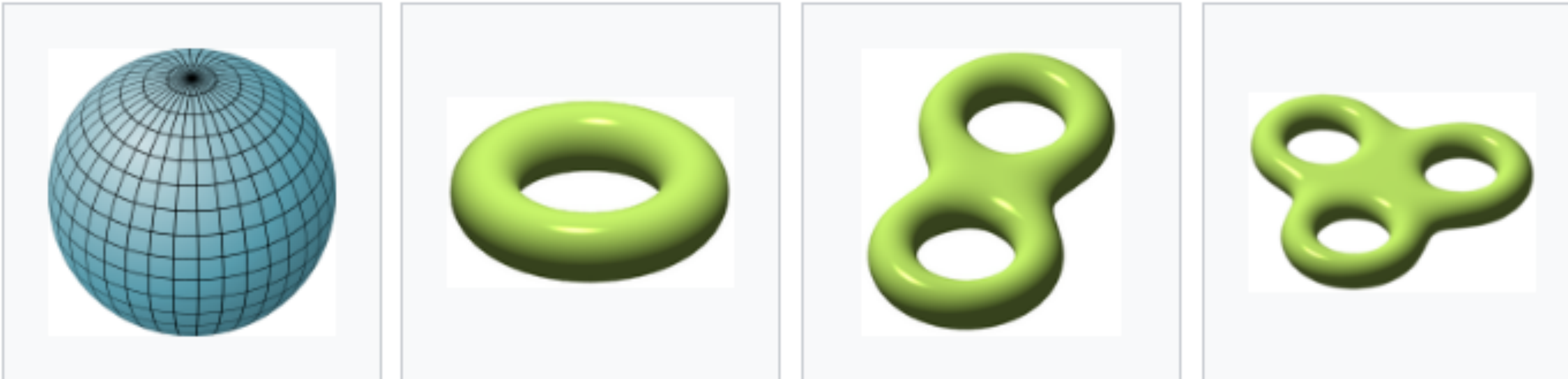
> *If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.*

Probably the best known equivalent of Euclid's parallel postulate, contingent on his other postulates, is Playfair's axiom, named after the Scottish mathematician John Playf

> *In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.*[4]

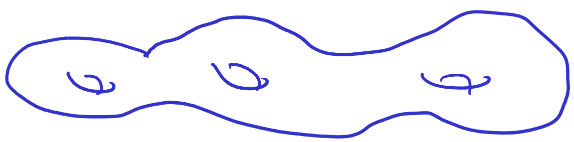
The parallel postulate seems natural enough, but it's the kind of statement that seems like it should be a theorem—something we prove using other axioms and postulates—rather than a postulate. For 2000 years, mathematicians tried to prove this, to show that the parallel postulate could be derived from the other axioms and postulates. (For the

The resulting geometries were later developed by Lobachevsky, Riemann and Poincaré into hyperbolic geometry (the acute case) and elliptic geometry (the obtuse case). The independence of the parallel postulate from Euclid's other axioms was finally demonstrated by Eugenio Beltrami in 1868.



genus 0 genus 1 genus 2 genus 3

Uniformization of surfaces - closed orient surface M_g genus g
or non-orient surface N_g genus g



S surface

	$\mathbb{S}^2$	$\mathbb{E}^2$	$\mathbb{H}^2$
$\chi(S)$	>0	$=0$	<0
Eg	$\mathbb{RP}^2$	T^2	M_g, N_g

most surfaces are hyp.

3-dim geometries: 8 Geometries:

$\mathbb{S}^3$ $\mathbb{E}^3$ $\mathbb{H}^3$

$\mathbb{S}^2 \times \mathbb{E}$ $\mathbb{H}^2 \times \mathbb{E}$ products

Nil $\widetilde{SL_2}$ twisted products

Sol

understood in terms of surfaces as Seifert fibered spaces (SFS) or torus semi-bundles

Geometrization of 3-mfds (Thurston, Perelman)

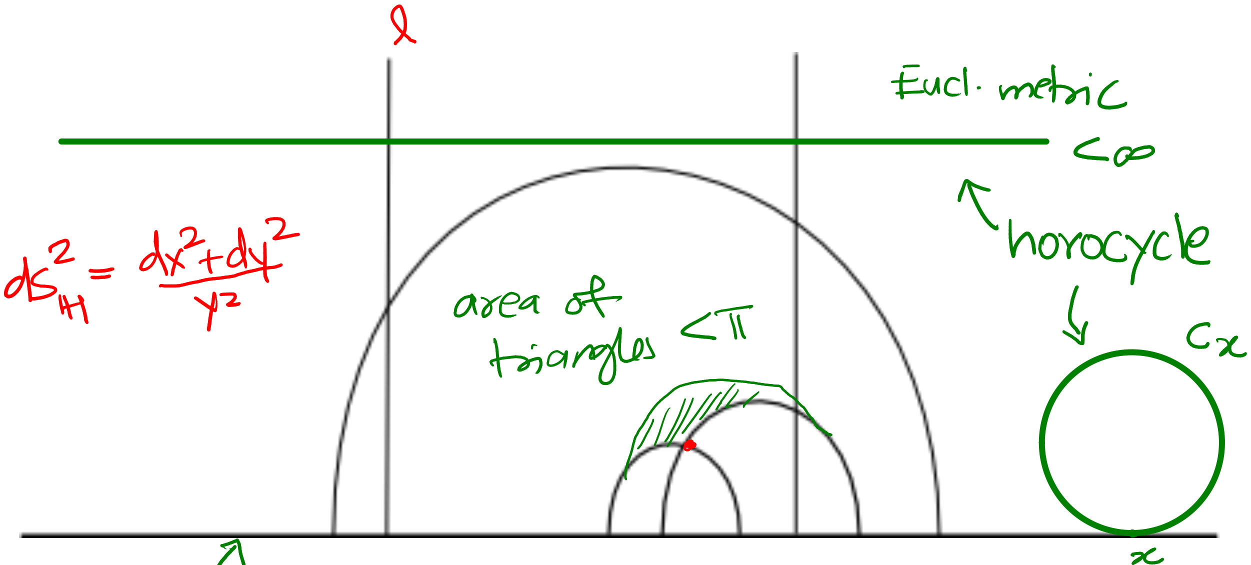
Most interesting geometry is $\mathbb{H}^3$.

2 Hyp. plane $\mathbb{H}^2$

Upper half-plane

Poincare disk

Crochet by Diana Tiamina



$$ds^2_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}$$

$\mathbb{H}^2 = \{(x,y) \mid y > 0\}$

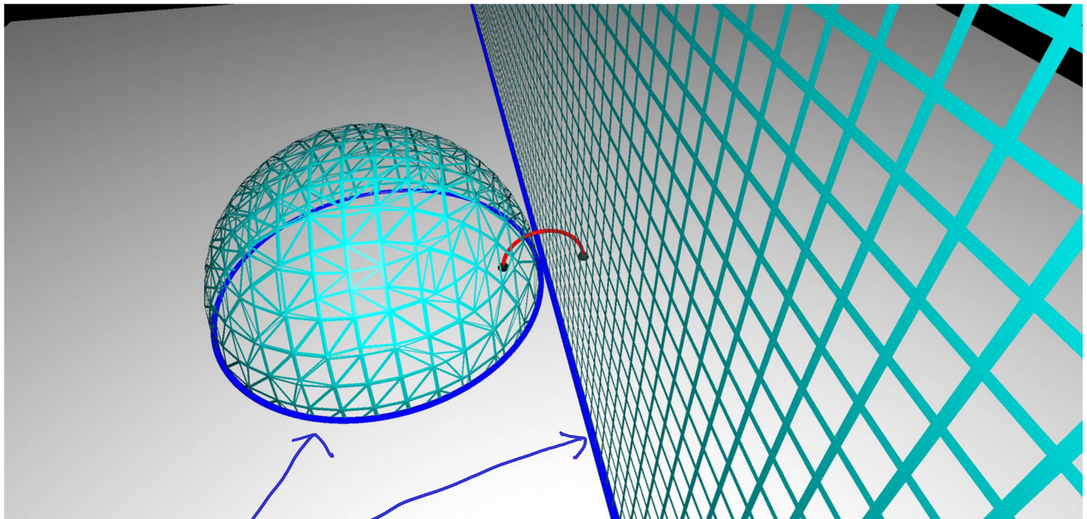
Upper half plane

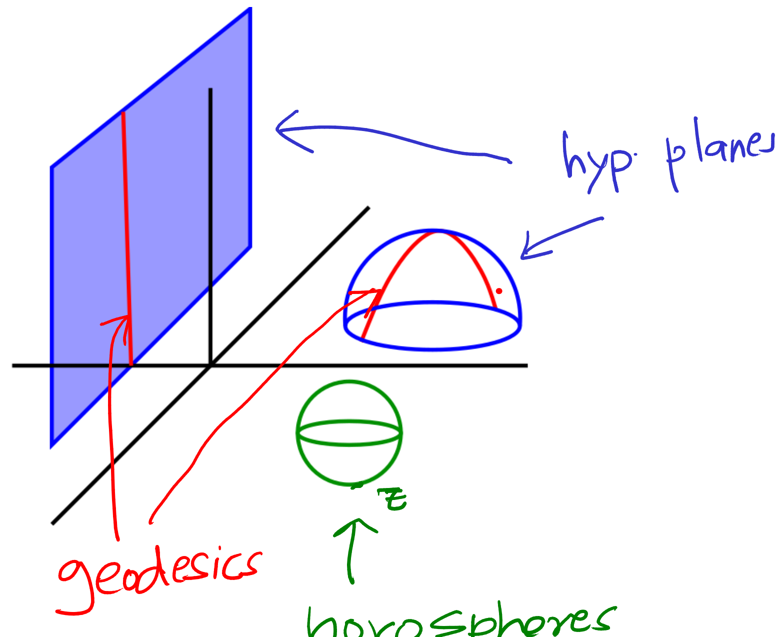
Geodesics - verticle half lines or semi-circles centered on x-axis

$\text{Isom}^+(\mathbb{H}^2) \cong PSL(2,\mathbb{R})$ acts as Möbius transf.

$$= \left\{ \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$$

$\mathbb{H}^3$ - hyp 3-space $\mathbb{H}^3 = \{(x,y,t) / t>0\}$ Upper half space





lines & circles on $\mathbb{C}\cup\{\infty\}$

xy-plane $\cup\{\infty\}$ = sphere at infinity = $\mathbb{C}\cup\{\infty\}$

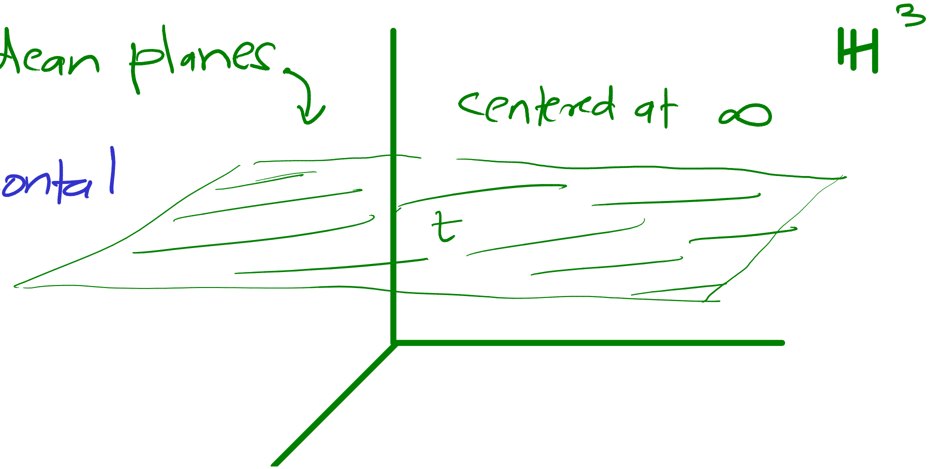
$\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$ acts as Möbius transf on $\mathbb{C}\cup\{\infty\}$

Horospheres - scaled Euclidean planes

$$ds^2_{\mathbb{H}} = \frac{dx^2+dy^2+dt^2}{t^2}$$

horizontal plane

t = fixed



② Ideal tetrahedra – geodesic hyp tet with all vertices on $\mathbb{C}\cup\{\infty\}$

$$\Delta = z_1, z_2, z_3, z_4$$

An Ideal Tetrahedron
in Poincare ball model

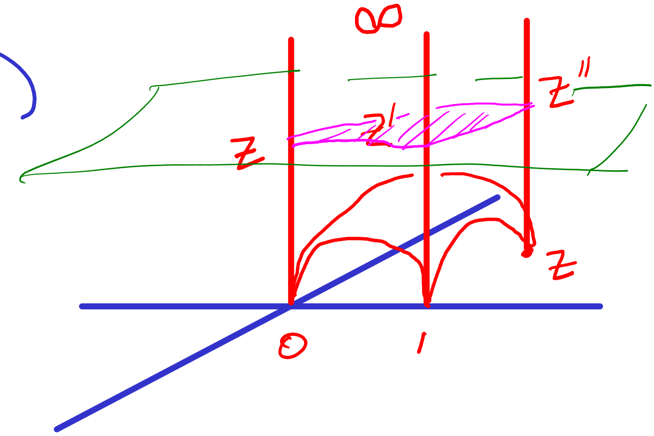
Cross ratio : $[z_1,z_2,z_3,z_4] = \dfrac{z_4 - z_1}{z_4 - z_2}\,\dfrac{z_3 - z_2}{z_3 - z_1}$

is a "geometric invariant" of ideal tet.

$\varphi : (z_1,z_2,z_3,z_4) \longmapsto (0,\infty,1,z)$

Möbius transf

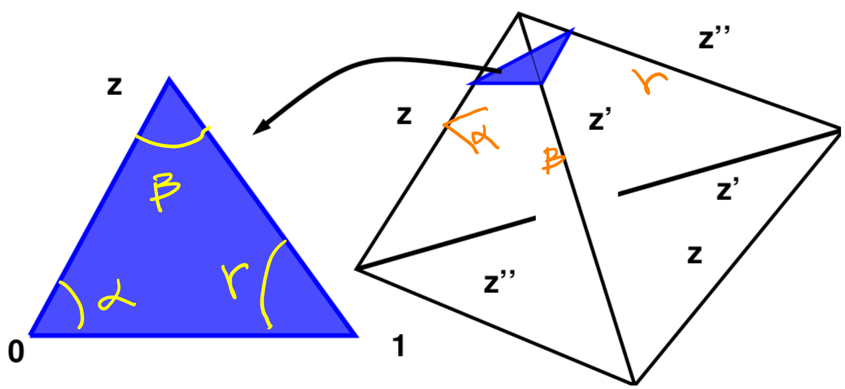
z = cross ratio



edge parameters for ideal tet.

$$z, \quad z' = \frac{1}{1-z}, \quad z'' = \frac{z-1}{z}$$

cross ratio changes like this
when you change order of vert keeping orient.



α, β, γ are dihedral angles of Δ

$$z = \frac{\sin\gamma}{\sin\beta}\, e^{i\alpha}$$

Angles of Δ $\longleftrightarrow$ Sim classes of Eucl. triangles $\longleftrightarrow$ Isom classes of ideal tet $\longleftrightarrow$ $\{z \mid \mathrm{Im}(z) > 0\}$

Volume of ideal tet.

Lobachevsky fn $\Lambda(\theta) = -\int_0^\theta \log|2\sin t|\,dt$

Thm Let $T_{\alpha,\beta,\gamma}$ be ideal tet with dihedral angles α, β, γ. The $\mathrm{vol}(T_{\alpha,\beta,\gamma}) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$

Eg. 1) Ideal tet of max volume $= T_{\pi/3,\pi/3,\pi/3}$ = reg. ideal tet

$$\mathrm{vol} = 3\Lambda(\pi/3) \approx 1.0149\ldots = V_{tet}$$

Dilogarithm: $\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$

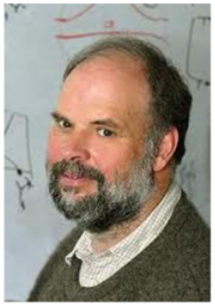
(Kummer) $\mathrm{Im}(\mathrm{Li}_2(z)) = \Lambda(\arg z) + \Lambda(\arg z') + \Lambda(\arg z'') - \log|z| \arg(1-z)$

Bloch-Wigner dilog: $D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z) = \mathrm{vol}(\Delta(z))$

Motivation

In 1980s, the discovery of two different types of knot invariants revolutionized research in knot theory.

Vaughan Jones used representation theory and discovered the Jones polynomial, which resulted in the theory of quantum invariants of knots.



Bill Thruston used hyperbolic geometry to introduce geometric invariants, which resulted in geometric invariants of knots.

Relating quantum knot invariants to geometric invariants of knots is a very active area of research.

3 Hyp 3-mfds

M be fin vol., complete, orient hyp 3-mfd

M hyp :

1. $M = \mathbb{H}^3/\Gamma$ $\quad \Gamma < PSL(2,\mathbb{C})$ disc & torsion free
2. $p : \mathbb{H}^3 \to M$ covering map + local isometry
3. $\rho_0 : \pi_1 M \to PSL(2,\mathbb{C})$ discrete + faithful

Tomorrow - Structure on hyperbolic 3-manifolds, Hyperbolic knots, Thurston's gluing equations, Examples, SnapPy