Role of Symmetries in Rotational Spectra

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Outline

- Basics concepts of symmetries in quantum systems
- Unitary transformations, degeneracy and multiplets
- Discrete symmetries in nuclei
- Consequences of discrete symmetries
-

Beauty ⟺ Symmetry

How the two are linked is difficult to define

Yet, both are intimately linked

Also linked are,

Conserved quantities ⟺ Symmetries

Quantities which remain invariant play an important role in defining the properties of a system

This is also the origin of quantum numbers

Conserved quantities

- Some quantities remain unchanged with the passage of time
- Many conserved quantities are related to *Global symmetries*
- Energy, which is linked to invariance under time translation
- Linear momentum, which is linked to invariance under space translation
- Angular momentum, which is linked to invariance under rotation in space
  - These are all continuous symmetries of space-time, and are valid in both the classical and the quantum world.

Theorem of Emmy Noether

Each continuous symmetry is related to a conserved quantity

www.EmmyNoether.com

Unitary Transformation, Degeneracy and Multiplets

- A useful unitary transformation arising from the Hamiltonian H is the time evolution operator

$$U = \exp(-iHt/\hbar)$$

- An operator Q evolves from time *t*=0 to *t* as,

$$U^{\dagger} Q U = Q_0$$

- While a state $|\psi(t)\rangle$ evolves as

$$|\psi(t)\rangle = U(t)|\psi(t=0)\rangle$$

- If Q is conserved i.e. $dQ/dt = 0$, we have

$$[Q, H] = QH - HQ = 0$$

If the symmetry transformations represented by Q is unitary then $Q^{\dagger} H Q = H$,and hamiltonian remains invariant.

- As an example, a very useful group of transformations which leaves the hamiltonian invariant is rotation, represented by the rotation operator. It generator is the angular momentum operator *L* or, *J*.

Rotation Group

- Rotation about z-axis can be represented by

$$\Psi' = e^{-i\theta j_z/\hbar}\Psi = (1 - i\theta J_z/\hbar)\Psi$$

- It is a Lie group whose algebra is defined by

$$[J_k, J_l] = i\varepsilon_{klm} J_m$$

- Casimir operator, which commutes with all the generators of the group, is

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

- The states $|j, m\rangle$ are simultaneous eigenstates of J^2 and J_z.

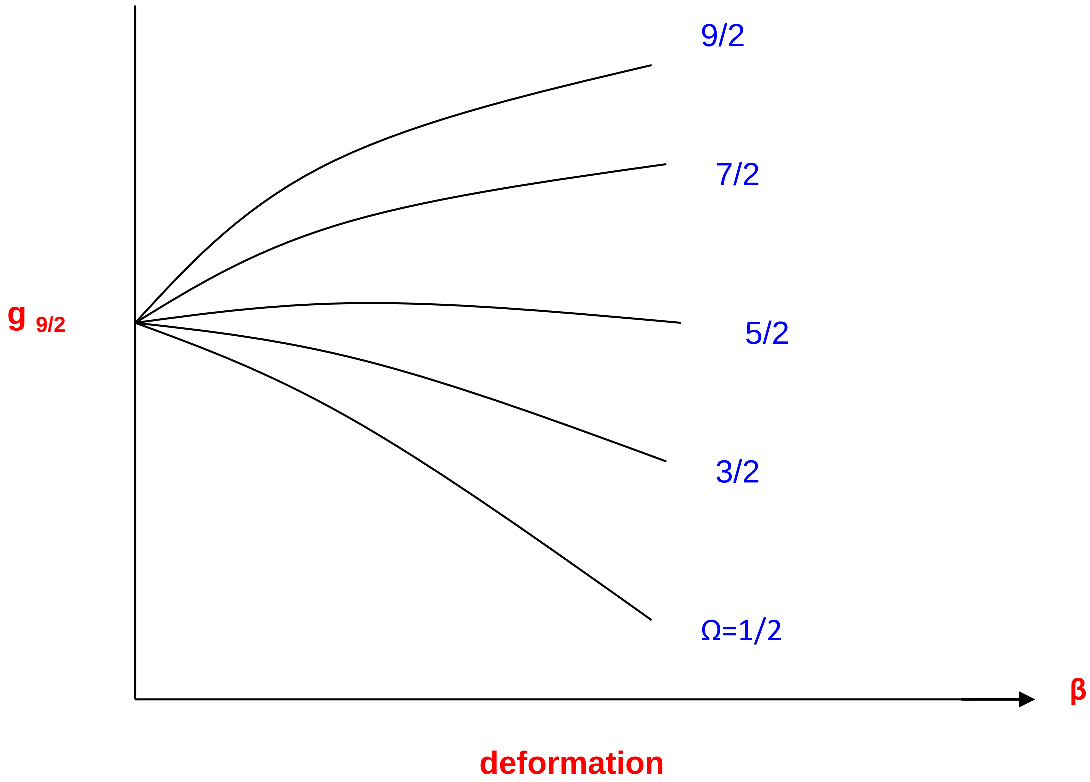
Degeneracy and Multiplets

- If $H\Psi = E\Psi$ then $H\Psi' = H(1+i\theta J_z)\Psi = E\Psi'$

  Thus Ψ and Ψ' are both simultaneous eigenstates with the same eigenvalue.
- A degeneracy arises in the m-substates.
- An energy eigenstate can have *n*-fold degeneracy if *n*-fold rotation of Ψ in some space leaves it invariant.
- Degeneracy is lifted if the corresponding symmetry is broken and a multiplet arises

Example: Transition from Shell Model to Nilsson Model

J is a good quantum number for spherically symmetric potential. Even a small deformation breaks the symmetry. If the deformation has an axial symmetry about the z-axis, J_z is the only conserved quantity. The (2j+1) fold degeneracy is lifted and multiplet arises. Therefore, the quantum number Ω is used to label the states.



The basic nucleon-nucleon interaction must exhibit invariance under translation, rotation, reflection in space and time etc.

THE MEAN FIELD EMERGING FROM A COLLECTION OF NUCLEONS IN A CONFIGURATION MAY BREAK ONE OR MORE OF THESE SYMMETRIES

Breaking of spherical symmetry

- Rotational motion becomes possible
- Rotational Bands built on intrinsic states make an appearance
- Therefore, patterns of rotational bands are typical of a given type of symmetry
- A given rotational band can be labeled by the conserved quantum numbers

Patterns in spectra

PATTERN RECOGNITION

is an important step towards

IDENTIFICATION OF

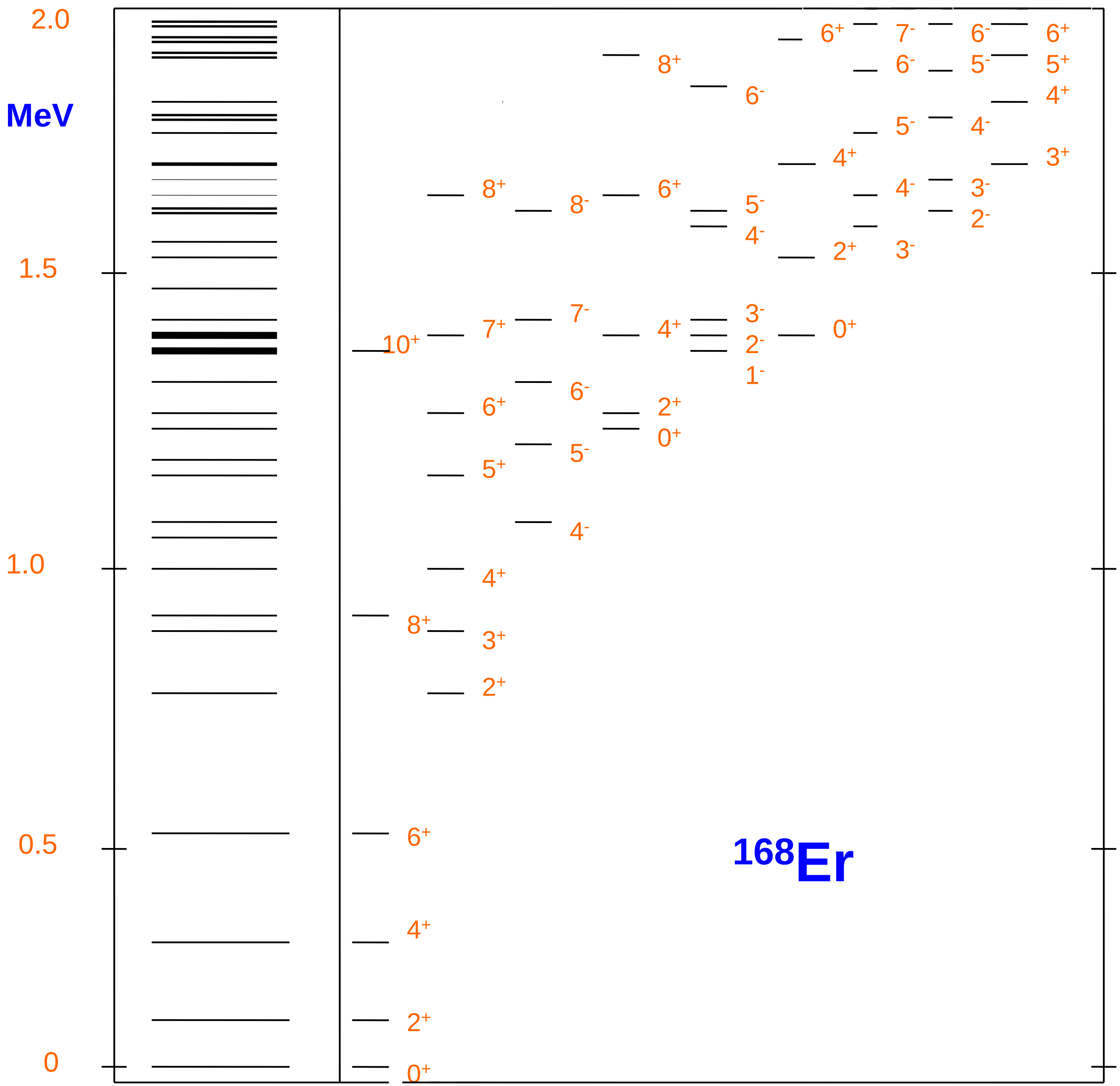
SYMMETRIES

or,

BREAKING OF SYMMETRIES

EXAMPLE:

Spectrum of 168Er



2.0
MeV
1.5
1.0
0.5
0
10+
8+
6+
4+
2+
0+
7+
5+
3+
8-
7-
6-
5-
4-
1-
2-
3-
0+
168Er

Symmetries

in the quantum world lead to discreteness in conserved quantities and are local

- Point Group Symmetries in Crystal structure
- Space Inversion – Parity
- Time Reversal – Kramer’s Degeneracy
- Rotation by any angle about the symmetry axis of a spheroid
- Rotation by 180 about an axis normal to the symmetry axis of a spheroid
- Reflection about a plane of a pear shape

NUCLEAR SHAPES

Radius vector of an arbitrarily deformed surface

$$R(\theta,\varphi) = R_0[1 + \sum_{\lambda,\mu} \alpha_{\lambda,\mu} Y^{*}{}_{\lambda\mu}(\theta,\varphi)]$$

- Different spherical harmonics have different geometric symmetries and may be present in the mean field of a nucleus.
- Most common is the $\lambda = 2$ quadrupole term. Higher order terms also occur in specific situations in many nuclei.

- Empirical evidence exists for the
- ***quadrupole,***
- ***quadrupole + hexadecapole,***
- ***quadrupole + octupole***.
- While **axial shapes** are most common, evidence exists for **non-axial shapes** also.

Axial and Reflection symmetry

- Most well known result for the axially symmetric e-e nuclei is the K=0 assignment to the ground rotational band – since no rotation is possible about axis of symmetry
- Additional restriction for the quadrupole axial symmetric shapes is the $R_1(\pi)$ symmetry which forbids odd-I states. This is evident in the observation of I=0,2,4,…levels in a K=0 band.

Discrete Symmetries in Nuclei

- **Most commonly encountered discrete symmetries in nuclei are**

1. Parity *P*
2. Rotation by π about the body fixed *x, y, z* axes,
3. $R_x (\pi), R_y (\pi,), R_z (\pi)$
4. Time reversal *T*
5. $TR_x (\pi), TR_y (\pi), TR_z (\pi)$.

- **These are all two fold discrete symmetries, and their breaking causes a doubling of states.**

See *Dobaczewski et al* (Phys. Rev. C62, 014310, and 014311 (2000)) for a complete classification

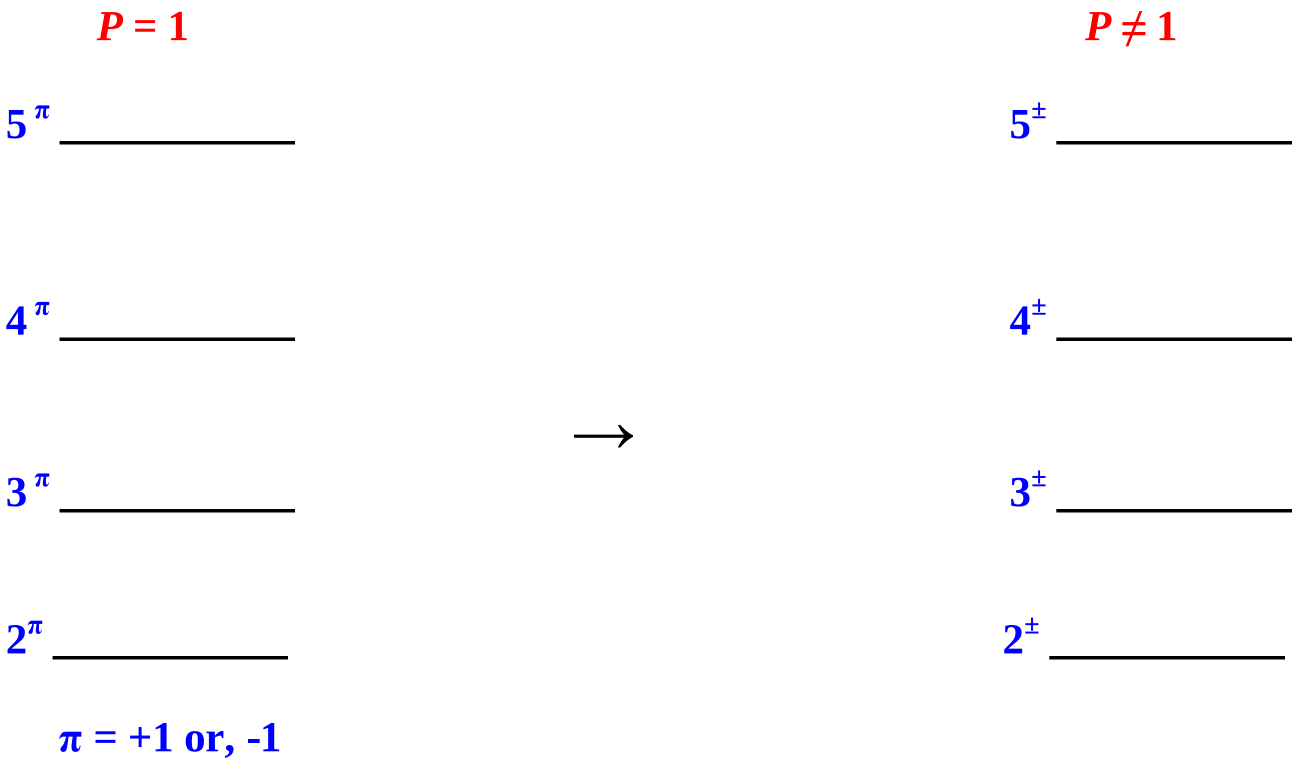
Simple rules to work out the consequences of these symmetries

1. When *P* is broken, we observe a parity doubling of states. A sequence like *4+, 5+, 6+, …* turns into *4±, 5±, 6±, …* . (in e-e nuclei)

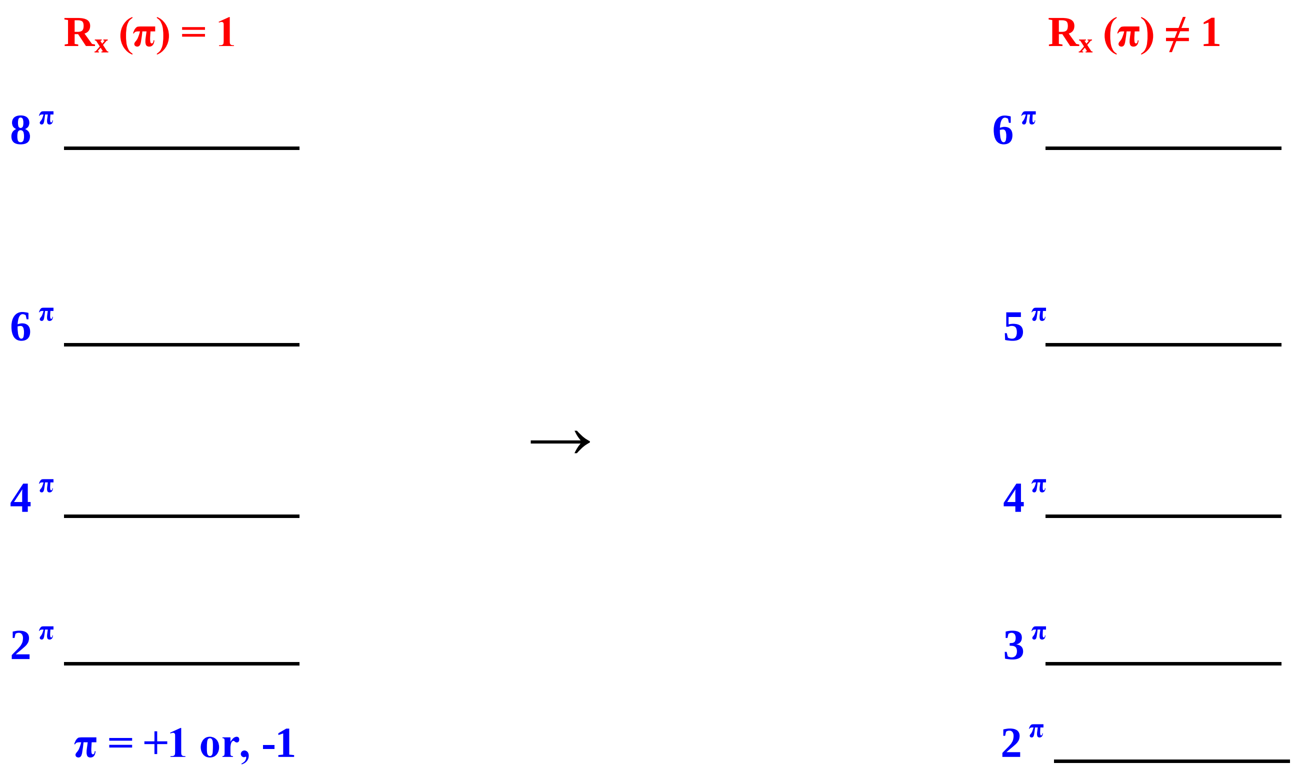
2. When R_X *(π)* is broken, states of both the signatures occur. The two sequences like *1/2, 5/2,…*etc. and *3/2, 7/2, …*etc. having different signatures, merge into one sequence like *1/2, 3/2, 5/2, 7/2 …* etc. (in odd-A nuclei)

3. When R_Y *(π)* T is broken, a doubling of states of the allowed angular momentum occurs. A sequence like *I, I+2, I+4, …* etc. becomes *2(I), 2(I+2), 2(I+4), …*, each state now occurring twice (chiral doubling).

4. When *P=*R_X *(π)*, the two signature partners will have different parity. Thus states of alternate parity occur. We obtain a sequence like *2+, 3-, 4+, 5- …* etc.

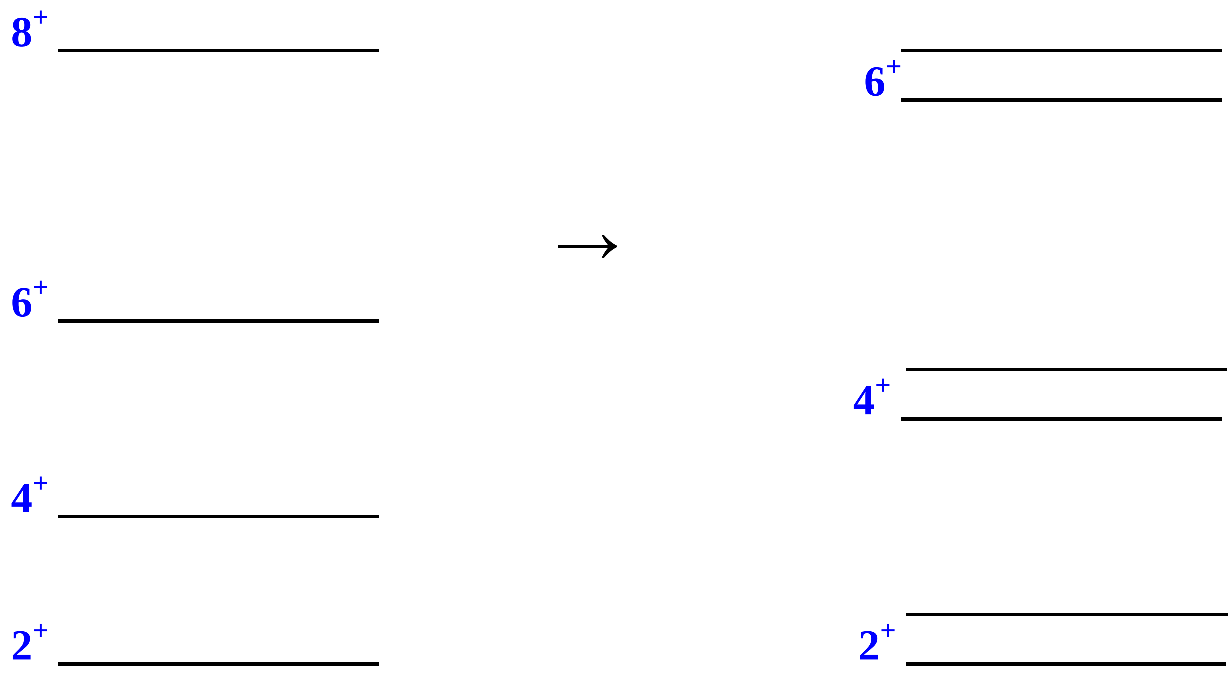


RULE NO. 1


$R_x(\pi) = 1$
8^{π}
6^{π}
4^{π}
2^{π}
$\pi = +1$ or, -1
$\rightarrow$
$R_x(\pi) \neq 1$
6^{π}
5^{π}
4^{π}
3^{π}
2^{π}

RULE NO.2

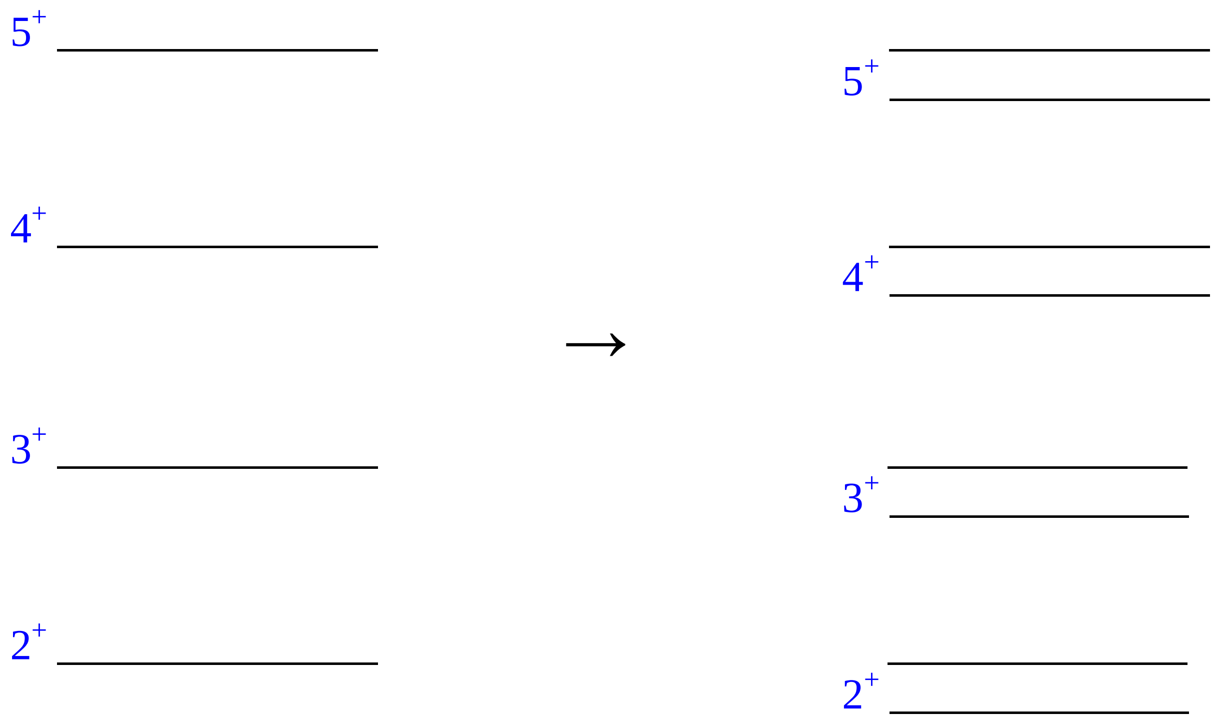
$R_y(\pi)\,T = 1,\ R_x(\pi) = 1$ $\longrightarrow$ $R_y(\pi)\,T \neq 1,\ R_x(\pi) = 1$



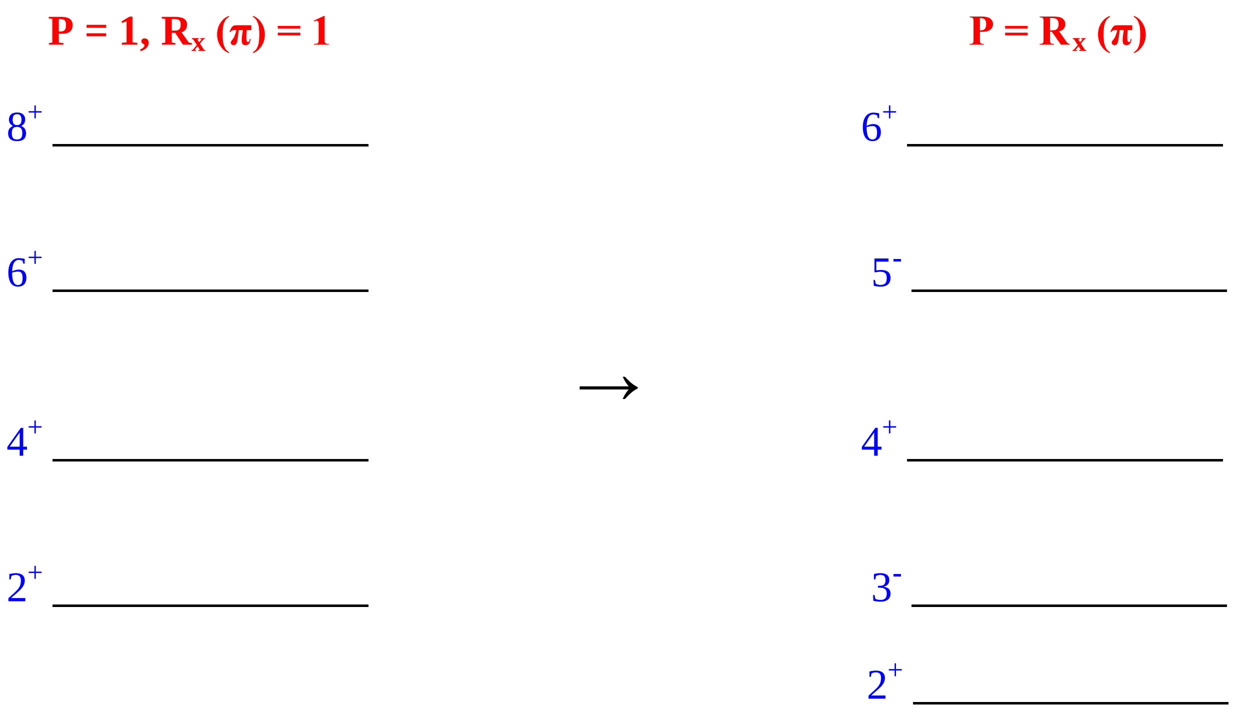
RULE NO. 3

$R_y(\pi)\,T = 1,\ R_x(\pi) \neq 1$

$R_y(\pi)\,T \neq 1,\ R_x(\pi) \neq 1$



RULE NO. 3



RULE NO. 4

Parity and Signature

- Total wave-function is a product of intrinsic and rotational part $|IMK\rangle = D^{I}_{MK}\chi_{K}$.
- Information of the parity of a state resides in the intrinsic part and not the rotational part.
- Besides axial symmetry, a spheroid also has a reflection symmetry in the 1-2 plane, which is represented by $R_1(\pi)$. It operates on the intrinsic and the rotational part differently and invariance of the wave-function leads to the signature quantum number.
- For K=0 intrinsic states, $R_1(\pi)\chi_{K=0} = r\chi_{K=0}, \quad r = \pm 1.$

Intrinsic Wavefunction and its consequences

The total wavefunction

$$\Psi^{I}_{MK} = \chi_{\Omega} D^{I}_{MK}$$

where χ_{Ω} is the intrinsic part. Axial symmetry is assumed so that $K = \Omega$

The total wavefunction must remain invariant under $R_x(\pi)$ and $R_e(\pi)$ acting on internal and collective coordinates with the condition that

$$R_x(\pi) = R_e(\pi)$$

Since j is not a good q.n.,

$$\chi_K = \sum_J C_J \chi_{JK}$$

For K=0, the following holds

$$R_x(\pi)\chi_{K=0} = r\chi_{K=0},$$

$$R_x^2(\pi)\chi_{K=0} = r^2\chi_{K=0},$$

so that $r^2 = 1$ and $r = \pm 1$

One may also write these expressions as

$$R_x(\pi)\chi_{K=0} = e^{-i\pi J_x}\chi_{K=0} = e^{-i\pi\alpha}\chi_{K=0}$$

which leads to values α=0 and α=1 corresponding to r = +1 and r = -1.

Both α and r are termed as the signature quantum number.

Signature and angular momentum get connected by the relation

$$R_e(\pi)D^I_{MK=0} = e^{-i\pi I}Y^I_M = (-1)^I Y^I_M$$

Therefore,

$$r = (-1)^I,$$

and the K=0 band can be classified as

$$\alpha = 0,\ r = +1, \quad I = 0,\ 2,\ 4, \ldots\ldots$$

$$\alpha = 1,\ r = -1, \quad I = 1,\ 3,\ 5, \ldots\ldots\ldots$$

The GSB of even-even nuclei is the best example of r=+1 signature.

A K=0 band in an odd-odd nucleus has both r=+1 and r=-1 signature.

For K ≠ 0, the intrinsic states are two fold degenerate because of $\hat{R}_x(\pi)$ symmetry. Note that $\hat{R}_x(\pi)$ and time-reversal *T have* the same effect on the wave-function. The time-reversed state $\overline{K}$ has the negative value of j_z, so that

$$\chi_{\overline{K}} = R_x^{-1}\chi_K$$

and

$$\chi_{\overline{K}} = e^{i\pi j_x}\cdot\chi_K = \sum_j (-1)^{j+K}\cdot\chi_{j-K}$$

The rotational wave function changes as

$$R_e D^I_{MK} = e^{-i\pi I} D^I_{MK} = (-1)^{I+K} D^I_{M-K}$$

so that a rotationally invariant wave function may be constructed as

$$\Psi^I_{MK} = \left(\frac{2I+1}{16\pi^2}\right)^{\frac{1}{2}}\left[\chi_K D^I_{MK} + (-1)^{I+K}\chi_{\overline{K}} D^I_{M-K}\right]$$

For odd-A nuclei, $$R_x^2 \chi_K = (-1)^{2j} \chi_K,$$ *2j* odd.

Now, $$R_x = e^{-i\pi j_x} = e^{-i\pi\alpha}$$

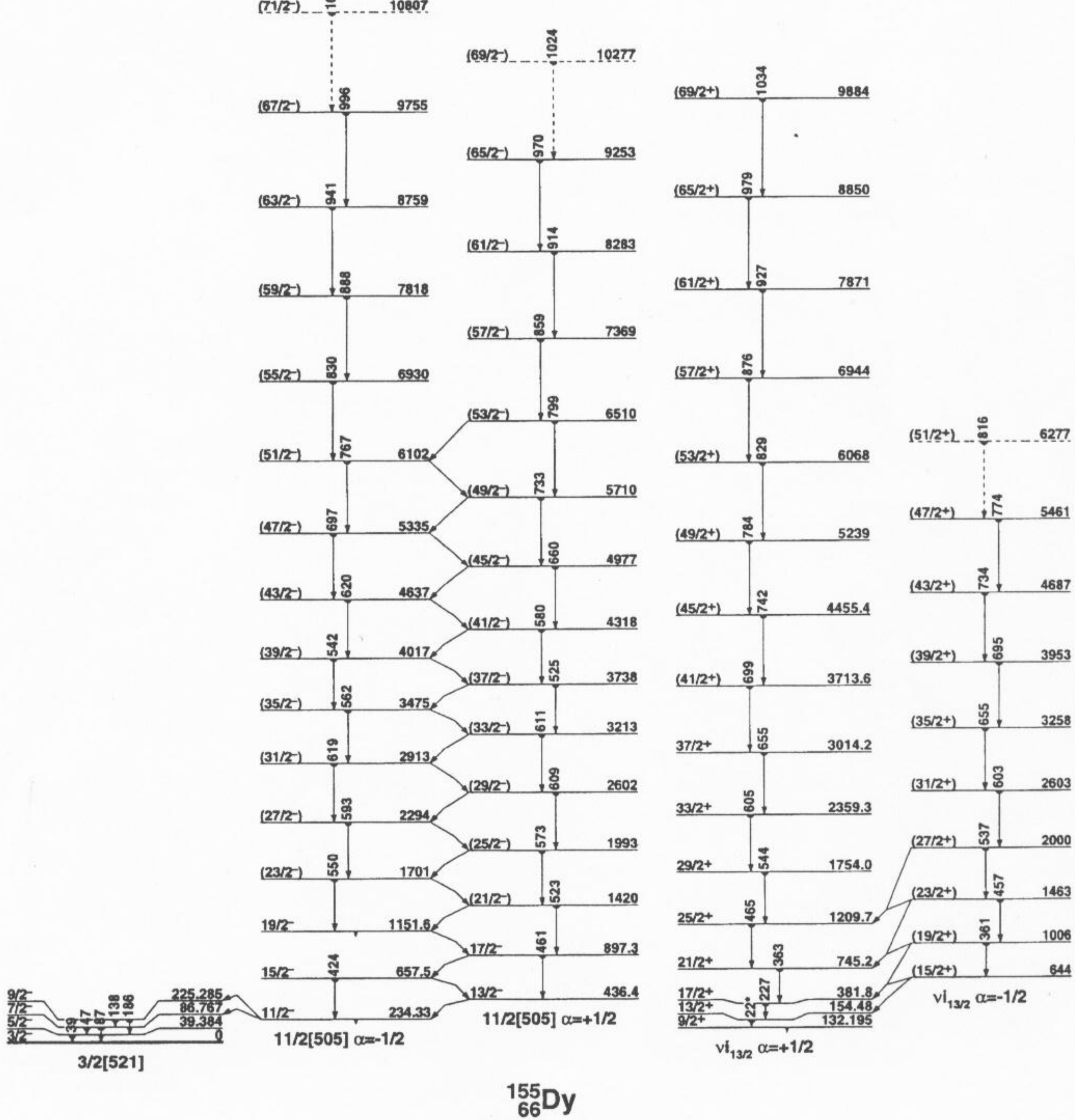
and, $$R_e = e^{-i\pi I}$$

together with, $$R_x^{-1} R_e = 1$$ lead us to the classification

$$I = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\ldots\ldots\ldots, \quad \alpha = \frac{1}{2}, \quad r = -i,$$

$$I = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots\ldots\ldots\ldots, \quad \alpha = -\frac{1}{2}, \quad r = +i$$

In general, **I = (α + even number)**.



(71/2−) 10807
(67/2−) 996 9755
(63/2−) 941 8759
(59/2−) 888 7818
(55/2−) 830 6930
(51/2−) 767 6102
(47/2−) 697 5335
(43/2−) 620 4637
(39/2−) 542 4017
(35/2−) 562 3475
(31/2−) 619 2913
(27/2−) 593 2294
(23/2−) 550 1701
19/2− 1151.6
15/2− 424 657.5
11/2− 234.33
11/2[505] α=-1/2
(69/2−) 1024 10277
(65/2−) 970 9253
(61/2−) 914 8283
(57/2−) 859 7369
(53/2−) 799 6510
(49/2−) 733 5710
(45/2−) 660 4977
(41/2−) 580 4318
(37/2−) 525 3738
(33/2−) 611 3213
(29/2−) 609 2602
(25/2−) 573 1993
(21/2−) 523 1420
17/2− 461 897.3
13/2− 436.4
11/2[505] α=+1/2
(69/2+) 1034 9884
(65/2+) 979 8850
(61/2+) 927 7871
(57/2+) 876 6944
(53/2+) 829 6068
(49/2+) 784 5239
(45/2+) 742 4455.4
(41/2+) 699 3713.6
37/2+ 655 3014.2
33/2+ 605 2359.3
29/2+ 544 1754.0
25/2+ 465 1209.7
21/2+ 363 745.2
17/2+ 227 381.8
13/2+ 22* 154.48
9/2+ 132.195
νi13/2 α=+1/2
(51/2+) 816 6277
(47/2+) 774 5461
(43/2+) 734 4687
(39/2+) 695 3953
(35/2+) 655 3258
(31/2+) 603 2603
(27/2+) 537 2000
(23/2+) 457 1463
(19/2+) 361 1006
(15/2+) 644
νi13/2 α=-1/2
9/2− 225.285
7/2− 86.767
5/2− 39.384
3/2− 0
39 47 87 138 186
3/2[521]
155 66 Dy

Consequences of the signature q.n.

- For K=0 bands, r=+1 corresponds to I=0,2,4,… and r=-1 corresponds to I=1,3,5,…
- Only r=+1 sequence is seen in even-even nuclei, while both r=±1 are observed in odd-odd nuclei.
- However, its full advantage is seen in odd-A (and odd-odd nuclei also) nuclei.
- In odd-A, K=1/2 bands, the decoupling term plays the role of splitting the two signatures of a band.
- Sign and value of the decoupling parameter decides the odd-even splitting
- For high-j states, the splitting could be so large that the unfavored signature is not seen at all.

Fixed point structure from SCM

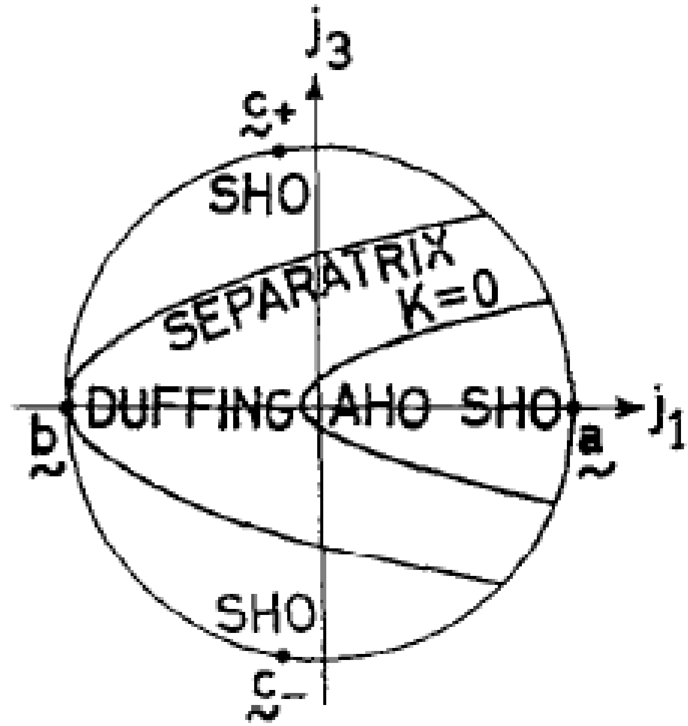


Fig. 1. Schematic diagram showing the projections of ε-cylindrical parabolae and j-sphere constituting the invariant region for $\omega_c/2Qj < 1$. The j-space is seen to be divided into four distinct regions – three arising from the separatrix and one from the critical parabola ($K = 0$). The four fixed points are shown by the labels $\boldsymbol{a}$, $\boldsymbol{b}$ and $c_\pm$.