Optimal Control problem for Burgers equation

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Overview

The inclusion of data into the models

- Data assimilation : - a method for combining the observations with the results of previous numerical simulations to produce 'the best' estimate of the system
- Some methods for Data Assimilation :
  - **Variational Method : 4 D -VAR** :
    - Minimize Cost Function J :
      Measures the "distance" or misfit between the model solution and the observation
  - **Sequential or Statistical Methods** :
    - Bayesian Approach
    - Kalman filter, extended KF
    - Ensemble KF

Burgers equation

- The mathematical model for $U(x,t)$ for $(x,t) \in (0,1) \times (0,T)$:

$$\frac{\partial U}{\partial t}(x,t) + \frac{1}{2}\frac{\partial U^2}{\partial x}(x,t) = \mu \frac{\partial^2 U}{\partial x^2}(x,t)$$

- With boundary conditions :

$$U(0,t) \; = 0 = \; U(1,t)$$

- Initial condition : $U(x,0) = u(x)$
- One dimensional form of Navier-Stokes equation

Existence and uniqueness

- For a given $u \in V := H_0^1(0,1)$, there exists a unique solution $U(x,t)$ in

$$W(0,T) = \{ v \in L^2(0,T;V) \ : \ v_t \in L^2(0,T;H) \ \}$$

- Map of control u to state $U(u)$ is continuous and differentiable
- Observation of the state is given by $z \in L^2(0,T;Z)$ for a Hilbert space Z
- C is continuous linear map from V to Z
- Define $J(u) = \frac{1}{2}\int_0^T \int_0^1 |C(U(u)) - z|^2 \ dx \ dt + \frac{\alpha}{2}\int_0^1 |u - u^b|^2 dx$
- Then for $T < T_0$ depending on μ and bound on initial conditions, there exists a unique minimum for J.

Control formulation of the Problem

- Mathematical model for U : initial value problem for Burger's equation
- Given: data z of observations of the state and an approximate initial condition u^b
- **Qn :** How to find an optimal initial condition u which minimizes the distance between the state U and the observation z ?
- Cost Functional :

$$J(u) = \frac{1}{2}\int_0^T \int_0^1 |C(U(u)) - z|^2 \, dx \; dt + \frac{\alpha}{2}\int_0^1 |u - u^b|^2 dx$$

- Find u that minimizes J

Optimality Conditions

- Euler-Lagrange equations : set of necessary conditions to be satisfied by the minimizer
- $P = P(x,t)$, the adjoint state corresponding to the solution U of Burger's equation satisfies, when $C = I, \alpha = 0$:

$$\frac{\partial P}{\partial t} + U \frac{\partial P}{\partial x} + \mu \frac{\partial^2 P}{\partial x^2} = z - U$$

- For $(x,t) = (0,1) \times (0,T)$; boundary conditions same as U , $P(x,T) = 0$
- Differentiating J, $(DJ(u)v) = \int_0^T < U - z, (D_u U)v >_H \;\; dt$
- Using adjoint equation, $(DJ(u), v) =< P(0), v >_H$

Numerical Schemes

- Implicit centered scheme :

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} + \frac{1}{4\Delta x}\left((u_{j+1}^m)^2 - (u_{j-1}^m)^2\right)$$

$$= \frac{\mu}{(\Delta x)^2}\left(u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}\right)$$

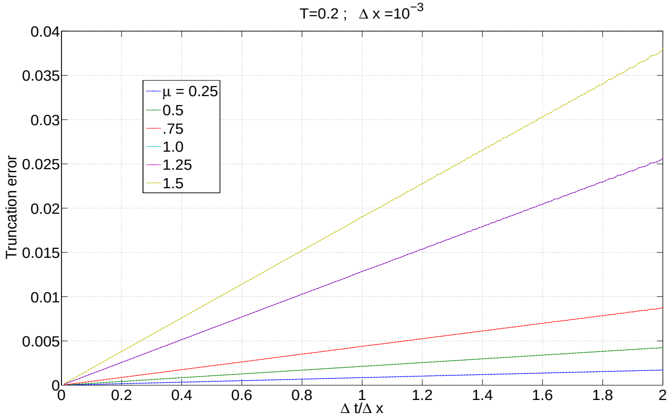
- Lax -Freidrichs scheme:

$$\frac{u_j^{m+1} - \frac{u_{j+1}^m + u_{j-1}^m}{2}}{\Delta t} + \frac{1}{4\Delta x}((u_{j+1}^n)^2 - u_{j-1}^m)^2$$

$$= \frac{\mu}{(\Delta x)^2}\left(u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}\right) \quad 1 \leq j \leq n-1,$$

Relative Error for the first scheme

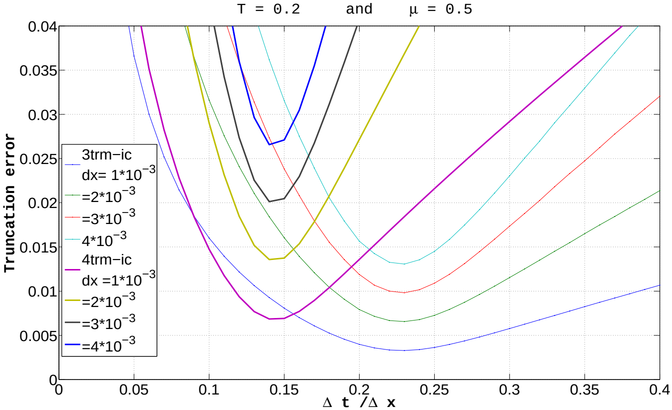
Maximum error E : $E \leq (C_1 \Delta t + C_2 \Delta x^2)$



Relative Error for Lax-Freidrichs scheme

Maximum error E :

$$E \leq (C_1 \Delta t + C_2 \Delta x^2 + C_3 \frac{\Delta x^2}{\Delta t})$$



Discrete time observations

- Observations are taken at a finite number M of points in "space" x and finitely many times.
- Discrete Cost Function:

$$J(u) = \sum \sum |z_i^m - U_i^m|^2 + \frac{\alpha}{2} \sum |u_i - u_i^b|^2$$

- Discretization of the constraint equation

$$(I - A)U^m = BU^{m-1}$$

for matrix A, B and vectors U^m and U^{m-1}

- Adjoint of the discretized problem

$$(I - A^*)P^m = B_1 P^{m+1} + (Z - U)^{m+1}$$

- $DJ(u) = (I - A^*)P^0$

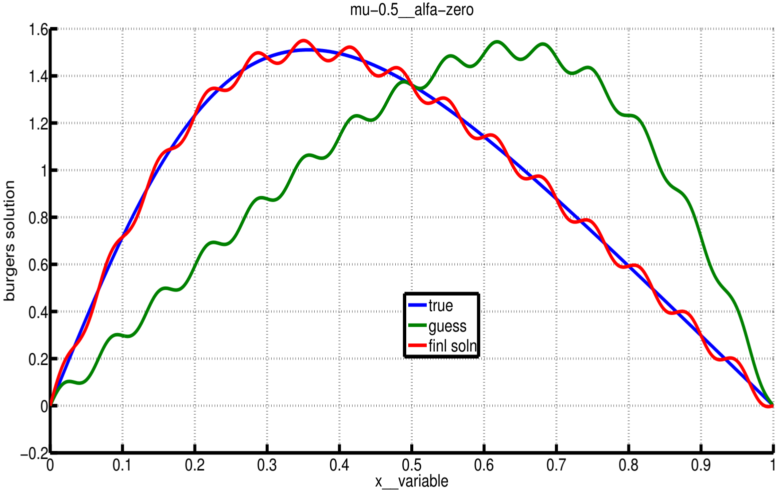
Steepest descent method

- Start with initial guess u_0
- Iterative algorithm :

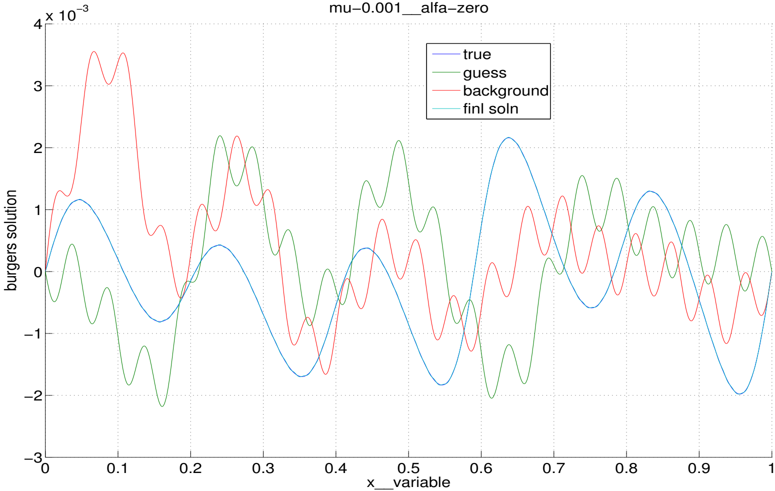
$$u_n = u_{n-1} + \rho_n D_n$$

  with D_n as the gradient descent direction and ρ_n as the step size
- In each iteration first solve the forward problem to get U and then solve the backward problem for P to get the gradient direction

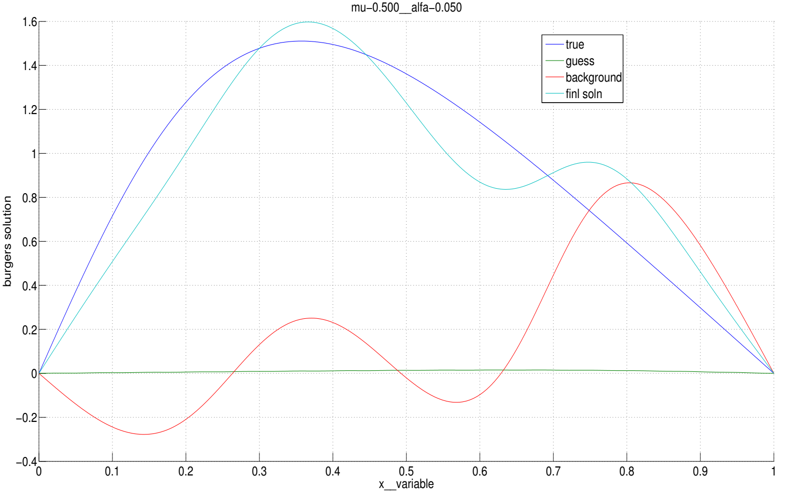
Comparison with exact initial condition: "ill posed " case



Comparison with exact initial condition: "ill posed " case



Comparison with exact initial condition: "regularized" case



Discussions

- **Summary**
  - set up the optimal control formulation of the data assimilation problem for the Burger's equation model
  - Numerically computed the solution when data is given at discrete times
- **Future Work**
  - Extend the computations to other cases
  - Use Sequential methods also and compare, combine
  - Use realistic Ocean models with real data

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