Nonlinear Stochastic Modelling, Critical Phenomena and Entropy

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- Real life systems: Physical, engineering biological or social systems.
  - Nonlinear and Stochastic.
- Linear framework: System behavior in terms of unique equilibrium state, Linear superposition principle.

Modeling of Nonlinear System

- Capable of capturing both qualitative and quantitative aspects of complex systems.
- Boundaries between different fields of study have become fuzzy
- Striking feature contributing to this is the emergence of fields:

  - ❖ Synergetics(Haken), Dissipative structures(Prigogine)
  - ❖ Catastrophe theory(Thom), Dynamical systems
  - ❖ Bifurcation theory, Chaos theory
  - ❖ Advances in nonlinear-stochastic modeling and estimation
  - ❖ Complexity theory

Within the framework of these development we are able to capture both qualitative and quantitative aspects of systems

Stochastic Modeling

- INTRINSIC STOCHASTICITY
- ENVIRONMENTAL STOCHASTICITY (SDE with Multiplicative Noise)

INTRINSIC STOCHASTICITY

- Arises due to discreteness of variables of the system
- Master Equation Description

❖ Chapman-Kolmogorov Equation (Continuous Markov process)

- $w(x \mid x')dx$ = Transition probability density per unit time.
- $$\frac{\partial p(x,t \mid x_0)}{\partial t} = \int \{p(x',t \mid x_0)w(x \mid x') - p(x,t \mid x_0)w(x' \mid x)\}dx'$$
$$p(x,0) = \delta(x - x_0)$$
- writing
$$x - x' = \Delta x, \quad w(x \mid x') = W(x', \Delta x)$$

- Kramers-Moyal Expansion

$$\frac{\partial p(x,t\mid x_0)}{\partial t} = \sum_{n=1}^{\infty} \frac{A_n}{n!}\left(-\frac{\partial}{\partial x}\right)^n p(x,t\mid x_0)$$

- Note $\quad A_n(x) = \int_{-\infty}^{\infty} d(\Delta x)(\Delta x)^n W(x,\Delta x)$

- Fokker-Planck Equation

$$\frac{\partial}{\partial t} p(x,t\mid x_0) = -\frac{\partial}{\partial x}[A_1(x)p(x,t\mid x_0)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[A_2(x)p(x,t\mid x_0)]$$

Discrete Markov Process

$$\frac{d}{dt}P(j,t\mid i) = \sum_{\substack{i=1,\\ k\neq j}}^{N} w_{jk}P(k,t\mid i) - w_{kj}P(j,t\mid i) \qquad 1\le j\le N$$

Intrinsic Stochasticity

- Relative Fluctuations of variables: $O(N^{-1/2})$, N system size.
- Central limit theorem: Far from Critical Point
- Breakdown of CLT: At critical point

Thus for a large system, one may even adopt deterministic approach.

- Asymptotic results:

$\{X_N(t), N = 1, 2, \ldots\}$ is a sequence of birth and death processes

$$Z_N(t) = \frac{\{X_N(t) - \phi(t) N\}}{\sqrt{N}}$$

Converges in distribution to Ornstein-Uhlenbeck process as $N \to \infty$

System size Expansion: (van Kampen)

$$X(t) = N\phi(t) + N^{1/2} Z(t)$$, Far from Critical Point

$$X(t) = N\phi(t) + N^{\nu} Z(t), \quad \nu \neq \frac{1}{2}$$, at Critical Point

Environmental Stochasticity

- Environment in which system is embedded-endowed with a large number of degrees of freedom fluctuating randomly.
  - ❖ Makes the parameters of the process themselves stochastic
- Ecological Models: Parameters involved are subject to random variables (e.g. fluctuations in temperature, humidity, age composition, ecological factors…)
- For a realistic Modeling: One must incorporate into the model "environmental stochasticity " as well as "intrinsic stochasticity"
  - ❖ The two being conceptually different from one another

- Note: In contrast to the effects of intrinsic stochasticity, the effect of environmental stochasticity are generally independent of the size of the system.
- For systems of large size, it is the environment stochasticity which contributes to the observed fluctuations.

- Mathematical formulation

Consider a process where deterministic version is governed by differential equation,

$$\frac{dX}{dt} = f(X,t) + \gamma(t)\, b(X,t), \qquad X(t=0) = X_0$$

X: Variable of interest

$\gamma(t)$: parameter of the problem, liable to vary with the state of the environment

- Random variation in the environment

$$\gamma(t) = \overline{\gamma}(t) + \gamma_\gamma(t).$$

- $\gamma_\gamma(t)$ is assumed to be stationary, Gaussian white noise, i.e.

$$\langle \gamma_\gamma(t) \rangle = 0, \qquad \langle \gamma_\gamma(t_1)\gamma_\gamma(t_2) \rangle = \sigma_\gamma^2 \delta(t_1 - t_2)$$

- SDE (with multiplicative noise)

$$\frac{dx}{dt} = A(x,t) + \gamma_\gamma(t) B(x,t)$$

SDE can be interpreted in two different ways : Ito sense and
Stratonovich sense

Wong and Zakai(1965): The stratonovich prescription preferable for modeling physical process.

- Fokker-Planck Equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[(A(x,t)+\frac{1}{2}\sigma_{\gamma}^{2}B(x,t)\frac{dB(x,t)}{dx})p(x,t)]+\frac{\sigma_1}{2}\frac{\partial}{\partial x}[B^2(x,t)p(x,t)],$$

$$p(x,0)=\delta(x-x_0)$$

- Boundary conditions: Reflecting Barriers
- Stationary pdf :

$$p_{s,t}(x) = \frac{c}{b(x)}\exp[2\int^{x}\frac{a(\xi)}{b(\xi)}d\xi]$$

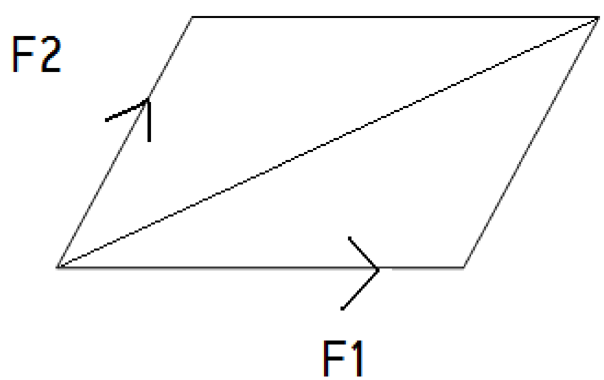
$$a(\xi) = A(x) + \frac{1}{2}\sigma^2 B(x)\frac{dB(x)}{dx}$$

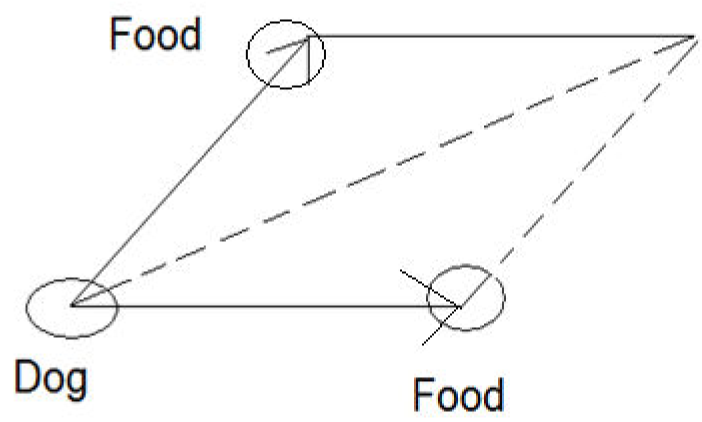
$$b(\xi) = \sigma_{\gamma}^{2} B^{2}(x)$$

Critical Phenomena in Social System

Social Systems

- Essentially nonlinear in nature
- Exhibit critical phenomena in the form of phase transformation as qualitative changes
- Changes manifest as a result of bifurcation, hysteresis and multiple equilibria



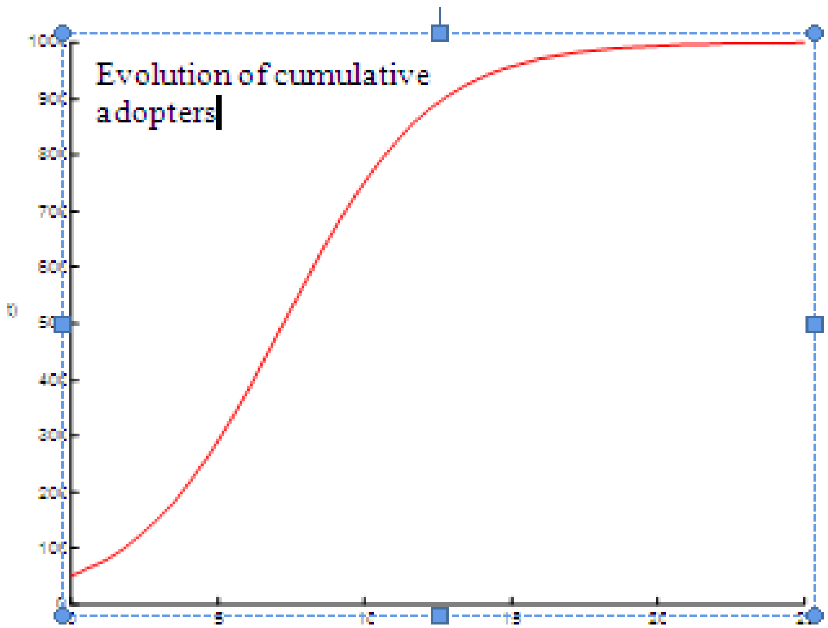


•**An Innovation** is an idea, practice or object that is perceived as new by an individual or other unit of adoption. This newness of an innovation may be expressed in terms knowledge, persuasion or a decision to adopt (Rogers 1995).

•A phenomenon of considerable interest

•It has been studied across wide ranging disciplines to explain the dissemination of new ideas, rumours, news, practices and new products throughout a social system.

S-shaped curve



The Generalized model (Bartholomew JMS (1976))

$N \equiv$ Size of the Homogeneously mixing population

$\alpha \equiv$ Transmission intensity of the source

$\beta \equiv$ Propagation intensity through interpersonal contact

$\mu^{-1} \equiv$ Mean duration of time for which a spreader remains active

- Deterministic Analysis

We have $$\frac{d}{dt}n(t) = \{\underbrace{\alpha(N-n)}_{\text{(Linear)}} + \underbrace{\frac{\beta n}{N}(N-n)}_{\text{(Nonlinear)}}\} - \underbrace{\mu n}_{\text{(Linear)}}$$

1.Bass Model (New product – Innovation Diffusion)

$\mu = 0$ (Absence of loss of interest)

$$\boxed{\frac{dn}{dt} = (\alpha + \beta n)(N - n), \quad n(t=0) = n_0}$$

2.Threshold Phenomenon

Define:

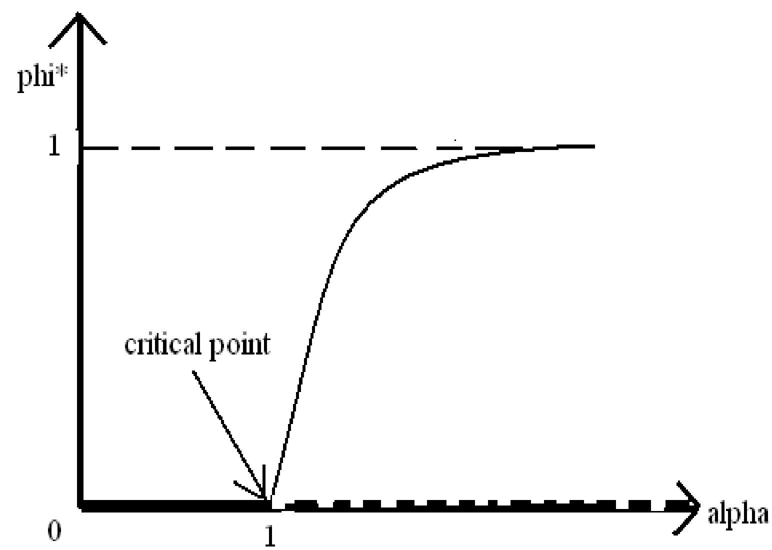
$$\omega = \frac{\alpha}{\beta}, d = \frac{N\beta}{\mu}, \tau = \mathrm{N}\beta t, \ \phi(\tau) = \frac{n(\tau)}{N}$$

No external sources $\alpha = 0 (\omega = 0)$

$$\frac{d}{dt}\phi(\tau) = \left(1 - \frac{1}{d}\right)\phi - \phi^2$$

and

$$n_{eq} = \begin{cases} 0, & \mathrm{d} < 1; \\ N\left(1 - \dfrac{1}{d}\right) & \mathrm{d} \geq 1 \end{cases}$$



The case of Homogeneous Population

- Basic Innovation Diffusion Model is based on the Twin process:
  - Mass mediated process
  - Interpersonal contact-interactive process (non-linear)
- Model with Loss of interest (discontinuation)

$$P\{n \to n+1 \ in \ (t, t+\delta t)\} = \{\alpha(N-n) + \beta \frac{n}{N}(N-n)\}\delta t + o(\delta t)$$

$$P\{n \to n-1 \ in \ (t, t+\delta t)\} = \mu_n \delta t + o(\delta t)$$

Defining

$$P(n,t) = \Pr\{\text{Probability of } n \text{ adopters at time t}\},$$

$$\lambda_n = \{\alpha(N-n) + \beta \frac{n}{N}(N-n)\} \ and \ \mu_n = n\mu,$$

$$\frac{dP}{dt}(n,t) = -(\lambda_n + \mu_n)P(n,t) + \lambda_{n-1}P(n-1,t) + \mu_{n+1}P(n+1,t)$$

Equilibrium solution: p_n^*
(Detailed balance $\mu_{n+1} p_{n+1}^* = \mu_n p_n^*$)

$$p_n^* \equiv p(n, \infty) = \mathcal{N} \binom{N}{n} \left(\frac{N}{d}\right)^{N-n} \Gamma(n+\omega)$$

Analysis of the model $\omega = 1$ (A very weak mass mediating source)

Case I: $d < 1$

$p_n^* = (1-d)\, d^n$ (Geometric distribution)

Relative fluctuations: $\dfrac{\sigma_n^2}{\bar{n}^2} = 1 + \dfrac{1}{\bar{n}} = O(1)$

Case II: $d > 1$

$p_n^* = \mathcal{N}_1 \dfrac{(N/d)^{N-n}}{(N-n)!}$ (Quasi–Poissonian distribution)

Relative fluctuations: $\dfrac{\sigma_n^2}{\bar{n}^2} = \dfrac{d}{(d-1)^2} \dfrac{1}{N} \ll 1$

Case III $d = d_c = 1$: $\bar{n} = \sqrt{\frac{2}{\pi} N}$, $\sigma_n^2 = N \frac{(\pi - 2)}{\pi}$

✔ Order–Disorder transition

Degree of order:

$$S = -\sum p_n \log p_n = E\left[-\log p_n\right]$$

1. $$s = \frac{S}{\bar{n}} = \ln\left(\frac{1}{d}\right) - \frac{1-d}{d}\ln(1-d)$$
$$= O(1), \qquad d < 1$$

2. $$s = \frac{S}{\bar{n}} = \sqrt{\frac{\pi}{8N}}\left[\ln\left(\frac{\pi N}{2}\right) + 1\right]$$
$$= O\left(N^{-1/2}\ln N\right) \approx 0, \qquad d = 1$$

3. $$s = \frac{S}{\bar{n}} = \frac{d}{2N(d-1)}\left[\ln\left(\frac{2\pi N}{d}\right) + 1\right]$$
$$= O\left(N^{-1}\ln N\right) \approx 0, \qquad d > 1$$

Below threshold system is in a state of *relative disorder*, whereas above threshold it attains a state *relative disorder*.

Stochastic Analysis : Hierarchy of Moment equation

$$\dot{P}(n,t) = -(\lambda_n+\mu_n)P(n,t)+\lambda_{n-1}P(n-1,t)+\mu_{n+1}P(n+1,t)$$

$$E[n(t)] = \sum_{n=0}^{N} n\, p(n,t)$$

$$\frac{d}{dt}E[n] = \alpha N + (\beta N - \mu - \alpha)E[n] - \beta E\left[n^2\right]$$

$$\frac{d}{dt}E\left[n^2\right] = \alpha N + (\beta N - 2\alpha N + \mu - \alpha)E[n] + (2\beta N - \beta - 2\mu - 2\alpha)E\left[n^2\right] - 2\beta E\left[n^3\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

☞ Hierarchy of moment equations.

$P(n,t) = Pr\{$Probability of finding n spreaders/adopters at time $t\}$

Critical Behavior of Nonlinear Stochastic Models: Diffusion of Information

(Sharma, Pathria, and Karmeshu, Phy. Rev. A ,1982)

Birth-Death process (Open model, Closed model)

$$\lambda_n = a + bn + c\frac{n^2}{\Omega}, \quad \mu_n = a' + b'n + c\frac{n^2}{\Omega}, \quad a > a' \text{ and } c < c'$$

Writing

$$n(t) = \Omega\,\phi(t) + \Omega^{v} x(t), \qquad \mathrm{x(t)} = \mathrm{O}(1)$$

First moment equation

$$\frac{d\langle n\rangle}{dt} = (a - a') + (b - b')\langle n\rangle + (c - c')\frac{\langle n^2\rangle}{\Omega}$$

Critical point b=b'

$$\langle n^2\rangle_{ST,C} = \left(\frac{a - a'}{c' - c}\right)\Omega = O(\Omega)$$

$$\langle n\rangle_{ST,C} = O(\Omega^{1/2})$$

Critical Region: $\langle n\rangle_{ST,C} = O(\Omega^{1/2})$

Nonlinear FPE

$$\frac{\partial \Pi}{\partial t} = \Omega^{-1/2}\left[-\frac{\partial}{\partial x}(\{(a-a^{'}) + [(b-b^{'}) + 2(c-c^{'})\phi]\Omega^{1/2}x + (c-c^{'})x^2\}\Pi) + \frac{1}{2}(b+b^{'})\frac{\partial^2}{\partial x^2}[(\phi\Omega^{1/2}+x)\Pi] \right]$$

Critical slowing down:

$$\rho = \frac{n}{\Omega} = \phi + \Omega^{-1/2}x = O(\Omega^{-1/2})$$

$$\Pi(\rho) = \mathrm{N}\rho^{\omega-1}\exp[\eta\Omega^{1/2}\rho - \frac{1}{2}\xi\Omega\rho^2]$$

$$\omega = \frac{2(a-a^{'})}{b+b^{'}} \simeq \frac{a-a^{'}}{b^{'}} > 0,$$

$$\xi = \frac{2(c-c^{'})}{b+b^{'}} \simeq \frac{c-c^{'}}{b^{'}} > 0$$

$$\text{and} \quad \eta = \frac{2(b-b^{'})}{b+b^{'}}\Omega^{1/2} \simeq \frac{b-b^{'}}{b^{'}}\Omega^{1/2} = O(1)$$

At Critical Point : $(\eta=0)$

$$\langle\rho\rangle = \frac{\omega}{(\Omega\xi)^{1/2}} \frac{D_{-(\omega+1)}(-\eta/\sqrt{\xi})}{D_{-\omega}(-\eta/\sqrt{\xi})}, \quad D_P(Z) \text{ parabolic cylinder function}$$

and

$$\langle\rho^2\rangle = \frac{\omega(\omega+1)}{(\Omega\xi)} \frac{D_{-(\omega+2)}(-\eta/\sqrt{\xi})}{D_{-\omega}(-\eta/\sqrt{\xi})}$$

$$\frac{\partial\langle\rho\rangle}{\partial\eta} = \Omega^{1/2}(\langle\rho^2\rangle - \langle\rho\rangle^2) = \Omega^{1/2} \ \text{var}\,\rho$$

- Rate of growth of order parameter $\langle\rho\rangle$ is directly prportional to variance of ρ.
- It elucidates the role played by fluctuations.
- Broader class of systems displaying identical behaviour in critical region.

Innovation diffusion model with multiple adoption levels:

(Karmeshu, Sharma & Jain;JSIR,1992)

- It has been assumed that parameters remain constant throughout the unfolding of the process.
- Esingwood, Mahajan and Muller have suggested that the time varying nature of the interaction parameter
- can be incorporated by specifying the internal influence as varying with adoption level:

$$\lambda_n = \alpha(N-n) + \beta\left(\frac{n}{N}\right)^{\delta}(N-n), \qquad \delta > 1$$

$$\mu_n = \mu n$$

For large N, we can associate SDE given by

$$dn(t) = \left[\alpha(N-n) + \beta(n/N)^2(N-n) - \mu n\right]dt$$

$$\left[\alpha(N-n) + \beta(n/N)^2(N-n)\right]^{1/2} dW_1(t) - \left[\mu n\right]^{1/2} dW_2(t), \quad \delta=2$$

SDE to be interpreted in Ito sense

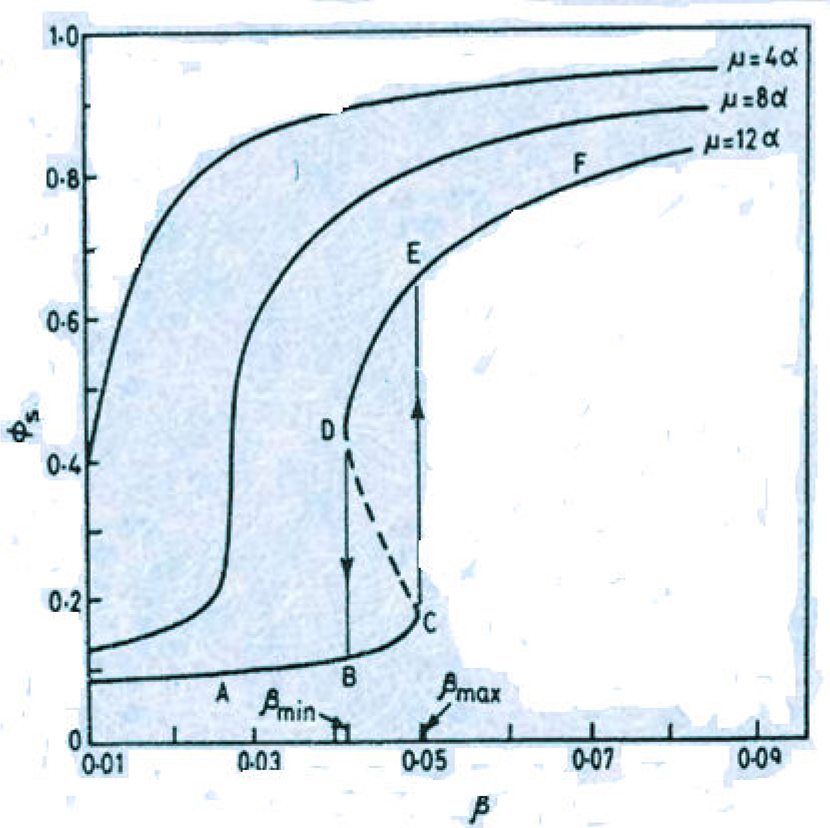
We write for large N,

$$n(t) = N\phi(t) + N^{\nu}x(t);$$

Comparing leading powers of N,

$$\boxed{\frac{d\phi}{dt} = (1-\phi)(\alpha + \beta\phi^2) - \mu\phi}$$

Hysteresis Effect



—Steady-state diagram representing the proportional adoption level ϕ_s versus the parameter β for different values of the parameter $\mu = 4\alpha$, 8α and 12α with $\alpha = 0.001$. The model exhibits adoption level transformation for $\beta_{min} < \beta < \beta_{max}$. The dotted part CD corresponds to unstable branch

<u>Multiply steady states</u>

$$\frac{dn}{dt} = [\alpha + \beta\left(\frac{n}{N}\right)^2](N - n) - \mu n =$$

$$f_0(n) = -\frac{\partial V}{\partial n}$$

The system's behaviour is charaterized by the shape of the potential function

$$(\Phi_s) = \frac{\beta\Phi_s^{\ 4}}{4} - \frac{\beta\Phi_s^{\ 3}}{3} + \frac{\alpha + \mu}{2}\frac{\Phi_s^{\ 2}}{4} - \alpha\Phi_s$$

Comparing next order of terms : the fluctuations are governed by Fokker-Planck Equation

$$\frac{\partial p(x,t)}{\partial t} = \varepsilon^2 f_3(\Phi_c)\frac{\partial}{\partial t}\left[x^3 p\right] + \frac{1}{2}N^{1-2\nu} g(\phi_c)\frac{\partial^2}{\partial x^2} p(x,t)$$

$$f_3(\phi_c) = -27\alpha,$$

$$g(\phi_c) = 16\,\alpha\!/\!_3\,.$$

Drift and diffusion are of comparable significance when ε^2 and $N^{1-2\nu}$ are of same order of magnitude.

This leads to

$$2(\nu - 1) = 1 - 2\nu$$

$$\boxed{\nu = \frac{3}{4}}$$

$$\frac{\partial}{\partial} p(x,t) = N^{1/2}\left[27\alpha\frac{\partial}{\partial x}\left(x^3 p\right) + \frac{8\alpha}{3}\frac{\partial^2}{\partial x^2} p\right]$$

FPE with nonlinear drift term

- Critical slowing down: The approach of the system towards its steady state is slowed by a factor $N^{-1/2}$.
- This corresponds to lengthening of relaxation time
- FPE can be written as

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left[a(x)p(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[Dp(x,t)\right]$$

$$p_S(x) = \exp(-V(x))$$

$V(x)$ is the potential function (Quartic function)

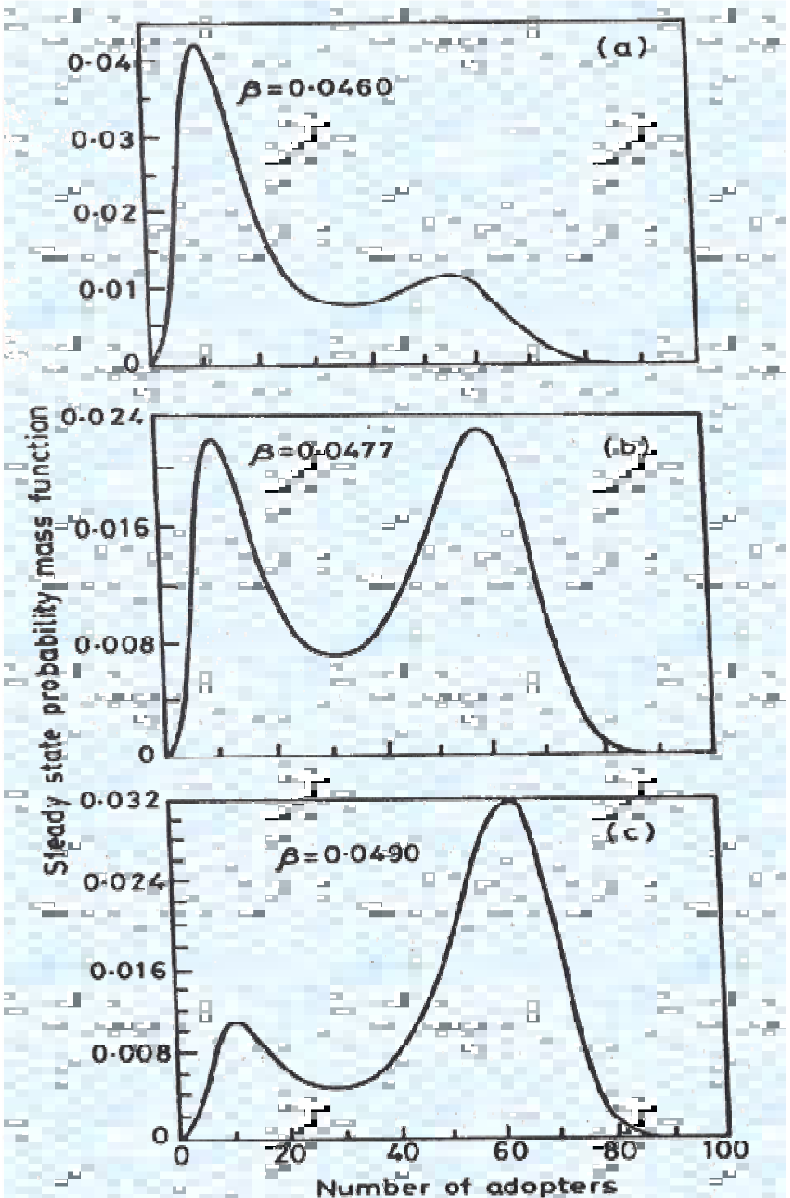


Fig. 2—Steady-state bimodal probability mass function $\mu_s(n)$ versus the number of adopters n_s for different values of $\beta=0.0460$, 0.0477 and 0.0490 with $\alpha=0.001$, $\mu=12\alpha$ and $N=100$ [The points representing the probability mass function are connected by continuous curves in diagrams (a), (b), (c)]. As the parameter β increases, the peak (the greater of the two ordinates at the modes) shifts from low level of adoption to high level of adoption.

Implications

To design control policies for promoters of innovation in the form of strategies requiring parameter trajectories to remain outside or within the cusp depending on the level of adoption.

Increasing returns to adoption

Arthur: "Where there are multiple equilibriua and a central authority favors a particular one, it can of course attempt to tilt the market towards this outcome. Timing is crucial here – there are only 'narrow windows' in which the policy is effective.

Role of Environmental Stochasticity

Noise Induced bistability in Innovation Diffusion

(Karmeshu)

- As a result of random variations on the environment the parameters may not be regarded as deterministic, they rather assume form

$$\beta(t) = \underset{(\mathrm{I})}{\bar{\beta}} + \underset{(\mathrm{II})}{\gamma(t)}$$

(I)- Average state of the environment:

(II)-Represents the fluctuations from average state of the environment

- Customarily $\gamma(t)$ is supposed to be stationary, Gaussian, White noise process i.e.

$$E[\gamma(t)] = 0, \quad E[\gamma(t_1)\gamma(t_2)] = \sigma^2 \delta(t_1 - t_2)$$

- The Stochastic differential equation is

$$dx(t) = \left[\left(\overline{\alpha} + \overline{\beta}x\right)(1-x) - \mu x\right]dt + \sigma x(1-x)dW(t)$$

- The stationary density function

$$\overset{*}{p}_s(x) = \frac{C}{x(1-x)}\left(\frac{x}{1-x}\right)^k e^{-\frac{a}{1-x}-\frac{b}{x}}$$

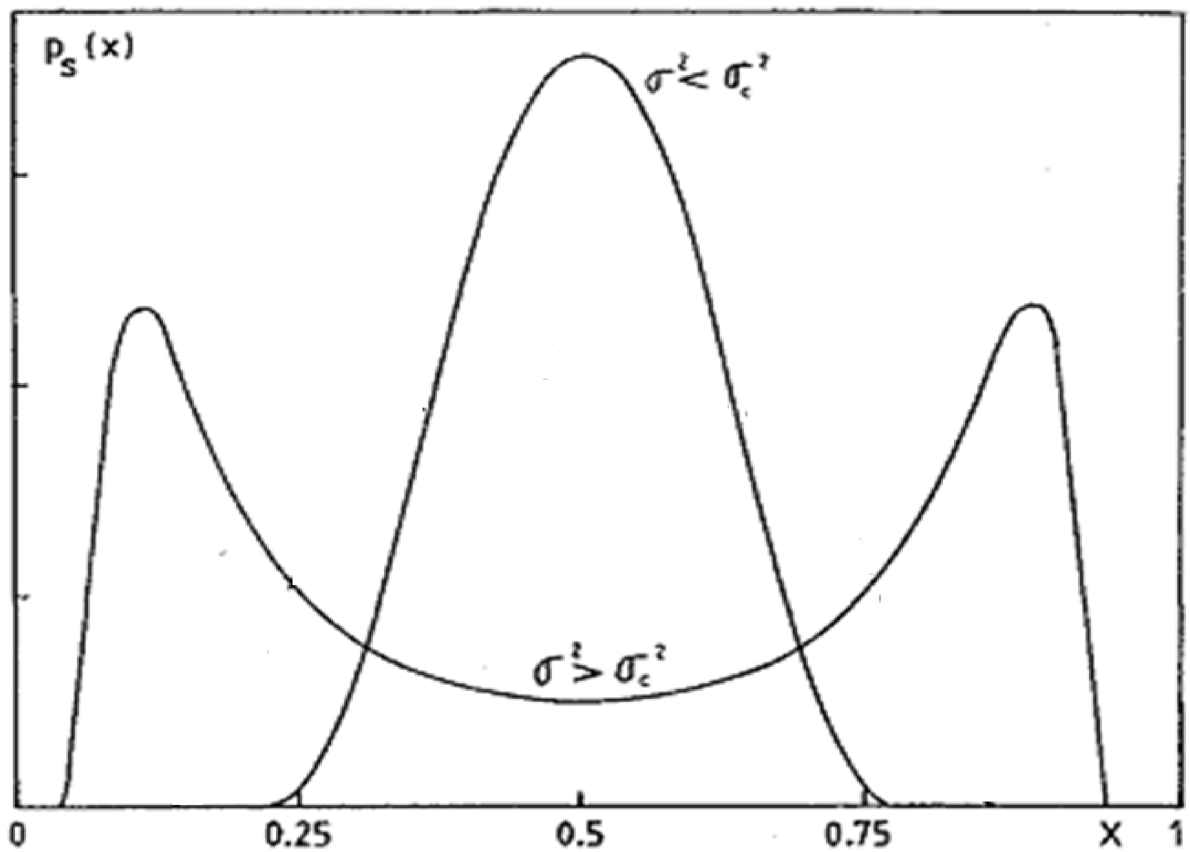
$$k = \frac{2(\alpha+\beta-\mu)}{\sigma^2},$$

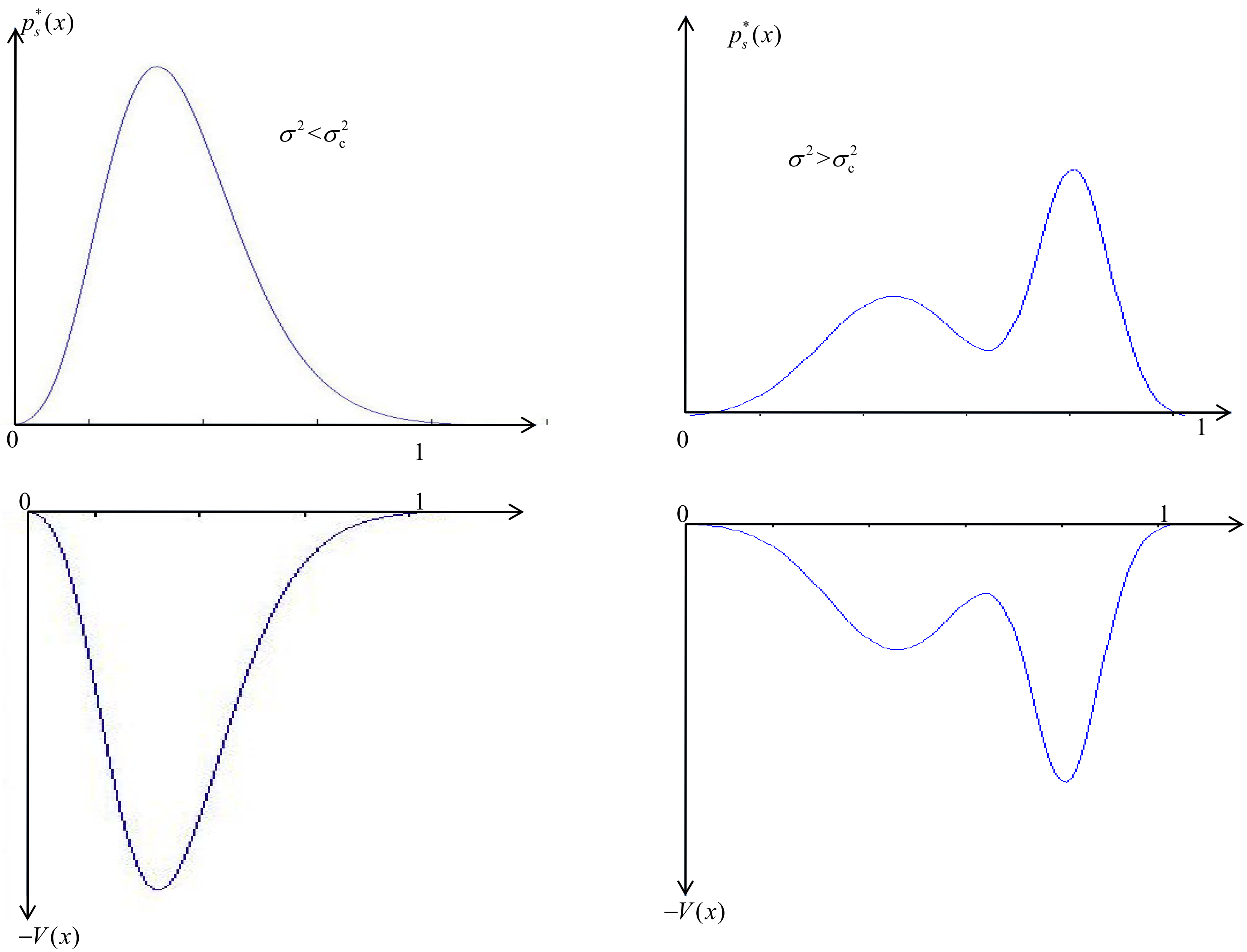
$$a = \frac{2\mu}{\sigma^2},$$

$$b = \frac{2x}{\sigma^2},$$

$$C^{-1} = 2\left(\frac{\alpha}{\mu}\right)^{k/2} K_{-k}\left(\frac{4\sqrt{\alpha\mu}}{\sigma^2}\right) e^{2(\alpha+\mu)/\sigma^2}.$$

- Role of environmental stochasticity





$p_s^*(x)$
$\sigma^2 < \sigma_c^2$
0
1
$p_s^*(x)$
$\sigma^2 > \sigma_c^2$
0
1
0
1
$-V(x)$
0
1
$-V(x)$

Random Differential Equation

Consider

$$\frac{dx}{dt} = rx, \qquad x(t=0) = x_0$$

Growth rate r is a random variable, such that,

$$r = \bar{r} + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2)$$

$$E[x(t)] = x_0 e^{\bar{r}t + \frac{\sigma^2 t^2}{2}}$$

$$\frac{dE[x(t)]}{dt} = (\bar{r} + \sigma^2 t) E[x(t)],$$

Non-autonomous system

Random Differential Equation

Transient Bimodality and Catastrophic Jumps in Innovation Diffusion (Karmeshu and Goswami, IEEE Trans. SMC)

Behavioral characteristics of individuals :

Tastes, perceptions, preferences, demographic and environmental factors, etc.

Vary randomly

The modified Bass model is

Nonlinear random differential equation

$$\frac{d}{dt}X(t|\alpha,\beta)=f(X;\alpha,\beta,t), \qquad t \geqslant t_0;\ X(t_0)=X_0 \qquad (1)$$

where

$$f(X;\alpha,\beta,t) = (\alpha+\beta X)(1-X). \qquad (2)$$

By denoting the solution process of (1) as

$$X(t|\alpha,\beta) = h(X_0,\alpha,\beta;t) \qquad (3)$$

the pdf of the adoption level $X(t)$ is

$$P(x,t) = \iint_{\Omega_{\alpha\times\beta}} g_{\alpha,\beta}(\alpha,\beta)\delta\left(x-h(X_0,\alpha,\beta;t)\right)d\alpha\, d\beta,$$
$$(x,t)\in\Omega_{X\times T}. \qquad (4)$$

Here, $g_{\alpha,\beta}(\cdot,\cdot)$ denotes the joint distribution of the random variables [20]. By using the solution process $h(X_0,\alpha,\beta;t)$, (4) can be further simplified to

$$P(x,t)=\int_{\Omega_z} g_{\alpha,\beta}\left(\frac{y_s z}{1+z},\frac{y_s}{1+z}\right)\frac{\left|\ln\frac{(1-X_0)(z+x)}{(1-x)(z+X_0)}\right|}{|t^2(1-x)(1+z)(x+z)|}dz. \qquad (5)$$

Write $\alpha = p\gamma$ and $\beta = q\gamma$, where p and q are fixed parameters, and γ is a random variable having a distribution $g_\gamma(\cdot)$. Thus, (1) reduces to

$$\boxed{\frac{d}{dt}X(t|\gamma) = \gamma(p + qX)(1 - X), \qquad t \geqslant t_0; \quad X(t_0) = X_0.} \tag{6}$$

The corresponding solution conditioned for a given value of random variable γ becomes

$$X(t|\gamma) = h(X_0, \gamma; t) = \frac{1 - \mathcal{A}_1 e^{-(p+q)\gamma t}}{1 + \mathcal{A}_2 e^{-(p+q)\gamma t}} \tag{7}$$

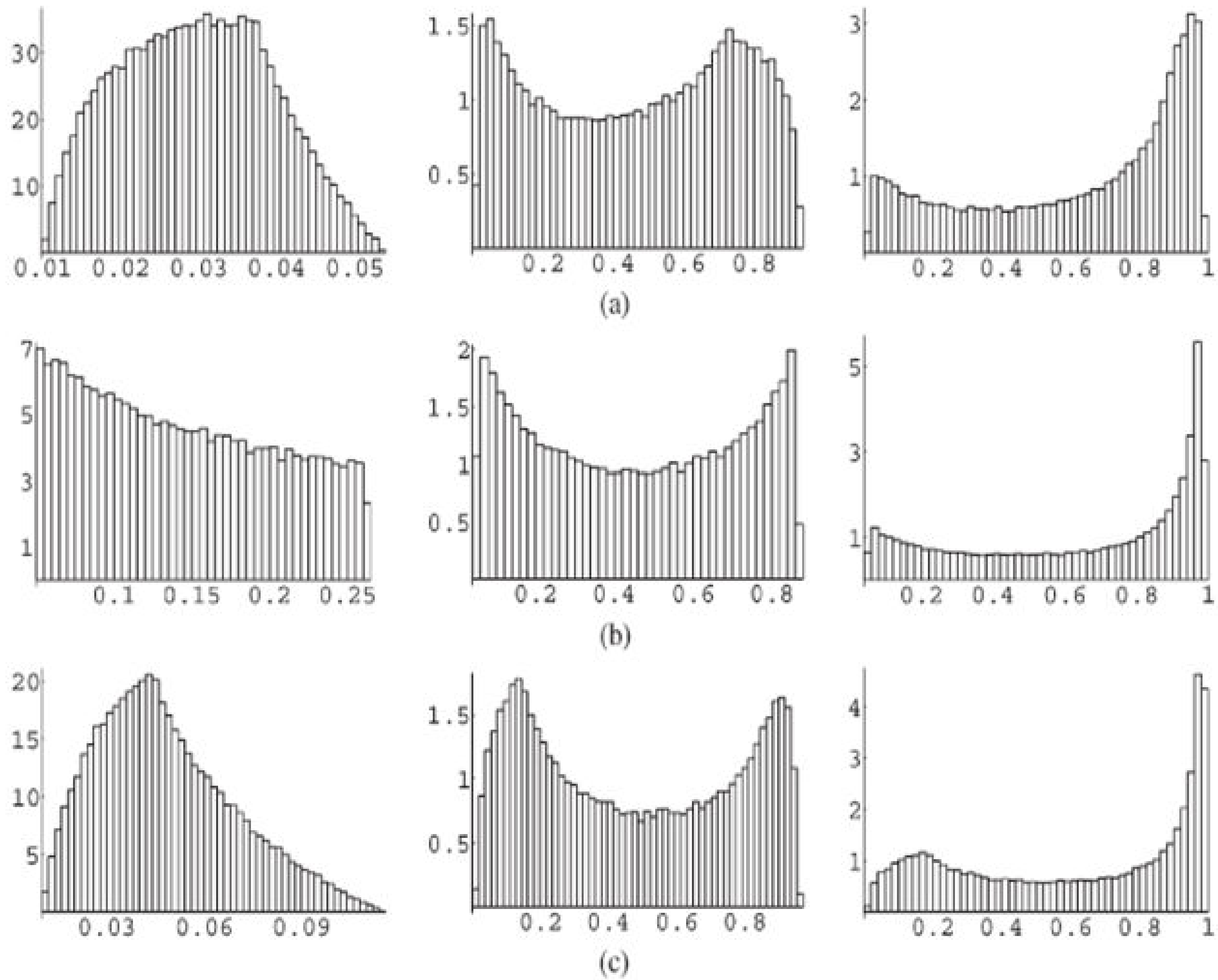
with

$$\mathcal{A}_1 = \frac{p(1 - X_0)}{p + qX_0} \qquad \mathcal{A}_2 = \frac{q(1 - X_0)}{p + qX_0}. \tag{8}$$

$$P(x,t) = \int_0^\infty g_\gamma(\gamma)\delta\left(x - h(X_0,\gamma;t)\right) d\gamma. \qquad (9)$$

Parameter γ (Gamma distributed)

$$\boxed{\begin{aligned} P(x,t) &= \frac{\lambda^{-n}}{\Gamma(n)(p+q)t}\frac{\mathcal{A}_1+\mathcal{A}_2}{(1-x)^2}\left[\frac{\mathcal{A}_1+x\mathcal{A}_2}{1-x}\right]^{-\left(1+\frac{1}{\lambda(p+q)t}\right)} \\ &\times\left[\frac{1}{(p+q)t}\log\left(\frac{\mathcal{A}_1+x\mathcal{A}_2}{1-x}\right)\right]^{n-1}, \qquad t > t_0 \quad (14) \end{aligned}}$$



Probability histograms constructed at different time points using 10^5 realizations of the process $X(t)$ under the proportionate effect model for random variable γ distributed as (a) triangular with parameters (0, 3, 4), $\bar{\alpha} = 0.003$, $\bar{\beta} = 0.78$, and $X_0 = 0.01$ and (b) uniform with parameters (0, 2), $\bar{\alpha} = 0.03$, $\bar{\beta} = 0.38$, and $X_0 = 0.05$, and for the general model with independent α and β distributed as (c) uniform with parameters (0, 0.06), (0, 2.76), $\bar{\alpha} = 0.03$, $\bar{\beta} = 1.38$, and $X_0 = 0.01$.

Multiplicative Noise and Non-extensive Statistical Mechanics

(Anteneodo & Tsallis , J.Math.Physics, 44, 5194 (2003))

- Boltzmann-Gibbs(BG) statistical mechanics

$$S_1 = -k \int p(u) \ln p(u) du$$

(these systems like equally to live everywhere in the phase space) - ergodic

- There are other systems- may prefer particular subspace – (a hierarchical or multifractal structure)

- Non-extensive statistical mechanics

$$S_q = k [1 - \int du [p(u)]^q] / (q - 1)$$

$$q - \text{Entropy Index}$$

- Large family of models with multiplicative noise belongs to non extensive class

Dynamical Foundation of Non-extensive Statistical Mechanics

(Beck, 2001, Phy. Rev.Lett.)

Linear Langevin eqation

$$\dot{u} = \gamma u + \sigma L(t), \quad \gamma > 0 \qquad (1)$$

L(t): Gaussian white noise

Define $\boxed{\beta = \gamma / \sigma^2}$ Random Variable

Assume γ or σ or both fluctuate randomly in such a way $\beta = \gamma / \sigma^2$ is χ^2 distributed

$$F(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{n}{2\beta_0}\right)^{n/2} \beta^{(n/2)-1} \exp\left[-\frac{n\beta}{2\beta_0}\right]$$

Probability density function

$$p(u) = \int p(u \mid \beta) f(\beta)$$

$$= \frac{\Gamma(\mathrm{n}+\frac{1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\frac{\beta_0}{\Pi n}} \frac{1}{\left(1+\frac{\beta_0}{n} u^2\right)^{(n/2)+1/2}}, \quad -\infty < u < \infty.$$

Hence, SDE (1) with χ^2 distributed $\beta = \gamma / \sigma^2$ generates

$$\boxed{p(u) \sim \frac{1}{[1 + \frac{1}{2} \tilde{\beta} (q - 1) u^2]^{1/(q-1)}}}$$

$$q = 1 + \frac{2}{n+1}, \qquad \tilde{\beta} = \frac{2}{3 - q} \beta_0$$

Motion of a particle in a velocity dependent random force

(Karmeshu, J. App. Prob., 1976)

$$\frac{du(t)}{dt}+\left(\beta+\alpha(t)\right)u(t)=m^{-1}F(t)$$

$$\langle F(t)\rangle=0, \qquad \langle F(t_1)F(t_2)\rangle=S\delta(t_1-t_2)$$

$$\langle \alpha(t)\rangle=0, \qquad \langle \alpha(t_1)\alpha(t_2)\rangle=2\lambda\delta(t_1-t_2)$$

Stochastic Liouville Equation

Introduce the function $\rho(u,t)=\delta\left(u-u(t)\right)$,

$$\frac{\partial\rho(u,t)}{\partial t}+\frac{\partial\left(\dot{u}(t)\rho\right)}{\partial u}=0$$

Velocity Distribution Function

$$P(u,t)=\langle \rho(u,t)\rangle$$

Fokker-Planck Equation

$$\frac{\partial P}{\partial t}=\frac{\partial(\beta uP)}{\partial u}+\lambda\frac{\partial}{\partial u}\left[u\frac{\partial}{\partial u}(uP)\right]+\frac{S}{2m^2}\frac{\partial^2 P}{\partial u^2}$$

$$\frac{\partial P}{\partial t}=\frac{\partial^2}{\partial u^2}\left[\left(\lambda u^2+\frac{S}{2m^2}\right)P\right]+\frac{\partial}{\partial u}\left[(\beta-\lambda)uP\right]$$

$$P(u,t;u_0)=\sqrt{\left(\frac{2m^2\lambda}{S}\right)}\left(1+\frac{2m^2\lambda}{S}u^2\right)^{-\left(\alpha+\frac{1}{2}\right)}$$

$$.\left\{\frac{1}{\pi}\sum_{n=0}^{N}\frac{\alpha-n}{n!\,\Gamma(2\alpha+1-n)}e^{-n\lambda(2\alpha-n)t}\theta_n\left(\sqrt{\left(\frac{2m^2\lambda}{S}\right)}u_0\right)\right.$$

$$.\theta_n\left(\sqrt{\left(\frac{2m^2\lambda}{S}\right)}u\right)+\frac{1}{2\pi}\int_0^\infty e^{-\lambda\left(\alpha^2+\mu^2\right)t}\left[\psi\left(\mu,\sqrt{\left(\frac{2m^2\lambda}{S}\right)}u_0\right)\right.$$

$$.\psi\left(-\mu,\sqrt{\left(\frac{2m^2\lambda}{S}\right)}u\right)$$

$$\left.\left.+\psi\left(-\mu,\sqrt{\left(\frac{2m^2\lambda}{S}\right)}u_0\right)\psi\left(-\mu,\sqrt{\left(\frac{2m^2\lambda}{S}\right)}u\right)\right]d\mu\right\}$$

In the limit of large times, we get the stationary pdf

$$P(u,\infty)=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha)}\left(1+\frac{2m^2\lambda}{S}\mu^2\right)^{-(\alpha+1/2)}\sqrt{\frac{2m^2\lambda}{S}}$$

For $\alpha=\frac{1}{2}$, it yields Cauchy pdf

$$P(u)=\sqrt{\left(\frac{2m^2\lambda}{S}\right)}\frac{1}{\pi}\left(1+\frac{2m^2\lambda}{S}u^2\right)^{-1}$$

Stability of moment in a simple neutronic system with stochastic parameter Nuclear science and engineering

(Karmeshu & Bansal, Nuclear Sci., Eng.(1975))

The equations for one delayed group of reaction kinetics

$$\frac{dP(t)}{dt} + A(t)P(t) = R(t)$$

$$where$$

$$P(t) = \begin{bmatrix} N(t) \\ C(t) \end{bmatrix}$$

$$A(t) = -\begin{bmatrix} \frac{\rho(t)-\beta}{l} & \lambda \\ \beta / l & \lambda \end{bmatrix}$$

$$R(t) = \begin{bmatrix} S(t) \\ 0 \end{bmatrix}$$

$$\rho(t) = \rho_0\left[1+\varepsilon\Delta(t)\right], \quad \Delta(\mathrm{t}) : \mathrm{DMP}$$

$$P(t) = G(t,0)P(0) + \int_0^t G(t,t_1)R(t_1)dt_1$$

N(t): Neutron density

C(t) : Delayed neutron-precursor density

ρ(t) : Reactivity

λ: Mean delayed neutron decay constant

S(t): Source term

l : generation time

- Consider processes subject to additive and multiplicative noise

$$\frac{du}{dt} = f(u) + g(u)\zeta(t) + \eta(t),$$

$\zeta(t)$ and $\eta(t)$ are uncorrelated and Gaussian distributed white noises

Fokker Planck Equation

$$\frac{\partial p(u,t)}{\partial t} = -\frac{\partial}{\partial u}[f(u)p(u,t)] + M\frac{\partial}{\partial u}\{g(u)\frac{\partial}{\partial u}[g(u)p(u,t)]\} + A\frac{\partial^2 p}{\partial t^2}$$

Whenever deterministic drift proportional the noise induced one

$$f(u) = -\tau g(u)g'(u)$$

Stationary Solution

$$p(u,\infty) \propto \{1-(1-q)\beta[g(u)]^2\}^{1/(1-q)}$$

$$q = \frac{\tau + 3M}{\tau + M}, \beta = \frac{\tau + 3M}{2A}.$$

This distribution is obtained by

Max S_q

Subject to

$$\langle [g(u)]^2 \rangle = const$$

Nonlinear Fokker-Planck Equation

(L. Borland, Phy. Rev. Lett.,1998)

- Anomalous diffusion (Mean square displacement $\propto t^{\theta}$)

  (Super diffusive $\theta>1$, Sub diffusive $\theta<1$)
- Self-gravitating systems
- Linear response theory
- The explicit form of Nonlinear FPE

$$\frac{\partial f^{\mu}}{\partial t} = -\frac{\partial}{\partial x}(Kf^{\mu}) + Q\frac{\partial^2}{\partial x^2} f^{\upsilon} \qquad (1)$$

K: drift coeffecient , Q: diffusion ,μ *and* υ are real numbers

The Kinetic equation (SDE)

$$\dot{x} = K(x) + \sqrt{Q}\, f(x,t)^{(\upsilon-\mu)/2}\, \eta(t) \qquad (2)$$

f satifies nonlinear FPE

Stationary solution

$$f_q(x) = \frac{1}{z_q}[1-\beta(1-q)V(x)]^{1/(1-q)} \qquad (3)$$

where z_q take care of normalization

$\mathrm{V(x)} = \int \mathrm{K}(x')dx'$ is the Potential

$$\beta = \frac{2}{\mathrm{Q}} \frac{Z_q^{q-1}}{(2-q)}$$

Probability distribution (3) maximizes Tsallis entropy

$$\mathrm{S_q} = \frac{1 - \int dx\ [f(x)]^q}{q-1}$$

Pearson System of Distribution

(as stationary solution of diffusion process)

- **Pearson(1895) introduced four parameter unimodal pdf governed by differential equation**

$$\frac{df(x)}{dx} = \left(\frac{x+b}{a_0 + a_1 x + a_2 x^2}\right) f(x)$$

The four parameter pdf is appealing : Data fitting

$$b = a_1 = -0.5\left\{\frac{\mu_3(3\mu_2^2 + \mu_4)}{9\mu_2^3 - 5\mu_2\mu_4 + 6\mu_3^2}\right\}, a_0 = -0.5\left\{\frac{\mu_2(4\mu_2\mu_4 - 3\mu_3^2)}{9\mu_2^3 - 5\mu_2\mu_4 + 6\mu_3^2}\right\},$$

$$a_2 = -0.5\left\{\frac{(6\mu_2^3 - 2\mu_2\mu_4 + 3\mu_3^2)}{9\mu_2^3 - 5\mu_2\mu_4 + 6\mu_3^2}\right\}$$

The nature of Pearson type is determined by parameter κ

$$\kappa = \frac{S_k^2 (K_u + 3)^2}{4(4K_u - 3S_k^2)(2K_u - 3S_k^2 - 6)}, \qquad S_k = \frac{\mu_3}{\mu_2^{3/2}}; K_u = \frac{\mu_4}{\mu_2^2}$$

$$Type\ I(\kappa < 0)$$

$$f(x) = A_0(1+\frac{x}{c_1})^{g_1}\ \ (1-\frac{x}{c_2})^{g_2}, \qquad -c_1 < x < c_2$$

$$Type\ IV(0 < \kappa < 1)$$

$$f(x) = A_0(1+(\frac{x}{c})^2)^{-g_1} e^{(-\alpha \arctan(x/c))}, \qquad -\infty < x < \infty$$

$$Type\ \mathrm{V}I(1 < \kappa)$$

$$f(x) = A_0(x-c)^{g_2}(x)^{-g_1}, \qquad c \le x < \infty$$

E.Wong (1964)

Consider FPE

$$\frac{\partial^2}{\partial x^2}\left[B(x)p\right] - \frac{\partial}{\partial X}\left[A(x)p\right] = \frac{\partial p}{\partial t}, \quad p\left(x\middle|x_0, t\right)$$

on an interval with end points x_1 *and* x_2.

Boundary condition

$$\frac{\partial}{\partial x}\left[B(x)p\right] - A(x)p = 0 \qquad x = x_1, x_2$$

(Reflecting Barrier)

Consider the class of first-order pdf W(x) which satisfy pearson equation

$$\frac{\mathrm{dW(x)}}{\mathrm{dx}} = \frac{ax+b}{cx^2+dx+e} W(x)$$

Class of Markov process is constructed for which

$$\lim_{t \to \infty} p\left(x \middle| x_0, t\right) = W\left(x\right).$$

"Pearson family serves a natural role in briging the gap between first order statistical description characterized by W(x) and transitional properties represented by $p\left(x \middle| x_0, t\right)$"

Pearson Diffusion

(Forman and Sorensen (2008))

Pearson family is a stationary solution of SDE

$$dx(t) = -\theta(x(t) - \mu) + \sqrt{2\theta(ax_t^2 + bx_t + c)}\,dW(t), \theta > 0$$

Invariant distribution belongs to Pearson family

$$\frac{dm(x)}{dx} = -\frac{(2a + c)x - (\mu - b)}{ax^2 + bx + c}m(x)$$

- Classification of stationary solution
- Diffusion with heavy-tailed

Optimal Parameter Estimation in SDEs: An Application to Random Multiaccess Communication Systems

Introduction

- Stochastic differential equations (SDEs) play a major role in the modeling and analysis of many real life engineering problems such as those arising in communication and wireless networks, aerospace systems and portfolio optimization.
- Problems of control and optimization in systems governed by SDEs arise naturally in many applications.
- It is however the case that solving SDEs analytically in most situations is an impossible task though they are a useful means for modelling highly random phenomena.
- We propose an algorithmic procedure for finding the optimal parameter trajectory in parameterized SDEs and extend the same to a procedure for optimal control of SDEs.
- We then show applications in the context of random multiaccess communication systems.

The SDE Framework

Bhatnagar, Karmeshu and Mishra
(TOMACS, 2009)

- Consider a process $X(t)$, $t \geq 0$ whose sample paths are governed by the parameterized SDE

$$dX(t) = b(X(t), \theta(t))dt + \sigma(X(t), \theta(t))dW(t). \tag{1}$$

- Here $\theta(t)$, $t \geq 0$ is the associated parameter process. Let $\theta(t) \in \mathcal{U} \subset \mathcal{R}$, for all $t \geq 0$ where $\mathcal{U}$ is compact.
- Also, $X(t) \in \mathcal{R}^d$, for some $d \geq 1$ for all $t \geq 0$ and $W(\cdot)$ is a 1-dimensional Brownian motion.
- Further, $b : \mathcal{R}^d \times \mathcal{R} \to \mathcal{R}^d$ and $\sigma : \mathcal{R}^d \times \mathcal{R} \to \mathcal{R}^d$ are the drift and diffusion terms, respectively, that we assume are both Lipschitz continuous functions.

The Cost Criterion

- Let $[0, T]$ for some $T > 0$ be the interval over which we consider the evolution of the SDE. Let $\bar{g} : [0, T] \times \mathcal{R}^d \to \mathcal{R}$ represent the associated cost function.
- Define by $\bar{J}_{X_0}(\theta(\cdot))$, the quantity

$$\bar{J}_{X_0}(\theta(\cdot)) = E\left[\int_0^T \bar{g}(t, X(t))dt \mid X(0) = X_0\right], \tag{2}$$

  that we assume is well defined and finite for all feasible parameter trajectories $\{\theta(t), 0 \leq t \leq T\}$.
- Objective: Find a function $\theta^* : [0, T] \to \mathcal{R}$ with $\theta^*(t) \in \mathcal{U}$, $\forall t \in [0, T]$ that minimizes (2) given the initial state X_0.

The Discretized Problem

- For discrete-valued states, Ito's formula gives the Euler-Milstein scheme (where $X_j = X(jh)$):

$$X_{j+1} = X_j + b(X_j, \theta_j)h + \sigma(X_j)\sqrt{h}Z_{j+1} + \frac{1}{2}\sigma'(X_j)\sigma(X_j)h(Z_{j+1}^2 - 1). \quad (3)$$

- Let $g_j(X_j) \equiv \bar{g}(jh, X(jh))$ denote the cost rate at instant jh. Given X_0, our aim is to find parameters $\theta_0, \theta_1, \ldots, \theta_{N-1} \in \mathcal{U}$ that minimize

$$J_{X_0}(\theta_0, \ldots, \theta_{N-1}) \stackrel{\triangle}{=} E\left[\sum_{j=1}^{N} g_j(X_j) \mid X(0) = X_0\right] h. \quad (4)$$

- AIM: Find an optimal parameter trajectory $\{\theta_0^*, \theta_1^*, \ldots, \theta_{N-1}^*\}$ that minimizes objective.

The Algorithm

- The algorithm is as follows: For $j = 1, \ldots, N$,

$$Y_j(n+1) = Y_j(n) + b(n)\left(\frac{\eta_1(n)}{2\beta}\left(\sum_{i=j}^{N}(g_i(X_i^+(n)) - g_i(X_i^-(n)))\right) - Y_j(n)\right).$$

- Next for $1 \leq j \leq N$, we have

$$\theta_j(n+1) = \Gamma_j(\theta_j(n) - a(n)Y_j(n)). \tag{5}$$

- Here, $X_j^+(n)$, $X_j^-(n)$, are respectively governed by $\theta + \beta\eta_j(n)$ and $\theta - \beta\eta_j(n)$. Also, $\Gamma_j : \mathcal{R} \to \mathcal{R}$ are suitable projection operators to the feasible region.

Slotted Aloha – An SDE Formulation

- Let $K - t$ be number of backlogged nodes at time t. For $X^m(t) \equiv K_{[mt]}/m$, it is argued in Kaj [2002] that for a large number m of users, the behaviour of the system can be approximated by the following SDE:

$$dX^m(t) = \mu(X^m(t))dt + \frac{1}{\sqrt{m}}\sigma(X^m(t))dW(t), \tag{6}$$

where the drift and diffusion terms are given by

$$\mu(X^m(t)) = mp(1 - X^m(t)) -$$

$$(mp(1 - X^m(t)) + mqX^m(t)) \exp(-mp(1 - X^m(t)) - mqX^m(t)),$$

$$\sigma^2(X^m(t)) = (mp)^2(1 - X^m(t))^2 + mp(1 - X^m(t))$$

$$-(mp(1 - X^m(t)) - mqX^m(t)) \exp(-mp(1 - X^m(t)) - mqX^m(t)).$$

- Our algorithm shows excellent throughput and delay performance.

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- [2] Karmeshu, Bhatnagar, S. and Mishra, V.K. (2011) “An optimized SDE model for slotted Aloha”, *Accepted in IEEE Transactions on Communications*.
- [3] Kaj, I. (2002) *Stochastic Modeling in Broadband Communication Systems*, SIAM Monographs on Mathematical Modeling and Computation, SIAM, Philadelphia.

Maximum Entropy Principle (MEP)

- **MEP:** ''The most objective probability distribution is that which maximizes the entropy subject to constraints supplied by the given information.'' (Jaynes' 1957)

1. Discrete random variable:

First moment prescribed

$$\text{Max} H_n(P) = -\Sigma p_i \log p_i$$

s.t

$$\sum_{i=1}^{n} p_i = 1 \qquad \sum_{i=1}^{n} i p_i = m$$

$$p_i = ab^i, \qquad i = 1, 2, ..., n$$

2. Continuous random variable:

First two moments prescribed

$$\text{Max} S = -\int f(x) \ln f(x) dx$$

s.t

$$\int f(x) = 1$$

$$\int x f(x) = \mu$$

Tsallis Entropy

- Generalized entropy measure (multifractal dynamical systems)

  $$S_q = K \frac{1 - \sum_{n=0}^{\infty} p_n^q}{q - 1}$$

  where
  p_n: probability of the system in state n
  q: Tsallis parameter, measure of degree of nonextensivity
  K: a positive constant

  Limiting form: Tsallis Entropy $\longrightarrow$ Shannon Entropy as $q \longrightarrow 1$

- **Non-Additivity:** For two independent systems A and B

  $$S_q(AB) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)$$

- Successfully applied to variety of problems

- **Limitation of Shannon entropy framework:** Power Law cannot be captured in natural manner (Montroll and Shlesinger 1984)

  Probability distribution with power law would require moments to be prescribed as E[ln X]

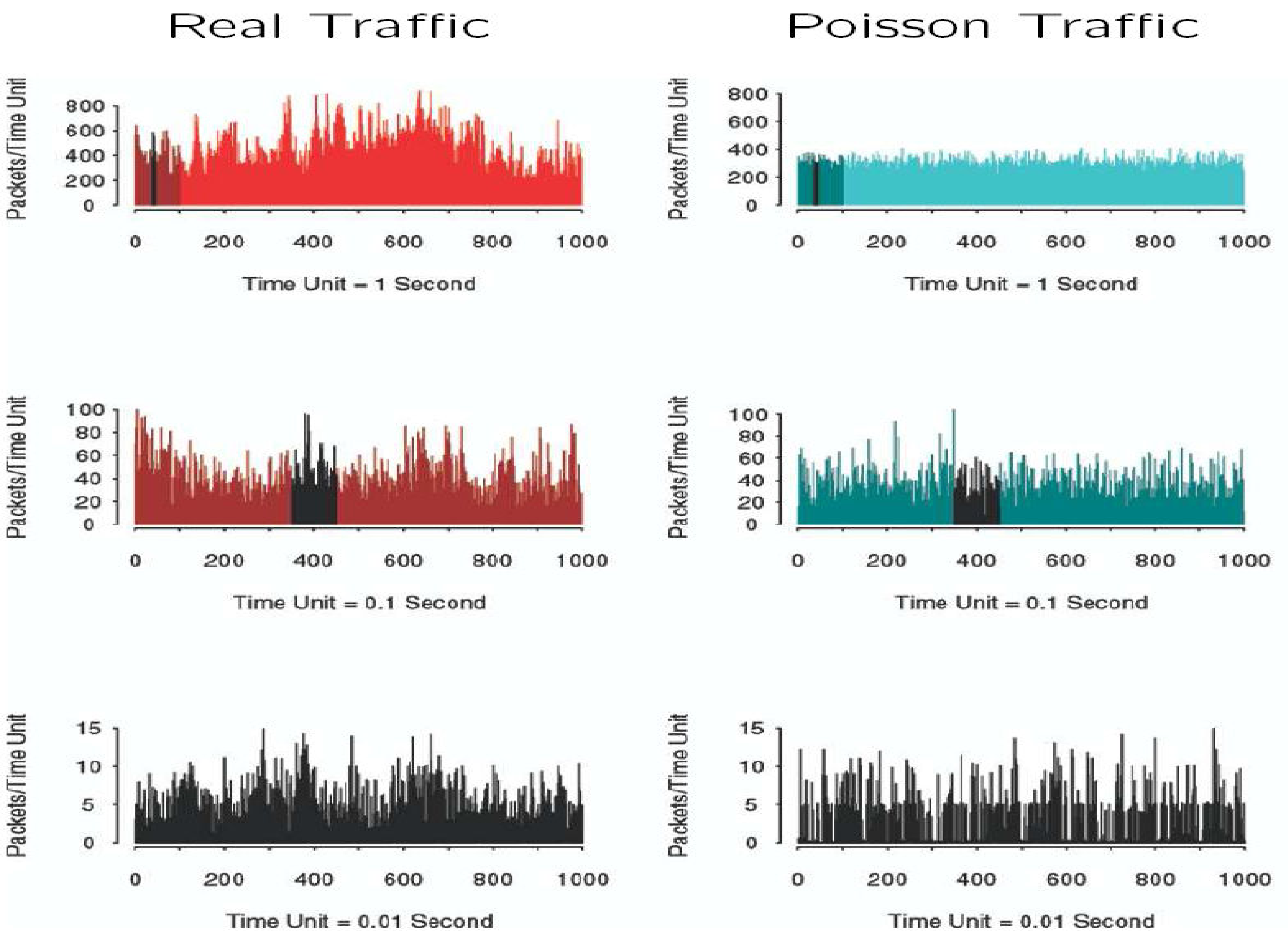
- **Appropriate framework:** **Tsallis parametric entropy** (Tsallis 1988)- Potential to generate power law behavior

Some Illustrative Examples

- To develop queueing models for broadband networks

- To study effects of LRD traffic on QoS

- This in turn will help in better understanding of design issues of high speed networks



(Source: Willinger et al., IEEE/ACM Trans. on Networking, vol. 5, no. 1, 1997. - adapted)

Queue Length Distribution of Network Packet Traffic

(Karmeshu and Shachi Sharma, IEEE Communication Letters, Vol. 10, no.1, 2006)

- Distributions like Pareto used to model LRD traffic

- First moment may not exist but probability distribution exists

Problem Formulation

$$\text{Max } S_q = \text{Max } K \frac{1 - \sum_{n=0}^{\infty} p_n^q}{q-1}$$

subject to

Fractional moment constraint

$$\sum_{n=0}^{\infty} n^k p_n = A_k, \qquad 0 < k < 1$$

Normalization constant

$$\sum_{n=0}^{\infty} p_n = 1$$

- **Queue length distribution**

$$p_n = Z^{-1}[1 + \beta_k (1-q) n^k]^{\frac{1}{q-1}}$$

- Power law behavior

$$p_n \sim n^{\frac{-k}{1-q}}, \qquad k/(1-q) > 1$$

- Limiting case ($q \longrightarrow 1$):

$$p_n = \frac{e^{-\beta n^k}}{\sum_{n=0}^{\infty} e^{-\beta n^k}}$$

- **Overflow probability**

$$P(n > \omega) = \frac{Z^{-1} q}{(k+q-1)\Gamma\left[\frac{1}{1-q}\right]}$$
$$\left[\frac{\omega^{-k}}{\beta_k(1-q)}\right]^{-\frac{q-1+k}{k(q-1)}} \left[\frac{1}{\beta_k(1-q)}\right]^{\frac{1}{k}} \Gamma\left[\frac{q}{1-q}\right]$$
$${}_2F_1\left[\frac{1}{1-q} - \frac{1}{k}, \frac{1}{1-q}, 1 - \frac{1}{k} + \frac{1}{1-q}, \frac{\omega^{-k}}{\beta_k(q-1)}\right]$$
$$- Z^{-1} k \beta_k \int_{\omega}^{\infty} \frac{(t - [t])\, t^{k-1}}{\left[1 + \beta_k(1-q)t^k\right]^{\frac{2-q}{1-q}}} dt$$

- Large buffer size, one finds,

$$P(n > \omega) \sim \omega^{-\frac{k}{1-q}+1}\ , \qquad \frac{k}{1-q} > 1$$

Packet Distribution in Queueing Network and Power Law Behavior
(Karmeshu and Shachi Sharma, IEEE Communication Letters, Vol. 10, no.8, 2006)

- Packet switching network can be modelled as the network of queues

- Well-known result: Jackson's product Form solution for Markovian queueing networks

- **What is the consequence of power law characteristics of traffic on queueing networks**?

- A queueing network with m nodes

- S denotes the state space of the system with a particular realization
$\underline{\mathbf{n}} = (n_1, n_2, ..., n_m) \in S$

- $f_{ij}(\underline{\mathbf{n}}), i = 1, 2, ..., m;\ \ j = 1, ..., r$ be the collection of functions defined on the state space S where r is the number of constraints on each node

- $p(\underline{\mathbf{n}}) = p(n_1, n_2, ..., n_m)$ denotes the joint probability of there being $n_1, n_2, ..., n_m$ units in the m nodes

- The Escort probabilities are defined as

$$P(\underline{\mathbf{n}}) = \frac{[p(\underline{\mathbf{n}})]^q}{\sum_{\underline{\mathbf{n}} \in S} [p(\underline{\mathbf{n}})]^q}$$

- Partial information in the form of q-mean value constraints are known

Determination of joint distribution $p(\underline{\mathbf{n}})$,

$$\text{Max } S_q = \text{Max } K \frac{1 - \sum_{\underline{\mathbf{n}} \in S} [p(\underline{\mathbf{n}})]^q}{q - 1}$$

subject to normalization constraint

$$\sum_{\underline{\mathbf{n}} \in S} p(\underline{\mathbf{n}}) = 1$$

and q-mean value constraints

$$\sum_{\underline{\mathbf{n}} \in S} P(\underline{\mathbf{n}}) f_{ij}(\underline{\mathbf{n}}) = E[f_{ij}], \quad j = 1, 2, ..., r$$

- **Joint distribution of queue length**

$$p(\underline{\mathbf{n}}) = \frac{exp_q g_1(n_1) \otimes_q \ldots \otimes_q exp_q g_m(n_m)}{\sum_{\underline{\mathbf{n}} \in S} exp_q g_1(n_1) \otimes_q \ldots \otimes_q exp_q g_m(n_m)}$$

where q-exponential $exp_q(x)$ is defined as

$$exp_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}}$$

The q-product is

$$x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}$$

- **Note: usual product form result is retrieved as q tends to one.**