Fake Projective Planes

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Outline

Classify algebraic varieties up to connected moduli

Nonsingular projective algebraic curves $/\mathbb{C}$ (compact Riemann surfaces) are classified by the " mighty" genus

$g(C) :=$ (the number of "holes" of C) $= \dim_{\mathbb{C}} H^0(C, \Omega^1_C) = \frac{1}{2}\dim_{\mathbb{Q}} H_1(C, \mathbb{Q})$.

$$g(C) = 0 \iff C \cong \mathbf{P}^1 \cong (\textit{Riemann sphere}) = \mathbb{C} \cup \{\infty\}.$$

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In dimension > 1, many invariants: Hodge numbers, Betti numbers

$$h^{i,j}(X) = \dim H^j(X, \Omega^i_X), \;\; b_i(X) := \dim H^i(X, \mathbb{Q}).$$

Given Hodge numbers (and even fixing fundamental group), hard to describe the moduli, in general.

Smooth Algebraic Surfaces with $p_g = q = 0$

Long history : Castelnuovo's rationality criterion, Severi conjecture, ...

Here, the geometric genus and the irregularity

$$p_g(X) := \dim H^n(X, \mathcal{O}_X) = \dim H^0(X, \Omega_X^n) = h^{0,n}(X) = h^{n,0}(X),$$

$$q(X) := \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega_X^1) = h^{0,1}(X) = h^{1,0}(X).$$

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Max Nöther(1844-1921) said [in the book of Federigo Enriques(1871-1946)] :

"Algebraic curves are created by god,

algebraic surfaces are created by devil."

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Enriques-Kodaira classification of algebraic surfaces (1940's):

- $\mathbf{P}^2$, rational ruled surfaces;
- Enriques surfaces;
- properly elliptic surfaces with $p_g = q = 0$;
- surfaces of general type with $p_g = 0$ (these have $K^2 = 1, 2, \ldots, 9$);
- blow-ups of the above surfaces.

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Smooth algebraic surfaces with minimal invariants, that is, with

$$b_1 = b_3 = 0,\ b_0 = b_2 = b_4 = 1\ (\Rightarrow p_g = q = 0)$$

are

- $\mathbf{P}^2$;
- fake projective planes (= surfaces of general type with $p_g = 0$, $K^2 = 9$).

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Remark. FPPs are ball quotients, so not simply connected.
Exotic $\mathbf{P}^2$ does NOT exist in complex geometry.

$\mathbb{Q}$-homology $\mathbf{P}^2$

Definition

A normal projective surface S is called a $\mathbb{Q}$-homology $\mathbf{P}^2$ if $b_i(S) = b_i(\mathbf{P}^2)$ for all i, i.e. $b_1 = b_3 = 0, b_0 = b_2 = b_4 = 1$.

- If S is smooth, then $S = \mathbf{P}^2$ or a fake projective plane.
- If S has A_1-singularities only, then $S \cong (w^2 = xy) \subset \mathbf{P}^3$.
- If S has A_2-singularities only, then S has $3A_2$ or $4A_2$ and $S \cong \mathbf{P}^2/G$ or FPP$/G$, where $G \cong \mathbb{Z}/3$ or $(\mathbb{Z}/3)^2$.

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In this talk, we assume S has at worst quotient singularities.
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Then S is a $\mathbb{Q}$-homology $\mathbf{P}^2$ if $b_2(S) = 1$.

For a minimal resolution $S' \to S$,

$$p_g(S') = q(S') = 0.$$

Trichotomy: $K_S =$ ample, $-$ample, num. trivial

Let S be a $\mathbb{Q}$-hom $\mathbf{P}^2$ with quotient singularities.

- $-K_S$ is ample
  - log del Pezzo surfaces of Picard number 1, e.g. $\mathbf{P}^2/G$, $\mathbf{P}^2(a,b,c)$, ...
  - $\kappa(S') = -\infty$.
- K_S is numerically trivial.
  - log Enriques surfaces of Picard number 1.
  - $\kappa(S') = -\infty, 0$.
- K_S is ample.
  - e.g. all quotients of fake projective planes, suitable contraction of a suitable blowup of $\mathbf{P}^2$, some Enriques surface, ...
  - $\kappa(S') = -\infty, 0, 1, 2$.

Problem

Classify all $\mathbb{Q}$-homology $\mathbf{P}^2$'s with quotient singularities.

The Maximum Number of Quotient Singularities

Question

How many singular points on S, a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities?

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How many singular points on S, a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities?

- $|Sing(S)| \leq 5$ by the orbifold Bogomolov-Miyaoka-Yau inequality (Sakai, Miyaoka, Megyesi for K nef)

$$\frac{1}{3}K_S^2 \leq e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \left(1 - \frac{1}{|\pi_1(L_p)|}\right).$$

(Keel-McKernan for $-K$ nef)

$$0 \leq e_{orb}(S).$$

- Many examples with $|Sing(S)| \leq 4$ (cf. Brenton, 1977)
- If $-K_S$ is ample, $|Sing(S)| \leq 4$ (Belousov, 2008).

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The case with $|Sing(S)| = 5$ were classified by Hwang-Keum.

Theorem (D.Hwang-Keum, JAG 2011)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities. Then $|Sing(S)| \leq 4$ except the following case:
S has 5 singular points of type $3A_1 + 2A_3$, and its minimal resolution S' is an Enriques surface.

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Corollary

Every $\mathbb{Z}$-homology $\mathbf{P}^2$ with quotient singularities has at most 4 singular points.

Remark

(1) Every $\mathbb{Z}$-cohomology $\mathbf{P}^2$ with quotient singularities has at most 1 singular point. If it has, then the singularity is of type E_8 [Bindschadler-Brenton, 1984].
(2) $\mathbb{Q}$-homology $\mathbf{P}^2$ with rational singularities may have arbitrarily many singularities, no bound.

$\mathcal{C}^\infty$-action of $\mathbf{S}^1$ on $\mathbf{S}^m$

$$\mathbf{S}^1 \subset Diff(\mathbf{S}^m).$$

The identity element $1 \in \mathbf{S}^1$ acts identically on $\mathbf{S}^m$.

Each diffeomorphism $g \in \mathbf{S}^1$ is homotopic to the identity map $1_{\mathbf{S}^m}$.
By Lefschetz Fixed Point Formula,

$$e(Fix(g)) = e(Fix(1)) = e(\mathbf{S}^m).$$

If m is even, then $e(\mathbf{S}^m) = 2$ and such an action has a fixed point, so the foliation by circles degenerates.

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If m is even, then $e(\mathbf{S}^m) = 2$ and such an action has a fixed point, so the foliation by circles degenerates. **Assume** $m = 2n - 1$ **odd.**

Definition

A $\mathcal{C}^\infty$-action of $\mathbf{S}^1$ on $\mathbf{S}^{2n-1}$

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

is called a pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$ if it is free except for finitely many orbits (whose isotropy groups $\mathbb{Z}/a_1, \ldots, \mathbb{Z}/a_k$ have pairwise prime orders).

Pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$

Example (Linear actions)

$$\mathbf{S}^{2n-1} = \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : |z_1|^2 + |z_2|^2 + ... + |z_n|^2 = 1\} \subset \mathbb{C}^n$$

$$\mathbf{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \subset \mathbb{C}.$$

Positive integers $a_1, ..., a_n$ pairwise prime.

$$\mathbf{S}^1 \times \mathbf{S}^{2n-1} \to \mathbf{S}^{2n-1}$$

$$(\lambda, (z_1, z_2, ..., z_n)) \to (\lambda^{a_1} z_1, \lambda^{a_2} z_2, ..., \lambda^{a_n} z_n).$$

- In this linear action

$$\mathbf{S}^{2n-1}/\mathbf{S}^1 \cong \mathbb{CP}^{n-1}(a_1, a_2, ..., a_n).$$

- The orbit of the i-th coordinate point $e_i \in \mathbf{S}^{2n-1}$ is exceptional iff $a_i \geq 2$.
- The orbit of a non-coordinate point of $\mathbf{S}^{2n-1}$ is NOT exceptional.
- This action has at most n exceptional orbits.
- The quotient map $\mathbf{S}^{2n-1} \to \mathbb{CP}^{n-1}(a_1, a_2, ..., a_n)$ is a Seifert fibration.

Pseudofree $\mathbf{S}^1$-action on $\mathbf{S}^{2n-1}$

- For $n = 2$ Seifert (1932) showed that each pseudo-free $\mathbf{S}^1$-action on $\mathbf{S}^3$ is linear and hence has at most 2 exceptional orbits.
- For $n = 4$ Montgomery-Yang (1971) showed that given arbitrary collection of pairwise prime positive integers $a_1, \ldots, a_k$, there is a pseudofree $\mathbf{S}^1$-action on a homotopy $\mathbf{S}^7$ whose exceptional orbits have exactly those orders.
- Petrie (1974) generalised the above M-Y for all $n \geq 5$.

Conjecture (Montgomery-Yang problem, Fintushel-Stern 1987)

A pseudo-free $\mathbf{S}^1$-action on $\mathbf{S}^5$ has at most 3 exceptional orbits.

- This problem is wide open. F-S withdrew their paper [$O(2)$-actions on the 5-sphere, Invent. Math. 1987].

- Pseudo-free $\mathbf{S}^1$-actions on a manifold Σ have been studied in terms of the orbit space $\Sigma/\mathbf{S}^1$.
- The orbit space $X = \mathbf{S}^5/\mathbf{S}^1$ of such an action is a 4-manifold with isolated singularities whose neighborhoods are cones over lens spaces $S^3/\mathbb{Z}_{a_i}$ corresponding to the exceptional orbits of the $\mathbf{S}^1$-action.

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- Easy to check that X is simply connected and $H_2(X, \mathbb{Z})$ has rank 1 and intersection matrix $(1/a_1 a_2 \cdots a_k)$.
- An exceptional orbit with isotropy type $\mathbb{Z}/a$ has an equivariant tubular neighborhood which may be identified with $\mathbb{C} \times \mathbb{C} \times \mathbf{S}^1$ with a $\mathbf{S}^1$-action

$$\lambda \cdot (z, w, u) = (\lambda^r z, \lambda^s w, \lambda^a u)$$

where r and s are relatively prime to a.

The following 1-1 correspondence was known to Montgomery-Yang, Fintushel-Stern, and revisited by Kollár(2005).

Theorem

There is a one-to-one correspondence between:

1. *Pseudo-free* $\mathbf{S}^1$*-actions on* $\mathbb{Q}$*-homology 5-spheres* Σ *with* $H_1(\Sigma, \mathbb{Z}) = 0$.
2. *Compact differentiable* 4*-manifolds* M *with boundary such that*
   1. $\partial M = \bigcup_i L_i$ *is a disjoint union of lens spaces* $L_i = S^3/\mathbb{Z}_{a_i}$,
   2. *the* a_i*'s are pairwise prime,*
   3. $H_1(M, \mathbb{Z}) = 0$,
   4. $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$.

Furthermore, Σ *is diffeomorphic to* $\mathbf{S}^5$ *iff* $\pi_1(M) = 1$.

Algebraic Montgomery-Yang Problem

This is the M-Y Problem when $\mathbf{S}^5/\mathbf{S}^1$ attains a structure of a normal projective surface.

Conjecture (J. Kollár)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with at worst quotient singularities. If $\pi_1(S^0) = \{1\}$, then S has at most 3 singular points.

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There are infinitely many examples S with $H_1(S^0, \mathbb{Z}) = 0$, $\pi_1(S^0) \neq \{1\}$, $|Sing(S)| = 4$.

These examples obtained from the classification of surface quotient singularities [E. Brieskorn, Invent. Math. 1968].

Example (coming from Brieskorn's classification of surface singularities)

$I_m \subset GL(2, \mathbb{C})$ the $2m$-ary icosahedral group $I_m = \mathbb{Z}_{2m}.\mathcal{A}_5$.

$$1 \to \mathbb{Z}_{2m} \to I_m \to \mathcal{A}_5 \subset PSL(2, \mathbb{C})$$

I_m acts on $\mathbb{C}^2$. This action extends naturally to $\mathbf{P}^2$. Then

$$S := \mathbf{P}^2 / I_m$$

is a $\mathbb{Z}$-homology $\mathbf{P}^2$ with $-K_S$ ample,

- S has 4 quotient singularities:
  one non-cyclic singularity of type I_m (the image of $O \in \mathbb{C}^2$), and
  3 cyclic singularities of order $2, 3, 5$ (on the image of the line at infinity),
- $\pi_1(S^0) = \mathcal{A}_5$, hence $H_1(S^0, \mathbb{Z}) = 0$.

Call these surfaces Brieskorn quotients.

Progress on Algebraic Montgomery-Yang Problem

Theorem (D.Hwang-Keum, MathAnn 2011)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with quotient singularities, not all cyclic, such that $\pi_1(S^0) = \{1\}$. Then $|Sing(S)| \leq 3$.

More precisely

Theorem (D.Hwang-Keum, MathAnn 2011)

Let S be a $\mathbb{Q}$-homology $\mathbf{P}^2$ with 4 or more quotient singularities, not all cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$. Then S is isomorphic to a Brieskorn quotient.

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Let S be a $\mathbb{Q}$*-homology* $\mathbf{P}^2$ *with quotient singularities, not all cyclic, such that* $\pi_1(S^0) = \{1\}$*. Then* $|Sing(S)| \leq 3$*.*

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More Progress on Algebraic Montgomery-Yang Problem:

Theorem (D.Hwang-Keum, 2013, 2014)

Let S be a $\mathbb{Q}$*-homology* $\mathbf{P}^2$ *with cyclic singularities such that* $H_1(S^0, \mathbb{Z}) = 0$*. If either S is not rational or* $-K_S$ *is ample, then* $|Sing(S)| \leq 3$*.*

The Remaining Case of Algebraic M-Y Problem:

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There are such surfaces. Examples given by

- Keel and Mckernan (Mem. AMS 1999),
- Kollár (Pure Appl. Math. Q. 2008) — an infinite series of examples with $|Sing(S)| = 2$.
- D. Hwang and Keum (Proc. AMS 2012) — infinite series of examples with $|Sing(S)| = 1, 2, 3$.

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Problem

Are there such surfaces S with $|Sing(S)| = 4$?
No examples known yet.

Kollár's examples

$$Y = Y(a_1, a_2, a_3, a_4) := (x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_4 + x_4^{a_4} x_1 = 0)$$

in $\mathbf{P}(w_1, w_2, w_3, w_4)$. Y has 4 singularities, two each on

$$C_1 := (x_1 = x_3 = 0), \quad C_2 := (x_2 = x_4 = 0).$$

Contracting C_1 and C_2 we get $X(a_1, a_2, a_3, a_4)$, a $\mathbb{Q}$-homology $\mathbf{P}^2$ with 2 singularities

$$[\underbrace{2, \ldots, 2}_{a_4 - 1}, a_3, a_1, \underbrace{2, \ldots, 2}_{a_2 - 1}]$$

$$[\underbrace{2, \ldots, 2}_{a_3 - 1}, a_2, a_4, \underbrace{2, \ldots, 2}_{a_1 - 1}].$$

K_X is ample iff $\sum a_j > 12$ and $a_i \geq 3$ for all i.

X can be obtained by blowing up $\mathbf{P}^2$, $\sum a_j$ times inside 4 lines, then contracting all negative curves with self-intersection ≤ -2 (Hwang-Keum 2012, also Urzua-Yanez 2016). The number of such curves is $\sum a_j$.

More examples

can be obtained by blowing up $\mathbf{P}^2$ many times

(1) inside the union of 3 lines and a conic (total degree 5), then contracting all negative curves with self-intersection ≤ -2
$\Longrightarrow$ infinite series of examples with $|Sing(S)| = 2, 3$;

(2) inside the union of 4 lines and a nodal cubic (total degree 7), then contracting all negative curves with self-intersection ≤ -2
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(2) inside the union of 4 lines and a nodal cubic (total degree 7), then contracting all negative curves with self-intersection ≤ -2
$\Longrightarrow$ infinite series of examples with $|Sing(S)| = 1$.

Problem

Are there any $\mathbb{Q}$-homology $\mathbf{P}^2$ which is a rational surface S with K_S ample and with $|Sing(S)| = 4$?

Symplectic Montgomery-Yang Problem

This is the M-Y Problem when $\mathbf{S}^5/\mathbf{S}^1$ attains a structure of a symplectic orbifold,
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Question

Bogomolov inequality holds for symplectic compact 4-manifolds?

$$c_1^2 \leq 4c_2$$

Fake Projective Planes

A compact complex surface with the same Betti numbers as $\mathbf{P}^2$ is called a fake projective plane if it is not biholomorphic to $\mathbf{P}^2$.

A FPP has ample canonical divisor K, so it is a smooth proper (geometrically connected) surface of general type with $p_g = 0$ and $K^2 = 9$ (this definition extends to arbitrary characteristic.)

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Keum FPP and Mumford FPP belong to the same class, in the sense that both fundamental groups are contained in the same maximal arithmetic subgroup of $PU(2,1)$, the isometry group of the complex 2-ball.

FPP's have Chern numbers $c_1^2 = 3c_2 = 9$ and are complex 2-ball quotients by Aubin (1976) and Yau (1977). Such ball quotients are strongly rigid by Mostow's rigidity theorem (1973), that is, determined by fundamental group up to holomorphic or anti-holomorphic isomorphism.

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FPP's come in complex conjugate pairs by Kharlamov-Kulikov (2002) and have been classified as quotients of the two-dimensional complex ball by explicitly written co-compact torsion-free arithmetic subgroups of $PU(2, 1)$ by Prasad-Yeung (2007, 2010) and Cartwright-Steger (2010).

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Key ingredients:

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There are exactly 100 fake projective planes total, corresponding to 50 distinct fundamental groups Cartwright-Steger (2010).

Interesting problems on fake projective planes:

- Exceptional collections in $D^b(coh(X))$
- Bicanonical map
- Explicit equations
- Bloch conjecture on zero cycles
- Modular forms

Explicit equations of a Fake Projective Plane

It has long been of great interest since Mumford to find equations of an FPP.

With Lev Borisov (Duke M.J. 2020), we find equations of a conjugate pair of fake projective planes by using the geometry of the quotients of such FPP [Keum, 2008].

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This pair has the most geometric symmetries among the 50 pairs, in the sense that
(i) $Aut \cong G_{21} = \mathbb{Z}_7 : \mathbb{Z}_3$, the largest (Keum's FPPs);
(ii) the $\mathbb{Z}_7$-quotient has a smooth model of a $(2, 4)$-elliptic surface, not simply connected.

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The universal double cover of this elliptic surface is an $(1,2)$-elliptic surface, has the same Hodge numbers as K3, but Kodaira dimension 1.

$$
\begin{array}{ccccccccccc}
\mathcal{B}^2 & & & & \widehat{\mathbf{P}^2_{\text{fake}}} & & & & X & \xrightarrow{\pi} & \mathbf{P}^3 \\
& \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\
& & \mathbf{P}^2_{\text{fake}} & & & & Y & & & & \mathbf{P}^1 \\
& & & \searrow & & \swarrow & & \searrow & & \swarrow & \\
& & & & \mathbf{P}^2_{\text{fake}}/\mathbb{Z}_7 & & & & \mathbf{P}^1 & &
\end{array}
\qquad (1)
$$

$\mathcal{B}^2$ is the complex 2-ball. $\mathbf{P}^2_{\text{fake}}$ is our FPP.
$Y \to \mathbf{P}^1$ is a $(2,4)$-elliptic surface with one I_9-fibre and three 4-sections.
$X \to \mathbf{P}^1$ is an $(1,2)$-elliptic surface with two I_9-fibres and six 2-sections.

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Using these 24 smooth rational curves on X we find a linear system which gives a birational map

$$\pi : X \to \mathbf{P}^3.$$

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The image is a sextic surface, highly singular.
Its equation is computed explicitly using the elliptic fibration structure $X \to \mathbf{P}^1$.

84 Equations of the fake projective plane

$$
\begin{aligned}
eq_1 &= U_1U_2U_3 + (1-\mathrm{i}\sqrt{7})(U_3^2U_4 + U_1^2U_5 + U_2^2U_6) + (10-2\mathrm{i}\sqrt{7})U_4U_5U_6 \\
eq_2 &= (-3+\mathrm{i}\sqrt{7})U_0^3 + (7+\mathrm{i}\sqrt{7})(-2U_1U_2U_3 + U_7U_8U_9 - 8U_4U_5U_6) \\
&\quad + 8U_0(U_1U_4 + U_2U_5 + U_3U_6) + (6+2\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) \\
eq_3 &= (11-\mathrm{i}\sqrt{7})U_0^3 + 128U_4U_5U_6 - (18+10\mathrm{i}\sqrt{7})U_7U_8U_9 \\
&\quad + 64(U_2U_4^2 + U_3U_5^2 + U_1U_6^2) + (-14-6\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) \\
&\quad + 8(1+\mathrm{i}\sqrt{7})(U_1^2U_8 + U_2^2U_9 + U_3^2U_7 - 2U_1U_2U_3) \\
eq_4 &= -(1+\mathrm{i}\sqrt{7})U_0U_3(4U_6 + U_9) + 8(U_1U_2U_3 + U_1U_6U_9 + U_5U_7U_9) \\
&\quad + 16(U_5U_6U_7 - U_1^2U_5 - U_3U_5^2) \\
eq_5 &= g_3(eq_4) \\
eq_6 &= g_3^2(eq_4) \\
&\ \ \vdots
\end{aligned}
$$

On the coordinates $(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9)$ of $\mathbf{P}^9$

$$g_7 := (U_0 : \zeta^6 U_1 : \zeta^5 U_2 : \zeta^3 U_3 : \zeta U_4 : \zeta^2 U_5 : \zeta^4 U_6 : \zeta U_7 : \zeta^2 U_8 : \zeta^4 U_9)$$

$$g_3 := (U_0 : U_2 : U_3 : U_1 : U_5 : U_6 : U_4 : U_8 : U_9 : U_7)$$

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It can be verified that the variety

$$Z \subset \mathbf{P}^9$$

defined by the 84 equations is indeed a FPP. Use Magma and Macaulay 2.

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Take a prime $p = 263$. Then $\sqrt{-7} = 16 \mod p$.
Magma calculates the Hilbert series of Z

$$h^0(Z, \mathcal{O}_Z(k)) = \frac{1}{2}(6k-1)(6k-2) = 18k^2 - 9k + 1, \ \ k \geq 0.$$

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Smoothness of Z is a subtle problem.
The 84×10 Jacobian matrix has too many 7×7 minors.
By adding suitably chosen 3 minors to the ideal of 84 cubics, the Hilbert polynomial drops from $18k^2 - 9k + 1$ to linear, then to constant, then to 0.
If the equations generate the ring modulo 263, then they also generate it with exact coefficients.

Thus Z is a smooth surface with a very ample divisor class $D = \mathcal{O}_Z(1)$. From the Hilbert polynomial we see that

$$D^2 = 36, DK_Z = 18, \chi(Z, \mathcal{O}_Z) = 1.$$

In part, $Z \not\cong \mathbf{P}^2$.

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Macaulay 2 calculates the projective resolution of $\mathcal{O}_Z$ as

$$\begin{array}{c} 0 \to \mathcal{O}(-9)^{\oplus 28} \to \mathcal{O}(-8)^{\oplus 189} \to \mathcal{O}(-7)^{\oplus 540} \to \mathcal{O}(-6)^{\oplus 840} \\ \to \mathcal{O}(-5)^{\oplus 756} \to \mathcal{O}(-4)^{\oplus 378} \to \mathcal{O}(-3)^{\oplus 84} \to \mathcal{O} \to \mathcal{O}_Z \to 0. \end{array}$$

By semicontinuity, the resolution is of the same shape over **C**.

Since all the sheaves $\mathcal{O}(-k)$ are acyclic, we see that

$$h^1(Z, \mathcal{O}_Z) = h^2(Z, \mathcal{O}_Z) = 0.$$

Macaulay also calculates (again working modulo 263)

$$\chi(Z, 2K_Z) = 10.$$

This implies $K_Z^2 = 9$. Thus Z is a FPP.

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Z can be further identified with the FPP which we started with.

Fake Projective Spaces

$\bar{G}$: a semisimple real algebraic group, connected, with trivial center

Assume

$$\bar{G} = PU(n, m).$$

Z : its symmetric space, i.e.
Z : the space of the maximal arithmetic subgroups of $\bar{G}(\mathbb{R})$

If $m = 1$, then $Z = \mathcal{B}^n$ and the compact dual of Z is $Gr(1, 1+n) = \mathbf{P^n}$.

For a cocompact torsion-free arithmetic subgroup Π : of $\bar{G}$ the quotient

$$X = Z/\Pi$$

is called an arithmetic fake $Gr(m, m+n)$ if X has the same Betti numbers as $Gr(m, m+n)$, but not biholomorphic to the compact dual.

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Theorem (Prasad-Yeung (2008))

An arithmetic fake $\mathbf{P^n}$ *exists only if* $n = 2$ *or* 4*, and an arithmetic fake* $Gr(m, m + n)$ *exists, with* $m + n \geq 5$ *odd, only if* $m + n = 5$*.*

They provide 4 examples of arithmetic fake $\mathbf{P^4}$.