EXPLICIT EQUATIONS OF A FAKE PROJECTIVE PLANE

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To our teacher and friend Igor Dolgachev on the occasion of his 75th birthday

Abstract

Fake projective planes are smooth, complex surfaces of general type with Betti numbers equal to those of the usual projective plane. They come in complex conjugate pairs and have been classified as quotients of the 2*-dimensional ball by explicitly written arithmetic subgroups. In the following, we find equations of a projective model of a conjugate pair of fake projective planes by studying the geometry of the quotient of such surface by an order* 7 *automorphism.*

1. Introduction

A compact, complex surface with the same Betti numbers as the usual complex projective plane is called *a fake projective plane* if it is not isomorphic to the complex projective plane. A fake projective plane has ample canonical divisor, so it is a smooth (and geometrically connected proper) surface of general type with geometric genus $p_g = 0$ and self-intersection of canonical class $K^2 = 9$ (this definition extends to arbitrary characteristic). The existence of a fake projective plane was first proved by Mumford in [22]. His method was based on the theory of 2-adic uniformization, and it led Ishida and Kato [12] to prove the existence of two more via the 2-adic approach. Recently Allcock and Kato [1] used a lattice with torsion in the 2-adic method to construct another fake projective plane. The second author in [13] gave a construction of a fake projective plane as a Galois cover of a singular model of Ishida elliptic surface which, as described by Ishida [11], is covered (non-Galois) by the Mumford fake projective plane.

Fake projective planes have Chern numbers $c_1^2 = 3c_2 = 9$ and are complex 2-ball quotients (see Aubin [2] and Yau [25]). Such ball quotients are strongly rigid by Mostow's rigidity theorem [21, Theorem A] (i.e., determined by fundamental group

DUKE MATHEMATICAL JOURNAL
Vol. 169, No. 6, © 2020 DOI 10.1215/00127094-2019-0076
Received 23 April 2018. Revision received 15 October 2019.

2010 *Mathematics Subject Classification*. Primary 14J29; Secondary 14F05, 32Q40, 32N15.

up to holomorphic or anti-holomorphic isomorphism). Fake projective planes come in complex conjugate pairs (see [20]) and they have been classified as quotients of the 2-dimensional complex ball by explicitly written cocompact torsion-free arithmetic subgroups of $\mathrm{PU}(2,1)$ (see [23], and also [3], [4]). The arithmeticity of their fundamental groups was proved by Klingler in [19]. There are exactly one hundred fake projective planes in total, corresponding to fifty distinct fundamental groups. Cartwright and Steger also computed the automorphism group of each fake projective plane X, which is given by $\mathrm{Aut}(X) \cong N(X)/\pi_1(X)$, where $N(X)$ is the normalizer of $\pi_1(X)$ in its maximal arithmetic subgroup of $\mathrm{PU}(2,1)$. In particular $\mathrm{Aut}(X) \cong \{1\}$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or G_{21}, where $\mathbb{Z}_n$ is the cyclic group of order n and G_{21} is the unique non-Abelian group of order 21. Among the fifty pairs exactly thirty-three admit nontrivial automorphisms: three pairs have $\mathrm{Aut} \cong G_{21}$, three pairs have $\mathrm{Aut} \cong \mathbb{Z}_3^2$, and twenty-seven pairs have $\mathrm{Aut} \cong \mathbb{Z}_3$. It turns out, for example, that the Mumford fake plane and the Keum fake plane have fundamental groups in the same maximal arithmetic subgroup of $\mathrm{PU}(2,1)$, but the former has $\mathrm{Aut} \cong \{1\}$ and the latter $\mathrm{Aut} \cong G_{21}$.

On the other hand, in [14] all possible quotients of fake projective planes were classified—that is, the $\mathbb{Z}_7$-quotient of a fake projective plane with $\mathrm{Aut} \cong G_{21}$ is a singular model of an elliptic surface with two multiple fibers and one I_9-fiber and three I_1-fibers. The three pairs of fake projective planes with $\mathrm{Aut} \cong G_{21}$ produce in this way three such elliptic surfaces, up to complex-conjugacy, with induced $\mathbb{Z}_3$-action: a $(2,3)$-elliptic surface whose $\mathbb{Z}_3$-quotient is a singular model of Ishida elliptic surface, another $(2,3)$-elliptic surface, and a $(2,4)$-elliptic surface (see also [16], [17] for further details).

In the present article, we find equations of a projective model of a conjugate pair of fake projective planes by studying the geometry of the quotient of such surface by an order 7 automorphism. The equations are given explicitly by eighty-four cubics in $\mathbb{P}^9$ with coefficients in the field $\mathbb{Q}[\sqrt{-7}]$. Their complex conjugate equations define the complex conjugate surface. This pair has the most geometric symmetries among the fifty pairs, in the sense that its automorphism group is G_{21} and its $\mathbb{Z}_7$-quotient is a singular model of a $(2,4)$-elliptic surface, which is not simply connected. The universal double cover of the $(2,4)$-elliptic surface has only one multiple (double) fiber and has the same Hodge numbers as K3 surfaces but with Kodaira dimension 1. This pair is different from those of Mumford and Keum fake planes, and was discussed in [15].

It is an open problem to determine whether the bicanonical map of a given fake projective plane gives an embedding into $\mathbb{P}^9$. It has been confirmed affirmatively for several pairs of fake projective planes, including the one we discuss here, by the vanishing result of [5], [18], and [17], and by the theorem of Reider [24, Theorem 1] (see also [7], [10], where the authors use the term "Keum's fake projective planes" for all fake projective planes with $\mathrm{Aut} \cong G_{21}$). The equations in this article also provide an explicit proof for the embeddability for the pair.

Our work is organized as follows. We describe our main result in Section 2 by presenting the equations of a subscheme in $\mathbb{CP}^9$, and we indicate the computer calculations that allow one to verify that this subscheme is a fake projective plane. In Section 3, we start the explanation of the process that led us to the equations. Specifically, we discuss the geometry of the minimal resolution of the quotient of a certain fake projective plane by $\mathbb{Z}_7$ and its universal double cover X. Section 4 describes the breakthrough calculation that allowed us to identify the image of X under a certain map to $\mathbb{CP}^3$ as a specific singular sextic surface. In Sections 5 and 6, we describe additional features of the surface X, and we explain how we found the field of rational functions of the fake projective plane. In Section 7, we finally explain how we obtained the eighty-four cubic equations of Section 2. We add a final, minor comment in Section 8.

Computer files for our computation are available as supplemental materials with the online version of this article (see https://doi.org/10.1215/00127094-2019-0076).

2. Equations

In this section, we write down eighty-four explicit degree 3 equations in ten variables. We argue that they cut out a fake projective plane Z with $\mathrm{Aut}(Z) \cong G_{21}$ and $H_1(Z, \mathbb{Z}) = \mathbb{Z}_2^4$. Here $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

The eighty-four equations with complex-conjugate coefficients cut out another fake projective plane that is complex-conjugate to the former. We identify this pair as the pair of fake projective planes which is $(a = 7, p = 2, \emptyset, D_3 2_7)$ in the Cartwright–Steger classification [4, registerofgps.txt], or as one of the three pairs in the class $(k = \mathbb{Q}, \ell = \mathbb{Q}(\sqrt{-7}), p = 2, \mathcal{T}_1 = \emptyset)$ (see [3], [23]). This pair does not belong to the class $(a = 7, p = 2, \{7\})$ which contains Mumford fake plane $(a = 7, p = 2, \{7\}, 7_{21})$ and Keum fake plane $(a = 7, p = 2, \{7\}, D_3 2_7)$.

Let $\mathbb{CP}^9$ be a projective space with homogeneous coordinates denoted by $(U_0, U_1, \dots, U_9)$. Consider the non-Abelian group G_{21} which is a semidirect product of $\mathbb{Z}_7$ and $\mathbb{Z}_3$. We define its action on $\mathbb{CP}^9$ by its action on the homogeneous coordinates by

$$\begin{aligned} &g_7(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9) \\ &\qquad := (U_0 : \xi^6 U_1 : \xi^5 U_2 : \xi^3 U_3 : \xi U_4 : \xi^2 U_5 : \xi^4 U_6 : \xi U_7 : \xi^2 U_8 : \xi^4 U_9) \\ &g_3(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9) \\ &\qquad := (U_0 : U_2 : U_3 : U_1 : U_5 : U_6 : U_4 : U_8 : U_9 : U_7) \end{aligned} \tag{2.1}$$

where $\xi = \exp(\frac{2\pi i}{7})$ is the primitive seventh root of 1.

Table 1. Equations of the fake projective plane 1–24.

$$eq_1 = U_1U_2U_3 + (1-\mathrm{i}\sqrt{7})(U_3^2U_4 + U_1^2U_5 + U_2^2U_6) + (10-2\mathrm{i}\sqrt{7})U_4U_5U_6$$

$$\begin{aligned} eq_2 ={}& (-3+\mathrm{i}\sqrt{7})U_0^3 + (7+\mathrm{i}\sqrt{7})(-2U_1U_2U_3 + U_7U_8U_9 - 8U_4U_5U_6) \\ &+8U_0(U_1U_4 + U_2U_5 + U_3U_6) + (6+2\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) \end{aligned}$$

$$\begin{aligned} eq_3 ={}& (11-\mathrm{i}\sqrt{7})U_0^3 + 128U_4U_5U_6 - (18+10\mathrm{i}\sqrt{7})U_7U_8U_9 + 64(U_2U_4^2 + U_3U_5^2 + U_1U_6^2) \\ &+(-14-6\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) + 8(1+\mathrm{i}\sqrt{7})(U_1^2U_8 + U_2^2U_9 + U_3^2U_7 - 2U_1U_2U_3) \end{aligned}$$

$$eq_4 = -(1+\mathrm{i}\sqrt{7})U_0U_3(4U_6 + U_9) + 8(U_1U_2U_3 + U_1U_6U_9 + U_5U_7U_9) + 16(U_5U_6U_7 - U_1^2U_5 - U_3U_5^2)$$

$$eq_5 = g_3(eq_4)$$

$$eq_6 = g_3^2(eq_4)$$

$$\begin{aligned} eq_7 ={}& (12+4\mathrm{i}\sqrt{7})U_1U_2U_3 + (4+4\mathrm{i}\sqrt{7})(U_3U_5U_8 - U_0U_2U_5 + 4U_4U_5U_6) + (3-\mathrm{i}\sqrt{7})U_0U_1U_7 \\ &+8(U_2U_4U_7 + U_6U_7U_8 - U_1^2U_8 - 2U_4U_6U_8) + (2+2\mathrm{i}\sqrt{7})(U_3U_8^2 - U_0U_2U_8) \end{aligned}$$

$$eq_8 = g_3(eq_7)$$

$$eq_9 = g_3^2(eq_7)$$

$$\begin{aligned} eq_{10} ={}& (2+6\mathrm{i}\sqrt{7})U_1U_2U_3 + 4(-5+\mathrm{i}\sqrt{7})U_5(U_1^2 + 2U_4U_6) - 8U_0(U_2U_5 + U_3U_6) + 8(-1+\mathrm{i}\sqrt{7})U_3U_5^2 \\ &+2(3-\mathrm{i}\sqrt{7})U_0U_1U_7 - 8U_1^2U_8 + (-1-\mathrm{i}\sqrt{7})U_8(U_0U_2 + 4U_4U_9) + 8(1+\mathrm{i}\sqrt{7})U_3U_5U_8 - 32U_4U_6U_8 \\ &+2(1-\mathrm{i}\sqrt{7})(2U_6U_7U_8 + 4U_5U_7U_9 + 4U_5U_6U_7 + U_7U_8U_9) + 2(3+\mathrm{i}\sqrt{7})U_3U_8^2 - 16U_4U_5U_9 + 4U_1U_9^2 \end{aligned}$$

$$eq_{11} = g_3(eq_{10})$$

$$eq_{12} = g_3^2(eq_{10})$$

$$\begin{aligned} eq_{13} ={}& -8\mathrm{i}\sqrt{7}U_1^2U_3 + (-7+5\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(-7+\mathrm{i}\sqrt{7})U_0U_6^2 + 4U_0^2U_7 + (8-8\mathrm{i}\sqrt{7})U_1U_4U_7 \\ &+4(-5-\mathrm{i}\sqrt{7})U_2U_5U_7 + (8+8\mathrm{i}\sqrt{7})U_3U_6U_7 + (-1-5\mathrm{i}\sqrt{7})U_1U_7^2 - 8U_2U_7U_8 + (6+6\mathrm{i}\sqrt{7})U_3U_7U_9 \end{aligned}$$

$$\begin{aligned} eq_{14} ={}& 8U_1^2U_3 + 2(3-\mathrm{i}\sqrt{7})U_0U_1U_5 + 16U_3U_4U_6 - 16U_5^2U_6 + 2(1+\mathrm{i}\sqrt{7})U_2U_5U_7 - 8U_3U_6U_7 \\ &+2(-1-\mathrm{i}\sqrt{7})U_3^2U_8 + 2(-1+\mathrm{i}\sqrt{7})U_0U_6U_9 + (-5-\mathrm{i}\sqrt{7})U_3U_7U_9 \end{aligned}$$

$$\begin{aligned} eq_{15} ={}& 2(-3-\mathrm{i}\sqrt{7})U_1^2U_3 + 2(3-\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(-1+\mathrm{i}\sqrt{7})U_0U_1U_5 + 4(-1-\mathrm{i}\sqrt{7})U_3^2U_5 \\ &+8U_1U_2U_6 + 4(1+\mathrm{i}\sqrt{7})U_0U_6^2 - 4U_0^2U_7 + (1+\mathrm{i}\sqrt{7})U_1U_7^2 + 2(-1+\mathrm{i}\sqrt{7})U_0U_1U_8 + 4U_3U_7U_9 \end{aligned}$$

$$\begin{aligned} eq_{16} ={}& (-3+\mathrm{i}\sqrt{7})U_2^3 + (-3+\mathrm{i}\sqrt{7})U_1^2U_3 + 4U_0U_2U_3 + (-2-2\mathrm{i}\sqrt{7})U_0^2U_4 + 8U_1U_4^2 + 8U_0U_1U_5 \\ &+(-5-\mathrm{i}\sqrt{7})U_1U_2U_6 + (4+4\mathrm{i}\sqrt{7})U_3U_4U_6 + 2U_0U_1U_8 + (3-\mathrm{i}\sqrt{7})U_2U_7U_8 + (2+2\mathrm{i}\sqrt{7})U_3U_4U_9 \end{aligned}$$

$$\begin{aligned} eq_{17} ={}& 4(-1-\mathrm{i}\sqrt{7})U_2^3 + (5+\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(3-\mathrm{i}\sqrt{7})U_3^2U_5 + 16(1-\mathrm{i}\sqrt{7})U_2U_4U_5 \\ &+4(-1-\mathrm{i}\sqrt{7})U_2U_5U_7 - 8U_1U_2U_9 + 4(1+\mathrm{i}\sqrt{7})U_3U_4U_9 - 32U_5^2U_9 - 16U_5U_8U_9 \end{aligned}$$

$$\begin{aligned} eq_{18} ={}& 8U_1^2U_3 + (-5-\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(1+\mathrm{i}\sqrt{7})U_3^2U_5 + 4(1+\mathrm{i}\sqrt{7})U_1U_2U_6 \\ &+16(-1+\mathrm{i}\sqrt{7})U_5^2U_6 + 8U_2U_5U_7 - 16U_3U_6U_7 + 8(-1+\mathrm{i}\sqrt{7})U_5U_6U_8 - 8U_3U_7U_9 \end{aligned}$$

$$\begin{aligned} eq_{19} ={}& (-5-\mathrm{i}\sqrt{7})U_0^2U_4 - 8U_2U_5U_7 + (-1-\mathrm{i}\sqrt{7})U_1U_7^2 + 4U_0U_1U_8 - 4U_2U_7U_8 + (-5+\mathrm{i}\sqrt{7})U_1U_2U_9 \\ &+2(1-\mathrm{i}\sqrt{7})U_3U_4U_9 + 2(1-\mathrm{i}\sqrt{7})U_0U_6U_9 + 4U_3U_7U_9 + 2U_8^2U_9 + 2U_0U_9^2 \end{aligned}$$

$$\begin{aligned} eq_{20} ={}& 4(1+\mathrm{i}\sqrt{7})U_1^2U_3 + 2(1-\mathrm{i}\sqrt{7})U_0U_2U_3 - 8U_0^2U_4 + 4(-3-\mathrm{i}\sqrt{7})U_1U_4^2 - 8\mathrm{i}\sqrt{7}U_0U_1U_5 \\ &+8(1-\mathrm{i}\sqrt{7})U_2U_4U_5 + (5-\mathrm{i}\sqrt{7})U_0U_1U_8 + 2(-5+\mathrm{i}\sqrt{7})U_3^2U_8 + 16U_5U_8U_9 + 8U_8^2U_9 \end{aligned}$$

$$\begin{aligned} eq_{21} ={}& (1-\mathrm{i}\sqrt{7})U_1^2U_3 - 4U_0U_1U_5 - 8U_3U_4U_6 - 8U_0U_6^2 + 4U_1U_4U_7 + (2-2\mathrm{i}\sqrt{7})U_2U_5U_7 \\ &+2U_1U_7^2 - 2U_0U_1U_8 + (1+\mathrm{i}\sqrt{7})U_3^2U_8 + (1-\mathrm{i}\sqrt{7})U_2U_7U_8 + (-1+\mathrm{i}\sqrt{7})U_3U_7U_9 \end{aligned}$$

$$\begin{aligned} eq_{22} ={}& -8U_1^2U_3 + 16U_2U_4U_5 - 8U_1U_2U_6 + 4(1+\mathrm{i}\sqrt{7})U_0U_6^2 + (1+\mathrm{i}\sqrt{7})U_0U_1U_8 \\ &+8U_2U_4U_8 - 8U_5U_6U_8 + 4U_1U_2U_9 - 8U_3U_4U_9 + 2(1+\mathrm{i}\sqrt{7})U_0U_6U_9 \end{aligned}$$

$$\begin{aligned} eq_{23} ={}& (-3+\mathrm{i}\sqrt{7})(U_2^3 + U_1^2U_3) + 4(-1-\mathrm{i}\sqrt{7})U_1U_4^2 + (-1+3\mathrm{i}\sqrt{7})U_1U_2U_6 + 2(-1-\mathrm{i}\sqrt{7})U_1U_4U_7 \\ &+(1+\mathrm{i}\sqrt{7})U_3^2U_8 + 8U_2U_4U_8 + 4(-1+\mathrm{i}\sqrt{7})U_5U_6U_8 + 4U_2U_7U_8 + 4U_1U_2U_9 \end{aligned}$$

$$\begin{aligned} eq_{24} ={}& 2U_0U_2U_3 + (-1-\mathrm{i}\sqrt{7})U_0^2U_4 + 2(1-\mathrm{i}\sqrt{7})U_0U_1U_5 + 2(1-\mathrm{i}\sqrt{7})U_1U_2U_6 + 2U_0U_1U_8 \\ &-4U_3^2U_8 - 4U_2U_4U_8 + 2(1-\mathrm{i}\sqrt{7})U_5U_6U_8 + 4U_5U_8U_9 + 2U_8^2U_9 \end{aligned}$$

THEOREM 2.1

Eighty-four cubic equations of Tables 1 and 2 give equations of a fake projective plane Z in $\mathbb{CP}^9$ embedded by its bicanonical linear system.

Table 2. Equations of the fake projective plane 25–84.

$$\begin{aligned}
eq_{25} = {} & (-1+3\mathrm{i}\sqrt{7})U_0^2U_1+(44-4\mathrm{i}\sqrt{7})U_2^2U_3+64U_3U_4U_5+(36-12\mathrm{i}\sqrt{7})U_1U_3U_6+(16+16\mathrm{i}\sqrt{7})U_4^2U_6\\
& +(-4-4\mathrm{i}\sqrt{7})U_0U_2U_7-32U_3U_4U_8+(4+4\mathrm{i}\sqrt{7})U_0U_6U_8-16U_3U_7U_8+(8-8\mathrm{i}\sqrt{7})U_1U_3U_9+16U_4U_7U_9\\
eq_{26} = {} & (-1+3\mathrm{i}\sqrt{7})U_0^2U_1+(-4-4\mathrm{i}\sqrt{7})U_2^2U_3+(40-8\mathrm{i}\sqrt{7})U_1U_2U_5+(4-12\mathrm{i}\sqrt{7})U_1U_3U_6+96U_4^2U_6\\
& +(-24-8\mathrm{i}\sqrt{7})U_2U_6^2+16U_1^2U_7+(-2+2\mathrm{i}\sqrt{7})U_0U_2U_7+64U_4U_6U_7+(20-4\mathrm{i}\sqrt{7})U_1U_2U_8-8U_0U_6U_8\\
& +16U_4U_7U_9\\
eq_{27} = {} & (5+\mathrm{i}\sqrt{7})U_0^2U_1+(-4-4\mathrm{i}\sqrt{7})U_2^2U_3+(16-16\mathrm{i}\sqrt{7})U_3U_4U_5+(-20-4\mathrm{i}\sqrt{7})U_1U_3U_6+32U_4^2U_6\\
& +32U_0U_5U_6+8U_0U_6U_8-16U_1U_3U_9+16U_0U_5U_9+8U_0U_8U_9\\
eq_{28} = {} & 8U_2^2U_3+(-3+\mathrm{i}\sqrt{7})U_0U_3^2+(-4-4\mathrm{i}\sqrt{7})U_1U_2U_5+(4+4\mathrm{i}\sqrt{7})U_3U_4U_5+32U_5^3+(4+4\mathrm{i}\sqrt{7})U_3U_5U_7\\
& +16U_5^2U_8+(3-\mathrm{i}\sqrt{7})U_1U_3U_9+8U_2U_6U_9\\
eq_{29} = {} & (-3+\mathrm{i}\sqrt{7})U_2^2U_3+(5+\mathrm{i}\sqrt{7})U_0U_2U_4+8U_1U_2U_5-8U_2U_6^2+2U_0U_2U_7+(-1-\mathrm{i}\sqrt{7})U_1U_2U_8+8U_5^2U_8\\
& +(3-\mathrm{i}\sqrt{7})U_1U_3U_9+(4+4\mathrm{i}\sqrt{7})U_4^2U_9-8U_2U_6U_9+(2+2\mathrm{i}\sqrt{7})U_4U_7U_9-2U_0U_8U_9+(-3+\mathrm{i}\sqrt{7})U_2U_9^2\\
eq_{30} = {} & 8U_2^2U_3+(4-4\mathrm{i}\sqrt{7})U_1^2U_4+(-12-4\mathrm{i}\sqrt{7})U_1U_2U_5+(-4-12\mathrm{i}\sqrt{7})U_4^2U_6+(12+4\mathrm{i}\sqrt{7})U_2U_6^2\\
& +(2-2\mathrm{i}\sqrt{7})U_1^2U_7-8U_1U_2U_8-16U_3U_4U_8+(1+3\mathrm{i}\sqrt{7})U_0U_6U_8+(-3-\mathrm{i}\sqrt{7})U_3U_7U_8+4U_1U_3U_9\\
& +(6+2\mathrm{i}\sqrt{7})U_2U_6U_9\\
eq_{31} = {} & (-4+4\mathrm{i}\sqrt{7})U_1^2U_4-4U_1U_2U_5+(-4+4\mathrm{i}\sqrt{7})U_3U_4U_5+16U_5^3+(-8+8\mathrm{i}\sqrt{7})U_4^2U_6+(2+2\mathrm{i}\sqrt{7})U_0U_5U_6\\
& -4U_1^2U_7+(2+2\mathrm{i}\sqrt{7})U_6U_7^2+8U_3U_4U_8-4U_0U_6U_8-4U_5U_8^2+(1+\mathrm{i}\sqrt{7})U_7^2U_9\\
eq_{32} = {} & (-5-\mathrm{i}\sqrt{7})U_0^2U_1+(-6+2\mathrm{i}\sqrt{7})U_0U_3^2+(-24+8\mathrm{i}\sqrt{7})U_3U_4U_5+(20+4\mathrm{i}\sqrt{7})U_1U_3U_6-32U_4^2U_6\\
& -32U_0U_5U_6+32U_2U_6^2+(2+2\mathrm{i}\sqrt{7})U_0U_2U_7+(4+4\mathrm{i}\sqrt{7})U_1U_2U_8-8U_0U_6U_8+(10+2\mathrm{i}\sqrt{7})U_1U_3U_9\\
& +16U_2U_6U_9\\
eq_{33} = {} & (7-5\mathrm{i}\sqrt{7})U_0^2U_1+(-56-24\mathrm{i}\sqrt{7})U_1^2U_4+32\mathrm{i}\sqrt{7}U_1U_2U_5+(28+4\mathrm{i}\sqrt{7})U_1U_3U_6+(28+28\mathrm{i}\sqrt{7})U_0U_5U_6\\
& +(-84-4\mathrm{i}\sqrt{7})U_1^2U_7+(7+7\mathrm{i}\sqrt{7})U_0U_2U_7-56U_3U_5U_7+56U_6U_7^2+24\mathrm{i}\sqrt{7}U_1U_2U_8+56U_0U_6U_8\\
& +(14-18\mathrm{i}\sqrt{7})U_1U_3U_9+28U_7^2U_9\\
eq_{34} = {} & (-5-\mathrm{i}\sqrt{7})U_0^2U_1+48U_1U_2U_5+(-16-16\mathrm{i}\sqrt{7})U_3U_4U_5+32U_4^2U_6+(2+10\mathrm{i}\sqrt{7})U_1^2U_7\\
& +(-48+16\mathrm{i}\sqrt{7})U_4U_6U_7+(28-4\mathrm{i}\sqrt{7})U_1U_2U_8+(-12-12\mathrm{i}\sqrt{7})U_3U_4U_8+(-16-8\mathrm{i}\sqrt{7})U_0U_6U_8\\
& +(-22+2\mathrm{i}\sqrt{7})U_1U_3U_9+(-8-8\mathrm{i}\sqrt{7})U_2U_6U_9+(-8+8\mathrm{i}\sqrt{7})U_4U_7U_9\\
eq_{35} = {} & (10+2\mathrm{i}\sqrt{7})U_2^2U_3+(-11+\mathrm{i}\sqrt{7})U_0U_2U_4-16U_1U_2U_5+(20+4\mathrm{i}\sqrt{7})U_3U_4U_5-16U_2U_6^2\\
& +(-1-\mathrm{i}\sqrt{7})U_0U_2U_7+(-2-2\mathrm{i}\sqrt{7})U_1U_2U_8-16U_5^2U_8+(-4-4\mathrm{i}\sqrt{7})U_4^2U_9+(3-\mathrm{i}\sqrt{7})U_2U_9^2\\
eq_{36} = {} & (2+2\mathrm{i}\sqrt{7})U_0U_3^2+(-6+2\mathrm{i}\sqrt{7})U_0U_2U_4+(4-4\mathrm{i}\sqrt{7})U_1U_3U_6+32U_4^2U_6+(-12-4\mathrm{i}\sqrt{7})U_2U_6^2+2U_0U_2U_7\\
& +16U_4U_6U_7+(7-\mathrm{i}\sqrt{7})U_1U_2U_8-8U_5^2U_8+4U_1U_3U_9+(4-4\mathrm{i}\sqrt{7})U_0U_5U_9+4U_4U_7U_9-2\mathrm{i}\sqrt{7}U_0U_8U_9\\
eq_k = {} & g_3(eq_{k-24}), \quad k=37,\dots,60\\
eq_k = {} & g_3^2(eq_{k-48}), \quad k=61,\dots,84
\end{aligned}$$

Proof

Let Z be the subscheme of $\mathbb{CP}^9$ cut out by these eighty-four equations. We use Magma to calculate the Hilbert series of Z to give

$$\dim H^0\big(Z,\mathcal{O}(k)\big)=18k^2-9k+1$$

for all $k\geq 0$.

We then use reduction modulo 263 with $\mathrm{i}\sqrt{7}=16 \bmod 263$ (which is chosen just because it is a prime of decent size with a clear root of -7). We calculate (by Macaulay2) the projective resolution of $\mathcal{O}_Z$ as

$$\begin{aligned}
0 \to \mathcal{O}(-9)^{\oplus 28} \to \mathcal{O}(-8)^{\oplus 189} \to \mathcal{O}(-7)^{\oplus 540} \to \mathcal{O}(-6)^{\oplus 840} \to\\
\to \mathcal{O}(-5)^{\oplus 756} \to \mathcal{O}(-4)^{\oplus 378} \to \mathcal{O}(-3)^{\oplus 84} \to \mathcal{O} \to \mathcal{O}_Z \to 0.
\end{aligned}$$

By semicontinuity, the resolution is of the same shape over $\mathbb{C}$. Since all of the sheaves $\mathcal{O}(-k)$, $k = 3, \dots, 9$, are acyclic, we see that, for all $i \geq 0$,

$$h^i(Z, \mathcal{O}_Z) = h^i(\mathbb{CP}^9, \mathcal{O}).$$

That is, $h^1(Z, \mathcal{O}_Z) = h^2(Z, \mathcal{O}_Z) = 0$ and $h^0(Z, \mathcal{O}_Z) = 1$, which implies that the scheme Z is connected. Since the Hilbert polynomial has degree 2, its irreducible components have dimension at most 2.

We also verify that Z is smooth. It is a somewhat delicate calculation. In theory, one can take the 7×7 minors of the 84×10 matrix of partial derivatives of the equations and verify that, together with the equations themselves, they generate the ideal which coincides with $\mathbb{C}[U_0, \dots, U_9]$ for large degrees. In practice, such direct calculation is impossible, since the number of minors is too large. Instead, we pick three 7×7 minors of the Jacobian matrix and show that they have no common zeros on Z by a Hilbert polynomial calculation. The minors are chosen so that they do not vanish at the fixed points of the automorphism g_7—namely, at the three points

$$(U_0, \dots, U_7, U_8, U_9) \in \big\{(0, \dots, 0, 0, 1), (0, \dots, 0, 1, 0), (0, \dots, 1, 0, 0)\big\}.$$

The subsets of equations and variables that define the minors are given in Table 3. This calculation can be performed in Magma software package modulo 263 with $\mathrm{i}\sqrt{7} = 16$. The Hilbert polynomial of the quotient drops from $18k^2 - 9k + 1$ to $504k - 3654$, then to 7056 and finally to 0 as one adds the three minors to the ideal. If the equations generate the ring modulo 263, then they also generate it with exact coefficients. This calculation means that all geometric points of Z have tangent space of dimension at most 2, which together with $h^0(\mathcal{O}_Z) = 1$ implies that Z is a smooth surface.

Thus we have a smooth surface Z and a very ample divisor class

$$D := \mathcal{O}_Z(1)$$

on it. The Hilbert polynomial together with the Riemann–Roch implies that

$$D^2 = 36, \qquad DK_Z = 18, \qquad \chi(Z, \mathcal{O}_Z) = 1.$$

Note that this shows that Z is not isomorphic to $\mathbb{CP}^2$. We also know that $h^{0,1}(Z) = h^{0,2}(Z) = 0$, so it remains to prove that $h^{1,1}(Z) = 1$.

Table 3. Three 7×7 minors used to verify smoothness.

$\{8, 19, 29, 43, 55, 61, 79\}; \{U_0, U_1, U_2, U_3, U_5, U_6, U_7\}$
$\{7, 19, 31, 37, 55, 67, 77\}; \{U_0, U_1, U_2, U_3, U_4, U_5, U_9\}$
$\{9, 13, 31, 43, 53, 67, 79\}; \{U_0, U_1, U_2, U_3, U_4, U_8, U_9\}$

To figure out this last Hodge number, we use Macaulay to calculate $\chi(Z, \mathcal{O}(2K_Z)) = 10$ (again working modulo 263). For this calculation, we use the resolution to compute the canonical bundle K_Z as in Hartshorne's book, as Ext from the canonical bundle of the ambient space to $\mathcal{O}_Z$, and then we tensored it with itself to get $2K_Z$ and then calculated the Hilbert polynomial of the corresponding graded module to get $\chi(Z, 2K_Z)$. Now by Riemann–Roch $\chi(Z, 2K_Z) = K_Z^2 + \chi(Z, \mathcal{O}_Z)$ and we know that $\chi(Z, \mathcal{O}_Z) = 1$, and thus $K_Z^2 = 9$. Now Noether's formula finishes the proof that Z is a fake projective plane.

We see that $2K$ is numerically equivalent to D. We calculate

$$\operatorname{Hom}\bigl(\mathcal{O}(K), \mathcal{O}(D)\bigr) = 0$$

by working modulo 263 and semicontinuity. This implies that

$$h^0\bigl(Z, \mathcal{O}(D-K)\bigr) = 0 = h^2\bigl(Z, \mathcal{O}(2K-D)\bigr).$$

This implies that $h^0(Z, \mathcal{O}(2K-D)) \geq 1$, so $\mathcal{O}(2K) \simeq \mathcal{O}(D)$. So the fake projective plane Z is embedded via a bicanonical embedding. ☐

Remark 2.2

Consider the closed subscheme C of Z cut out by $U_0 = 0$ and the following eighteen quadrics, which fall into six orbits under the $\mathbb{Z}_3$-action

$$\begin{aligned}\Bigl\langle U_0, U_1^2 - U_6U_7 + \frac{1}{8}(-5 - \mathrm{i}\sqrt{7})U_7U_9, U_4U_6 - \frac{1}{8}(1 + \mathrm{i}\sqrt{7})U_3U_8,\\ U_2U_4 + \frac{1}{8}(1 + \mathrm{i}\sqrt{7})U_8U_9, U_1U_4 + U_3U_6 + \frac{1}{8}(5 + \mathrm{i}\sqrt{7})U_3U_9,\\ U_1U_2 + U_5U_8, U_4^2 + \frac{1}{8}(1 + \mathrm{i}\sqrt{7})U_2U_9 + \frac{1}{8}(5 + \mathrm{i}\sqrt{7})U_4U_7,\\ \text{twelve images of the six quadrics under } \mathbb{Z}_3\Bigr\rangle.\end{aligned}$$

By calculating its Hilbert polynomial, we see that it is 1-dimensional, with the total degree of 1-dimensional components equal to 18. This means that C is a (manifestly $\mathbb{Z}_7$-invariant) curve on Z. Moreover, by computing Hilbert polynomials of $\langle U_0\rangle + I^2$ and $\langle U_0\rangle$, we see that the square of this ideal I lies in $\langle U_0\rangle$. Therefore, we see that the bicanonical divisor $2K_Z$ is linearly equivalent to $2C$. By Lemma 2.3, this implies that $Z/\mathbb{Z}_7$ has a minimal model which is a $(2,4)$-elliptic surface. It also identifies Z as the pair of fake projective planes which is $(a = 7, p = 2, \emptyset, D_3 2_7)$ in the Cartwright–Steger classification (see [4]), or as one of the three pairs in the class $(k = \mathbb{Q}, \ell = \mathbb{Q}(\sqrt{-7}), p = 2, \mathcal{T}_1 = \emptyset)$ (see [3], [23]).

LEMMA 2.3

Let W be a fake projective plane with $\mathrm{Aut}(W) = \mathbb{Z}_7 : \mathbb{Z}_3$. *Then the following are equivalent:*

(1) *W contains an effective $\mathbb{Z}_7$-invariant curve C with $C^2 = 9$,*

(2) *the action of $\mathbb{Z}_7$ on W fixes a nontrivial element in $H_1(W,\mathbb{Z})$,*

(3) *$H_1(W,\mathbb{Z}) = \mathbb{Z}_2^4$,*

(4) *the minimal resolution of $W/\mathbb{Z}_7$ is a $(2,4)$-elliptic surface.*

Proof

On a fake projective plane, an effective curve C with $C^2 = 9$ is a member of the linear system $|K_W + t|$ for some nonzero $t \in \mathrm{Tor}\,\mathrm{Pic}(W) \cong H_1(W,\mathbb{Z})$. For a subgroup G of $\mathrm{Aut}(W)$, the linear system $|mK_W + t|$ is G-invariant if and only if t is also. For a cyclic subgroup G of $\mathrm{Aut}(W)$ a complete linear system is G-invariant if and only if a member of the system is G-invariant. This proves the equivalence of (1) and (2). These two are equivalent to (3) by [5, Corollary 3.4] and therefore to (4) by the classification of [14] on the possible geometric structures of quotients of fake projective planes. ☐

Furthermore, if $H_1(W,\mathbb{Z}) = \mathbb{Z}_2^4$, a $\mathbb{Z}_7$-invariant, nontrivial, 2-torsion is unique (see [5, Corollary 3.4]), then it is also $\mathrm{Aut}(W)$-invariant.

Remark 2.4

We have the following two comments.

(1) It is well known (see [9, Example 9.1.3(ii)]) that a local complete intersection subscheme of $\mathbb{P}^N$ is the scheme-theoretic intersection of $N+1$ hypersurfaces. Thus our surface Z can be defined scheme-theoretically by ten equations, in which case the eighty-four equations seem too many. However, it is important for constructing the resolution to cut it out ideal-theoretically. Moreover, Macaulay2 works smoothly with the saturated ideal generated by the eighty-four cubics.

(2) One of the referees kindly informed us that only fifteen among the eighty-four equations were enough to define Z scheme-theoretically—for example,

$$\{3, 7, 11, 19, 23, 31, 35, 40, 43, 53, 55, 67, 71, 79, 83\}.$$

3. Our explanation begins: The double cover of the resolution of the $\mathbb{Z}_7$-quotient of a fake projective plane

The equations of Section 2 appear quite mysterious, so we will spend the rest of this article explaining their origins. Our general construction can be roughly summarized in the following commutative diagram of morphisms, with notation that will be used throughout the remainder of the present article.

$\mathbb{P}^2_{\text{fake}}$: a fake projective plane with $\mathrm{Aut} = \mathbb{Z}_7 : \mathbb{Z}_3$ such that the minimal resolution Y of $\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$ is a $(2,4)$-elliptic surface;

X: the universal double cover of Y.

$$\begin{tikzcd}
\mathcal{B}^2 \arrow[dr] & & \widehat{\mathbb{P}^2_{\text{fake}}} \arrow[dl] \arrow[dr] & & X \arrow[r, "\pi"] \arrow[dl] \arrow[dr] & \mathbb{P}^3 \\
& \mathbb{P}^2_{\text{fake}} \arrow[dr] & & Y \arrow[dl] \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
& & \mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7 & & \mathbb{P}^1 &
\end{tikzcd} \tag{3.1}$$

In this section, we will describe the known results of [14], [15], and [17] on the quotients of fake projective planes with automorphism group of order 21 by the subgroup of order 7. Specifically, we describe the aspects of the geometry of Y and X in (3.1) that will be used later to find the equation of $\pi(X) \subset \mathbb{P}^3$.

Let $\mathbb{P}^2_{\text{fake}}$ be a fake projective plane with noncommutative automorphism group $G_{21} \cong \mathbb{Z}_7 : \mathbb{Z}_3$. Consider the quotient $\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$ of $\mathbb{P}^2_{\text{fake}}$ by the (normal) Sylow 7-subgroup of G_{21}. It is a singular surface of Kodaira dimension 1 with three quotient singular points of type $\frac{1}{7}(1,3)$ and it inherits an order 3 automorphism which permutes these singular points. The minimal resolution Y of $\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$ is an elliptic surface over $\mathbb{CP}^1$ with $h^{2,0}(Y) = h^{1,0}(Y) = 0$, and two multiple fibers with multiplicities $(2,3)$ or $(2,4)$, as shown in [14]. The Hodge numbers of Y are given by

$$h^{0,0}(Y) = h^{2,2}(Y) = 1, \qquad h^{1,1}(Y) = 10, \qquad h^{p,q}(Y) = 0 \quad \text{otherwise}.$$

Throughout the rest of the present work, we will consider the fake projective planes which lead to elliptic surfaces Y with multiple fibers of multiplicities $(2,4)$. By the classification of [3] and [23] there is exactly one such conjugate pair of fake projective planes. (The other two conjugate pairs with $\mathrm{Aut} \cong G_{21}$ lead to $(2,3)$-elliptic surfaces.) Let us denote by $4F_Y$ the multiplicity 4 fiber and by $2F_{2,Y}$ the multiplicity 2 fiber. We summarize the results of [14], [15], and [17].

The preimages of $\frac{1}{7}(1,3)$ singular points in Y are three pairwise disjoint chains of spheres

$$S - B - C, \qquad S' - B' - C', \qquad S'' - B'' - C''$$

with $S^2 = (S')^2 = (S'')^2 = -3$ and the squares of the rest equal to -2. The canonical class K_Y is numerically equivalent to F_Y, and the elliptic fibration $Y \to \mathbb{P}^1$ is given by the linear system

$$|4F_Y| = |2F_{2,Y}| = |4K_Y|,$$

(i.e., a general fiber is linearly equivalent to $4F_Y$). The curves S, S', and S'' are 4-sections of the fibration; that is,

$$F_Y S = F_Y S' = F_Y S'' = 1.$$

The curves B, C, and their translates are part of an I_9-fiber of $Y \to \mathbb{P}^1$ and the order 3 automorphism group acts fiberwise. There are three additional I_1-fibers, some of which may be the multiple fibers.

The structure of the I_9-fiber will be very important in what follows. We denote its nine components by

$$A - B - C - A' - B' - C' - A'' - B'' - C'' - A.$$

The curve S intersects B transversely and does not intersect C, B', C', B'', C''. The I_9-fiber is not a multiple fiber (i.e., it is equivalent to $4F_Y$ by [17, Theorem 2.3]), and thus we see that S must intersect A, A', and A'' in three points total. These intersection numbers determine the intersection numbers of S' and S'' with A, A', A'' because of the order 3 automorphism.

It is easy to see that the classes of the curves

$$A, B, C, A', B', C', A'', B'', C'', S, S', S''$$

generate a sublattice of rank greater than or equal to 10 inside the Picard lattice of Y, the Néron-Severi group of Y modulo torsion. (The first nine curves already generate a rank 9 sublattice.) By Poincaré duality, the Picard lattice of Y is unimodular of signature $(1, 9)$, thus the sublattice must have rank 10 and the discriminant a square integer, which puts strong restrictions on the intersection numbers. It was observed in [15, p. 1676] that the only possibilities are

$$\begin{aligned} &\text{Case 1:} \quad SA = 1, \qquad SA' = 0, \qquad SA'' = 2; \\ &\text{Case 2:} \quad SA = 0, \qquad SA' = 2, \qquad SA'' = 1. \end{aligned} \tag{3.2}$$

In the following, we will show that Case 1 cannot occur.

The fundamental group of a $(2, 4)$-elliptic surface is of order 2 (see [8]), and thus the surface Y has an unramified double cover X which is part of the diagram (3.1). It comes from a double cover $\mathbb{P}^1 \to \mathbb{P}^1$ of the base of the fibration ramified over the images of F_Y and $F_{2,Y}$. The preimage of the canonical divisor class K_Y is the canonical class K_X of X and is numerically equivalent to the preimage F of F_Y. Since on a simply connected surface a numerical equivalence is a linear equivalence, K_X is linearly equivalent to F. We will denote by F_2 the preimage of $F_{2,Y}$. Then the elliptic fibration $X \to \mathbb{P}^1$ is given by the linear system

$$|2F| = |F_2| = |2K_X|,$$

and has only one multiple fiber $2F$ (with multiplicity 2). In particular, X has Kodaira dimension 1. Since X is simply connected, $h^{1,0}(X) = 0$. Since $\chi(X, \mathcal{O}_X) =$

$2\chi(Y, \mathcal{O}_Y) = 2$, we get $h^0(X, K_X) = 1$. This implies that

$$h^{0,0}(X) = h^{2,2}(X) = h^{2,0}(X) = h^{0,2}(X) = 1, \qquad h^{1,1}(X) = 20,$$
$$h^{p,q}(X) = 0 \quad \text{otherwise},$$

(i.e., X has the Hodge numbers of K3 surfaces). Its Jacobian fibration is an elliptic surface over $\mathbb{P}^1$ with a section, with no multiple fiber, and with singular fibers of the same type as those of X (this is true for Jacobian fibration of any genus 1 fibration; see [6]), and thus has trivial canonical class and the sum of Euler numbers of singular fibers 24, and hence is a K3 surface.

The preimage under $X \to Y$ of the curve S is $S_1 + S_2$, where S_i are disjoint, smooth, rational curves with $S_i^2 = -2$. Each of the curves S_i is a 2-section of $X \to \mathbb{CP}^1$. Similarly, we define S_1', S_2', S_1'', and S_2''. The preimage of the I_9-fiber $A - B - \cdots - C'' - A$ consists of two disjoint I_9-fibers

$$A_1 - B_1 - \cdots - C_1'' - A_1, \qquad A_2 - B_2 - \cdots - C_2'' - A_2.$$

We arrange the indexing so that we get six $(-3) - (-2) - (-2)$ chains of $\mathbb{CP}^1$ curves

$$S_i - B_i - C_i, S_i' - B_i' - C_i', S_i'' - B_i'' - C_i'', \quad i \in \{1, 2\}.$$

As before, we would like to determine the possible intersection numbers of the twenty-four curves

$$S_1, \ldots, S_2'', A_1, \ldots, C_2''$$

with each other. These intersections are uniquely determined by the nonnegative integers S_1A_1, S_1A_1', S_1A_1'', S_2A_1, S_2A_1', S_2A_1'', which are subject to $S_i(A_1 + A_2) = SA$, $S_i(A_1' + A_2') = SA'$, $S_i(A_1'' + A_2'') = SA''$ from (3.2). The resulting intersection matrix has to have rank at most 20, because the rank of the Picard group does not exceed $h^{1,1}(X)$.

A simple computer calculation shows that only Case 2 of (3.2) is possible and, moreover, it holds that

$$S_1A_1 = S_2A_1 = 0, \qquad S_1A_1' = S_2A_1' = 1, \qquad S_1A_1'' = 0, \qquad S_2A_1'' = 1, \quad (3.3)$$

(i.e., S_1 intersects at one point exactly B_1 and A_1' of the first I_9-fiber, and exactly A_2' and A_2'' of the second; S_2 intersects exactly B_2 and A_2' of the second I_9-fiber, and exactly A_1' and A_1'' of the first). This gives a rank 19 intersection matrix. This rank is not the maximum possible $h^{1,1}(X) = 20$, and thus leaves a possibility that F or F_2 is of type I_2 (i.e., F_Y or $F_{2,Y}$ on Y is of type I_1).

The following is crucial in our approach.

PROPOSITION 3.1

Let D be the divisor $3F+S_1+S_2$ on X which is the pullback of the divisor $3F_Y+S$ from Y. Then $D^2=6$, $h^0(D)=4$ and the linear system $|D|$ is basepoint free. It gives a birational map $\pi: X\to\mathbb{CP}^3$ whose image is a sextic surface. Moreover,

(1) *F is an elliptic curve and maps $2:1$ onto a line;*

(2) *each I_9-fiber maps to a union of a conic and two lines;*

(3) *a general fiber maps birationally onto a plane quartic curve with nodes at the points $\pi(S_1)$ and $\pi(S_2)$;*

(4) *F_2, if irreducible, maps $2:1$ onto a conic.*

Proof

We see immediately that

$$D^2=(3F+S_1+S_2)^2=6FS_1+6FS_2+S_1^2+S_2^2=12-3-3=6.$$

Therefore, $\chi(D)=\frac{1}{2}D(D-F)+\chi(\mathcal{O}_X)=\frac{1}{2}(6-2)+2=4$.

Consider the short exact sequence

$$0\to\mathcal{O}(3F)\to\mathcal{O}(D)\to\mathcal{O}(D)|_{S_1}\oplus\mathcal{O}(D)|_{S_2}\to 0. \tag{3.4}$$

We know that the bicanonical map of X is the elliptic fibration and has $\mathbb{P}^1$ as its image. Thus the canonical ring of X is a polynomial ring with generators of weights 1 and 2, corresponding to F and F_2, so $h^0(3F)=2$. Together with the Euler characteristics calculation and $h^2(3F)=h^0(-2F)=0$, this implies that $h^1(3F)=0$.

Because of $(3F+S_i)S_i=3-3=0$, we know that the restrictions of the sheaf $\mathcal{O}(D)$ to either S_i is isomorphic to the structure sheaf. Thus the long exact sequence in cohomology associated to (3.4) implies that $\dim H^0(X,\mathcal{O}(D))=2+1+1=4$. The same long exact sequence implies $h^1(D)=h^2(D)=0$.

Let us now prove that $H^0(X,\mathcal{O}(D))$ is basepoint free. The long exact sequence associated to (3.4) implies that there are sections which restrict to nonzero constants on S_1 and S_2, and the base locus of $H^0(X,\mathcal{O}(D))$ is contained in that of $H^0(3F)$. We know that this space is generated by the sections with divisors $3F$ and $F+F_2$. Therefore, the base locus of $H^0(X,\mathcal{O}(D))$ is contained in F. Consider the short exact sequence

$$0\to\mathcal{O}(2F+S_1+S_2)\to\mathcal{O}(D)\to\mathcal{O}(D)|_F\to 0.$$

Since $S_i(2F+S_1+S_2)<0$, then either S_i is a base component of $|2F+S_1+S_2|$, and hence $h^0(2F+S_1+S_2)=h^0(2F)=2$. Since $h^2(2F+S_1+S_2)=0$, Riemann–Roch implies that $h^1(2F+S_1+S_2)=0$ and $h^0(\mathcal{O}(D)|_F)=2$. If F is irreducible, then it is an elliptic curve and the restriction of $\mathcal{O}(D)$ to F is the full linear system of degree 2, and hence it is basepoint free, which implies that so is $H^0(X,\mathcal{O}(D))$.

Furthermore, F maps $2:1$ onto a line, which passes through the points $\pi(S_1)$ and $\pi(S_2)$. If F is of type I_2 (i.e., $F = R_1 + R_2$ for two (-2)-curves R_i), then the restriction of $\mathcal{O}(D)$ to either R_i is the full linear system of degree one, hence it is basepoint free and so is $H^0(X, \mathcal{O}(D))$, and R_i maps $1:1$ onto a line L_i. Since R_1 and R_2 intersect at two distinct points, we see that $L_1 = L_2$, but then S_1 must intersect both R_1 and R_2, contradicting $S_1 F = 1$.

Looking at the intersection number of D with each component of the I_9-fibers, we easily see that each I_9-fiber maps to a union of a conic and two lines. Since the image of a fiber is contained in a hyperplane section of $\pi(X)$, the degree of $\pi(X)$ is at least 4, hence must be 6.

The restriction of D to a general smooth fiber H of $X \to \mathbb{P}^1$ gives the short exact sequence

$$0 \to \mathcal{O}(F + S_1 + S_2) \to \mathcal{O}(D) \to \mathcal{O}(D)|_H \to 0.$$

The corresponding long exact sequence shows that $H^0(X, \mathcal{O}(D))$ restricts to a 3-dimensional linear subspace of the 4-dimensional space of sections of a degree 4 line bundle on the elliptic curve H. The corresponding $\mathbb{P}^2$ contains the line which is the image of F. The images of the fibers H are degree 4 curves in $\mathbb{P}^2$ of genus 1, unless they are double covers of conics. In the latter case, $\pi(X)$ would have degree less than 6, which is a contradiction.

Assume that F_2 is irreducible. Then it is an elliptic curve, and the corresponding long exact sequence shows that $H^0(X, D)$ restricts to a 3-dimensional linear subspace of the 4-dimensional space $H^0(F_2, D|_{F_2})$. Let a, a', possibly $a = a'$, be the intersection points of F_2 and S_1. Then $F_2 \cap S_2 = \{a + t, a' + t\}$ for a fixed 2-torsion point $t \in F_2$, because the deck transformation of X acts freely on F_2 and switches S_1 and S_2. Let $\nu : F_2 \to \mathbb{P}^1$ be the double cover given by the degree 2 linear system $|a + a'|$ on F_2. We claim that

$$H^0(X, D)|_{F_2} = \nu^* H^0\big(\mathbb{P}^1, \mathcal{O}(2)\big)$$

as 3-dimensional subspaces of $H^0(F_2, D|_{F_2})$. To prove this, consider the subspace

$$W_1 := H^0(X, 3F + S_1) \times H^0(S_2) \subset H^0(X, D).$$

Since $h^0(3F) = 2$, $h^1(3F) = 0$, and $\mathcal{O}(3F + S_1)$ restricts to the structure sheaf of S_1, we see that $h^0(3F + S_1) = 3$, and hence $\dim W_1 = 3$. It is easy to compute $h^0(F + S_1) = 1$, $h^1(F + S_1) = 0$, which implies that $H^0(3F + S_1)$ restrict to the full linear system of $H^0(F_2, (3F + S_1)|_{F_2})$. The latter space equals the 2-dimensional space $H^0(F_2, a + a')$ of the degree 2 line bundle $\mathcal{O}_{F_2}(a + a')$. Thus W_1 restricts to the 2-dimensional space corresponding to the 1-dimensional linear system $|a + a'| + (a + t) + (a' + t)$. Since $(a + t) + (a' + t) \in |a + a'|$, this 1-dimensional linear system belongs to the linear system of $\nu^* H^0(\mathbb{P}^1, \mathcal{O}(2))$. Similarly, $W_2 := H^0(X, 3F + S_2) \times H^0(S_1) \subset H^0(X, D)$ restricts to the 2-dimensional space corresponding to the linear system $a + a' + |(a+t) + (a'+t)|$. The two 2-dimensional spaces $W_i|_{F_2}$ in $\nu^* H^0(\mathbb{P}^1, \mathcal{O}(2))$ have 1-dimensional intersection, which corresponds to the unique divisor $a + a' + (a + t) + (a' + t)$. The claim and the last assertion is proved. ☐

We remark that F_2, if reducible, maps onto a union of two conics.

Remark 3.2
We eventually expected that a fake projective plane can be identified as such once we have its explicit equations, as we did in Section 2. As a consequence, we felt free to pursue the most likely scenarios rather than try to exhaustively exclude all degenerate cases, since the justification of our approach will be in its final result. This liberating philosophy is similar to the physicists' approach to mathematics: anything goes as long as the final answers concur with experiments. In particular, we assume that F_2 is irreducible.

4. Breakthrough: Equation of the image of the double cover X

In this section, we describe the major breakthrough that allowed us to eventually write down the equations of the fake projective plane. Specifically, we describe the method that allowed us to find the $\mathbb{Z}_2$-invariant sextic in $\mathbb{CP}^3$, which gives a (highly singular) birational model of the double cover X of the resolution of the $\mathbb{Z}_7$-quotient.

The action of the covering involution σ on X leads to an involution on $H^0(X, D)$ which has 2-dimensional eigenspaces. We observe that there are two natural, up-to-scaling elements y_0 and y_1 of $H^0(X, \mathcal{O}(D))$ which correspond to divisors

$$F + F_2 + S_1 + S_2, \qquad 3F + S_1 + S_2,$$

respectively. We will linearize the action of the covering involution σ so that $\sigma(y_0) = y_0$ and $\sigma(y_1) = -y_1$. We pick other basis elements of the eigenspaces and denote them by y_2 and y_3.

We know that the images of S_1 and S_2 are disjoint points on $(0 : 0 : * : *)$ which are permuted by the involution. We can scale y_2 and y_3 to ensure that these are $(0 : 0 : -1 : 1)$ and $(0 : 0 : 1 : 1)$, respectively. For generic a, the divisor of $y_0 - a y_1$ is $F_1 + S_1 + S_2 + H_a$, where H_a is a fiber of $X \to \mathbb{CP}^1$. Note that S_1 and S_2 intersect H_a in two points each. These points need to map to the the same point in $\mathbb{CP}^3$, which leads to the statement in Proposition 3.1 that the image of H_a is a nodal plane quartic with two nodes. We also know that F_2 maps $2 : 1$ onto a conic.

Putting it all together, the geometry of $\pi : X \to \mathbb{CP}^3$ implies the following.

- The involution acts by $y_i \mapsto (-1)^i y_i$. The sextic f is invariant with respect to this involution.
- The sections y_0 and y_1 are zero on S_1 and S_2. These are automatically zero on F.
- The section $y_1 = 0$ corresponds to the divisor $3F + S_1 + S_2$ and the section $y_0 = 0$ corresponds to $F + F_2 + S_1 + S_2$. The image of F is $(0 : 0 : * : *)$. This is a $2 : 1$ cover, so $f = 0$ has singularities along $(0 : 0 : * : *)$.
- For $a \neq 0$ the restriction of f to $x_0 = ax_1$ is

  $$f(ay_1, y_1, y_2, y_3) = y_1^2 g_a(y_1, y_2, y_3)$$

  where $g_a = 0$ is a degree 4 curve which has nodes at $(0 : \pm 1 : 1)$.
- For $a \neq 0$ the quartic $g_a = 0$ is irreducible, except for $a = \pm 1$ that correspond to the images of the I_9 fibers. (We can fix $a = \pm 1$ for the location of I_9 fiber by scaling y_0 and y_1.)
- The restriction to $y_1 = 0$ is given by

  $$f(y_0, 0, y_2, y_3) = y_0^6.$$

  Indeed, we must have a multiple of F_1 (since S_1 and S_2 map to points). This means that this should be a multiple of y_0 and we can scale it to be y_0^6.
- The restriction of f to $y_0 = 0$ is given by

  $$f(0, y_1, y_2, y_3) = y_1^2 h_0^2(y_1, y_2, y_3)$$

  where $h_0 = 0$ is a σ-invariant conic that passes through $(0 : \pm 1 : 1)$. The surface $f = 0$ has singularities along $y_0 = h_0 = 0$.

There are additional restrictions on $f = 0$ that come from the geometry of the I_9 fibers. Without loss of generality, let us assume that the fiber at $y_0 = y_1$ corresponds to the image of the cycle of curves

$$A_1 - B_1 - C_1 - A_1' - B_1' - C_1' - A_1'' - B_1'' - C_1'' - A_1$$

and the $y_0 = -y_1$ fiber corresponds to the cycle $A_2 - \cdots - C_2'' - A_2$.

The intersection numbers (3.3) imply that

$$DA_1' = 2, \qquad DA_1'' = 1, \qquad DB_1 = 1$$

so the degree 4 genus 1 curve with two nodes degenerates into two lines $\pi(A_1'')$ and $\pi(B_1)$ and a conic $\pi(A_1')$. The other six rational curves of this I_9 fiber are contracted to singular points. The line $\pi(B_1)$ must pass through $\pi(S_1) = (0 : 0 : -1 : 1)$ as does the conic $\pi(A_1')$. The line $\pi(A_1'')$ passes through the other node $\pi(S_2) = (0 : 0 : 1 : 1)$.

These lines intersect at some point P which we can set to be $P = (1:1:0:0)$ by adding multiples of y_0 and y_1 to y_2 and y_3, respectively. Moreover, we see that

$$P = \pi(B_1'') = \pi(C_1'') = \pi(A_1)$$

so the surface $\pi(X)$ has at least an A_3 type singularity at P. In particular, the partial derivatives and the derivative of the Hessian matrix vanish at P. In addition, we have a singular point $\pi(C_1)$ at the intersection of the line $\pi(B_1)$ and the conic $\pi(A_1')$ which is different from $\pi(S_1) = (0:0:-1:1)$. We also have a singular point

$$\pi(B_1') = \pi(C_1')$$

at the intersection of the conic $\pi(A_1'')$ and the line $\pi(A_1'')$ that is different from $\pi(S_2) = (0:0:1:1)$.

We immediately see from the intersection numbers that

$$DS_1' = DS_2' = DS_1'' = DS_2'' = 3.$$

We focus specifically on S_1''. Note that S_1'' intersects both B_1'' and A_1, which means that $\pi(S_1'')$ passes through $(1:1:0:0)$ twice. Thus it should be a planar degree three rational nodal curve. This turned out to be a key observation that allowed us to get enough equations on the coefficients of f to solve for it.

PROPOSITION 4.1

The sextic equation $f(y_0, y_1, y_2, y_3) = 0$ *where*

$$\begin{aligned} f = {} & 28y_0^6 - (42 - 2\mathrm{i}\sqrt{7})y_0^4y_1^2 - 4\mathrm{i}\sqrt{7}y_0^2y_1^4 + 56y_0^2y_1^2y_2^2 - (14 + 22\mathrm{i}\sqrt{7})y_0^4y_1y_3 \\ & - (7 - 13\mathrm{i}\sqrt{7})y_0^2y_1^2y_3^2 - (77 + 17\mathrm{i}\sqrt{7})y_1^4y_3^2 \\ & + (21 - 31\mathrm{i}\sqrt{7})(y_0^3y_1y_2y_3 - y_0y_1^3y_2y_3) \\ & - (28 - 20\mathrm{i}\sqrt{7})y_1^3y_3(y_1^2 + y_2^2 - y_3^2) \\ & + (14 + 2\mathrm{i}\sqrt{7})y_1^2\bigl(y_1^4 + 2y_1^2y_2^2 + (y_2^2 - y_3^2)^2\bigr) \\ & + (42 + 2\mathrm{i}\sqrt{7})\bigl(y_0^2y_1^3y_3 + y_0y_1^2y_2(-y_0^2 + y_1^2 + y_2^2 - y_3^2)\bigr) \end{aligned}$$

cuts out a surface which has the same expected properties as the image of the double cover X *under the map given by* $|3F + S_1 + S_2|$.

Remark 4.2

It is clear that complex conjugation provides another surface with the same properties that comes from the complex-conjugate fake projective plane.

We remark that the formula of Proposition 4.1 was obtained by writing down a generic invariant sextic that satisfied the properties and then using *Mathematica* software package to write down equations on the coefficients. The equations are too complicated to be solved symbolically, but numerical solutions give values that "look like" algebraic numbers. This allow us to identify a putative equation, which can then be checked to give the desired properties.

We now describe the images of the twenty-four curves $S_1, \dots, S_2'', A_1, \dots, C_2''$ on $\pi(X)$. The curve S_1'' was found in the process of getting Proposition 4.1. The curve S_2'' is obtained by simply applying the involution σ. It took a bit of effort to find S_1'. The idea is that there should be an order 3 automorphism that acts fiberwise on $X \to \mathbb{CP}^1$ and sends $S_1 \to S_1' \to S_1''$. This automorphism is a lift of the order 3 automorphism acting on the quotient $\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$. Each of the curves S_1, S_1', and S_1'' have two points in the generic fiber, which give two orbits under addition of an element of order 3. Thus, if we parameterize S_1'' as $S_1''(t)$ there should be a point $S_1'(t)$ in the fiber so that

$$S_1'(t) + S_1''(t) = (S_1)_1 + (S_1)_2$$

where $(S_1)_i$ are two preimages of the node $\pi(S_1)$. Since the preimage of the class of the line in $\mathbb{CP}^2$ that contains the fiber is $(S_1)_1 + (S_1)_2 + (S_2)_1 + (S_2)_2$, we see that the fourth intersection point of the line through the node $\pi(S_2) = (0:0:1:1)$ and $S_1''(t)$ with the quartic image of the fiber should give parameterization of S_1'. We write the corresponding equations in Table 4.

Remark 4.3
The construction of S_1' has an additional advantage of providing us with a rational function on Y which has well-understood zeros and poles. Specifically, the section

$$\Big(y_0^3 - y_0^2 y_1 - y_0 y_1^2 + y_1^3 + \frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0 - y_1)y_1(y_2 - y_3) + \frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2-y_3)^2\Big)$$

defines a (nonnormal) cubic cone with vertex $(0:0:1:1)$ that contains S_1' and S_1''. Its symmetrization $f_{cones}(y_0, y_1, y_2, y_3)$ given by

$$\Big(y_0^3 - y_0^2 y_1 - y_0 y_1^2 + y_1^3 + \frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0 - y_1)y_1(y_2 - y_3) + \frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2-y_3)^2\Big)$$
$$\Big(y_0^3 + y_0^2 y_1 - y_0 y_1^2 - y_1^3 - \frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0 + y_1)y_1(y_2 + y_3) - \frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2+y_3)^2\Big)$$

Table 4. Images of curves on X under the map $\pi : X \to \mathbb{CP}^3$ (The equations are either parametric or nonparametric; the curves $\pi(S_2), \dots, \pi(C_2'')$ can be found by applying $\sigma(y_0 : y_1 : y_2 : y_3) = (y_0 : -y_1 : y_2 : -y_3)$ to $\pi(S_1), \dots, \pi(C_1'')$.)

Curves	Equations
$\pi(F)$	$y_0 = y_1 = 0$
$\pi(F_2)$	$y_0 = 0, y_1^2 + y_2^2 + \frac{1}{4}(-1 + 3\mathrm{i}\sqrt{7})y_1 y_3 - y_3^2 = 0$
$\pi(S_1)$	$(0 : 0 : -1 : 1)$
$\pi(S_1')$	$y_0 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7})t + \frac{1}{8}(-3 + \mathrm{i}\sqrt{7})t^3$ $y_1 = t^3$ $y_2 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7}) + \frac{1}{8}(-1 + 3\mathrm{i}\sqrt{7})t - \frac{1}{8}(5 + \mathrm{i}\sqrt{7})t^2 + \frac{1}{8}(3 - \mathrm{i}\sqrt{7})t^3,$ $y_3 = -\frac{1}{16}(9 + 5\mathrm{i}\sqrt{7}) + \frac{1}{16}(11 - \mathrm{i}\sqrt{7})t + \frac{1}{16}(21 + \mathrm{i}\sqrt{7})t^2 - \frac{1}{16}(7 - 5\mathrm{i}\sqrt{7})t^3$
$\pi(S_1'')$	$y_0 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7})t + \frac{1}{8}(-3 + \mathrm{i}\sqrt{7})t^3$ $y_1 = t^3$ $y_2 = \frac{1}{16}(-9 - 5\mathrm{i}\sqrt{7} + (11 - \mathrm{i}\sqrt{7})t)(-1 + t^2)$ $y_3 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7} + (-1 + 3\mathrm{i}\sqrt{7})t)(-1 + t^2)$
$\pi(A_1)$	$(1 : 1 : 0 : 0)$
$\pi(B_1)$	$y_0 = y_1, y_2 = -y_3$
$\pi(C_1)$	$(1 : 1 : -\frac{1}{4}(3 + \mathrm{i}\sqrt{7}), \frac{1}{4}(3 + \mathrm{i}\sqrt{7}))$
$\pi(A_1')$	$y_0 = y_1,$ $\frac{1}{2}(11 - \mathrm{i}\sqrt{7})y_1^2 + \frac{1}{4}(11 - \mathrm{i}\sqrt{7})y_1 y_2 + y_2^2 + \frac{1}{2}(-1 + 3\mathrm{i}\sqrt{7})y_1 y_3 - y_3^2 = 0$
$\pi(B_1')$	$(-1 : -1 : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}) : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}))$
$\pi(C_1')$	$(-1 : -1 : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}) : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}))$
$\pi(A_1'')$	$y_0 = y_1, y_2 = y_3$
$\pi(B_1'')$	$(1 : 1 : 0 : 0)$
$\pi(C_1'')$	$(1 : 1 : 0 : 0)$

gives a σ-invariant section of $H^0(X, 6D)$ which contains $S_1' + S_2' + S_1'' + S_2''$. In fact, we were able to show that its degree 36 intersection curve with $\pi(X)$ is fully accounted for by the curves from our list of 24 rational curves as well as F. The σ-invariant rational function

$$\frac{f_{\text{cones}}(y_0, y_1, y_2, y_3)}{(y_0^2 - y_1^2)^3}$$

on X gives a rational function on Y whose divisor is

$$2A - 3A' + A'' - B - B' + 2B'' - 2C + 2C'' - 2S + S' + S''.$$

The curves $A_1, \dots, C_1''$ are either contracted to points or map isomorphically to lines or conics in the plane $y_0 = y_1$, as indicated in Table 4.

An important part of our calculations will be based on finding a putative normalization of the ring

$$\mathbb{C}[y_0, y_1, y_2, y_3] / \langle f(y_0, y_1, y_2, y_3) \rangle.$$

PROPOSITION 4.4
The rational functions

$$\hat{y}_4 = \frac{y_0^3}{y_1}$$
$$\hat{y}_5 = \frac{(y_1^2 + y_2^2 + \frac{1}{4}(-1 + 3\mathrm{i}\sqrt{7})y_1 y_3 - y_3^2)y_1}{y_0}$$

lie in the normalization of $\mathbb{C}[y_0, y_1, y_2, y_3]/\langle f(y_0, y_1, y_2, y_3)\rangle$ *in its field of fractions. These elements are odd with respect to the involution* σ *and are homogeneous with grading* 2.

Proof
It is straightforward to see that $\hat{y}_4$ and $\hat{y}_5$ satisfy monic quadratic equations with coefficients in the ring. The parity and grading are obvious. ☐

Remark 4.5
We believe that $y_0, \ldots, y_3, \hat{y}_4, \hat{y}_5$ generate the normalization of the ring $\mathbb{C}[y_0, y_1, y_2, y_3]/\langle f(y_0, y_1, y_2, y_3)\rangle$ which is isomorphic to

$$\bigoplus_{k\geq 0} H^0\big(X, \mathcal{O}(kD)\big).$$

Moreover, we have calculated generators of the ideal of relations between $y_0, \ldots, y_3$, $\hat{y}_4, \hat{y}_5$. Since we do not need this information for our purposes, we will not present it in the present article. However, we do use the fact that $\hat{y}_4$ and $\hat{y}_5$ give odd sections of $H^0(X, \mathcal{O}(2D))$.

5. Order 3 automorphism

An important feature of X is an order 3 automorphism which is a lift of the order 3 automorphism acting on the quotient $\mathbb{P}^2_{\mathrm{fake}}/\mathbb{Z}_7$. In this section, we describe how to find an explicit formula for it in terms of the birational automorphism of the sextic surface $\pi(X) \subset \mathbb{CP}^3$.

PROPOSITION 5.1
Let $Y_0 = \frac{y_0}{y_1}$, $Y_2 = \frac{y_2}{y_1}$ *and* $Y_3 = \frac{y_3}{y_1}$ *be the generators of the field extension* $\mathrm{Rat}(X) \supset \mathbb{C}$. *The automorphism of order* 3 *sends* (Y_0, Y_2, Y_3) *to* (Y_0, Y_2', Y_3') *given by Table* 5. *Its inverse sends* (Y_0, Y_2, Y_3) *to* (Y_0, Y_2'', Y_3'') *given by Table* 6.

Table 5. Automorphism of order 3 : $(Y_0, Y_2, Y_3) \mapsto (Y_0, Y_2', Y_3')$.

$$\boxed{\begin{aligned}
Y_2' = {} & \tfrac{(3+i\sqrt{7})}{8} Y_0^{-1}((-21i + 31\sqrt{7})Y_2^2 + (-35i + 9\sqrt{7})Y_3^2))^{-1} \\
& (7(9i + 5\sqrt{7})Y_0^4 Y_3 + 2Y_0^2(4(21i + \sqrt{7})Y_2^2 - (7i + 11\sqrt{7})Y_3) \\
& +Y_3(-49i - 13\sqrt{7} - (49i + 13\sqrt{7})Y_2^2 + 8(-7i + 5\sqrt{7})Y_3 + (49i + 13\sqrt{7})Y_3^2) \\
& -Y_0((-21i + 31\sqrt{7})Y_2^3 + Y_2Y_3(112i + 48\sqrt{7} + 21iY_3 - 31\sqrt{7}Y_3))) \\
Y_3' = {} & \tfrac{(-3i+\sqrt{7})}{8} Y_0^{-1}((-21i + 31\sqrt{7})Y_2^2 + (-35i + 9\sqrt{7})Y_3^2))^{-1} \\
& ((-21 - 31i\sqrt{7})Y_2^3 + Y_0Y_2^2(-168 + 8i\sqrt{7} + 49Y_3 - 13i\sqrt{7}Y_3) \\
& +Y_0Y_3^2(56 + 40i\sqrt{7} - 49Y_3 + 13i\sqrt{7}Y_3) + Y_2(-21 - 31i\sqrt{7} \\
& +7(13 + 7i\sqrt{7})Y_0^4 + 8(21 - i\sqrt{7})Y_3 + (21 + 31i\sqrt{7})Y_3^2 \\
& +Y_0^2(-70 - 18i\sqrt{7} + (-56 - 40i\sqrt{7})Y_3)))
\end{aligned}}$$

Table 6. Inverse automorphism of order 3 : $(Y_0, Y_2, Y_3) \mapsto (Y_0, Y_2'', Y_3'')$.

$$\boxed{\begin{aligned}
Y_2'' = {} & (-20 - 4i\sqrt{7} - 4i(-9i + \sqrt{7})Y_0^5Y_2 + (34 - 30i\sqrt{7})Y_3 + (134 + 14i\sqrt{7})Y_3^2 \\
& -(15 - 43i\sqrt{7})Y_3^3 - 48Y_3^4 - (1 + 3i\sqrt{7})Y_3^5 + 4iY_0^6(5i - \sqrt{7} + 2\sqrt{7}Y_3) \\
& +Y_2^4(-20 - 4i\sqrt{7} + (-1 - 3i\sqrt{7})Y_3) + 2Y_0^3Y_2(36 + 4i\sqrt[7]{+}(3 + 5i\sqrt{7})Y_2^2 \\
& +(-5 + 15i\sqrt{7})Y_3 + (-16 + 2i\sqrt{7})Y_3^2) + Y_2^2(-40 - 8i\sqrt{7} + 33(1 - i\sqrt{7})Y_3 \\
& +(68 + 4i\sqrt{7})Y_3^2 + (2 + 6i\sqrt{7})Y_3^3) + 2Y_0^2(10 + 2i\sqrt{7} + 8Y_2^4 + (-26 + 10i\sqrt{7})Y_3 \\
& +(-29 - 9i\sqrt{7})Y_3^2 + (8 - 4i\sqrt{7})Y_3^3 - Y_2^2Y_3(17 + i\sqrt{7} + 8Y_3)) + Y_0^4(20 + 4i\sqrt{7} \\
& +2(9 + i\sqrt{7})Y_3 + 4(3 - i\sqrt{7})Y_3^2 + (7 + 5i\sqrt{7})Y_3^3 + Y_2^2(-48 + 16i\sqrt{7} - (7 + 5i\sqrt{7})Y_3)) \\
& +Y_0Y_2(-36 - 4i\sqrt{7} + (5 - i\sqrt{7})Y_2^4 + 10(1 - 3i\sqrt{7})Y_3 + 52Y_3^2 + (5 - i\sqrt{7})Y_3^4 \\
& +2iY_2^2(13i - 7\sqrt{7} + (5i + \sqrt{7})Y_3^2)))/(2Y_0(-3i - \sqrt{7} + (3i + \sqrt{7})Y_0^2 - 2iY_2^2 \\
& +(-5i + \sqrt{7})Y_3 + (i + \sqrt{7})Y_3^2 - Y_2(-5i + \sqrt{7} + (-i + \sqrt{7})Y_3)) \\
& (-3i - \sqrt{7} + (3i + \sqrt{7})Y_0^2 - 2iY_2^2 - (5i - \sqrt{7})Y_3 + (i + \sqrt{7})Y_3^2 + Y_2(-5i + \sqrt{7} \\
& +(-i + \sqrt{7})Y_3))) \\
Y_3'' = {} & (8i\sqrt{7}Y_0^6Y_2 + Y_0^5(-40 - 8i\sqrt{7} + (26 + 2i\sqrt{7})Y_3) + 2Y_0^3(40 + 8i\sqrt{7} \\
& +2(-17 + 11i\sqrt{7} + (1 - 2i\sqrt{7})Y_2^2)Y_3 + (-39 - 11i\sqrt{7})Y_3^2 + (11 - 3i\sqrt{7})Y_3^3) \\
& +2Y_0^2Y_2(-4i\sqrt{7} + (33 - 3i\sqrt{7})Y_3 + (25 + 9i\sqrt{7})Y_3^2 + 4i(i + \sqrt{7})Y_3^3 \\
& +4Y_2^2(-4 - i\sqrt{7} + (1 - i\sqrt{7})Y_3)) + Y_2(8i\sqrt{7} + (5 - i\sqrt{7})Y_2^4 + 2i(27i + \sqrt{7})Y_3 \\
& +(-23 - 17i\sqrt{7})Y_3^2 + (8 - 8i\sqrt{7})Y_3^3 + (5 - i\sqrt{7})Y_3^4 + Y_2^2(5 + 7i\sqrt{7} + 8i(i + \sqrt{7})Y_3 \\
& +2i(5i + \sqrt{7})Y_3^2)) + Y_0^4((7 - 3i\sqrt{7})Y_2^3 + iY_2(-8\sqrt{7} + 4(3i + \sqrt{7})Y_3 + (7i + 3\sqrt{7})Y_3^2)) \\
& +Y_0(-40 - 8i\sqrt{7} + (42 - 46i\sqrt{7})Y_3 + 2(83 + 7i\sqrt{7})Y_3^2 + (-14 + 46i\sqrt{7})Y_3^3 \\
& -48Y_3^4 + (-1 - 3i\sqrt{7})Y_3^5 + Y_2^4(-4 - 4i\sqrt{7} + (-1 - 3i\sqrt{7})Y_3) \\
& +2Y_2^2(-44 + 4i\sqrt{7} + (-6 - 16i\sqrt{7})Y_3 + (26 + 2i\sqrt{7})Y_3^2 + (1 + 3i\sqrt{7})Y_3^3))) \\
& /(2Y_0(-3i - \sqrt{7} + (3i + \sqrt{7})Y_0^2 - 2iY_2^2 - (5i - \sqrt{7})Y_3 + (i + \sqrt{7})Y_3^2 \\
& -Y_2(-5i + \sqrt{7} + (-i + \sqrt{7})Y_3))(-3i - \sqrt{7} + (3i + \sqrt{7})Y_0^2 - 2iY_2^2 - (5i - \sqrt{7})Y_3 \\
& +(i + \sqrt{7})Y_3^2 + Y_2(-5i + \sqrt{7} + (-i + \sqrt{7})Y_3)))
\end{aligned}}$$

Remark 5.2

While formulas of Tables 5 and 6 are not particularly inspiring, they are far preferable to some other formulas for the automorphism that we initially found.

Proof

It is a straightforward computer calculation to check that the formulas provide automorphisms. However, it takes too long to verify that the cube of it is identity symbolically. It is, however, trivial to do so heuristically by taking a random point on $\pi(X)$ calculated to high precision and iterating the automorphism three times.

To find the automorphism, we use the fact that Y_2 and Y_3 are rational functions with poles along $3F + S_1 + S_2$. So their transforms should be rational functions with poles along $3F + S_1' + S_2'$ and $3F + S_1'' + S_2''$. We also know that $Y_2 + Y_3$ is zero on S_1 and $Y_2 - Y_3$ is zero on S_2. This allows us to fix the transforms up to constants, which can then be recovered. ☐

6. Double cover of the fake projective plane

In this section, we explain how we found the function field of the fake projective plane. According to [15, p. 1676], we need to attach the seventh root of the rational function which has divisor

$$5S + B + 4C + 6S' + 4B' + 2C' + 3S'' + 2B'' + C''$$

up to multiples of 7. This divisor is divisible by 7 in the Picard group and corresponds to the third possibility for the divisor B in [ibid], where the curves A_1, A_2, E_1, B_1, B_2, E_2, C_1, C_2, E_3 correspond to C'', B'', S'', C', B', S', C, B, S in our notation. The first possibility for B was ruled out, because the I_9-fiber has multiplicity $\mu = 1$ by [17, p, 2 and Theorem 2.3], and the second possibility corresponds to Case 1 of (3.2), which was ruled out in Section 3. We found this function by looking at the equation of the cubic cone with vertex $(0:0:1:1)$ that contains S_1'' and S_1'. When divided by y_1^3 it gives a divisor whose zeros and poles occur only at the named divisors. By symmetrizing it via σ and using the automorphism, we were able to get the desired function. We denote the seventh root of this function by z; the function z^7 is given in Table 7.

To find the function field of the fake projective plane we simply need to take the invariants with respect to σ that preserves z and Y_3 and negates Y_0 and Y_2.

We also found a lift of the action of the order 3 automorphism to the field generated by Y_0, Y_2, Y_3, z. Specifically, the action on z is given in Table 8.

7. Embedding of the fake projective plane into $\mathbb{CP}^9$

Let us now describe the method that allowed us to construct the equations of the fake projective plane. By a Riemann–Roch calculation, the dimension of the bicanonical linear system on $\mathbb{P}^2_{\text{fake}}$ is 10.

The pullback of the ($\mathbb{Q}$-Cartier) canonical divisor via $\mu : Y \to \mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$ satisfies

$$K_Y = \mu^* K_{\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7} - \frac{3}{7}(S + S' + S'') - \frac{2}{7}(B + B' + B'') - \frac{1}{7}(C + C' + C''). \quad (7.1)$$

This shows that the preimage of $\mu(F_Y)$ on $\mathbb{P}^2_{\text{fake}}$ is numerically equivalent to a canonical divisor. (It is actually a section of a canonical line bundle twisted by an invertible

Table 7. Formula for z^7.

$$
\begin{aligned}
z^7 = {} & ((-315\mathrm{i}+47\sqrt{7})^2(-1+Y_0^2)^5(2795\mathrm{i}+287\sqrt{7}-5590\mathrm{i}Y_0\\
& -574\sqrt{7}Y_0+11573\mathrm{i}Y_0^2+2689\sqrt{7}Y_0^2\\
& -17556\mathrm{i}Y_0^3-4804\sqrt{7}Y_0^3+14357\mathrm{i}Y_0^4+5601\sqrt{7}Y_0^4-11158\mathrm{i}Y_0^5-6398\sqrt{7}Y_0^5\\
& +5579\mathrm{i}Y_0^6+3199\sqrt{7}Y_0^6+5590\mathrm{i}Y_2^2+574\sqrt{7}Y_2^2-5590\mathrm{i}Y_0Y_2^2-574\sqrt{7}Y_0Y_2^2\\
& +5994\mathrm{i}Y_0^2Y_2^2-510\sqrt{7}Y_0^2Y_2^2+5590\mathrm{i}Y_0^3Y_2^2+574\sqrt{7}Y_0^3Y_2^2+2795\mathrm{i}Y_2^4+287\sqrt{7}Y_2^4\\
& +1616\mathrm{i}Y_3-4336\sqrt{7}Y_3+5568\mathrm{i}Y_0Y_3+5824\sqrt{7}Y_0Y_3+3232\mathrm{i}Y_0^2Y_3-8672\sqrt{7}Y_0^2Y_3\\
& -448\mathrm{i}Y_0^3Y_3+11584\sqrt{7}Y_0^3Y_3-9968\mathrm{i}Y_0^4Y_3-4400\sqrt{7}Y_0^4Y_3+11584\mathrm{i}Y_0^2Y_2Y_3\\
& +64\sqrt{7}Y_0^2Y_2Y_3-17600\mathrm{i}Y_0^3Y_2Y_3+5696\sqrt{7}Y_0^3Y_2Y_3+1616\mathrm{i}Y_2^2Y_3\\
& -4336\sqrt{7}Y_2^2Y_3+7184\mathrm{i}Y_0Y_2^2Y_3+1488\sqrt{7}Y_0Y_2^2Y_3-17174\mathrm{i}Y_3^2-638\sqrt{7}Y_3^2\\
& +11606\mathrm{i}Y_0Y_3^2-5186\sqrt{7}Y_0Y_3^2-5994\mathrm{i}Y_0^2Y_3^2+510\sqrt{7}Y_0^2Y_3^2+5994\mathrm{i}Y_0^3Y_3^2\\
& -510\sqrt{7}Y_0^3Y_3^2-5590\mathrm{i}Y_2^2Y_3^2-574\sqrt{7}Y_2^2Y_3^2-1616\mathrm{i}Y_3^3+4336\sqrt{7}Y_3^3\\
& -7184\mathrm{i}Y_0Y_3^3-1488\sqrt{7}Y_0Y_3^3+2795\mathrm{i}Y_3^4+287\sqrt{7}Y_3^4)(2795\mathrm{i}+287\sqrt{7}+5590\mathrm{i}Y_0\\
& +574\sqrt{7}Y_0+11573\mathrm{i}Y_0^2+2689\sqrt{7}Y_0^2+17556\mathrm{i}Y_0^3+4804\sqrt{7}Y_0^3+14357\mathrm{i}Y_0^4\\
& +5601\sqrt{7}Y_0^4+11158\mathrm{i}Y_0^5+6398\sqrt{7}Y_0^5+5579\mathrm{i}Y_0^6+3199\sqrt{7}Y_0^6+5590\mathrm{i}Y_2^2\\
& +574\sqrt{7}Y_2^2+5590\mathrm{i}Y_0Y_2^2+574\sqrt{7}Y_0Y_2^2+5994\mathrm{i}Y_0^2Y_2^2-510\sqrt{7}Y_0^2Y_2^2-5590\mathrm{i}Y_0^3Y_2^2\\
& -574\sqrt{7}Y_0^3Y_2^2+2795\mathrm{i}Y_2^4+287\sqrt{7}Y_2^4+1616\mathrm{i}Y_3-4336\sqrt{7}Y_3-5568\mathrm{i}Y_0Y_3\\
& -5824\sqrt{7}Y_0Y_3+3232\mathrm{i}Y_0^2Y_3-8672\sqrt{7}Y_0^2Y_3+448\mathrm{i}Y_0^3Y_3-11584\sqrt{7}Y_0^3Y_3\\
& -9968\mathrm{i}Y_0^4Y_3-4400\sqrt{7}Y_0^4Y_3-11584\mathrm{i}Y_0^2Y_2Y_3-64\sqrt{7}Y_0^2Y_2Y_3-17600\mathrm{i}Y_0^3Y_2Y_3\\
& +5696\sqrt{7}Y_0^3Y_2Y_3+1616\mathrm{i}Y_2^2Y_3-4336\sqrt{7}Y_2^2Y_3-7184\mathrm{i}Y_0Y_2^2Y_3\\
& -1488\sqrt{7}Y_0Y_2^2Y_3-17174\mathrm{i}Y_3^2-638\sqrt{7}Y_3^2-11606\mathrm{i}Y_0Y_3^2+5186\sqrt{7}Y_0Y_3^2\\
& -5994\mathrm{i}Y_0^2Y_3^2+510\sqrt{7}Y_0^2Y_3^2-5994\mathrm{i}Y_0^3Y_3^2+510\sqrt{7}Y_0^3Y_3^2\\
& -5590\mathrm{i}Y_2^2Y_3^2-574\sqrt{7}Y_2^2Y_3^2-1616\mathrm{i}Y_3^3+4336\sqrt{7}Y_3^3+7184\mathrm{i}Y_0Y_3^3\\
& +1488\sqrt{7}Y_0Y_3^3+2795\mathrm{i}Y_3^4+287\sqrt{7}Y_3^4))/(4096Y_0^4(-4\mathrm{i}+4\mathrm{i}Y_0\\
& +4\mathrm{i}Y_0^2-4\mathrm{i}Y_0^3+2\mathrm{i}Y_2-2\sqrt{7}Y_2-2\mathrm{i}Y_0Y_2+2\sqrt{7}Y_0Y_2+\mathrm{i}Y_2^2+\sqrt{7}Y_2^2-2\mathrm{i}Y_3\\
& +2\sqrt{7}Y_3+2\mathrm{i}Y_0Y_3-2\sqrt{7}Y_0Y_3-2\mathrm{i}Y_2Y_3-2\sqrt{7}Y_2Y_3+\mathrm{i}Y_3^2+\sqrt{7}Y_3^2)^2(-4\mathrm{i}-4\mathrm{i}Y_0\\
& +4\mathrm{i}Y_0^2+4\mathrm{i}Y_0^3-2\mathrm{i}Y_2+2\sqrt{7}Y_2-2\mathrm{i}Y_0Y_2+2\sqrt{7}Y_0Y_2+\mathrm{i}Y_2^2+\sqrt{7}Y_2^2-2\mathrm{i}Y_3\\
& +2\sqrt{7}Y_3-2\mathrm{i}Y_0Y_3+2\sqrt{7}Y_0Y_3+2\mathrm{i}Y_2Y_3+2\sqrt{7}Y_2Y_3+\mathrm{i}Y_3^2+\sqrt{7}Y_3^2)^2(-21\mathrm{i}Y_2^2\\
& +31\sqrt{7}Y_2^2-35\mathrm{i}Y_3^2+9\sqrt{7}Y_3^2)^2)
\end{aligned}
$$

Table 8. Automorphism of order 3 : $(Y_0, Y_2, Y_3, z) \mapsto (Y_0, Y_2', Y_3', z'')$.

$$
\begin{aligned}
z'' = {} & z^2(-1+Y_0^2)^{-3}(1-Y_0-Y_0^2+Y_0^3-\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})(Y_2-Y_3)\\
& +\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})Y_0(Y_2-Y_3)+\tfrac{1}{4}(-1+\mathrm{i}\sqrt{7})(Y_2-Y_3)^2)(-1-Y_0+Y_0^2+Y_0^3\\
& -\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})(Y_2+Y_3)-\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})Y_0(Y_2+Y_3)+\tfrac{1}{4}(1-\mathrm{i}\sqrt{7})(Y_2+Y_3)^2)
\end{aligned}
$$

torsion line bundle). In particular, to calculate

$$H^0(\mathbb{P}^2_{\text{fake}}, 2K_{\mathbb{P}^2_{\text{fake}}})$$

we can look for rational functions on $\mathbb{P}^2_{\text{fake}}$ which have poles of order at most 2 on the curve F^{FPP} which is the preimage of $\mu(F_Y)$ and no other poles.

The action of $\mathbb{Z}_7$ splits the space of such functions into seven eigenspaces. Each eigenspace consists of functions of the form $z^i g$, where g is a function from the function field of Y, as i runs over residues modulo 7. The residual $\mathbb{Z}_3$ action allows us to reduce the calculation to that of $i = -1, 0, 1$.

The $i = 0$ case is easy. The only such function up to scaling is 1.

Now let us calculate such functions of the form zg. Consider the Cartesian product diagram below

$$\begin{tikzcd}
\widehat{\mathbb{P}^2_{\text{fake}}} \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathbb{P}^2_{\text{fake}} \arrow[r] & \mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7
\end{tikzcd}$$

where $\widehat{\mathbb{P}^2_{\text{fake}}}$ is the singular Galois cover of Y ramified at the nine curves $S, \dots, C''$ given by normalization of Y in the field of fractions of $\mathbb{P}^2_{\text{fake}}$. We can calculate the global sections of an invertible sheaf on $\mathbb{P}^2_{\text{fake}}$ in terms of the pullback of these sections on $\widehat{\mathbb{P}^2_{\text{fake}}}$.

In view of (7.1) we see that the pullback of $2F^{FPP}$ on $\widehat{\mathbb{P}^2_{\text{fake}}}$ is equal to twice its proper preimage $\widehat{F^{FPP}}$ plus

$$\frac{6}{7}(S + S' + S'') + \frac{4}{7}(B + B' + B'') + \frac{2}{7}(C + C' + C''),$$

where $\frac{1}{7}S$ is the reduced preimage of S under $\widehat{\mathbb{P}^2_{\text{fake}}} \to Y$, and similarly for the other eight curves. The divisor of z on $\widehat{\mathbb{P}^2_{\text{fake}}}$ is

$$-A + A' + \frac{5}{7}S - \frac{1}{7}S' - \frac{4}{7}S'' + \frac{1}{7}B + \frac{4}{7}B' - \frac{5}{7}B'' + \frac{4}{7}C + \frac{2}{7}C' - \frac{6}{7}C''.$$

This means that the divisor of g on $\widehat{\mathbb{P}^2_{\text{fake}}}$ must be greater than or equal to

$$\begin{aligned}
&-2\widehat{F^{FPP}} - \frac{6}{7}(S + S' + S'') - \frac{4}{7}(B + B' + B'') - \frac{2}{7}(C + C' + C'') - \operatorname{div}(z) \\
&\quad = -2\widehat{F^{FPP}} + A - A' - \frac{11}{7}S - \frac{5}{7}S' - \frac{2}{7}S'' - \frac{5}{7}B - \frac{8}{7}B' + \frac{1}{7}B'' \\
&\quad\quad - \frac{6}{7}C - \frac{4}{7}C' + \frac{4}{7}C''.
\end{aligned}$$

Since g is a rational function on Y, this translates into the condition that the divisor of g on Y is greater than or equal to

$$-2F_Y + A - A' - S - B' + B'' + C'';$$

in other words, it can be computed as a global section of the invertible sheaf

$$\mathcal{O}_Y(2F_Y + S - A + A' + B' - B'' - C'')$$

on Y, or equivalently σ-invariant sections of

$$\mathcal{O}_X(2F + S_1 + S_2 - A_1 - A_2 + A_1' + A_2' + B_1' + B_2' - B_1'' - B_2'' - C_1'' - C_2'').$$

Note that the rational function $Y_0^2 - 1$ on Y has pole of order 2 at F_Y and zeros of order 1 at the nine curves $A, \dots, C''$ of the I_9 fiber. As a result, the σ-invariant section $y_0^2 - y_1^2$ of $H^0(X, 2D)$ is $2F + I_9 + 2S_1 + 2S_2$. Since

$$\begin{aligned}(2F_Y + I_9 + 2S) - (2F_Y - A + A' + B' - B'' - C'' + S)\\ = S + 2A + A'' + B + 2B'' + C + C' + 2C'',\end{aligned}$$

we can find σ-invariant sections of

$$\mathcal{O}_X(2F + S_1 + S_2 - A_1 - A_2 + A_1' + A_2' + B_1' + B_2' - B_1'' - B_2'' - C_1'' - C_2'').$$

by looking at σ-invariant sections of $2D$ which vanish on $(S + 2A + A'' + B + 2B'' + C + C' + 2C'')$. By using the calculation of Table 4, it can be seen that such sections are multiples of $y_2^2 - y_3^2$, so the rational function in question is

$$\frac{(y_2^2 - y_3^2)z}{y_0^2 - y_1^2},$$

up to a multiplicative constant.

Similarly, for the $z^{-1}g$, we end up looking at g which are global sections of

$$\mathcal{O}_Y(2F_Y + A - A' - C + B'' + C'' + S' + S'').$$

We can construct these functions as

$$\frac{(y_0^2 - y_1^2)r(y_0, y_1, y_2, y_3)}{f_{\text{cones}}(y_0, y_1, y_2, y_3)}$$

where $r(y_0, y_1, y_2, y_3)$ is a σ-invariant section of $H^0(X, 4D)$ and f_{cones} is given in Remark 4.3. The denominator f_{cones} is a σ-invariant element of $H^0(X, 6D)$ which vanishes on $S' + S''$ given by

$$\begin{aligned}&\Big(y_0^3 - y_0^2y_1 - y_0y_1^2 + y_1^3 + \frac{1}{2}(1 + \mathrm{i}\sqrt{7})(y_0 - y_1)y_1(y_2 - y_3)\\ &\qquad + \frac{1}{4}(-1 + \mathrm{i}\sqrt{7})y_1(y_2 - y_3)^2\Big)\\ &\Big(y_0^3 + y_0^2y_1 - y_0y_1^2 - y_1^3 - \frac{1}{2}(1 + \mathrm{i}\sqrt{7})(y_0 + y_1)y_1(y_2 + y_3)\\ &\qquad - \frac{1}{4}(-1 + \mathrm{i}\sqrt{7})y_1(y_2 + y_3)^2\Big).\end{aligned}$$

We know that the section $(y_0^2 - y_1^2)$ of $H^0(Y, D)$ has divisor $2F_Y + 2S + I_9$ where $I_9 = A + \cdots + C''$ is the sum of the curves in the I_9 fiber. As a result, the section r should be vanishing on

$$\begin{aligned}(4F_Y+4S+2I_9)&+(2A-3A'+A''-B-B'+2B''\\&-2C+2C''-2S+S'+S'')\\&-(2F_Y+A-A'-C+B''+C''+S'+S'')\\&=2F_Y+2S+3A'+3A''+B+B'+3B''+C+2C'+3C''.\end{aligned}$$

Importantly, we need to use not just polynomial r but also elements of the normalization, namely, products of σ-anti-invariant degree 2 polynomials in y_i with $\hat{y}_4$ and $\hat{y}_5$ from Proposition 4.4.

This is a rather delicate calculation that led us to the results in Table 9. Note that these functions are only determined up to linear changes of variables. We have reduced the ambiguity a bit by requiring that the first of these sections vanishes at the fixed points of $\mathbb{Z}_7$ action on $\mathbb{P}^2_{\text{fake}}$ and have chosen constants in a noble but not very successful attempt to make the equations more palatable.

The rational functions we have constructed so far lead to the variables U_0, U_1, U_4, U_7 of Theorem 2.1. The other sections are obtained by applying the order 3 automorphism. We used *Mathematica* to tabulate numerically several dozen points on $\mathbb{P}^2_{\text{fake}}$ by first picking random values for Y_2 and Y_3, then solving for (one of the) values of Y_0, then solving for one of the values of z by taking a seventh root of z^7. Then we looked for degree 2 and 3 polynomial equations that vanish on these points. *Mathematica* is able to work with these numerical approximations by keeping accuracy estimates. As a result, it can give solutions of expected dimension to linear system whose coefficients are only known approximately by assuming that all minors within the accuracy bound of zero are in fact zero. After finding approximations of

Table 9. Rational functions $z^{-1}g$.

$(4\mathrm{i}(-1+Y_0)(1+Y_0)(-266\mathrm{i}Y_0+34\sqrt{7}Y_0+532\mathrm{i}Y_0^3-68\sqrt{7}Y_0^3-266\mathrm{i}Y_0^5+34\sqrt{7}Y_0^5 -70\mathrm{i}Y_2+46\sqrt{7}Y_2-126\mathrm{i}Y_0^2Y_2-58\sqrt{7}Y_0^2Y_2+196\mathrm{i}Y_0^4Y_2+12\sqrt{7}Y_0^4Y_2-469\mathrm{i}Y_0Y_2^2 +97\sqrt{7}Y_0Y_2^2-63\mathrm{i}Y_0^3Y_2^2-29\sqrt{7}Y_0^3Y_2^2-70\mathrm{i}Y_2^3+46\sqrt{7}Y_2^3+238\mathrm{i}Y_0Y_3+266\sqrt{7}Y_0Y_3 -238\mathrm{i}Y_0^3Y_3-266\sqrt{7}Y_0^3Y_3+259\mathrm{i}Y_2Y_3+41\sqrt{7}Y_2Y_3-259\mathrm{i}Y_0^2Y_2Y_3-41\sqrt{7}Y_0^2Y_2Y_3 +56\mathrm{i}Y_0Y_2^2Y_3+104\sqrt{7}Y_0Y_2^2Y_3+728\mathrm{i}Y_0Y_3^2-56\sqrt{7}Y_0Y_3^2-196\mathrm{i}Y_0^3Y_3^2-12\sqrt{7}Y_0^3Y_3^2 +70\mathrm{i}Y_2Y_3^2-46\sqrt{7}Y_2Y_3^2-56\mathrm{i}Y_0Y_3^3-104\sqrt{7}Y_0Y_3^3))/((-35\mathrm{i}+23\sqrt{7})Y_0(4-4Y_0-4Y_0^2 +4Y_0^3-2Y_2-2\mathrm{i}\sqrt{7}Y_2+(2+2\mathrm{i}\sqrt{7})Y_0(Y_2-Y_3)+2Y_3+2\mathrm{i}\sqrt{7}Y_3+\mathrm{i}(\mathrm{i}+\sqrt{7})(Y_2-Y_3)^2) (-4\mathrm{i}-4\mathrm{i}Y_0+4\mathrm{i}Y_0^2+4\mathrm{i}Y_0^3-2\mathrm{i}Y_0Y_2+2\sqrt{7}Y_0Y_2-2\mathrm{i}Y_0Y_3+2\sqrt{7}Y_0Y_3+2(-\mathrm{i}+\sqrt{7}) (Y_2+Y_3)+(\mathrm{i}+\sqrt{7})(Y_2+Y_3)^2)z)$
$(16\mathrm{i}(-1+Y_0)(1+Y_0)(-133\mathrm{i}+17\sqrt{7}+266\mathrm{i}Y_0^2-34\sqrt{7}Y_0^2-133\mathrm{i}Y_0^4+17\sqrt{7}Y_0^4 -133\mathrm{i}Y_0Y_2+17\sqrt{7}Y_0Y_2+133\mathrm{i}Y_0^3Y_2-17\sqrt{7}Y_0^3Y_2-217\mathrm{i}Y_2^2+37\sqrt{7}Y_2^2-49\mathrm{i}Y_0^2Y_2^2 -3\sqrt{7}Y_0^2Y_2^2+119\mathrm{i}Y_3+133\sqrt{7}Y_3-119\mathrm{i}Y_0^2Y_3-133\sqrt{7}Y_0^2Y_3+217\mathrm{i}Y_3^2-37\sqrt{7}Y_3^2 +49\mathrm{i}Y_0^2Y_3^2+3\sqrt{7}Y_0^2Y_3^2))/((-35\mathrm{i}+23\sqrt{7})(4-4Y_0-4Y_0^2+4Y_0^3-2Y_2-2\mathrm{i}\sqrt{7}Y_2 +(2+2\mathrm{i}\sqrt{7})Y_0(Y_2-Y_3)+2Y_3+2\mathrm{i}\sqrt{7}Y_3+\mathrm{i}(\mathrm{i}+\sqrt{7})(Y_2-Y_3)^2)(-4\mathrm{i}-4\mathrm{i}Y_0+4\mathrm{i}Y_0^2 +4\mathrm{i}Y_0^3-2\mathrm{i}Y_0Y_2+2\sqrt{7}Y_0Y_2-2\mathrm{i}Y_0Y_3+2\sqrt{7}Y_0Y_3+2(-\mathrm{i}+\sqrt{7})(Y_2+Y_3) +(\mathrm{i}+\sqrt{7})(Y_2+Y_3)^2)z)$

the resulting expressions by algebraic numbers, we arrived at eighty-four degree 3 equations of Theorem 2.1.

8. Concluding remarks

We have also calculated one-hundred-forty-seven degree 7 equations among sections of $4H$ on the unramified double cover of $\mathbb{P}^2_{\text{fake}}$. There were no degree 6 equations. It seems difficult to compute explicit equations of the unramified $\mathbb{Z}_2^4$-cover of $\mathbb{P}^2_{\text{fake}}$.

9. Computer files

Nine computer files used for our computations are available as supplemental materials with the online version of this article (see https://doi.org/10.1215/00127094-2019-0076).

The file “Magma84FinalFPPexact” contains the calculation of the Hilbert polynomial of the surface, as well as the check of Remark 2.2. This is done with exact coefficients (i.e., in $\mathbb{Z}[\sqrt{-7}]$).

The file “Magma84FinalFPPmodular” contains the smoothness calculation. Specifically, it calculates three size 7 minors of the Jacobian matrix and verifies that they have no common zeros on the surface. The calculation of each minor takes approximately one hour on our hardware.

The file “M284FinalFPP” is a Macaulay2 file. It computes the projective resolution over a finite field, calculates the canonical bundle using this resolution, calculates bicanonical bundle and its Euler characteristic, and, finally, calculates $H^0(Z, \mathcal{O}(D-K)) = 0$. The last calculation is time-consuming; it runs between one and two hours on our hardware.

The choices of Magma versus Macaulay are somewhat idiosyncratic. We are not experts in either language and we used what was accessible. It is conceivable that many of the calculations can be performed in either platform. Some linear algebra calculations appeared faster in Magma, which also allowed us to work with exact coefficients. On the other hand, some schemes and sheaf cohomology calculations were more natural in Macaulay.

Acknowledgments. Borisov’s work was partially supported by National Science Foundation grants DMS-1201466 and DMS-1601907. Keum’s work was partially supported by the National Research Foundation of Korea grant NRF-2019R1A2C3010487.

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