Spectra in Locally Symmetric Spaces

Alan Reid

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Length Spectrum: The set of lengths of closed geodesics in M also consists of a discrete set of real numbers ℓ_j with multiplicity. Define

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Perhaps the best known version of the question: *Does isospectral imply isometric?* is the informal formulation due to Mark Kac(1966), which is:

"Can you hear the shape of a drum?"

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In particular:
If M_1 and M_2 are closed hyperbolic surfaces, then M_1 and M_2 are isospectral if and only they are iso-length spectral.

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The methods of both Vignéras and Sunada also produce iso-length spectral manifolds.

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Question 1: *When does isospectrality imply commensurability?*

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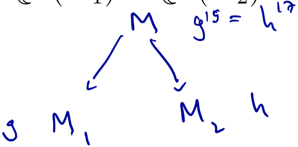
Note that if M_1 and M_2 are commensurable then

$$\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2).$$



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Two Riemannian manifolds M_1 and M_2 are length commensurable if $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$.

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Caution: Lubotzky-Vishne-Samuel's result tells us not always.

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Beautiful and deep work of Prasad and Rapinchuk provided a remarkably complete picture in the setting of:

Arithmetic locally symmetric spaces attached to absolutely simple real algebraic groups.

In particular the following papers clarify when "length commensurable implies commensurable" for arithmetic locally symmetric spaces arising from absolutely simple real algebraic groups of all types.

G. Prasad, A.S. Rapinchuk, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces*, Publ. IHES (2009).

Tour-de-force

G. Prasad, A.S. Rapinchuk, *A local-global principle for embeddings of fields with involution into simple algebras with involution*, Comment. Math. Helv. (2010).

G.Prasad, A.S.Rapinchuk, *On the fields generated by the lengths of closed geodesics in locally symmetric spaces*, Geom. Dedicata (2013).

The complete picture also relies on work of:

S. Garibaldi, *Outer automorphisms of algebraic groups and determining groups by their maximal tori*, Michigan Math. J. (2012).

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(2) The case of arithmetic hyperbolic 3-manifolds was proved in (Chinburg, Hamilton, Long, R, Duke, 2008).

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Open Question: *Can any of the manifolds provided by (2) of this Theorem be shown to be isospectral?*

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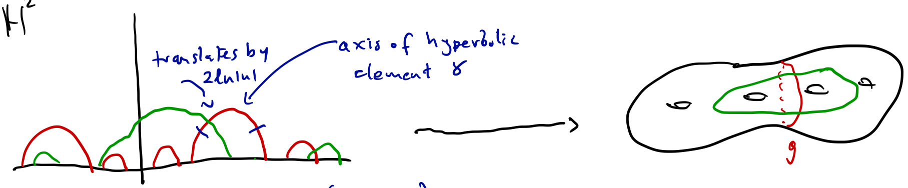
Example: Hyperbolic 2-manifolds



γ is conjugate to $\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$, $|u| > 1$

length of $g = 2\ln|u|$

$\{$lengths$\}$ $\longleftrightarrow$ $\{$eigenvalues of hyperbolic elements$\}$
semisimple

Suppose $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ are arithmetic hyperbolic 2-manifolds (ie Γ_1 and Γ_2 are torsion-free arithmetic subgroups of $\mathrm{SL}(2, \mathbb{R})$.

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Thus if $\gamma_1 \in \Gamma_1$ has trace $u_1 + u_1^{-1}$ and $\gamma_2 \in \Gamma_2$ has trace $u_2 + u_2^{-1}$, then:

$u_1^n + u_1^{-n} = u_2^m + u_2^{-m}$ and are algebraic integers by arithmeticity.

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1. $k(u)$ is a quadratic extension of k.

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Put another way: This condition means the subgroups generated by the eigenvalues have nontrivial intersection.

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Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be semi-simple algebraic groups defined over a field F of characteristic 0.

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Prasad and Rapinchuk: Length Commensurable implies Weakly Commensurable.

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But for $m \geq 3$, Γ^m has infinite index in Γ.

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e.g. In the Sample Theorem (2): *If* $n = 1 \pmod 4$ *then there are arithmetic hyperbolic* n*-manifolds that have weakly commensurable but not commensurable fundamental groups.*

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This in turn is closely related to the classical problem in algebra of identifying division algebras by their maximal subfields.

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Open question: *If Γ_1 is a finitely generated Zariski dense discrete subgroup of $\mathbb{G}_1(F)$ which is weakly commensurable to a finitely generated Zariski dense subgroup of $\mathbb{G}_2(F)$. Is Γ_2 discrete?*

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Caution: Recent work of Bader-Fisher-Miller-Stover implies these more general totally geodesic spectra only make sense (ie if non-empty are infinite) in the arithmetic setting!

Thank You

Happy 75th Birthday Gopal