Zero mean curvature surfaces in Euclidean and Lorentz-Minkowski 3-space III

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Maximal surfaces in $\mathbb{L}^3$

$\mathbb{L}^3$ the Lorentz-Minkowski 3-space. $\langle\ ,\ \rangle := dx_1^2 + dx_2^2 - dx_3^2$.

- $f : M \to \mathbb{L}^3$ is spacelike $\iff \langle df, df \rangle$ is positive definite.
- f is a (spacelike) maximal surface $\iff H \equiv 0$.

Theorem (Weierstrass-type representation (O. Kobayashi, 1983))

M a Riemann surface,
(g, η) a pair of meromorphic function and holomorphic 1-form on M such that $0 < \left(1 - |g|^2\right)^2 \eta\bar{\eta} < \infty$ on M.

$$\Longrightarrow \qquad f := \operatorname{Re} \int \left(1 + g^2,\ i\left(1 - g^2\right),\ 2g\right) \eta$$

gives a maximal surface in $\mathbb{L}^3$.

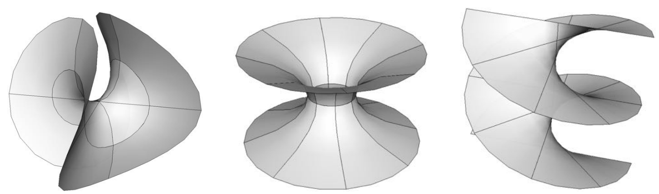
(g, η) the Weierstrass data, g the Gauss map.

Remark (Calabi, 1970 / Cheng-Yau, 1976)

The only complete maximal surface is a spacelike plane.

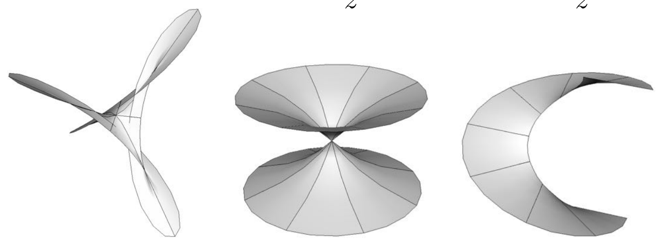
Examples: Enneper, catenoid, helicoid ($g = z$)

Min. in $\mathbb{R}^3$



$M = \mathbb{C}$	$M = \mathbb{C} \setminus \{0\}$	$M = \mathbb{C} \setminus \{0\}$
$\eta = dz$	$\eta = \dfrac{dz}{z^2}$	$\eta = i\dfrac{dz}{z^2}$

Max. in $\mathbb{L}^3$



Singularities

$f : M \to \mathbb{L}^3$ a maximal surface with Weierstrass data (g, η).
$f^* : M \to \mathbb{L}^3$ maximal surface with Weierstrass data $(g, i\eta)$.

f^* is called the conjugate surface of f.

Fact

- f has cuspidal edge at $p \in M \iff f^*$ has cuspidal edge at $p \in M$.
- f has swallowtail at $p \in M \iff f^*$ has cuspidal cross cap at $p \in M$.
- f has cuspidal cross cap at $p \in M \iff f^*$ has swallowtail at $p \in M$.
- f has cone-like sing. at $p \in M \iff f^*$ has fold sing. at $p \in M$.
- f has fold sing. at $p \in M \iff f^*$ has cone-like sing. at $p \in M$.

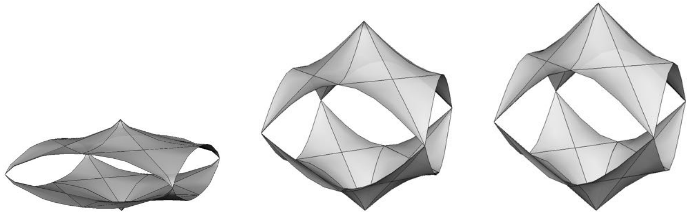
Examples: Schwarz-type maximal surfaces

$M_a := \{(z,w) \in (\mathbb{C}\cup\{\infty\})^2\,;\, w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\},\ (0 < a < 1)$

Schwarz P

$g = z$

$\eta = \dfrac{dz}{w}$

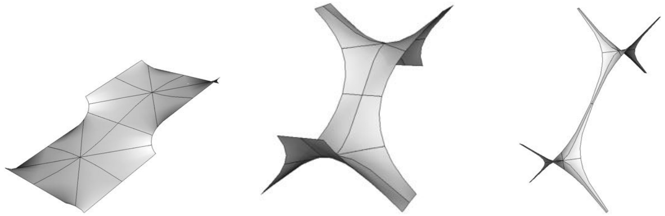


$a = 0.1$ $\quad a = (\sqrt{3}-1)/\sqrt{2}$ $\quad a = 0.9$

Schwarz D

$g = z$

$\eta = i\dfrac{dz}{w}$

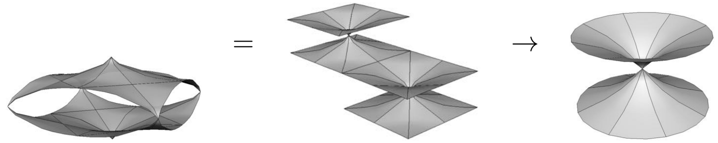


$a = 0.1$ $\quad a = (\sqrt{3}-1)/\sqrt{2}$ $\quad a = 0.9$

Limits of Schwarz-type surfaces: $a \to 0$

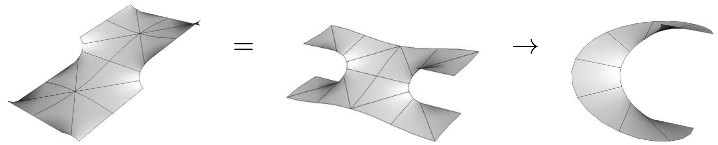
$M_a := \{(z,w) \in (\mathbb{C}\cup\{\infty\})^2\,;\, w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\},\ (0 < a < 1)$

Schwarz P



$a = 0.1$ $=$ $a = 0.1$ $\to$ catenoid

Schwarz D

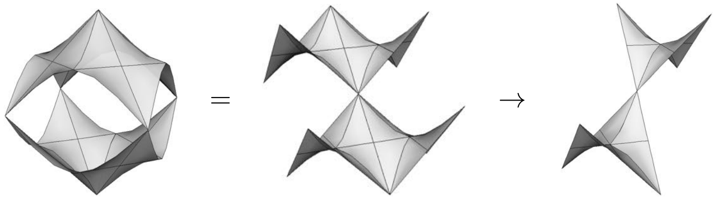


$a = 0.1$ $=$ $a = 0.1$ $\to$ helicoid

Limits of Schwarz-type surfaces: $a \to 1$

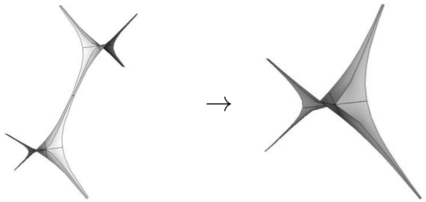
$M_a := \{(z,w) \in (\mathbb{C}\cup\{\infty\})^2\,;\, w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\},\ (0 < a < 1)$

Schwarz P



$a = 0.9$ $=$ $a = 0.9$ $\to$ Scherk 1

Schwarz D

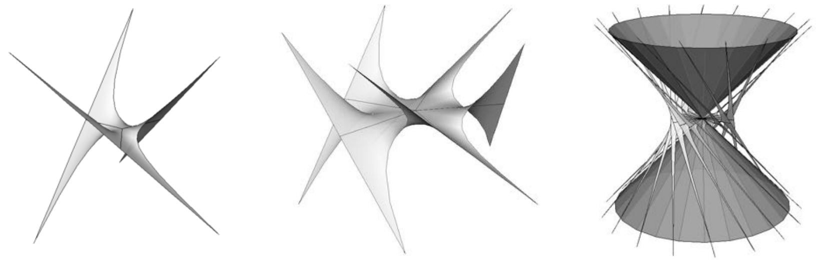


$a = 0.9$ $\to$ Scherk 2

Jorge-Meeks type maximal surfaces

$M = (\mathbb{C}\cup\{\infty\}) \setminus \{z \in \mathbb{C}\,;\, z^n = 1\}\ (n \geq 2)$,

$$g = z^{n-1}, \qquad \eta = \frac{i}{(z^n - 1)^2}dz.$$



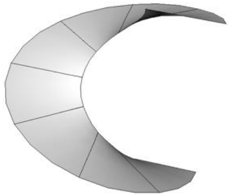
$n = 3$ $\quad n = 5$ $\quad n = 17$

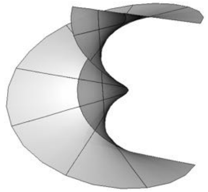
Remark

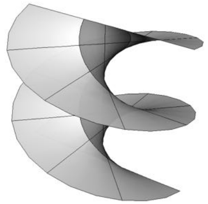
$S(f) = \{z \in M\,;\, |z| = 1\}$ consists of (non-deg.) fold singularities.

Extensions of maximal surfaces with fold singularities

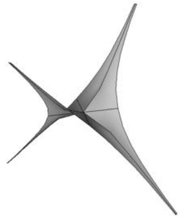
helicoid

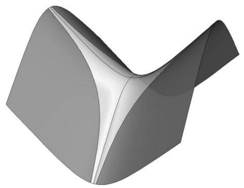

max. surf.


extension


more extension

Scherk


max. surf.


extension

Timelike minimal surfaces, zero mean curvature surfaces

- $f : M \to \mathbb{L}^3$ is a timelike surface $\Longleftrightarrow$ $\langle df, df \rangle$ is Lorentzian metric.
- f is a (timelike) minimal surface $\Longleftrightarrow$ $H \equiv 0$.

Remark

Graph of a function $t = \varphi(x, y)$ in $\mathbb{L}^3$ is a spacelike maximal surface (resp. timelike minimal surface) $\Longleftrightarrow$ φ satisfies

$$(1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0 \qquad (\bigstar)$$

and $1 - \varphi_x^2 - \varphi_y^2 > 0$ (resp. $1 - \varphi_x^2 - \varphi_y^2 < 0$).

Definition

($\bigstar$) is called the zero mean curvature equation and a graph $t = \varphi(x, y)$ satisfying ($\bigstar$) is called a zero mean curvature surface (in $\mathbb{L}^3$).

Extension of max. surf. to zero mean curvature surf.

Definition

Regular curve $\sigma : I (\subset \mathbb{R}) \to \mathbb{L}^3$ is called null curve if $\sigma'(p)$ is lightlike ($\forall p \in I$). Null curve σ is said to be non-degenerate if $\sigma'(p)$ and $\sigma''(p)$ are linearly independent ($\forall p \in I$).

Theorem (Gu, 1985 / Klyachin, 2003, cf. [FKKRSUYY])

$f : M \to \mathbb{L}^3$ a maximal surface with fold singularities,
$\gamma(t)$ $(t \in I)$: a set of fold sing. of f.
$\Longrightarrow$ $\hat{\gamma}(t) := f \circ \gamma(t)$ is non-degenerate null curve, and

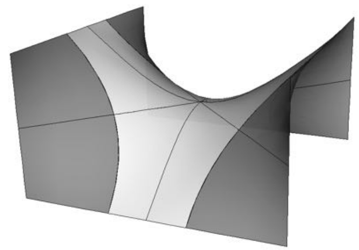
$$\tilde{f}(u, v) := \frac{1}{2}\left(\hat{\gamma}(u + v) + \hat{\gamma}(u - v)\right)$$

is real analytically connected to the image of f along γ as a timelike minimal surface.

Analytic extensions of Schwarz D-type maximal surfaces


maximal surface

$\to$


extension

By the analytic extensions of Schwarz D-type maximal surfaces, we have:

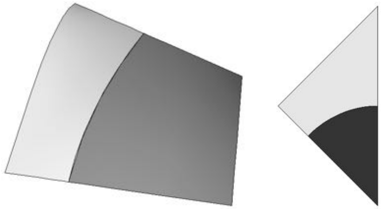
Theorem (FRUYY, 2014)

$\forall a \in (0, 1)$, $\exists \Sigma_a$: oriented closed 2-mfd of genus 3, Γ_a: 3-dim lattice,

$$\exists f_a : \Sigma_a \to \mathbb{L}^3/\Gamma_a \qquad \text{zero mean curvature embedding}$$

Idea of the proof of the embeddedness

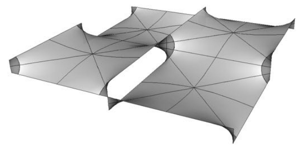
$$D^{\max} := \{(z,w) \in M_a \ ; \ |z| < 1, \ 0 \le \arg z \le \pi/4\},$$
$$D^{\min} := \{(u,v) \in \mathbb{R}^2 \ ; \ 0 \le u \le \pi/4, \ 0 < v \le \pi/2\}.$$



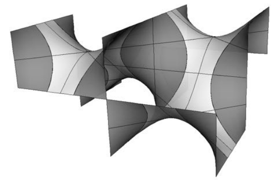
$f(D^{\max}) \cup \tilde{f}(D^{\min})$

- First we show the fundamental piece $f(D^{\max}) \cup \tilde{f}(D^{\min})$ is embedded and contained some vertical prism over a isosceles right triangle.
- After a reflection w.r.t. any boundary of $f(D^{\max}) \cup \tilde{f}(D^{\min})$, the original piece and its duplicate are not intersect each other.

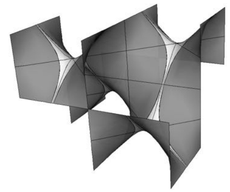
Schwarz D-type zero mean curvature embeddings



$a = 0.1$

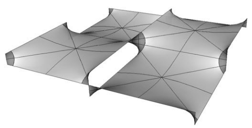


$a = (\sqrt{3}-1)/\sqrt{2}$



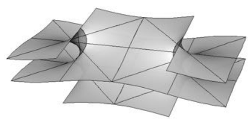
$a = 0.9$

Limits



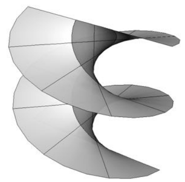
$a = 0.1$

$=$

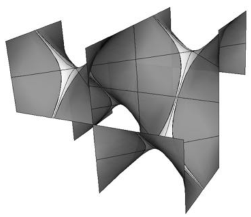


$a = 0.1$

$\overrightarrow{a \to 0}$

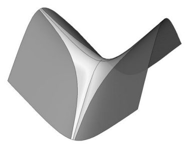


helicoid



$a = 0.9$

$\overrightarrow{a \to 1}$



Scherk

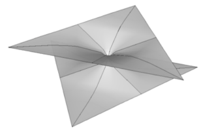
Other examples: Schwarz H-type surface

$$M_a := \left\{(z,w) \in (\mathbb{C} \cup \{\infty\})^2 \, ; \, w^2 = z^7 + (a^3 + a^{-3})z^4 + z\right\}, \ (0 < a < 1)$$

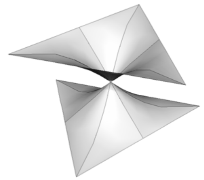
Schwarz H

$g = z$

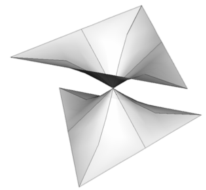
$\eta = \dfrac{dz}{w}$



$a = 0.1$



$a = 0.5$

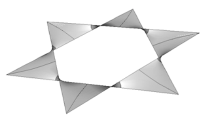


$a = 0.9$

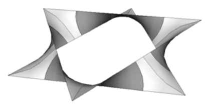
Schwarz HC

$g = z$

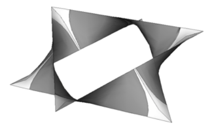
$\eta = i\dfrac{dz}{w}$



$a = 0.1$

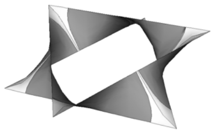


$a = 0.5$



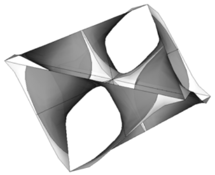
$a = 0.9$

Limit for Schwarz HC-type surface ($a \to 1$)



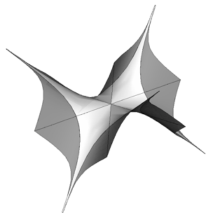
$a = 0.9$

$=$



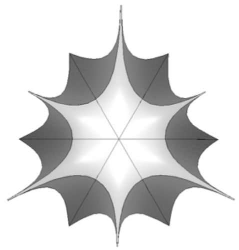
$a = 0.9$

$\overrightarrow{a \to 1}$

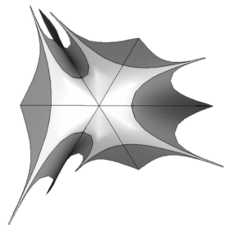


Karcher

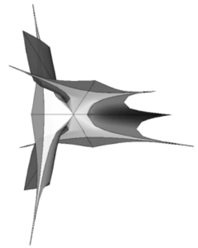
Other examples: Karcher-type, JM-type surfaces



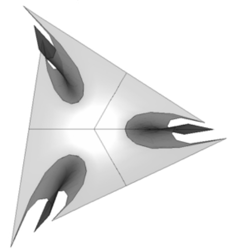
$\varphi = 1$



$\varphi = 0.5$



$\varphi = 0.2$

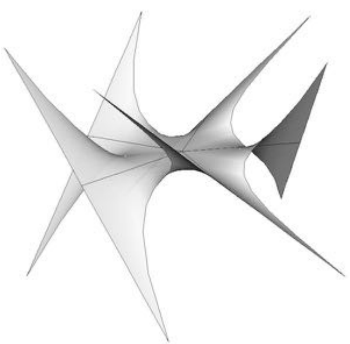


$\varphi = 0$

Jorge-Meeks-type surfaces

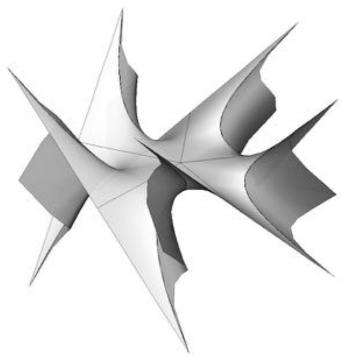
Theorem (FKKRUY, 2017)

For any $n \geq 2$, the analytic extension of Jorge-Meeks type n-noids are properly embedded ZMC surfaces.



maxface

$\longrightarrow$ ($n = 5$)



extension

Outline of proof

$$M = (\mathbb{C} \cup \{\infty\}) \setminus \{z \in \mathbb{C}\,;\, z^n = 1\} \ (n \geq 2),\ g = z^{n-1},\ \eta = \frac{i}{(z^n - 1)^2} dz.$$

$$f = \mathrm{Re} \int \left(1 + g^2,\, i\left(1 - g^2\right),\, -2g\right) \eta = (x_1,\, x_2,\, x_3)\,,$$

$$x_1 = -\frac{(r^{2n-1} + r)\sin\theta + (r^{n+1} + r^{n-1})\sin(n-1)\theta}{n(r^{2n} - 2r^n\cos n\theta + 1)}$$

$$+ \frac{n-1}{n^2}\sum_{j=1}^{n-1} \log\left(r^2 - 2r\cos\left(\theta - \frac{2\pi j}{n}\right) + 1\right)\sin\frac{2\pi j}{n},$$

$$x_2 = \frac{-(r^{2n-1} + r)\cos\theta + (r^{n+1} + r^{n-1})\cos(n-1)\theta}{n(r^{2n} - 2r^n\cos n\theta + 1)}$$

$$+ \frac{n-1}{n^2}\sum_{j=0}^{n-1} \log\left(r^2 - 2r\cos\left(\theta - \frac{2\pi j}{n}\right) + 1\right)\cos\frac{2\pi j}{n},$$

$$x_3 = \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n\cos n\theta + 1)}, \quad \text{where} \quad z = re^{i\theta}.$$

Outline of proof

We have $f(r,\theta) = f(1/r,\theta)$. Set $u := \dfrac{r + r^{-1}}{2}$. Then

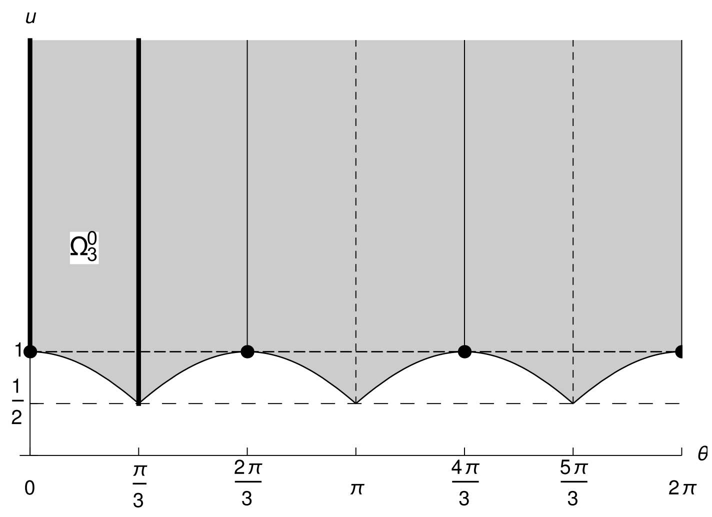
$$x_1 = -\frac{T_{n-1}(u)\sin\theta + u\sin(n-1)\theta}{n(T_n(u) - \cos n\theta)} + \frac{n-1}{n^2}\sum_{j=1}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\sin\frac{2\pi j}{n},$$

$$x_2 = \frac{-T_{n-1}(u)\cos\theta + u\cos(n-1)\theta}{n(T_n(u) - \cos n\theta)} + \frac{n-1}{n^2}\sum_{j=0}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\cos\frac{2\pi j}{n},$$

$$x_3 = \frac{\sin n\theta}{n(T_n(u) - \cos n\theta)},$$

where $T_n(u)$, $T_{n-1}(u)$ denote the first Chebyshev polynomials in the variable u of degree n, $n-1$, respectively.

Domain for the analyric extension of f is defined

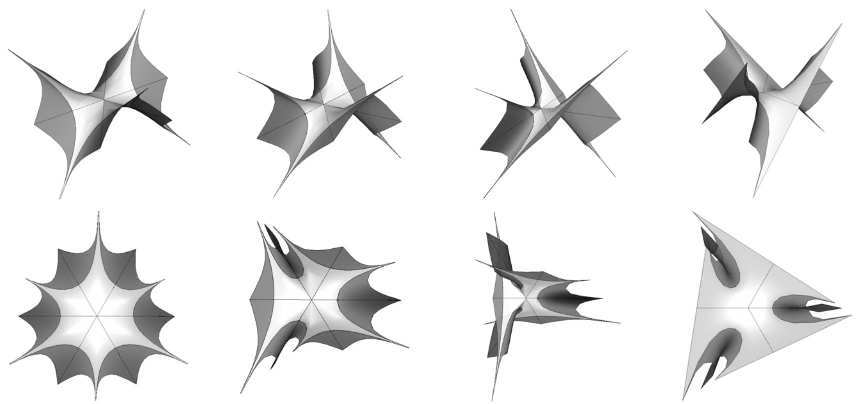


$n = 3$

Karcher type ZMC surfaces with $2n$ ends (non periodic)

$g = z^{n-1}$, $\eta = \dfrac{i}{z^{2n} - 2\cos(n\varphi)z^n + 1}dz$ $\quad (0 < \varphi \le \pi/(2n))$.

$n = 3$:

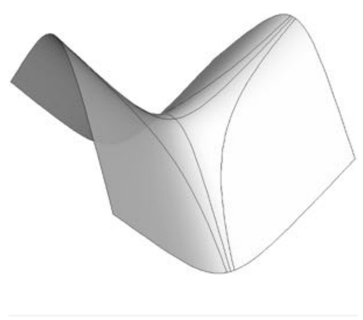


$\varphi = \pi/(2n)$ $\quad \varphi = \pi/(4n)$ $\quad \varphi = \pi/(10n)$ $\quad \varphi \to 0$

ZMC entire graphs

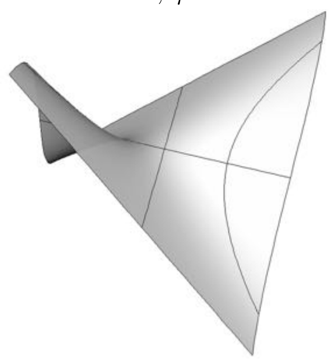
$g = z^{n-1}$, $\eta = \dfrac{i}{z^{2n} - 2\cos(n\varphi)z^n + 1}dz$ $\quad (0 < \varphi \le \pi/(2n))$.

$n = 2$, $\varphi = \pi/(2n)$



$t = \log(\cosh x/\cosh y)$

$n = 2$, $\varphi = 0$



$t = x\tanh y$

These surfaces were first found by O. Kobayashi (1983).

Kobayashi surfaces

Lemma

Let $0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{2n-1}$ be $2n$ real numbers ($n \geq 2$).
We set $M = (\mathbb{C} \cup \{\infty\}) \setminus \{e^{i\alpha_0}, \ldots, e^{i\alpha_{2n-1}}\}$, and

$$g = z^{n-1}, \qquad \eta = i\frac{e^{i(\alpha_0+\cdots+\alpha_{2n-1})/2}}{\prod_{j=0}^{2n-1}(z - e^{i\alpha_j})}dz.$$

$\Longrightarrow$ The maxface $f : M \to \mathbb{L}^3$ with the above W-data is well-defined on M, and the singular set $S(f) = \{z \in M\,;\, |z| = 1\}$ consists of (non-deg.) fold singularities.

Definition

We call a maxface given in this lemma an order n Kobayashi surface (of principal type), and $(\alpha_0, \ldots, \alpha_{2n-1})$ the angle data of f.

Kobayashi surfaces

Theorem (FKKRUY, 2016)

Let $f : M \to \mathbb{L}^3$ be an order n Kobayashi surface with the angle data $(\alpha_0, \ldots, \alpha_{2n-1})$, and $\tilde{f} : M \to \mathbb{L}^3$ its analytic extension.
We set $\alpha_{2n} = 2\pi$.
$\Longrightarrow$

1. If $|\alpha_j - \alpha_{j+1}| < 2\pi/(n-1)$ ($j = 0, \ldots, 2n-1$) hold and that $\alpha_0, \ldots, \alpha_{2n-1}$ are distinct, then $\tilde{f}$ is a proper immersion.
2. If $|\alpha_j - \alpha_{j+1}| < \pi/(n-1)$ ($j = 0, \ldots, 2n-1$) hold and that $\alpha_0, \ldots, \alpha_{2n-1}$ are distinct, then $\tilde{f}$ gives an entire graph.
3. When $n = 2$, $\tilde{f}$ is a properly embedded.

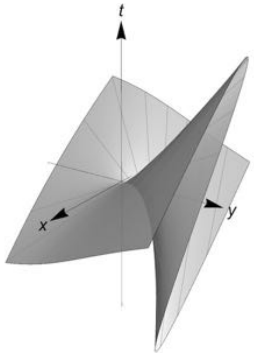
Problem

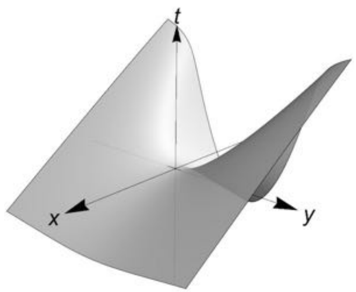
What is the condition for $\tilde{f}$ to be properly embedded?

Examples (n=2)

$$g = z, \qquad \eta = i\frac{e^{i(\alpha_0+\alpha_1+\alpha_2+\alpha_3)/2}}{\prod_{j=0}^{3}(z - e^{i\alpha_j})}dz.$$

$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, 0)$ $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 0, 0, \pi)$





$$\eta = \frac{i}{(z-1)^4}dz \qquad \eta = \frac{-1}{(z-1)^3(z+1)}dz$$

Relationship to fluid mechanics [FKKRSUYY]

Consider a 2-dim flow with velocity vector field $\boldsymbol{v} = (u, v)$, density ρ, pressure p.
Suppose the following conditions for the flow:

(1) barotropic. i.e. p depends only on ρ.
$c := \sqrt{dp/d\rho}$ is called the local speed of sound.

(2) steady. i.e. $\boldsymbol{v}$, ρ, p do not depend on time.

(3) no external forces.

(4) irrotational. i.e. $\mathrm{rot}\boldsymbol{v} (= v_x - u_y) = 0$.

By (2), the equation of continuity is reduced to

$$\mathrm{div}(\rho\boldsymbol{v}) = (\rho u)_x + (\rho v)_y = 0.$$

Hence $\exists\psi = \psi(x, y)$ s.t.

$$\psi_x = -\rho v, \qquad \psi_y = \rho u,$$

which is called the stream function of the flow.

Relationship to fluid mechanics [FKKRSUYY]

The stream function ψsatisfies the following equation:

$$(\rho^2c^2 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (\rho^2c^2 - \psi_x^2)\psi_{yy} = 0.$$

When $\rho c = 1$, this equation coincides with the ZMC equation (★).
Suppose now $\rho c = 1$.
$\Longrightarrow \exists\rho_0$ a positive constant s.t.

$$p = \text{const.} - \rho^{-1},$$

$$\rho = \rho_0|1 - \psi_x^2 - \psi_y^2|^{1/2}, \quad c = 1/\rho = \rho_0^{-1}|1 - \psi_x^2 - \psi_y^2|^{-1/2}.$$

Also, $\boldsymbol{v} = \rho^{-1}(\psi_y, -\psi_x)$.

Lemma

$|\boldsymbol{v}| > c$ (resp. $|\boldsymbol{v}| < c$) $\Longleftrightarrow 1 - \psi_x^2 - \psi_y^2 < 0$ (resp. $1 - \psi_x^2 - \psi_y^2 > 0$).

Relationship to fluid mechanics [FKKRSUYY]

Theorem

$\sigma(t) = (x(t), y(t)) \in \mathbb{R}^2$ a locally convex curve (t an arc-length).
$\Longrightarrow \exists\psi = \psi(x, y)$ s.t. $(x, y, \psi(x, y))$ is a zero mean curvature surface which change type across the non-degenerate null curve $(x(t), y(t), t)$.
i.e. ψ is the stream function of some flow with $\rho c = 1$.
Moreover, the velocity vector field $\boldsymbol{v} = \rho^{-1}(\psi_y, -\psi_x)$ of this flow satisfies:

- $|\boldsymbol{v}| \to \infty$ as (x, y) approaches $\sigma(t)$.
- The flow changes from being subsonic to being supersonic across σ.
- $\sigma''(t)$ points to the supersonic region.

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