Zero mean curvature surfaces in Euclidean and Lorentz-Minkowski 3-space I

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Volume of hypersurfaces

U a compact set in $\mathbb{R}^n$, $u = (u^1, \ldots, u^n) \in U$, $f : U \to \mathbb{R}^{n+1}$ an imm.
$g := \Sigma g_{ij} du^i du^j$ induced metric on U by f (i.e. $g(X,Y) := \langle f_* X, f_* Y\rangle$).

Definition

The volume $\mathrm{Vol}(f)$ of $f : U \to \mathbb{R}^{n+1}$ is defined by

$$\mathrm{Vol}(f) := \int_U dV, \qquad dV := \sqrt{\det(g_{ij})}\, du^1 \wedge \cdots \wedge du^n.$$

dV is called the volume element of f.

C^∞ map $F : I \times U \to \mathbb{R}^{n+1}$ ($I := (-\varepsilon, \varepsilon)$) with the following 3 conditions is called the smooth variation with fixed boundary of $f(U)$ (or simply variation of f).

(1) $f_t := F(t, \cdot) : U \to \mathbb{R}^{n+1}$ is immersion $\forall t \in I$.
(2) $f_0(u) = f(u)$ $(\forall u \in U)$.
(3) $f_t(u) = f(u)$ $(\forall t \in I, \forall u \in \partial U)$.

The first variation of the volume

$f : U \to \mathbb{R}^{n+1}$ an imm., $F : I \times U \to \mathbb{R}^{n+1}$ a variation of f ($f_t = F(t, \cdot)$), ν_t the unit normal v.f. on U along f_t, H_t the mean curvature of f_t, dV_t the volume element of f_t. $\Longrightarrow$

$$\frac{d}{dt}\mathrm{Vol}(f_t) = -n \int_U \left\langle (f_t)_* \frac{\partial}{\partial t}, H_t \nu_t \right\rangle dV_t.$$

As $t = 0$, we have:

The first variation formula for the volume of f

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(f_t) = -n \int_U \left\langle (f_t)_* \left.\frac{\partial}{\partial t}\right|_{t=0}, H\nu \right\rangle dV,$$

where $\nu := \nu_0$, $H := H_0$, $dV := dV_0$.

Theorem

$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(f_t) = 0$ for $\forall f_t$ variation of $f \iff H \equiv 0$.

The second variation of the volume

f minimizes the volume for given fixed boundary
$\Longrightarrow$ $\mathrm{Vol}(f) \le \mathrm{Vol}(f_t)$ $\forall f_t$ variation of f
$\Longrightarrow$ $\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(f_t) = 0$ $\forall f_t$ variation of f
$\iff$ $H \equiv 0$ (prev. Thm)

Thus the volume minimizing hypersurface satisfies $H \equiv 0$.
How about the converse?

Under $\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(f_t) = 0$ (i.e. $H \equiv 0$), compute $\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(f_t)$.

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Vol}(f_t) = -\int_U n \left.\frac{\partial H_t}{\partial t}\right|_{t=0} \left\langle (f_t)_* \left.\frac{\partial}{\partial t}\right|_{t=0}, \nu \right\rangle dV$$

The second variation of the volume

Set $\beta := \left\langle (f_t)_* \frac{\partial}{\partial t}\Big|_{t=0}, \nu \right\rangle$, we have

$$n \frac{\partial H_t}{\partial t}\Big|_{t=0} = \triangle_g \beta + \beta |A|^2,$$

where A is the second f.f. of f, $\triangle_g$ is the Laplacian w.r.t. g.
Therefore we have

The second variation formula for the volume of f

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(f_t) = -\int_U \beta \left(\triangle_g \beta + \beta |A|^2\right) dV = \int_U \left(|\nabla_g \beta|^2 - \beta^2 |A|^2\right) dV$$

where ∇_g is the gradient w.r.t. g.
Remark The variation with $\beta \equiv 0$ is called the trivial variation.

Stability

Definition

Suppose that the first variation of $f : U \to \mathbb{R}^{n+1}$ vanishes for any variation (i.e. $H \equiv 0$). f is stable if the socond variation of f is positive for any non-trivial variation.

Proposition

$f : U \to \mathbb{R}^{n+1}$ an imm. s.t. $H \equiv 0$.
$\nu : U \to S^n \subset \mathbb{R}^{n+1}$ the unit normal v.f. on U along f.
If $\nu(U)$ is contained a hemihypersphere, then f is stable.

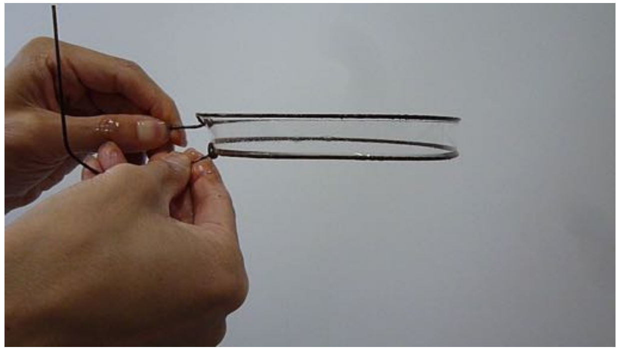
The key equation

$$\triangle_g \nu + |A|^2 \nu + n \nabla_g H = 0$$

Corollary

Graph hypersurface is always stable (in fact volume minimizing).

Stability of the catenoid

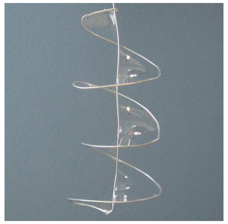


Soap film experiment

Minimal hypersurfaces

An immersion s.t. $H \equiv 0$ is called the minimal hypersurface.

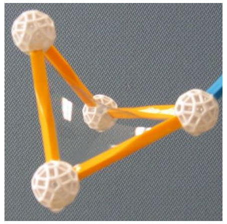
Catenoid



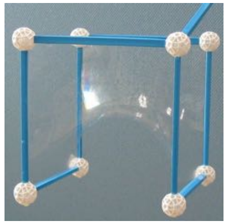
Helicoid

Möbius strip

Enneper (order 3)



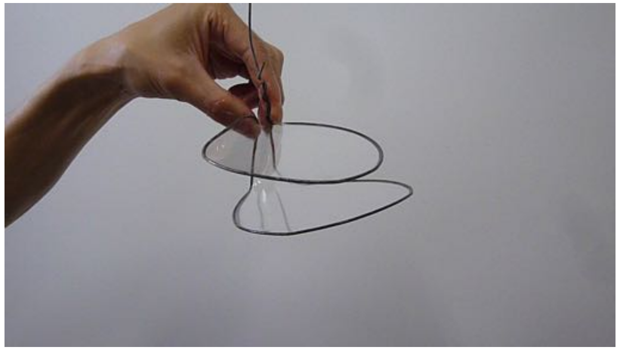
Schwarz D



Meeks superman

Stable minimal surface v.s. area-minimizing surface

Area-minimizing surface is stable minimal surface, but the converse is not true.



Soap film experiment

Graph hypersurface

U a domain in $\mathbb{R}^n$, $u = (u^1, \ldots, u^n) \in U$, $\varphi : U \to \mathbb{R}$.
The mean curvature H of the graph hypersurface φ satisfies

$$nH = \mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right).$$

Definition

$\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) = 0$ is called the minimal hypersurface equation.

Remark

When $n = 2$, set $(u^1, u^2) = (x, y)$. Then the above eqn. is equivalent to

$$\left(1+\varphi_y^2\right)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + \left(1+\varphi_x^2\right)\varphi_{yy} = 0.$$

(J. Lagrange, 1760). This eqn is called the minimal surface equation.

The Bernstein problem

Theorem (S. Bernstein, 1915)

The graph minimal surface φ defined on the entire $\mathbb{R}^2$ must be a plane.

Problem

Is the graph minimal hypersurface φ defined on the entire $\mathbb{R}^n$ a hyperplane?

This problem is affirmatively solved by S. Bernstein for $n = 2$ (1915), E. de Giorgi for $n = 3$ (1961), F. J. Almgren for $n = 4$ (1966), and J. Simons for $n \leq 7$ (1968), respectively, and negatively solved by E. Bombieri, E. de Giorgi, E. Giusti for $n \geq 8$ (1969).
(R. Schoen, L. Simon, S.-T. Yau (1974) gave alternative proof for $n \leq 5$.)

Examples of minimal surface

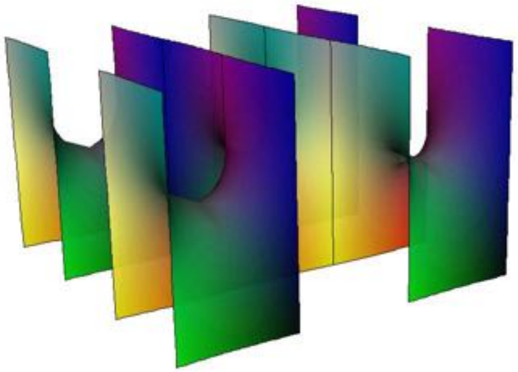
In the following, we assume $n = 2$.

- L. Euler (1744), proved that the catenoid is a minimal surface.
- J. Meusnier (1776), proved that the helicoid is a minimal surface.
- H. Scherk (1835) found new examples.

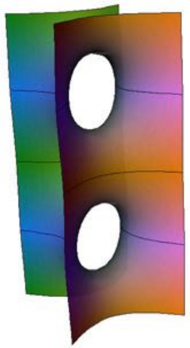
Proposition

- Minimal surface with constant Gaussian curvature must be a plane.
- Minimal surface of revolution must be a plane or a catenoid (O. Bonnet, 1860).
- Ruled minimal surface must be a plane or a helicoid (E. Catalan, 1842).
- Translation minimal surface must be a plane or doubly periodic Scherk (H. Scherk, 1835).

Scherk' minimal surfaces (before Weierstrass)



doubly periodic



singly periodic

Weierstrass representaion

$f : U \ni (x, y) \mapsto (f_1(x, y),\ f_2(x, y),\ f_3(x, y)) \in \mathbb{R}^3$ an imm.,
(x, y) the isothermal coordinate of f (i.e. $|f_x| = |f_y|$, $\langle f_x, f_y\rangle = 0$).
ν the unit normal v.f. on U along f. We have

$$f_{xx} + f_{yy} = 2H\nu.$$

Thus

$$f \text{ is minimal} \iff \text{each } f_j\ (j = 1, 2, 3) \text{ is harmonic fct.}$$

Remark

This implies that there does not exist a compact minimal surface without boundary.

Set $z = x + iy$, $\varphi_j = \dfrac{\partial f_j}{\partial z} dz$ $(j = 1, 2, 3)$, then

$$f \text{ is minimal} \iff \text{each } \varphi_j\ (j = 1, 2, 3) \text{ is holom. 1-form.}$$

Weierstrass representaion

φ_j satisfy

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0, \qquad |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 > 0.$$

Since $f_z dz = (\varphi_1,\ \varphi_2,\ \varphi_3)$,

$$f = 2\mathrm{Re} \int (\varphi_1,\ \varphi_2,\ \varphi_3).$$

Assume φ_3 is not identically zero, and set $g = -(\varphi_1 + i\varphi_2)/\varphi_3$, $\eta = \varphi_1 - i\varphi_2$ then g is meromorphic fct. on U and η is holomorphic 1-form on U.

$$f = \mathrm{Re} \int \left(\left(1 - g^2\right),\ i\left(1 + g^2\right),\ 2g \right) \eta.$$

This is the Weierstrass represeatation (K. Weierstrass 1866, R. Osserman 1964).

The first and second fundamental forms, the Gauss map

The first f.f. ds^2 of f, and the second f.f. A of f are given

$$ds^2 = \left(1 + |g|^2\right)^2 |\eta|^2, \qquad A = -2\,\mathrm{Re}\,Q, \qquad Q := \eta dg$$

respectively.
$\nu : U \to S^2$ the unit normal v.f. on U along f, $\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ the stereographic projection. Then

$$g = \sigma \circ \nu.$$

Hence we call g the Gauss map of f.

The period problem

M a Riemann surface. g a merom. fct on M, η a holom. 1-form on M s.t. $(1+|g|^2)^2|\eta|^2$ gives a complete Riemannian metric of finite total curvature on M. If M is not simply connected, then

$$f = \mathrm{Re}\int \left((1-g^2),\, i\,(1+g^2),\, 2g\right)\eta.$$

might not be well-defined on M. We set

$$\mathrm{Per}(f) := \left\{\mathrm{Re}\int_\gamma (\varphi_1,\, \varphi_2,\, \varphi_3) \;:\; \gamma \in H_1(M,\mathbb{Z})\right\}$$

Period problem

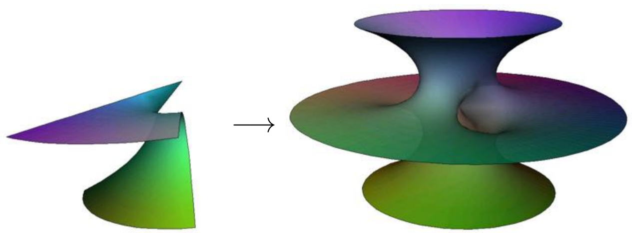
1. $\mathrm{Per}(f) = \{0\} \implies f : M \to \mathbb{R}^3$ is well-defined on M.
2. $\exists v \in \mathbb{R}^3 \setminus \{0\}$ such that $\mathrm{Per}(f) \subset \Lambda_1 := \{nv \;:\; n \in \mathbb{Z}\}$
   $\implies f$ is singly periodic (SPMS).
   f is well-defined in $\mathbb{R}^3/\Lambda_1 = \mathbb{R}^2 \times S^1$.
3. $\exists v_1, v_2 \in \mathbb{R}^3$ (lin. indep.) such that
   $\mathrm{Per}(f) \subset \Lambda_2 := \{\sum_{j=1}^2 n_j v_j \;:\; n_j \in \mathbb{Z}\}$
   $\implies f$ is doubly periodic (DPMS).
   f is well-defined in $\mathbb{R}^3/\Lambda_2 = T^2 \times \mathbb{R}$.
4. $\exists v_1, v_2, v_3 \in \mathbb{R}^3$ (lin. indep.) such that
   $\mathrm{Per}(f) \subset \Lambda_3 := \{\sum_{j=1}^3 n_j v_j \;:\; n_j \in \mathbb{Z}\}$
   $\implies f$ is triply periodic (TPMS).
   f is well-defined in $\mathbb{R}^3/\Lambda_3 = T^3$.

Symmetry

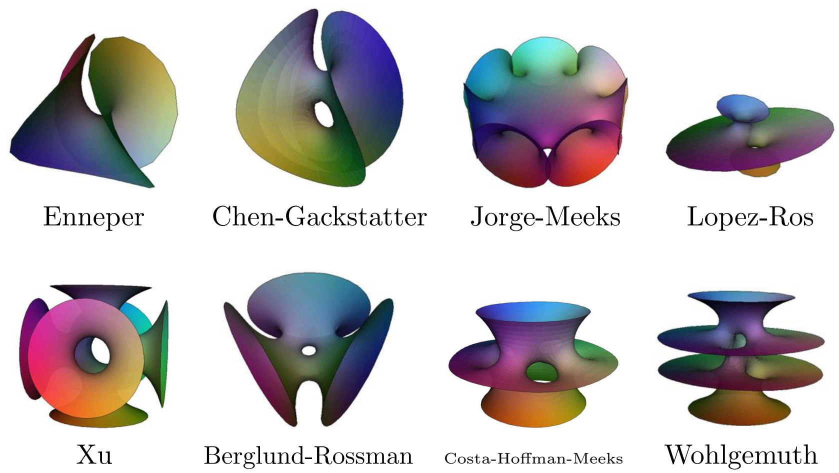
$f : M \to \mathbb{R}^3$ a minimal surface, $c : I \to M$ a geodesic on M. $Q = \eta dg$ the Hopf differential of f (Q is a holomorphic 2-differential on M). Appling the Schwarz' reflection principle, we have the following:

Proposition

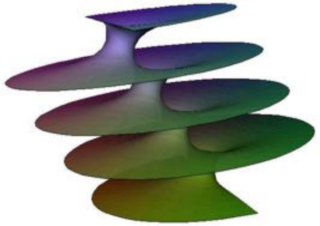
- If c satisfies $Q(c', c') \in i\mathbb{R}$, then $f \circ c(I)$ is a straight line. Moreover, $f(M)$ is symmetric w.r.t. this line.
- If c satisfies $Q(c', c') \in \mathbb{R}$, then $f \circ c(I)$ is contained some plane in $\mathbb{R}^3$. Moreover, $f(M)$ is symmetric w.r.t. this plane.

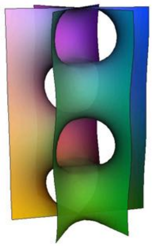


Example (minimal surfaces of finite total curvature)



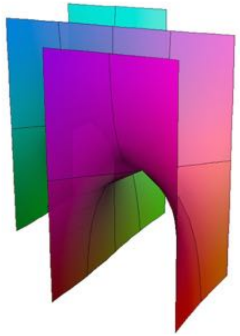
Example (Singly periodic minimal surfaces)

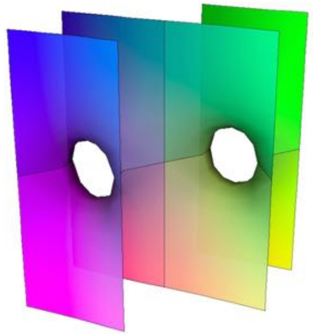

Riemann

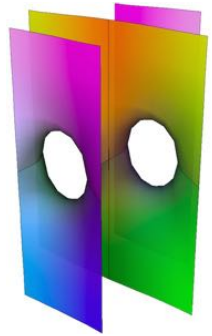

Karcher


Hoffman-Karcher-Wei

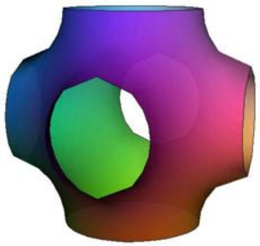
Example (Doubly periodic minimal surfaces)

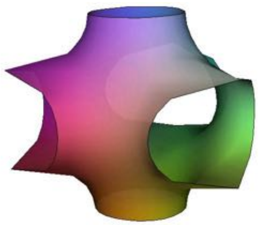

Scherk

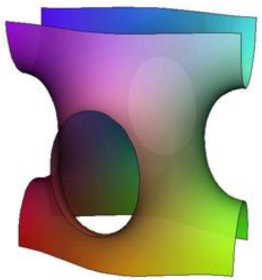

Karcher-Meeks-Rosenberg


Rodríguez

Example (Triply periodic minimal surfaces)

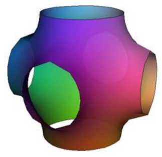

Schwarz P

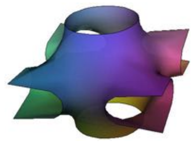

Schwarz H

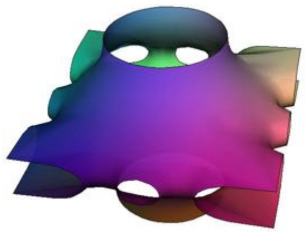

Schwarz CLP

New triply periodic minimal surfaces

As superpositioning two of known examples.


Schwarz (1860)


A. Schoen (1970)


Fujimori-Weber (2009)

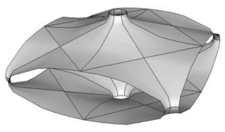
Schwarz P, Schwarz D

$$M_a := \left\{(z,w) \in (\mathbb{C}\cup\{\infty\})^2\,;\, w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\right\},\ (0<a<1)$$

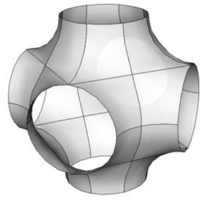
Schwarz P

$g = z$

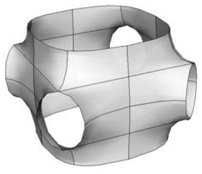
$\eta = \dfrac{dz}{w}$



$a = 0.1$



$a = (\sqrt{3}-1)/\sqrt{2}$

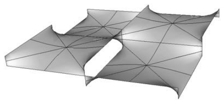


$a = 0.9$

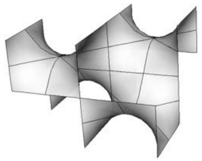
Schwarz D

$g = z$

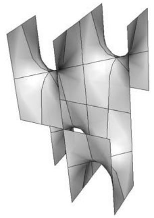
$\eta = i\dfrac{dz}{w}$



$a = 0.1$



$a = (\sqrt{3}-1)/\sqrt{2}$

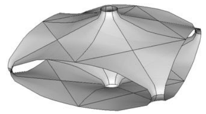


$a = 0.9$

Limit of Schwarz P, Schwarz D: $a \to 0$

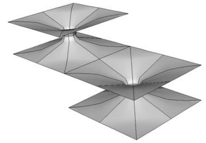
$$M_a := \left\{(z,w) \in (\mathbb{C}\cup\{\infty\})^2\,;\, w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\right\},\ (0<a<1)$$

Schwarz P



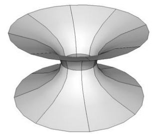
$a = 0.1$

$=$



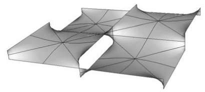
$a = 0.1$

$\to$



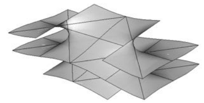
catenoid

Schwarz D



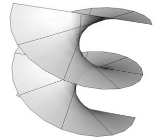
$a = 0.1$

$=$



$a = 0.1$

$\to$

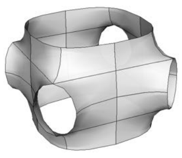


helicoid

Limit of Schwarz P, Schwarz D: $a \to 1$

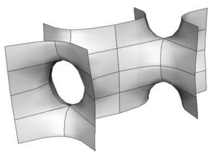
$$M_a := \left\{(z,w) \in (\mathbb{C}\cup\{\infty\})^2\,;\, w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\right\},\ (0<a<1)$$

Schwarz P



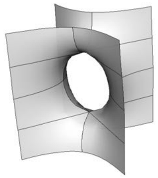
$a = 0.9$

$=$



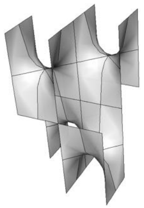
$a = 0.9$

$\to$



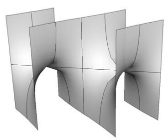
Scherk S

Schwarz D



$a = 0.9$

$\to$



Scherk D

Minimal surfaces of finite total curvature

In the following, $f: M \to \mathbb{R}^3$ a complete minimal surface of finite total curvature.

Theorem (Huber, 1957)

$f: M \to \mathbb{R}^3$ a complete minimal surface of f.t.c.

$\Longrightarrow$

$$^\exists\overline{M}_\gamma : \text{cpt Riemann surface of genus } \gamma, \quad ^\exists p_1, \dots, p_n \in \overline{M}_\gamma$$

such that

$$M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \qquad \text{(biholomorphic)}.$$

Moreover g, η extend meromorphically on $\overline{M}_\gamma$. $p_1, \dots, p_n$ correspond to the ends of f.

We set

$$M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \quad (\gamma = 0, 1, 2, \dots, \quad n = 1, 2, \dots).$$

The Osserman inequality

Theorem (Osserman, 1964)

Complete minimal surface of f.t.c. $f : M \to \mathbb{R}^3$ satisfies:

$$\frac{1}{2\pi}\int_M K dA \leq \chi(M) - n = \chi(\overline{M}) - 2n = 2(1-\gamma-n).$$

Moreover, "=" $\iff$ each end is embedded (Jorge-Meeks, 1983).

Remark. If M is a cpt Riemannian mfd, then

$$\frac{1}{2\pi}\int_M K dA = \chi(M) \quad \text{(Gauss-Bonnet, 1848)}.$$

If M is a non-cpt, complete Riemannian mfd of f.t.c, then

$$\frac{1}{2\pi}\int_M K dA \leq \chi(M) \quad \text{(Cohn-Vossen, 1935)}.$$

The Osserman inequality

Remark. Since The Gaussian curvature K of a minimal surface is non positive, The total absolute curvature $\tau(M)$ of M is

$$\tau(M) = \int_M (-K) dA.$$

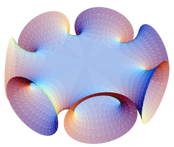
Remark. $(-K)ds^2$ gives the induced metric of the Fubini-Study metric on $\mathbb{C} \cup \{\infty\}$ by f, i.e.

$$(-K)ds^2 = \frac{4|dg|^2}{(1+|g|^2)^2}.$$

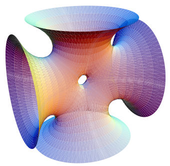
Therefore $\tau(M) = 4\pi \deg(g)$, and Osserman inequality can be written as

$$\deg(g) \geq \gamma + n - 1.$$

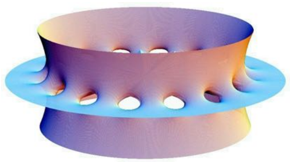
Examples which satisfy the equality ($n \geq 3$)



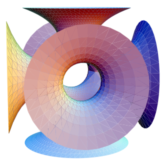
$(\gamma, n) = (0, 7)$



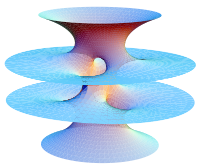
$(\gamma, n) = (1, 4)$



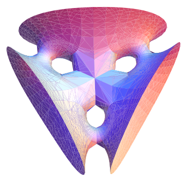
$(\gamma, n) = (14, 3)$



$(\gamma, n) = (0, 6)$



$(\gamma, n) = (2, 4)$

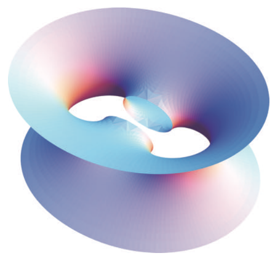


$(\gamma, n) = (3, 3)$

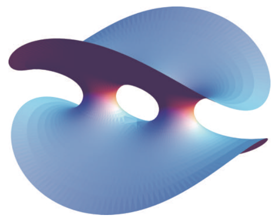
The case $n \leq 2$

There are a few complete minimal surface of f.t.c.

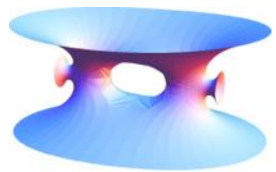
- $n = 1 \implies$ plane.
- $n = 2 \implies$ the catenoid (R. Schoen, 1983).



genus 1 catenoid



genus 1 wevy-catenoid



genus 1 fournoid

Nonorientable minimal surfaces

M' a nonorientable surface.

$f' : M' \to \mathbb{R}^3$ a nonorientable minimal surfaces $:\Longleftrightarrow$ M' the mean curvature w.r.t. any unit normal vanishes identically.

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
Take a double cover $\pi : M \to M'$ (M a orientable surface), then
$f := f' \circ \pi : M \to \mathbb{R}^3$ is an orientable minimal surface.
$\longrightarrow$ one can apply the Weierstrass rep.
(g, η): the Weierstrass data of f.
$I : M \to M$ the anti-holomorphic deck transf w.r.t. π. Then,

$$f \circ I(p) = f(p) \qquad (\forall p \in M).$$

Lemma

$$f \circ I = f \quad \Longleftrightarrow \quad g \circ I = -\frac{1}{\bar{g}} \quad \text{and} \quad I^*\eta = \overline{g^2\eta}.$$

The Gauss map

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
$g : M \to \mathbb{C} \cup \{\infty\}$ the Gauss map of $f = f' \circ \pi$.
$I : M \to M$ the anti-holomorphic deck transf w.r.t. π.
Then, $\exists 1\ \hat{g} : M' \to \mathbb{RP}^2$ s.t. the following diagram is commutative.

$$\begin{array}{ccc} M & \xrightarrow{g} & \mathbb{C} \cup \{\infty\} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_0} \\ M' & \xrightarrow{\hat{g}} & \mathbb{RP}^2 \end{array}$$

where $p_0 : \mathbb{C} \cup \{\infty\} \to \mathbb{RP}^2 = (\mathbb{C} \cup \{\infty\})/\langle I_0 \rangle$ is the natural projection,
$I_0(z) := -1/\bar{z}$.

Definition

The above $\hat{g}$ is called the Gauss map of a nonorientable minimal surface $f' : M' \to \mathbb{R}^3$.

Remark. Since $\deg(\pi) = \deg(p_0) = 2$, can define $\deg \hat{g}$: $\deg \hat{g} = \deg g$.

$\deg \hat{g}$

Theorem (Meeks, 1981)

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
$\hat{g}$ the Gauss map of f'. Then,

$$\deg \hat{g} \equiv \chi(\overline{M'}) \pmod 2.$$

Corollary (Meeks, 1981)

$f' : M' \to \mathbb{R}^3$ a complete nonorientable minimal surface of f.t.c.
$\hat{g}$ the Gauss map of f'. Then,

$$\deg \hat{g} \geq 3.$$

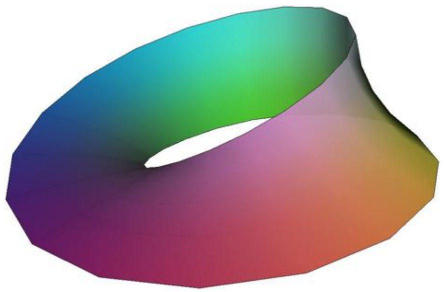
(Proof) Let $\pi : M \to M'$ the double cover.

- $\deg \hat{g} = 1 \Rightarrow M = S^2 - \{p, q\}$ (embedded ends) $\Rightarrow M$: catenoid.
- $\deg \hat{g} = 2 \Rightarrow M = T^2 - \{p, q\}$ (embedded ends) $\Rightarrow \nexists$. □

Example: Möbius strip ($\deg \hat{g} = 3$)

$M = \mathbb{C} - \{0\}$, $I(z) = -1/\bar{z}$, $M' = M/\langle I \rangle = \mathbb{RP}^2 - \{\pi(0)\}$,

$$g = z^2 \frac{z+1}{z-1}, \qquad \eta = i\frac{(z-1)^2}{z^4}dz.$$



Theorem (Meeks, 1981)

This is the unique example with $\deg \hat{g} = 3$.

Remark. There exists a Möbius strip with $\deg \hat{g}$ is odd (≥ 5).

Example: Klein bottle$-\{1 \text{ pt}\}$ ($\deg \hat{g} = 4$)

$$M = \left\{ (z,w) \in (\mathbb{C} \cup \{\infty\})^2 \,;\, w^2 = \frac{rz-1}{z+r} \right\} - \{(0,0),(\infty,\infty)\},$$

$$(r \in \mathbb{R}_+ - \{1\}),\ I(z,w) = \left(-\frac{1}{\bar{z}}, \frac{1}{\bar{w}} \right),\ g = w\frac{z+1}{z-1},\ \eta = i\frac{(z-1)^2}{z^2 w}dz.$$

Theorem (López, 1996)

This is the unique example with $\deg \hat{g} = 4$.