Optimal structures for transient growth

Ritabrata, Arjun Sharma (Cornell)
Rama Govindarajan

ICTS | International Centre for Theoretical Sciences
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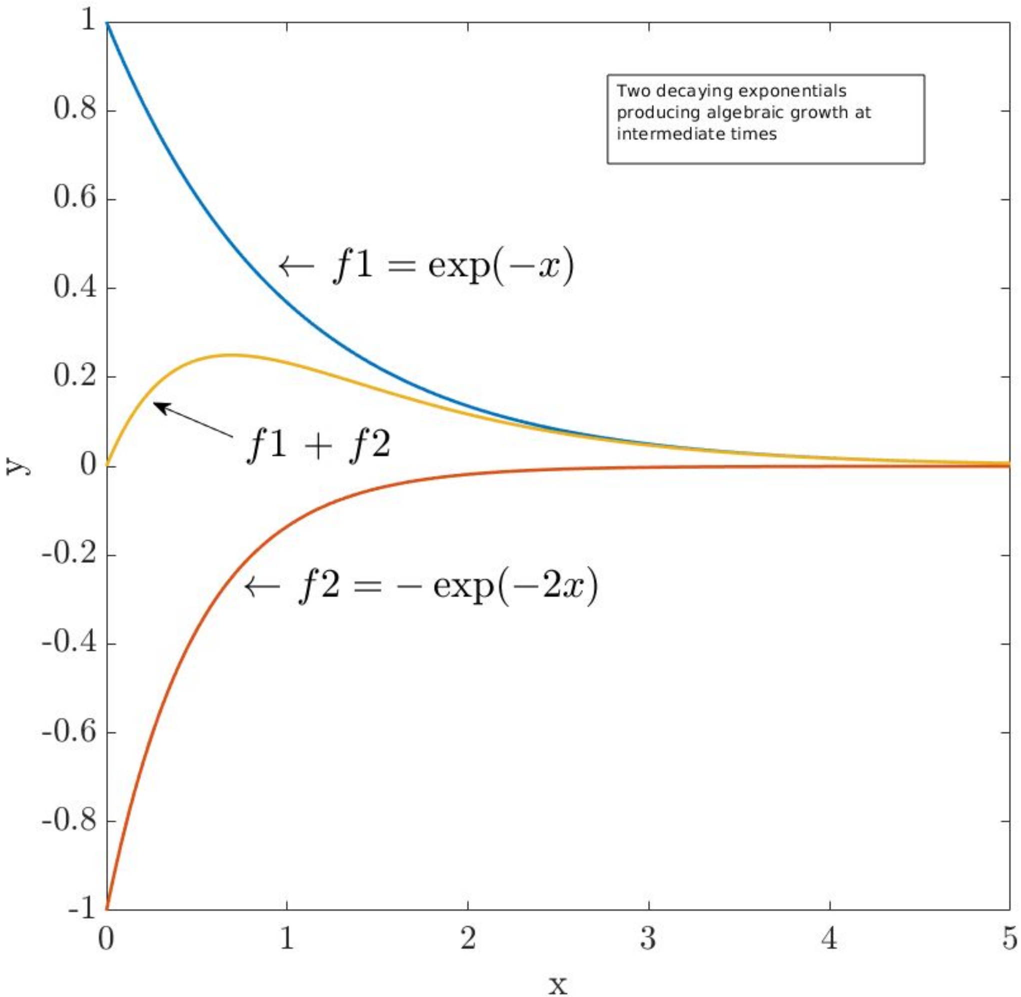
$$dX/dt = Mx$$

Non-normality condition

$$MM^T \neq M^T M$$



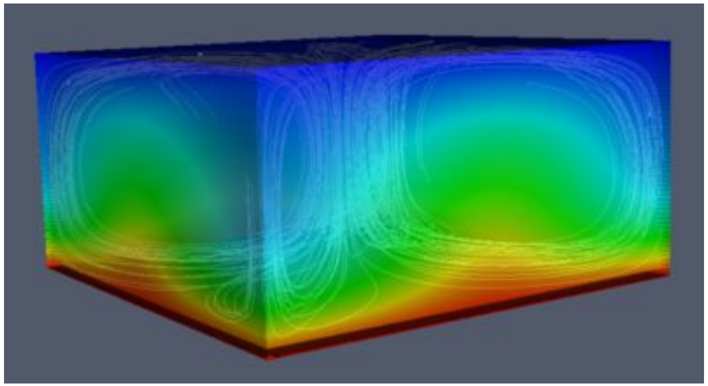
"Stability without eigenvalues"
(non-modal)

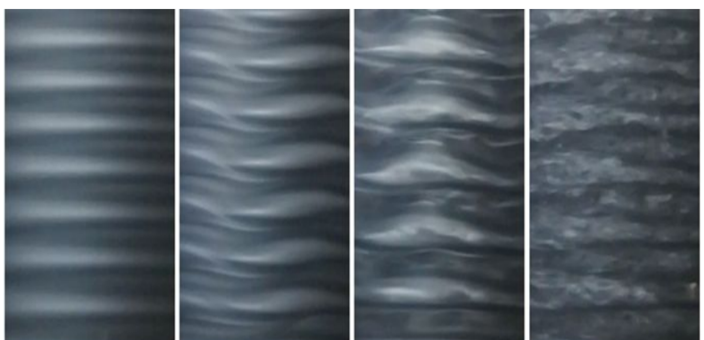
and

Transient growth

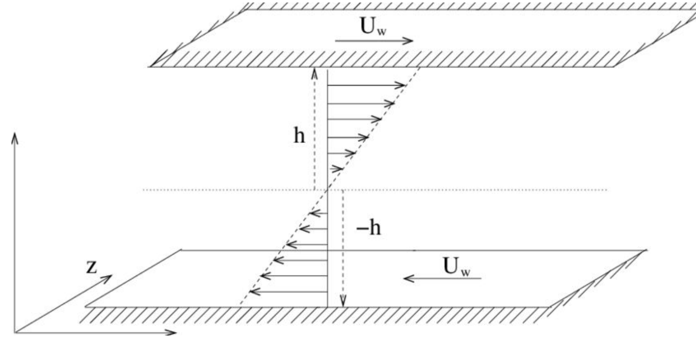
Butler and Farrell 1992, Trefethen et al 1993, Schmid 2007, Bale and Govindarajan 2010

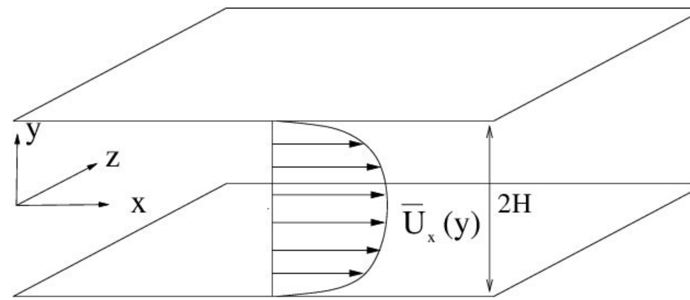
Normal systems

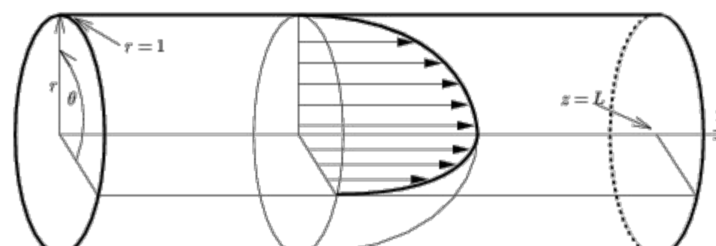


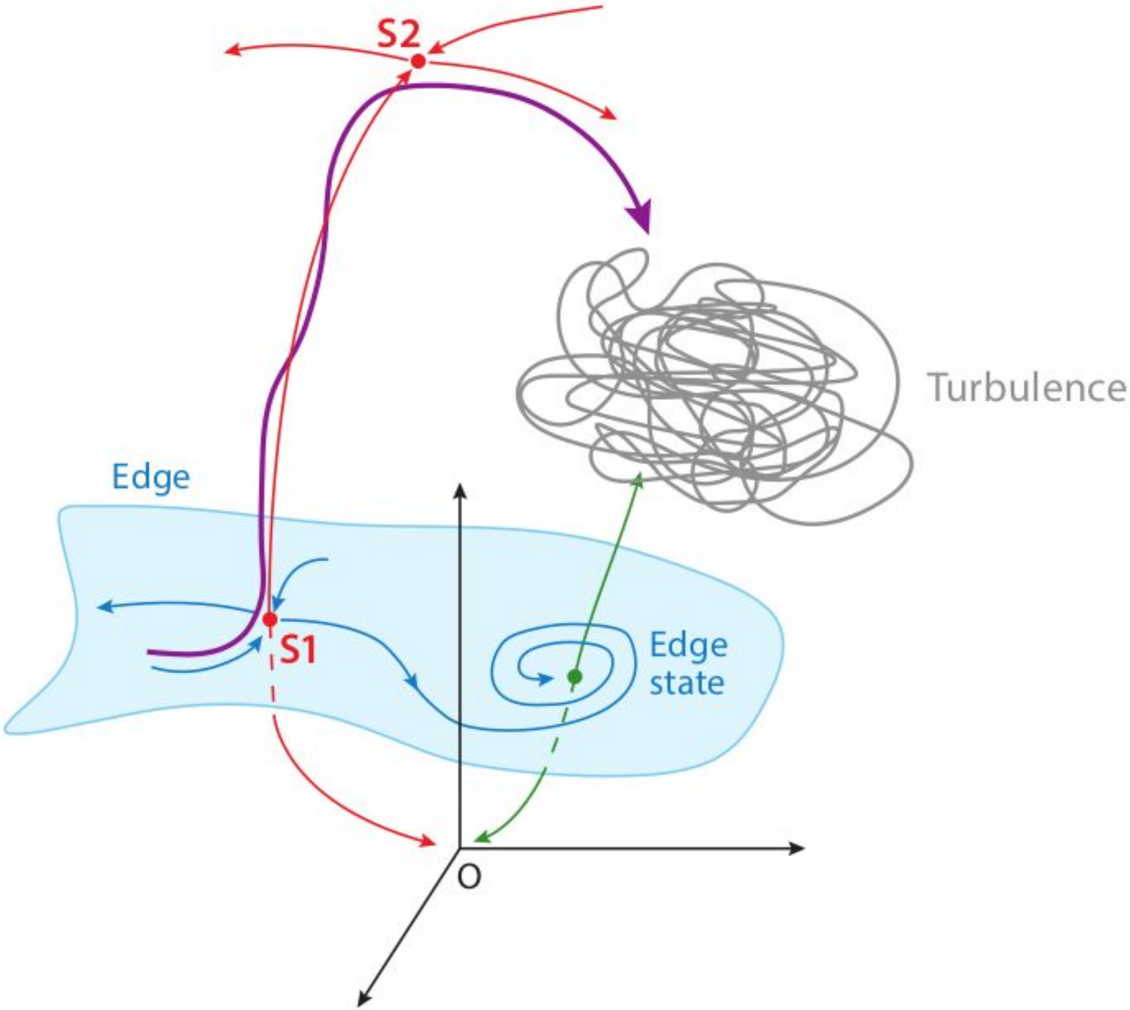


Non-normal systems



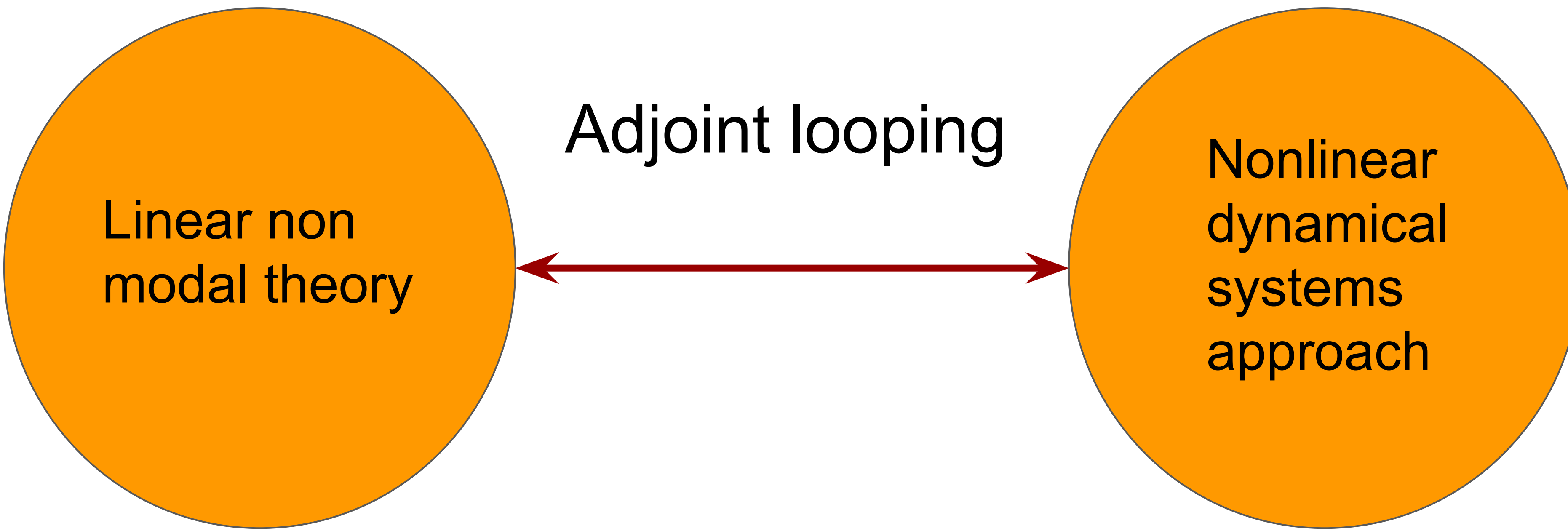






Kerswell, ARFM, 2018

Flow as a phase space


Adjoint looping
Linear non modal theory
Nonlinear dynamical systems approach

$$\frac{\partial u_i}{\partial x_i} = 0,$$

$$\frac{\partial u_i}{\partial t} + (U_j + u_j)\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial U_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + Ri_b T n_i + \frac{2}{Re}\frac{\partial}{\partial x_j}\Big(\mu\big(s_{ij} + S_{ij}\big) + \bar{\mu}s_{ij}\Big),$$

$$\frac{\partial T}{\partial t} + (U_j + u_j)\frac{\partial T}{\partial x_j} + u_j\frac{\partial(\overline{T} + T_0)}{\partial x_j} = \frac{1}{RePr}\frac{\partial^2 T}{\partial x_j^2}.$$

A cost function :
(your choice)

$$\mathcal{J}(\mathcal{T}) = A_1 \frac{E(\mathcal{T})}{E_0} + A_2 D + A_3 \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \frac{E(t)}{E_0} \mathrm{d}t$$

Initial Energy :

$$E_0 = \frac{d_1}{2} \Big\langle u_{0,i}, u_{0,i} \Big\rangle + \frac{d_2}{2} Ri_b \frac{\langle T_0, T_0 \rangle}{T_{ref}^2}.$$

NS

Scalar

$$\begin{aligned}
\mathcal{L} = & \mathcal{J}(\mathcal{T}) - \left[\frac{\partial u_i}{\partial t} + (U_j + u_j) \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - Ri_b T n_i \right. \\
& \left. - \frac{2}{Re} \frac{\partial}{\partial x_j} \Big(\mu \big(s_{ij} + S_{ij} \big) + \bar{\mu} s_{ij} \Big), v_i \right] - \left[\frac{\partial T}{\partial t} + (U_j + u_j) \frac{\partial T}{\partial x_j} + u_j \frac{\partial (\overline{T} + T_0)}{\partial x_j} - \frac{1}{RePr} \frac{\partial^2 T}{\partial x_j^2}, \tau \right] \\
& - \left[\frac{\partial u_i}{\partial x_i}, q \right] - \langle u_i(0) - u_{0,i}, v_{0,i} \rangle - \langle T(0) - T_0, \tau_0 \rangle - \frac{1}{2} \Big(d_1 \Big\langle u_{0,i}, u_{0,i} \Big\rangle + Ri_b d_2 \frac{\langle T_0, T_0 \rangle}{T_{ref}^2} - E_0 \Big) c.
\end{aligned}$$

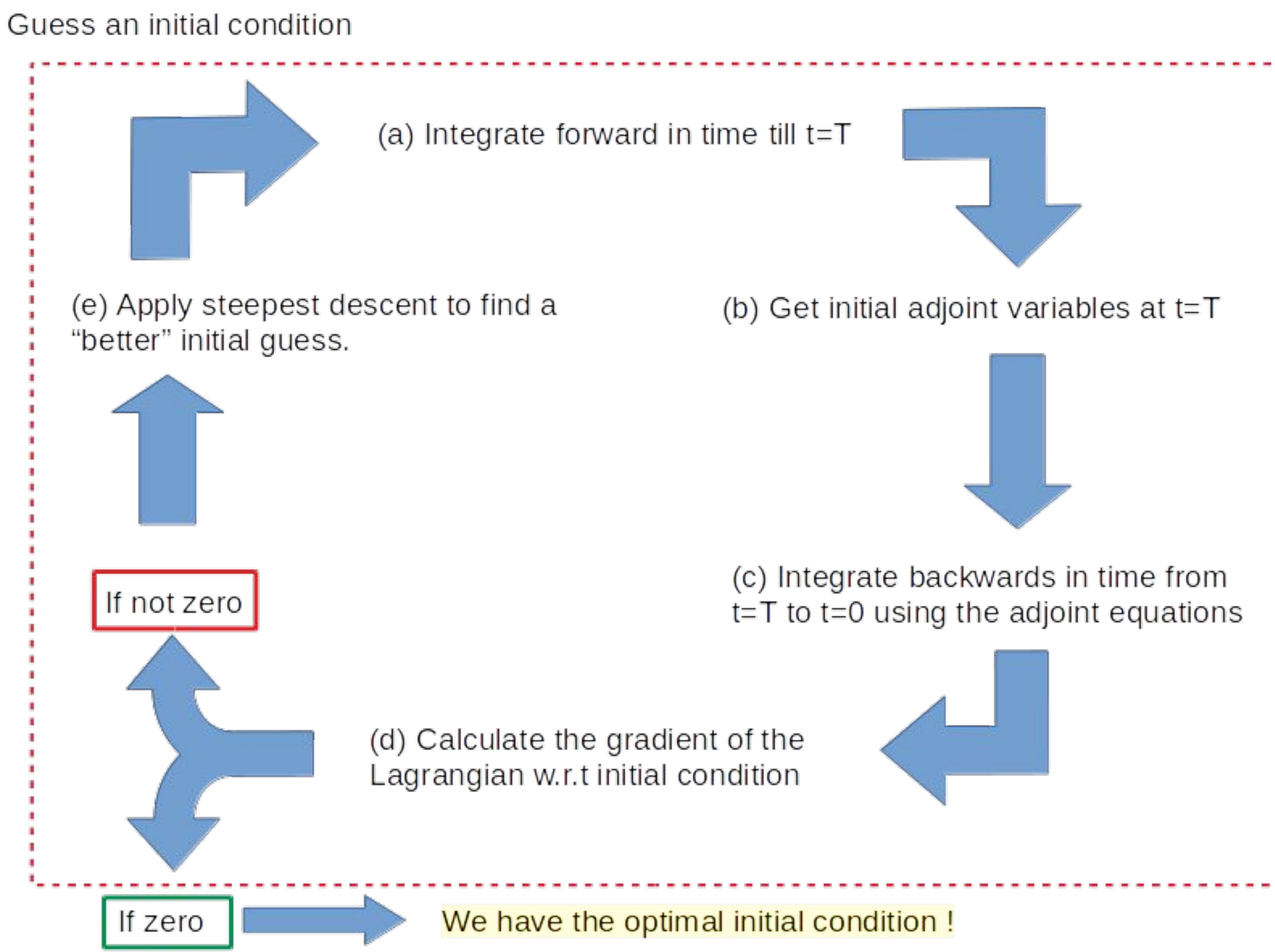
Incompressibility

Initial conditions

Initial perturbation energy

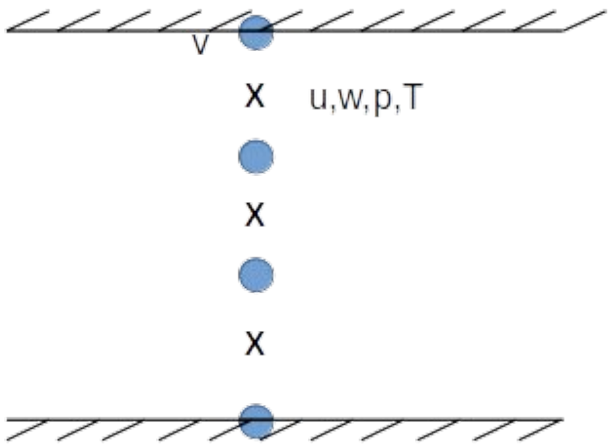
where,

$$[a_i, b_i] \equiv \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \langle a_i, b_i \rangle \mathrm{d}t, \quad \langle a, b \rangle \equiv \frac{1}{V} \int_V a_i b_i \mathrm{d}V,$$


Guess an initial condition
(a) Integrate forward in time till t=T
(b) Get initial adjoint variables at t=T
(c) Integrate backwards in time from t=T to t=0 using the adjoint equations
(d) Calculate the gradient of the Lagrangian w.r.t initial condition
If not zero
(e) Apply steepest descent to find a “better” initial guess.
If zero
We have the optimal initial condition !

Overview of the numerical scheme

Channel flow



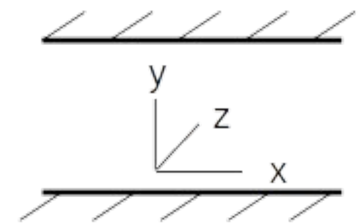


Fig: Model geometry

Spatial discretisation:

2 spectral directions (x, y)
(equispaced, unstaggered)
wall normal: second order central FD
(stretched and staggered)

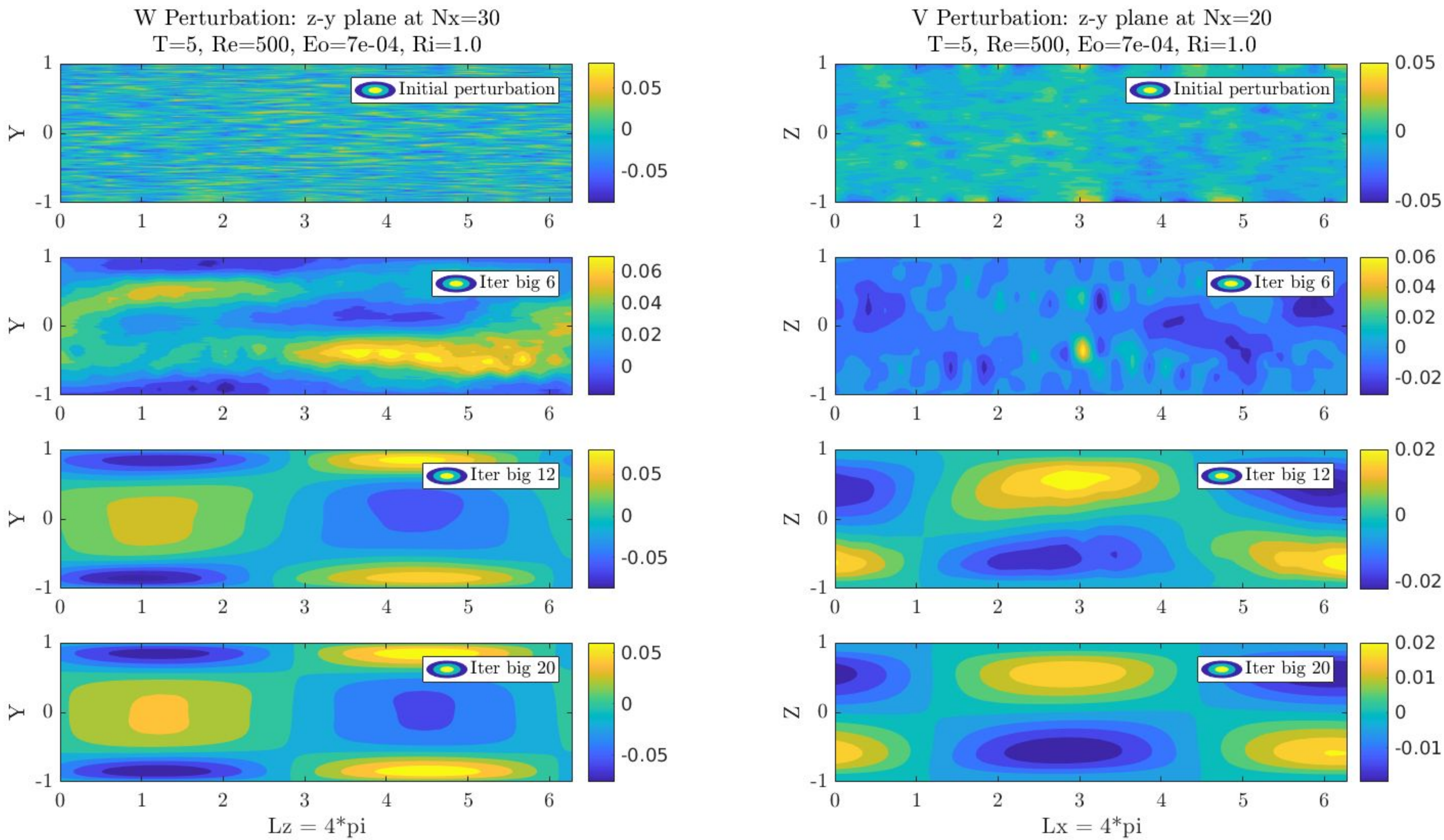
To capture wall dynamics

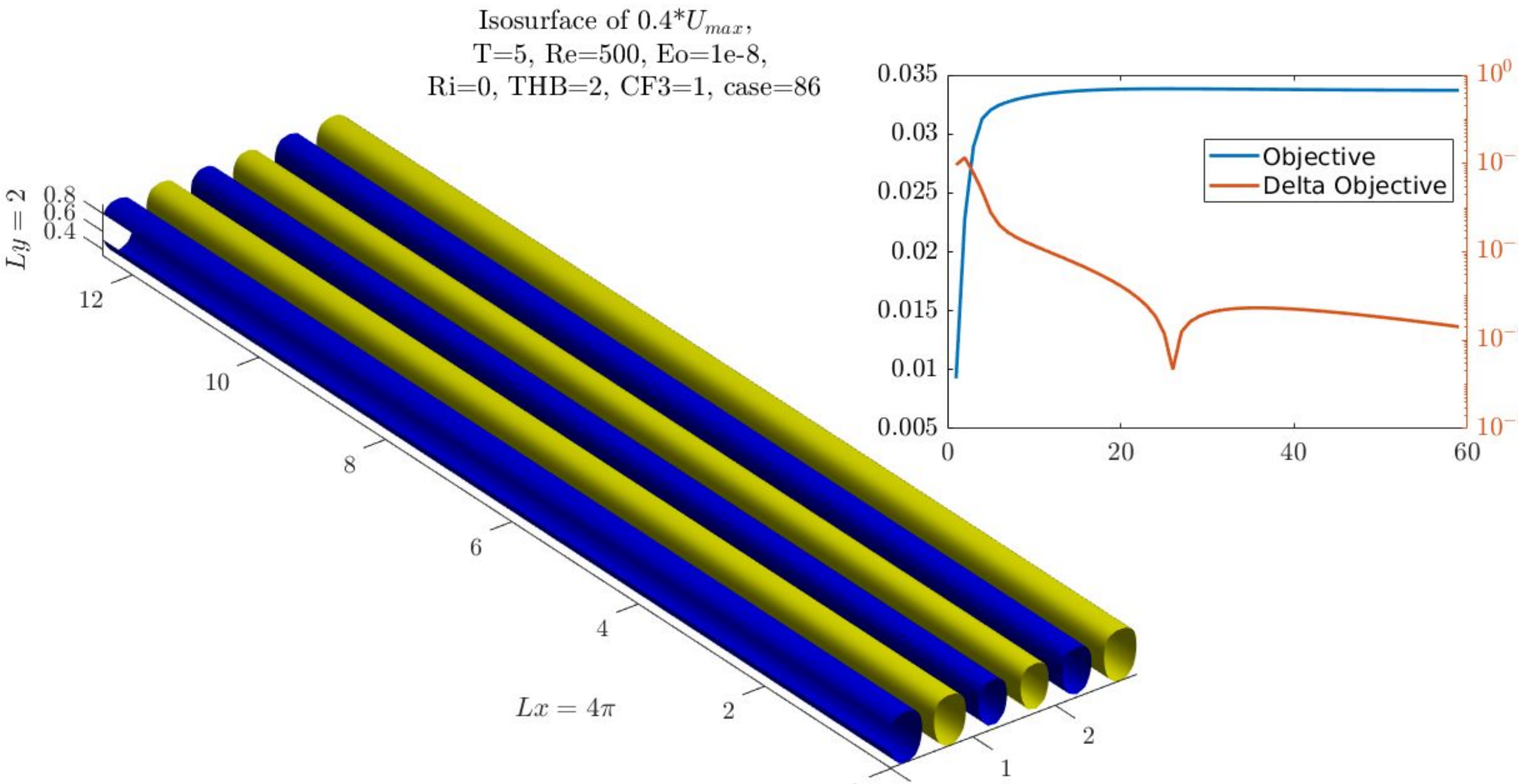
To tightly couple flow variables

Temporal discretisation:

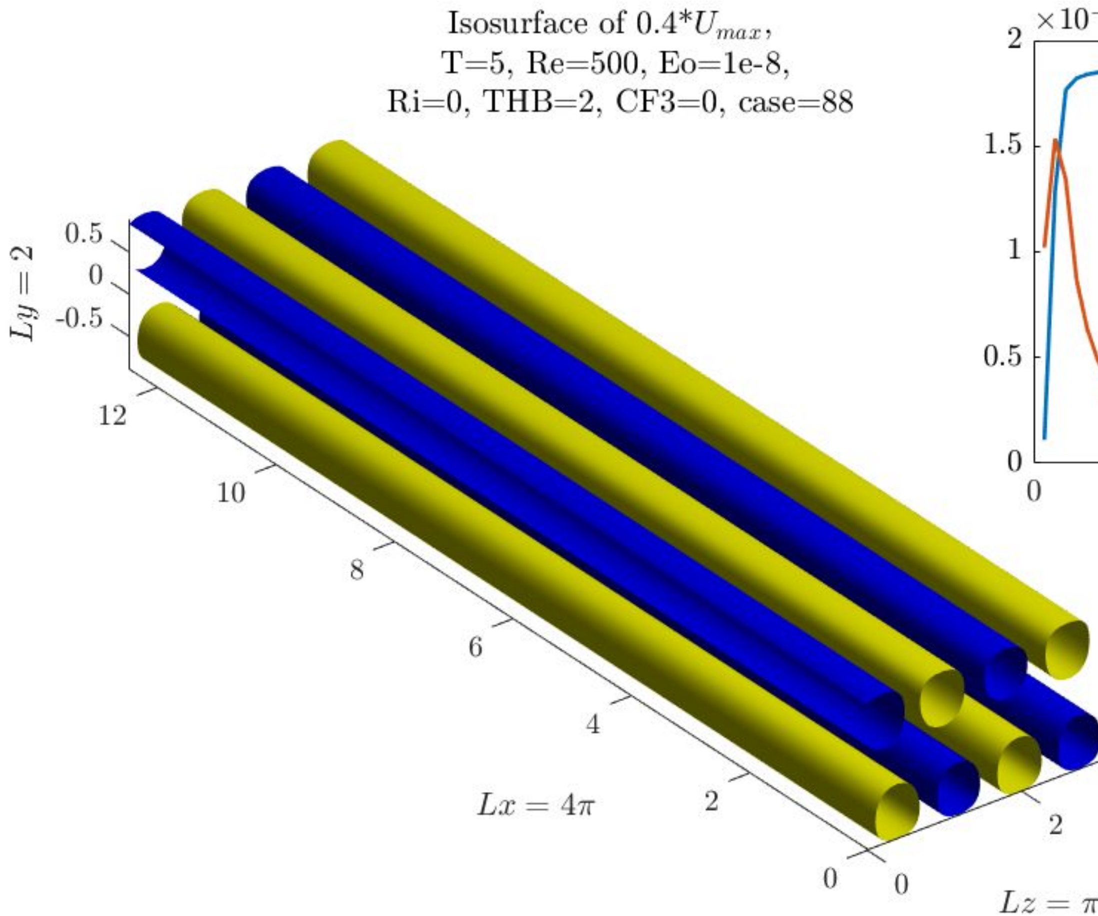
Nonlinear terms: RKW3
Viscous terms & y-derivatives: CN

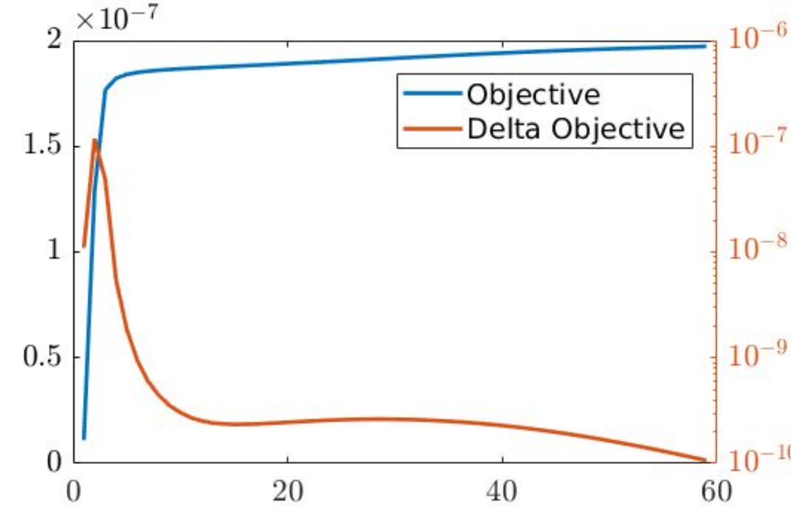
Numerical Renaissance, Thomas Bewley


W Perturbation: z-y plane at Nx=30
T=5, Re=500, Eo=7e-04, Ri=1.0
V Perturbation: z-y plane at Nx=20
T=5, Re=500, Eo=7e-04, Ri=1.0
Initial perturbation
Iter big 6
Iter big 12
Iter big 20
Y
Z
Lz = 4*pi
Lx = 4*pi

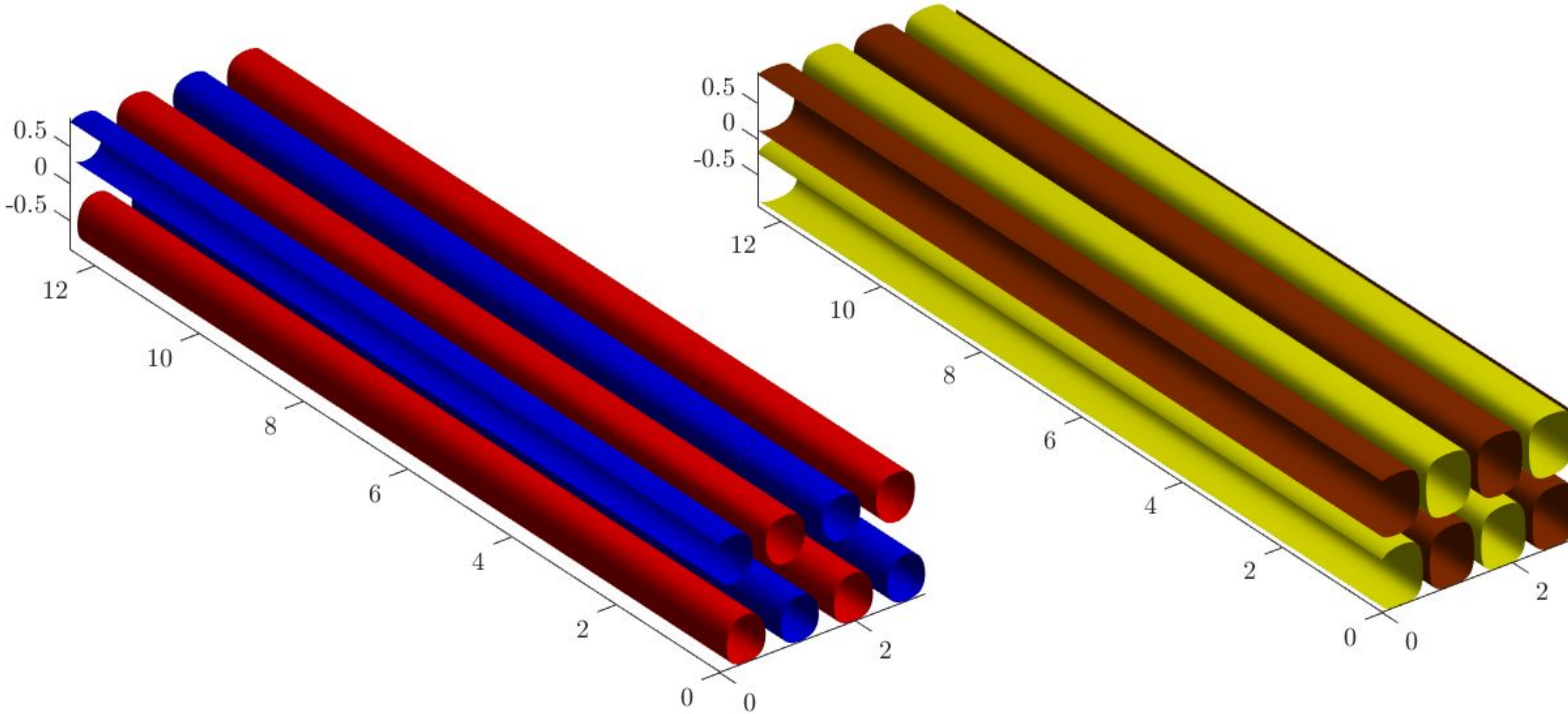


Integrated energy maximisation





Final time energy maximisation



T=5 T=10

Summary

a) Shear flows admit transient growth in energy

b) The optimal structures for transient growth are organised structures

ritabrata.thakur@icts.res.in