Curvature function renormalisation, topological phase transitions and multi-criticality

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Work in progress - Collaborators

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What are topological phases?

* Topological phases are characterised by quantised physical quantities, e.g., charge polarisation (defined by winding numbers), Hall conductance (defined by Chern numbers), etc
* Typically, topologically non-trivial phases have edge states - the bands touch each other at some points in the Brillouin zone, whereas trivial phases are gapped

Classification of phases

* Phases classified by symmetry, dimension, etc
* For non-interacting fermions in presence of internal symmetries, periodic table has been found
* Further extensions in the presence of crystalline symmetries, extension of K-theory classification

Topological phase transitions

* Can go between phases in different topological classes only when a gap closes somewhere in the Brillouin zone i.e. change in the number of edge states
* Change parameters of the theory for topological phase transitions - e.g., as a function of the magnetic field in the quantum Hall effect
* As parameters are tuned, bulk gap closes and reopens signalling phase transition

Motivation

* To understand topological phase transitions using ideas familiar from Landau theory of phase transitions, even though there is no order parameter
* Topological phase transitions implies discrete changes in an integer topological invariant - it signals gap closing in single particle or many-body spectrum
* Can we classify phase transitions by scaling behaviour of appropriate correlation functions near the transition?

Introduction to curvature function renormalisation group

W. Chen, M. Sigrist, A. P. Schnyder, R. Chitra, P.Molignini (2016-

* First define an appropriate curvature function $F(\mathbf{k}, M)$
* Function in the space of the Brillouin zone and M which is the set of all the tuning parameters in the Hamiltonian
* Integral of F over the Brillouin zone gives topological invariant

Examples

* In 1D, winding number is given by $\omega = \oint \frac{dk}{2\pi} A(k)$
* 2D system with broken time-reversal symmetry, quantum Hall system

$$C = \int_{BZ} \frac{d^2k}{2\pi} F_{xy}(\mathbf{k})$$

* Time-reversal invariant topological insulators

$$\nu = \frac{1}{2\pi i} \oint_{\partial(1/2)BZ} d \log Pf(\mathbf{k}, M) \bmod 2$$

* Point is that there is some function $F(\mathbf{k}, M)$ containing global geometric information about the band structure which is integrated over the Brillouin zone

* Idea is that close to the phase transition, as $M \to M_c$ the curvature function diverges at the gap closing momentum k_0 denoting a critical point where the topological invariant changes

* Tuning parameter M can be magnetic field, chemical potential, hopping parameters in the Hamiltonion, etc

Idea of the renormalisation group approach

* Scaling procedure renormalises the curvature function keeping the topological quantum number invariant

* Analogy of messy string - integrate to find number of knots or stretch it out until knots become obvious

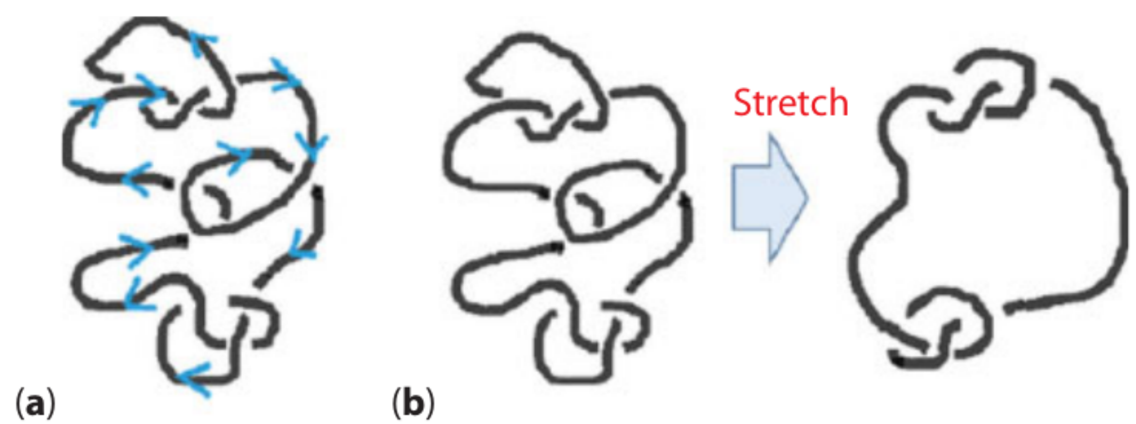


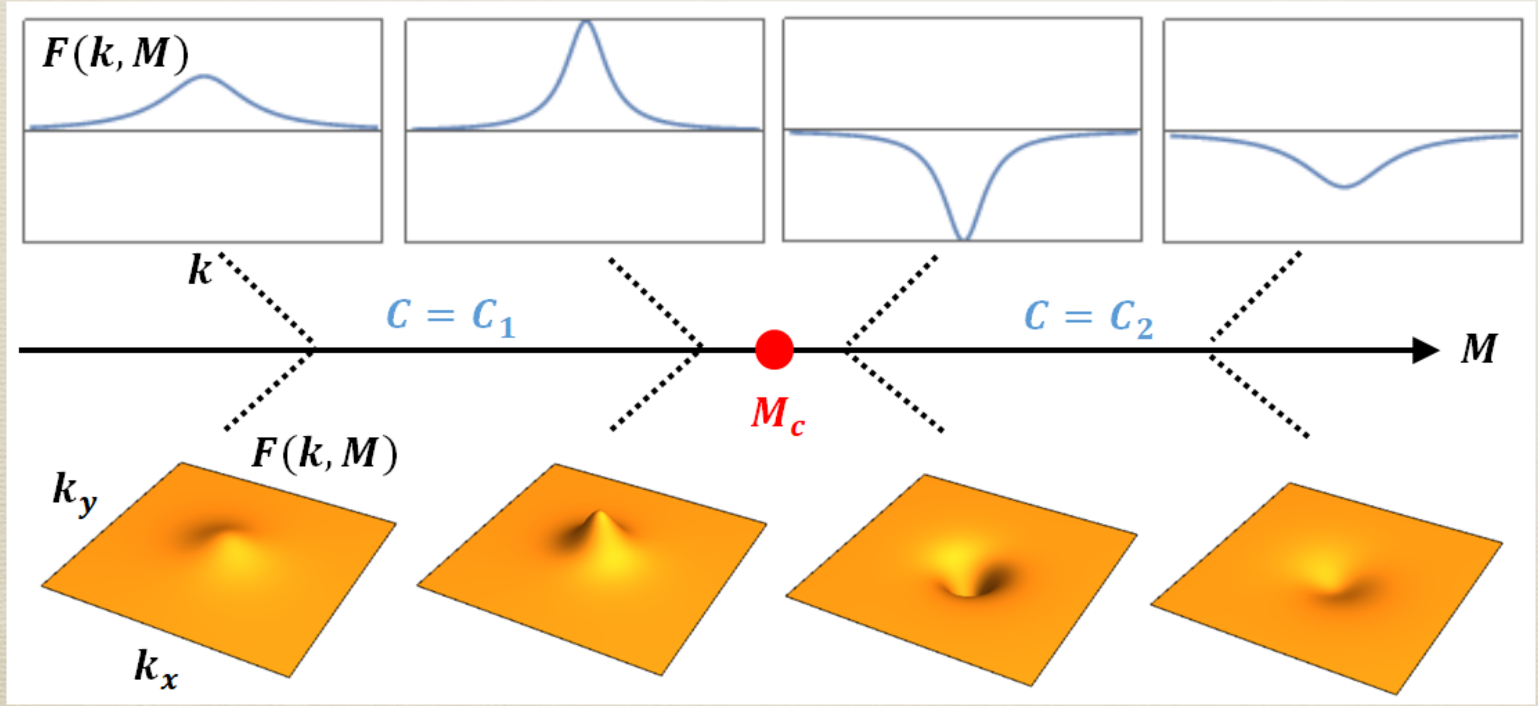
Figure: W. Chen, M. Sigrist, Advanced Topological Insulators, 239-280

Scaling mechanism and flow equation

- Idea is that as $M \to M_c$, the curvature function develops a divergence at some high symmetry points (HSP) where the gap closing takes place and then the curvature function changes sign
- Will consider gap closing at non-HSP later

Peak divergence scenario

- Function gradually peaks as function of k, as we approach M_c and changes sign at the phase transition



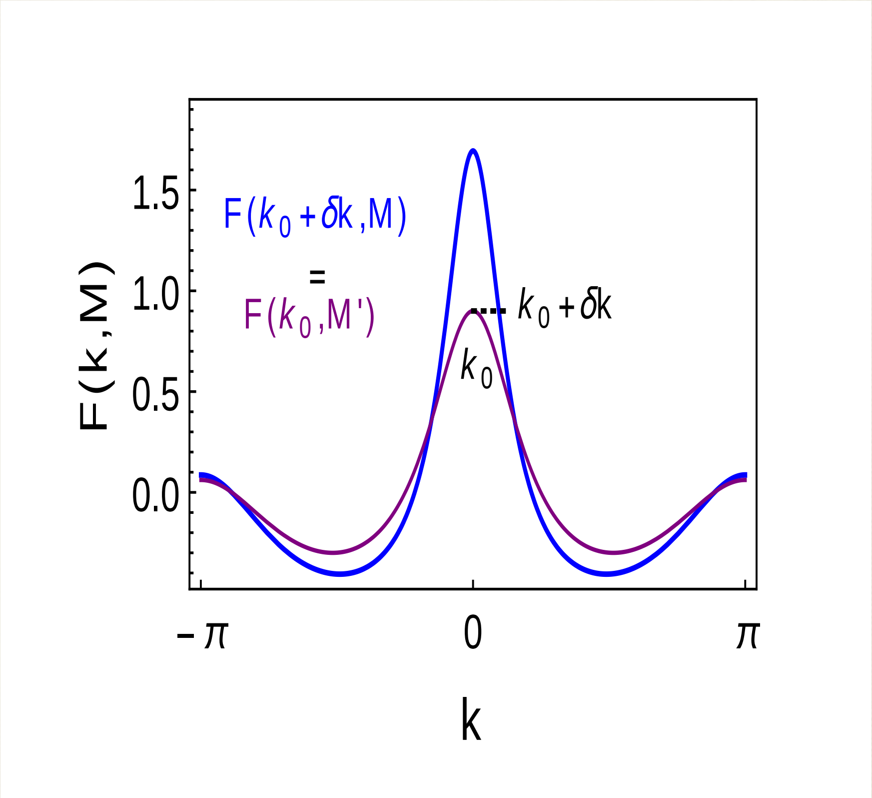
1D as function of single parameter M

2D

Flow equations

* Essential idea, change parameters in the theory and reduce divergence - repeat - flow stops at some point

$$F(k_0, M') = F(k_0 + \delta k, M)$$



* Close to the TPT,

$$\lim_{M \to M_c^+} F(k_0, M) = -\lim_{M \to M_c^-} F(k_0, M) = \pm\infty$$

* At the fixed point

$$F(k_0 + \delta k, M_0) = F(k_0, M_0)$$

* Same equation can be written as a differential equation - the RG equation in parameter space

$$\frac{dM}{dl} = \frac{1}{2}\frac{\partial_k^2 (F(k,M)|_{k=k_0}}{\partial_M F(k_0,M)}$$

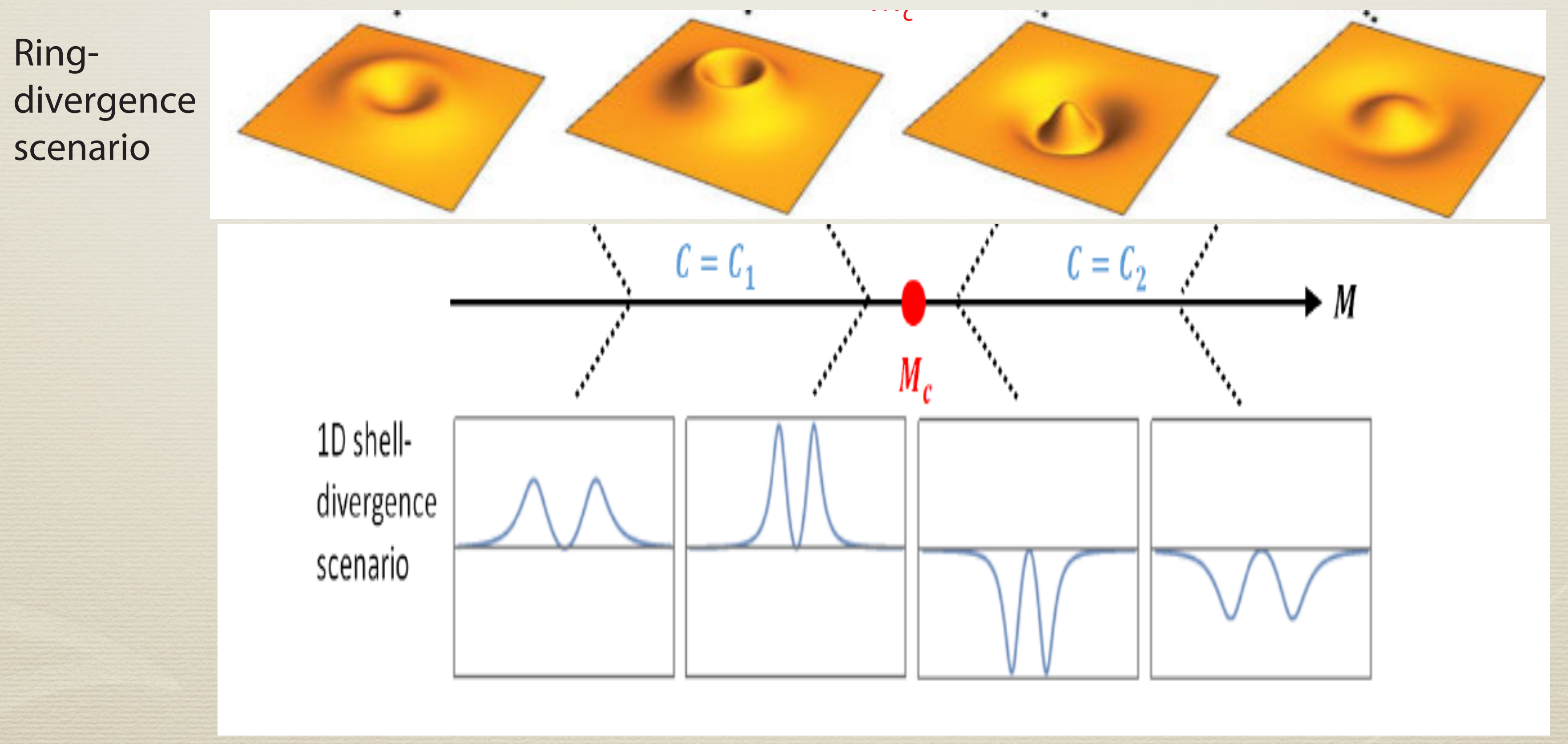
$$dl = dk^2$$

$$\text{Critical point :} \qquad \left|\frac{d\mathbf{M}}{dl}\right| \to \infty,$$

$$\text{Fixed point :} \qquad \left|\frac{d\mathbf{M}}{dl}\right| \to 0.$$

Ring divergence scenario

* Function has a ring shape whose radius reduces and magnitude increases as $M \to M_c$



- In this case, the divergence has a ring shape, whose radius reduces and magnitude increases as $M \to M_c$
- Extremum of ring changes sign across M_c
- Turns out to be an unstable fixed point $dM/dl = 0$ solution of the RG equation
- So points where dM/dl diverges and unstable fixed points denote topological phase transitions(TPT)

Length scales and critical exponents

* Curvature function in the vicinity of a high symmetry point (HSP) typically has Lorentzian form

$$F(k_0+\delta k, M) = \frac{F(k_0, M)}{1 \pm \xi_{k_0}^2 \delta k^2}$$

* Divergence of the curvature function at quantum critical point introduces exponents γ, ν

$$F(k, M) = |M - M_c|^{-\gamma} \quad \xi_{k_0} = |M - M_c|^{-\nu}$$

* Conservation of the topological invariant implies

$$C = \int d^D k F(\mathbf{k}, M) \propto \frac{F(\mathbf{k}_0, M)}{\xi^D}$$

* yields scaling law $\gamma = \nu D$

Correlation function characterising the TPT

* Can introduce a correlation function that decays with correlation length ξ
* In terms of Wannier state $|R, n >= \frac{1}{N} \sum_k e^{ik(r-R)} u_{nk}$
* Fourier transform of curvature function denotes overlap of Wannier functions a distance R apart

$$\lambda_R = \int dk e^{ikR} F(k, \mathbf{M}) = \sum_n < R, n|r|0, n >$$

* λ_0 - is precisely the topological invariant
* λ_R - is expected to be related to the correlation length ξ
* So close to the TPT, Wannier functions become extended and have overlaps over large regions

* To obtain RG flow, we require knowledge of the curvature function only at a few points
* To get topological invariant, need to integrate over the whole BZ, so need to know the curvature function over the whole BZ
* So if we can get the topological phase transitions from the RG equations, could be much more efficient

Simple example - Su-Schrieffer- Heeger model

* CRG procedure sufficient to obtain topological phase diagram in a simple way

$$H = \sum_i (t+\delta t) c^\dagger_{Ai} c_{Bi} + (t-\delta t) c^\dagger_{Ai+1} c_{Bi} + h.c.$$

$$M = \delta t$$

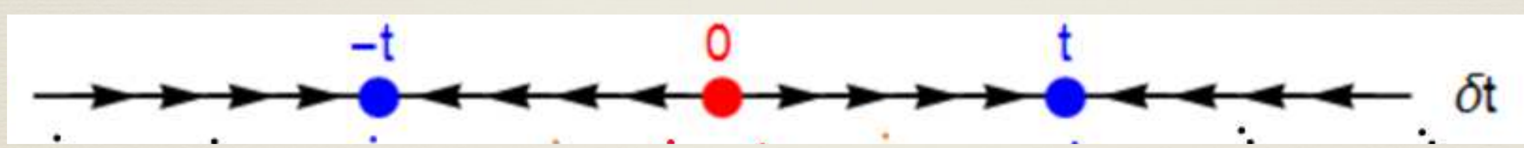
$$\frac{d\delta t}{dl} = \frac{\delta t}{4}\left(1-\frac{\delta t^2}{t^2}\right) \text{ if } k_0 = 0 \,,$$

$$\frac{d\delta t}{dl} = \frac{t^2}{4\delta t}\left(1-\frac{\delta t^2}{t^2}\right) \text{ if } k_0 = \pi \,,$$

2 HSP are $k=0$ and $k=\pi$

* In the first case, $\delta t = 0$ denotes an unstable fixed point (critical point) and in the second case $\delta t = 0$ is divergent and denotes a critical point

* $\delta t = \pm t$ - are fixed points



* Topological phase transition at $\delta t = 0$

* Exact solution shows that gap closes in the BZ at $k = \pi$
* Winding number = 0 for $\delta t > 0$ and 1 for $\delta t < 0$.
* Here, we know that there is only one phase transition at $\delta t = 0$ and we need to compute the winding number only at the fixed points $\delta t = t$ and $\delta t = -t$
* Easy to check that winding number at $\delta t = t$ is 0 and at $\delta t = -t$ is 1

Other examples

* Many other examples, 1D and 2D models, periodically driven models, weakly interacting models

W. Chen Journal of Physics: Condensed Matter **28** 055601 (2016).
W. Chen, M. Sigrist and A. P. Schnyder, Journal of Physics: Condensed Matter **28**, 365501 (2016).
W. Chen et. al Physical Review B **95**, 075116 (2017).
W. Chen, Physical Review B **97** 115130 (2018).
W. Chen, M. Sigrist, Advanced Topological Insulators, 239-280
P.Molignini, W. Chen and R. Chitra, Physical Review B98, 125129
P.Molignini, W. Chen and R. Chitra, 1906.10695
P.Molignini, R. Chitra and W. Chen, 1912.008819

Our work

* Motivation - explore this idea for more complex models
* Main results - find unstable fixed points (critical points) where the gap closes at non HSP
* Find 2 different length scales ξ_0 and ξ_π
* Find multi-critical points where three topological phases meet at a point

The Model

* Extended Kitaev model in one dimension

hoppings : t_1, t_2 pairings : λ_1, λ_2 chemical potential : g

$$H = -t_1 \sum_{i=1}^{N-1} \left(c_i^\dagger c_{i+1} + h.c \right) - t_2 \sum_{i=2}^{N-1} \left(c_{i-1}^\dagger c_{i+1} + h.c \right)$$

$$- \lambda_1 \sum_{i=1}^{N-1} \left(c_i^\dagger c_{i+1}^\dagger + h.c \right) - \lambda_2 \sum_{i=2}^{N-1} \left(c_{i-1}^\dagger c_{i+1}^\dagger + h.c \right)$$

$$+ g \sum \left(2 c_i^\dagger c_i - 1 \right).$$

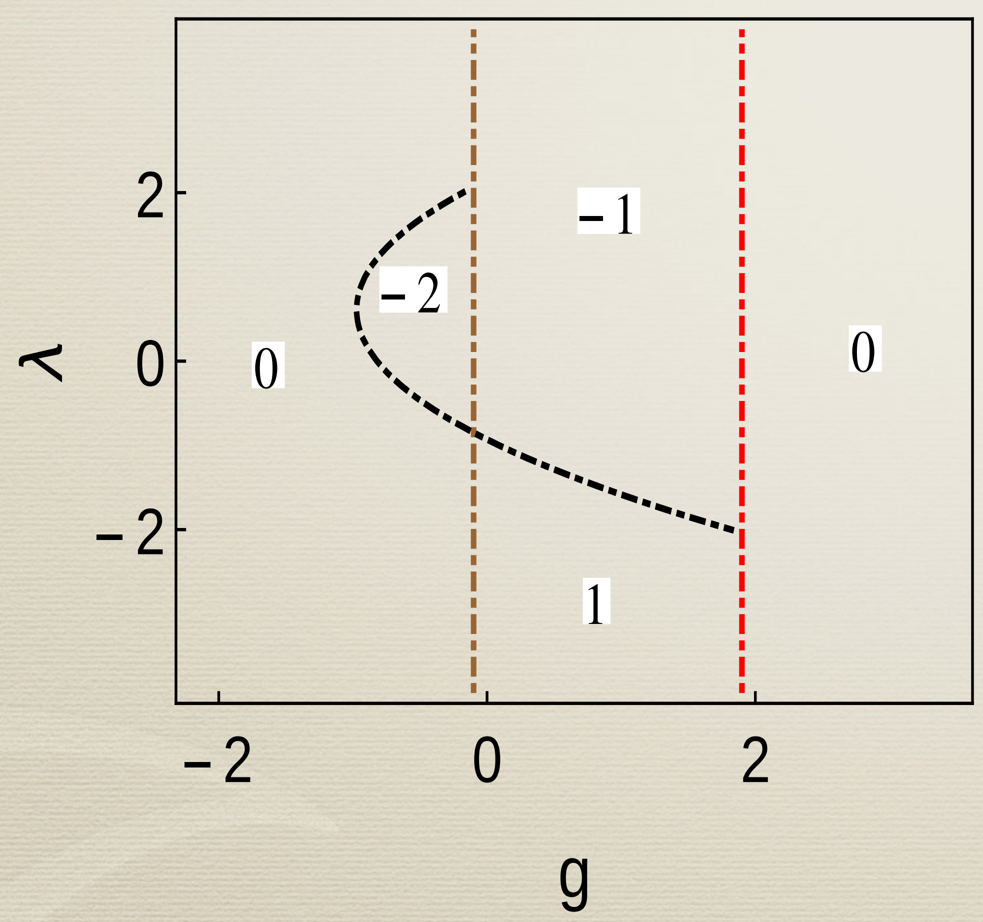
$$M = \{g, t_1, t_2, \lambda_1, \lambda_2\}$$

Scale by t_1 to get $M = \{g, t, \lambda, \lambda_2\}$ $\lambda = \lambda_1 / \lambda_2$

Y. Niu, S. B. Chung, C.H. Hsu, I. Mandal, S. Raghu and S. Chakravarty, Phys.Rev.B **85** 035110(2012).

Phase diagram from exact computation of gap closings

* For fixed t and λ_2



$$k_0 = \pi \qquad g = t + 1 = 1.9$$
$$k_0 = 0 \qquad g = t - 1 = -0.1$$

$$k = \frac{1}{2} \arccos\left[\frac{1}{t}\left(\frac{\lambda}{2} + g\right)\right]$$

$$g = \frac{t(\lambda^2 - 2)}{2} - \frac{\lambda}{2}$$
$$= \frac{0.9(\lambda^2 - 2)}{2} - \frac{\lambda}{2}$$

The renormalisation group equations

$$\frac{d\lambda}{dl} = \frac{-1}{2(g-t\mp 1)^2}\alpha(g,t,\lambda,\lambda_2)$$

$$\frac{d\lambda_2}{dl} = \frac{-\lambda_2}{2(g-t\mp 1)(\lambda\pm 2)}\alpha(g,t,\lambda,\lambda_2)$$

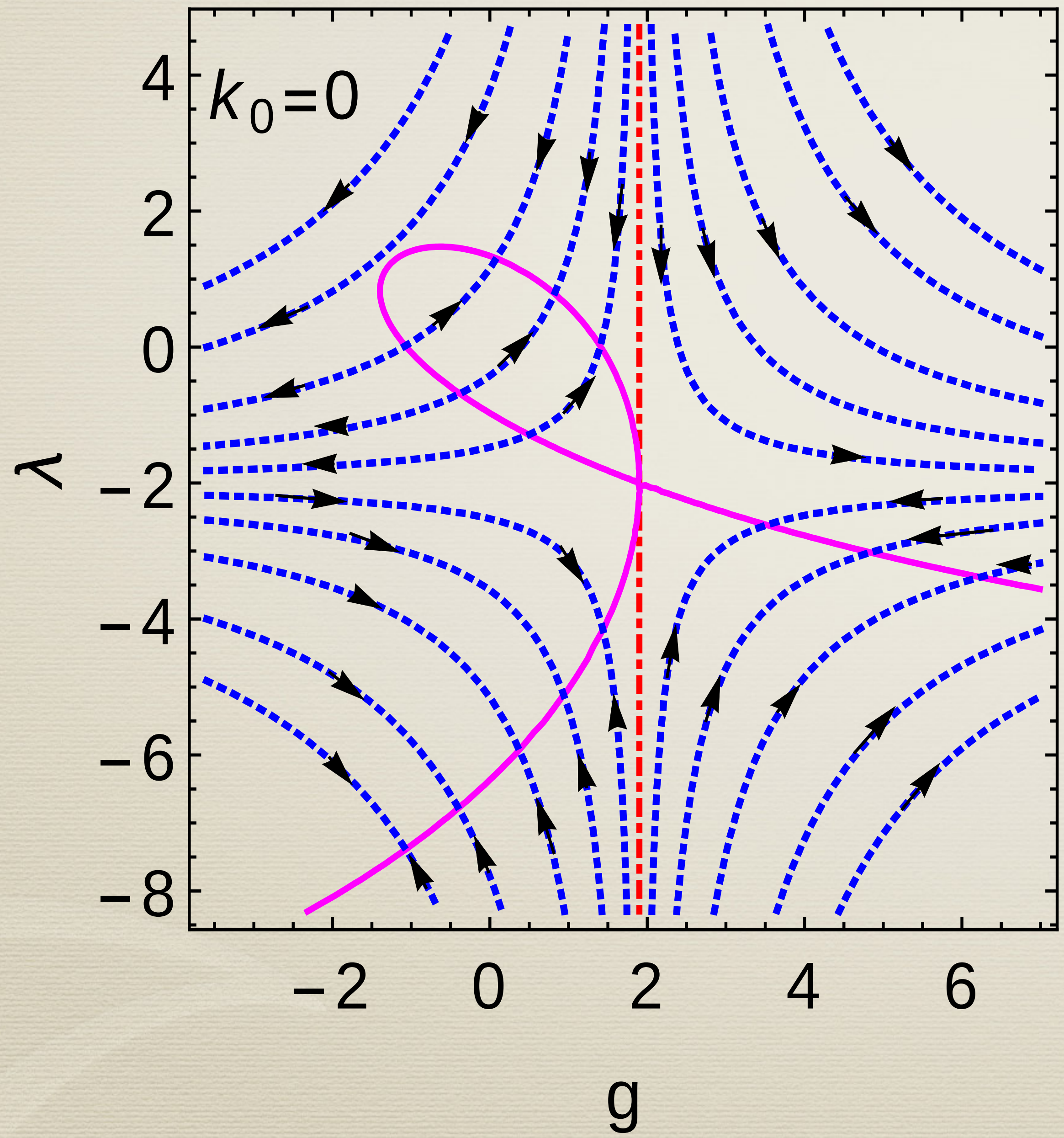
$$\frac{dg}{dl} = \frac{1}{2(g-t\mp 1)(\lambda\pm 2)}\alpha(g,t,\lambda,\lambda_2)$$

$$\frac{dt}{dl} = \frac{-1}{2(g-t\mp 1)(\lambda\pm 2)}\alpha(g,t,\lambda,\lambda_2)$$

$$\alpha(g,t,\lambda,\lambda_2) = (\lambda\pm 8)(g-t\mp 1)^2 + 2\lambda_2^2(\lambda\pm 2)^3 + 3(\lambda\pm 2)(g-t\mp 1)(4t\pm 1).$$

Upper/lower signs for $k_0 = 0/k_0 = \pi$

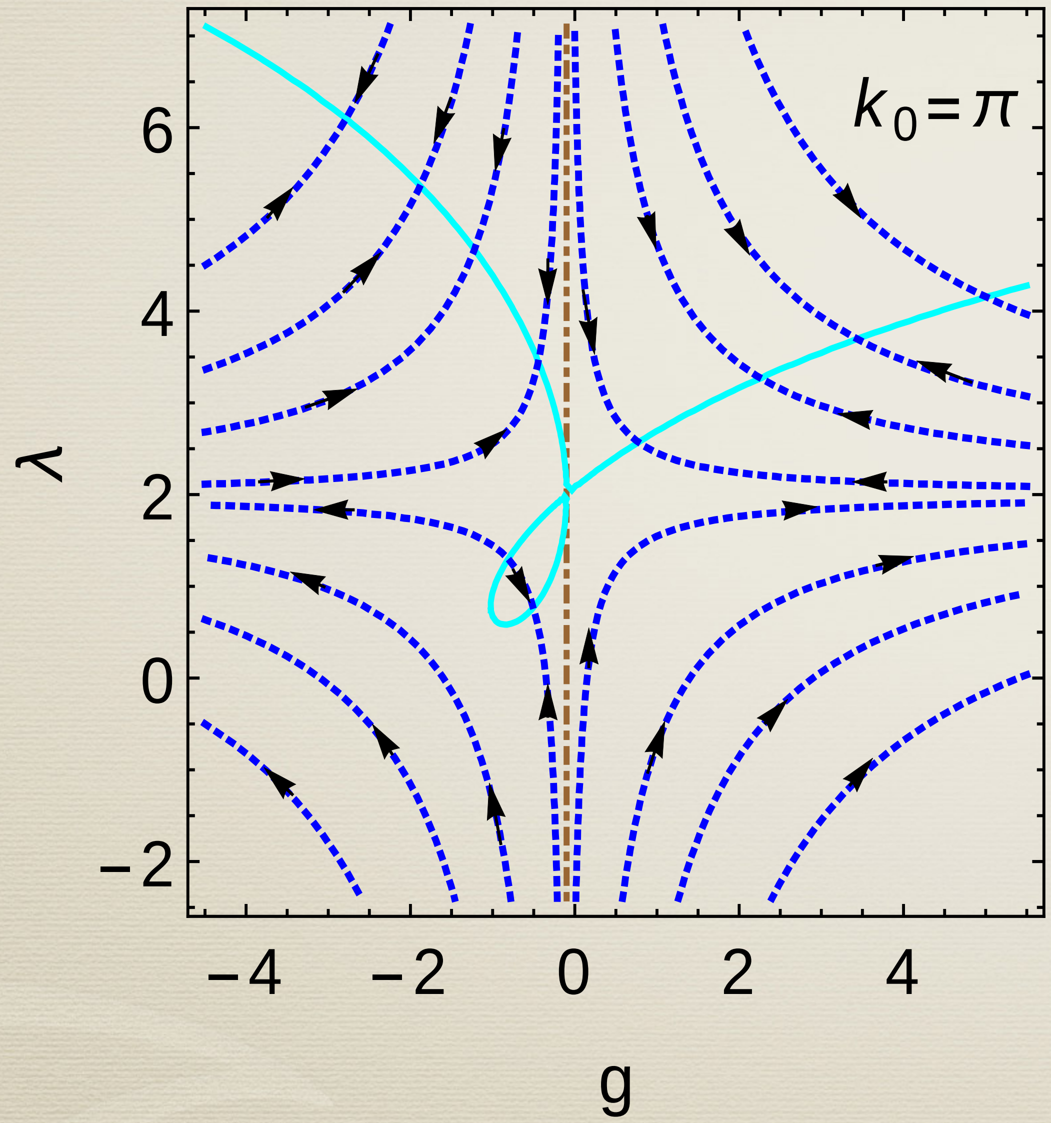
Flow diagram at HSP $k_0 = 0$



RG equations derived by setting $k_0 = 0$

Red dashed line and part of mauve line denoting unstable fixed point signify critical lines with TPT

Flow diagram at HSP $k_0 = \pi$



RG equations derived by setting $k_0 = \pi$

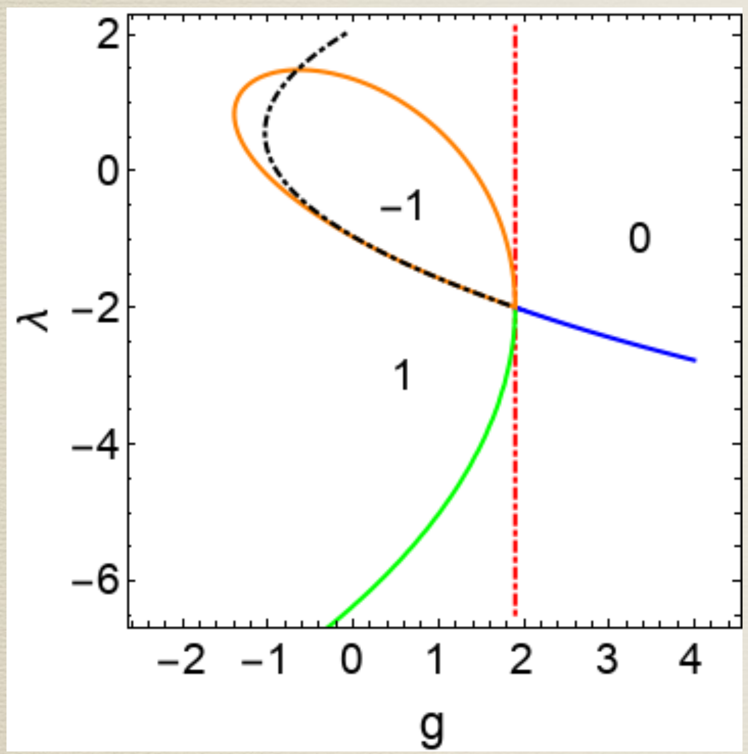
Brown dashed line and part of cyan line denoting unstable fixed point signify critical lines with TPT

Surprising Overlap with exact solution

Almost entire non HSP critical line has been obtained

Also change of winding number of 2 across non-HSP critical line can be predicted because of ring divergence which implies 2 flips of sign at TPT point

Tricritical point where three phases meet



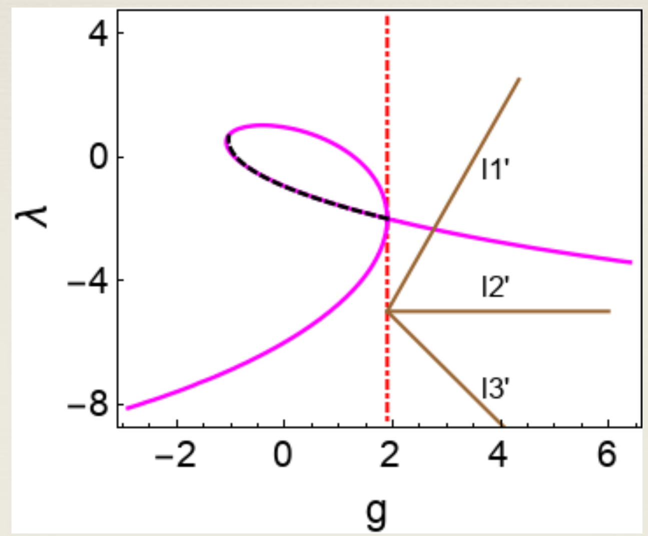
* Point where red dashed one and black dash-dot line meet has three different phases with

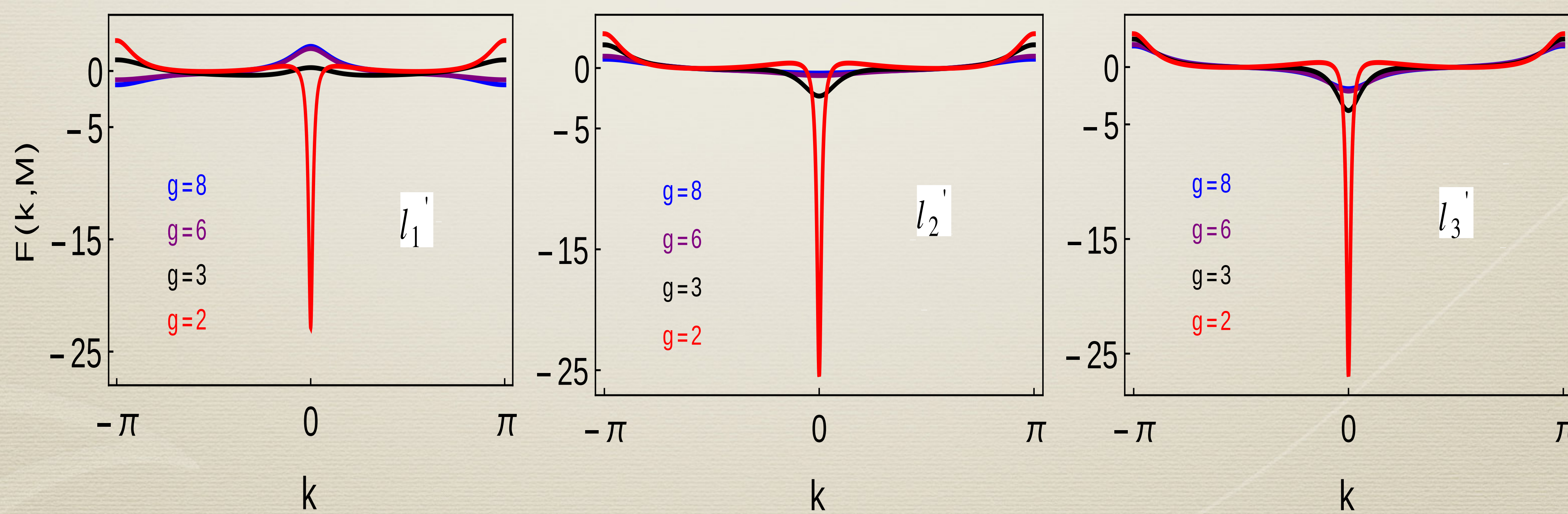
  C=0, C=-1 and C=1

* Precisely at this point, the curvature function is indeterminate

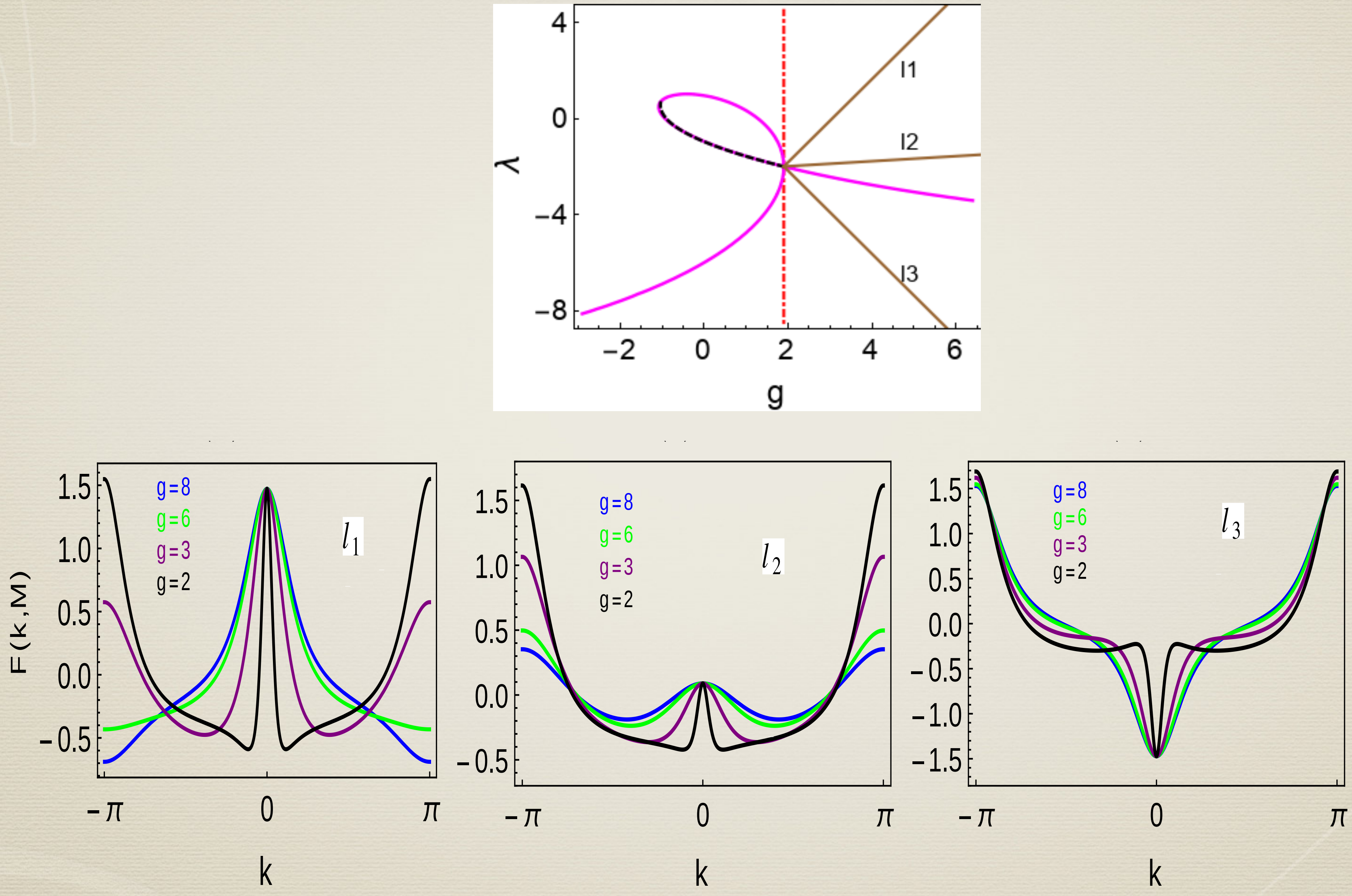
* Expect criticality to be different at this point

Approach to criticality for a critical point

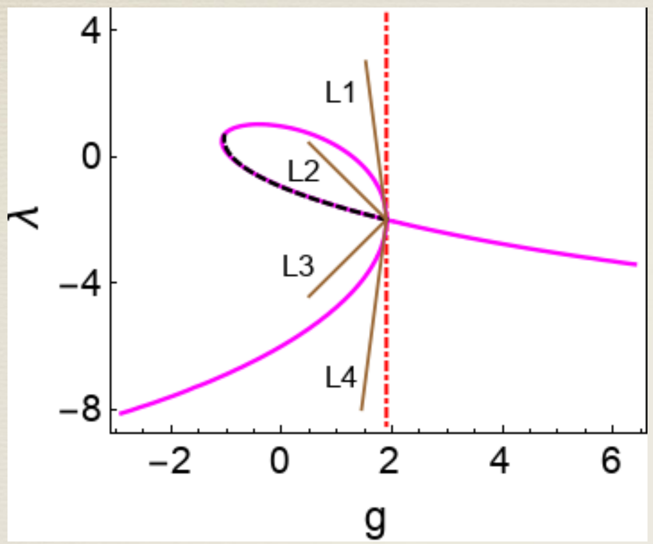


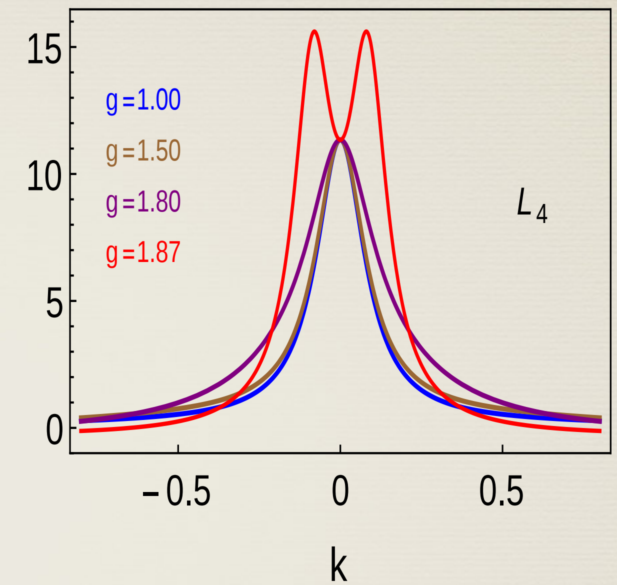


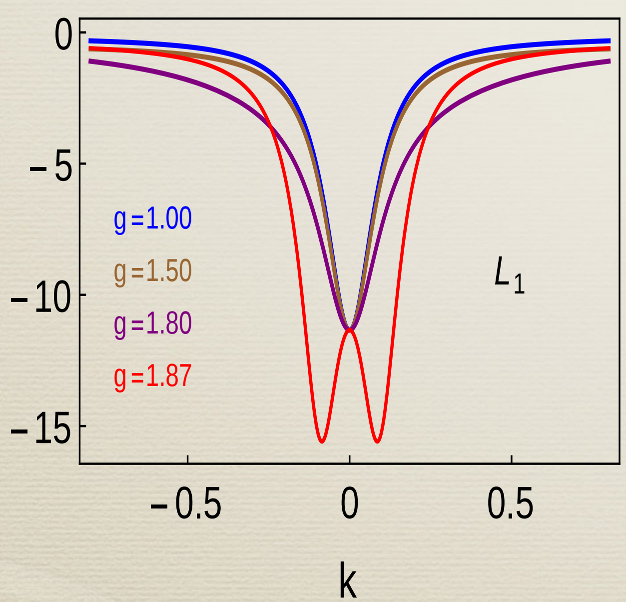
Approach to criticality for a multi critical point (from trivial side)

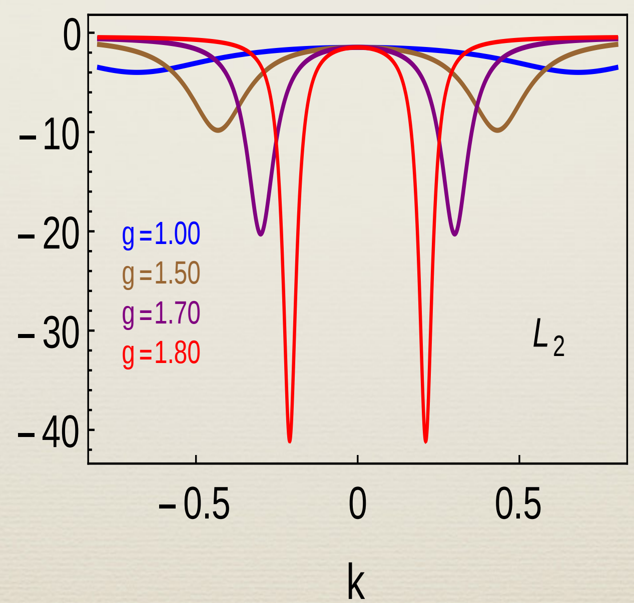


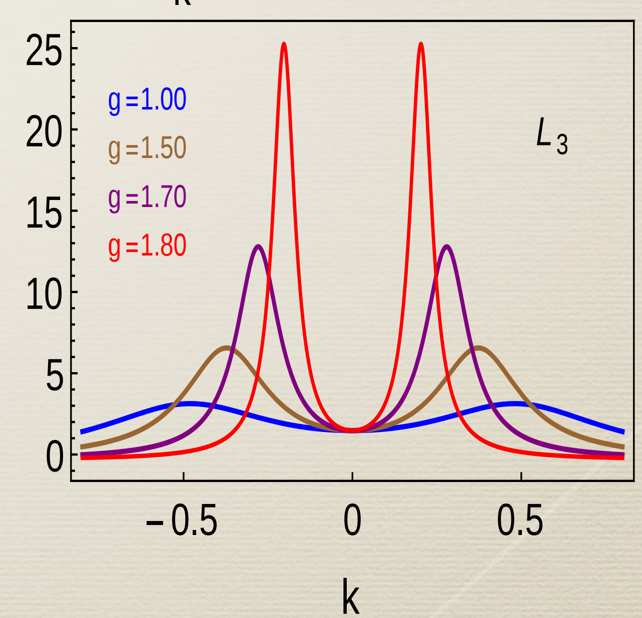
Approach to criticality for a multicritical point (from the topological side)











Universal features

* For critical point, as expected, function diverges as you approach the critical line and is path independent
* For the tricritical point, function does not diverge as we approach the critical point. In fact, approaches a path independent peak value, which can be obtained from the equations

Computation of critical exponents

- Easy to check that $F(k_0 = 0/\pi, \mathbf{M})$ diverges with the exponent $\gamma = 1$

$$F(k_0 = 0/\pi, \mathbf{M}) = \pm \frac{\lambda_2(\lambda \pm 2)}{g - t \mp 1}$$
$$= \left|\lambda_2(\lambda \pm 2)\right| \times \left|g - t \mp 1\right|^{-1}$$

- Also, $\xi_{0/\pi}$ diverges with exponent $\nu = 1$ and $\nu = \gamma$ as expected

$$\xi_{k_0}(\mathbf{M}) = \left|\frac{1}{2(g - t \mp 1)^2(\lambda \pm 2)} \times \beta(g, t, \lambda, \lambda_2)\right|^{1/2}$$
$$\underset{\mathrm{g \to t \pm 1}}{\propto} \left|\lambda_2(\lambda \pm 2)\right| \times \left|g - t \mp 1\right|^{-1},$$

Correlation length ξ and Correlation function λ_R

* Can compute the correlation function λ_R as the Fourier transform of the curvature function
* Model has divergences at HSP and non-HSP. However, divergences at HSP seem to capture all the phase transitions
* Choose to fit curvature function to Lorentzians at the HSP $k_0 = 0$ and $k_0 = \pm\pi$ and compute λ_R

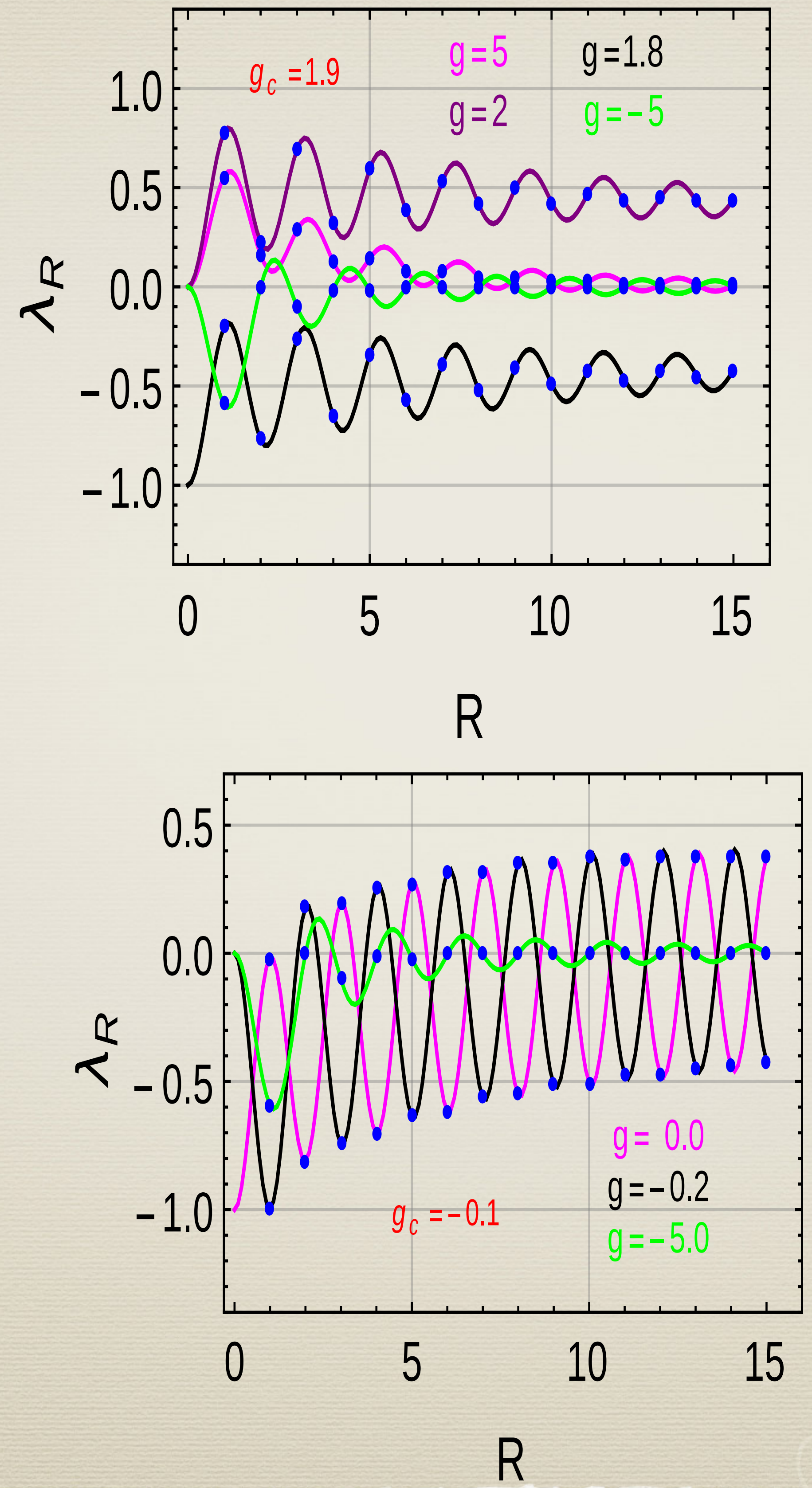
$$\lambda_R = \int_{BZ} dk e^{ikR} F(k, \mathbf{M})$$

$$\begin{aligned}\lambda_R &= \int_{-\infty}^{\infty} dk e^{ikR} F(k, \mathbf{M}) \\ &= 2\pi \frac{F(\pi, \mathbf{M})}{\xi_\pi(\mathbf{M})} \times \cos(\pi R) \times \exp\left[-\frac{|R|}{\xi_\pi(\mathbf{M})}\right] \\ &\quad + \pi \frac{F(0, \mathbf{M})}{\xi_0(\mathbf{M})} \times \exp\left[-\frac{|R|}{\xi_0(\mathbf{M})}\right].\end{aligned}$$

$$\lambda_0 = \text{topological invariant}$$

* Can show that when $M \to M_c$ close to the critical point where the gap closes at $k = 0$, the envelope of the oscillatory decay falls off slowly and the amplitude of the oscillations falls of quickly

* When $M \to M_c$ close to the critical point where the gap closes at $k = \pi$, the envelope of the oscillations falls of quickly but the amplitude of the oscillations continue and falls off slowly



g_c =1.9
g=5
g=1.8
g=2
g=−5
1.0
0.5
0.0
−0.5
−1.0
λ_R
0
5
10
15
R
0.5
0.0
−0.5
−1.0
λ_R
g= 0.0
g=−0.2
g_c =−0.1
g=−5.0
0
5
10
15
R

To conclude

* Expect this technique to be useful to characterise topological phase transitions in multi-dimensional parameter spaces, particularly in interacting and periodically driven systems - could be numerically efficient because phase space is so large.
* Still in the exploratory phase