Some unusual topological systems —
spin waves and Josephson junctions

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Outline

- Spin system on a Kagome lattice and phase diagram
- Band structure of magnons and Chern numbers
- Thermal Hall effect

R. Seshadri and D. Sen, Phys. Rev. B 97, 134411 (2018)

- Josephson junctions of three superconducting wires
- AC Josephson effect
- Shapiro plateaus

O. Deb, K. Sengupta and D. Sen, Phys. Rev. B 97, 174518 (2018)

What is topology ?

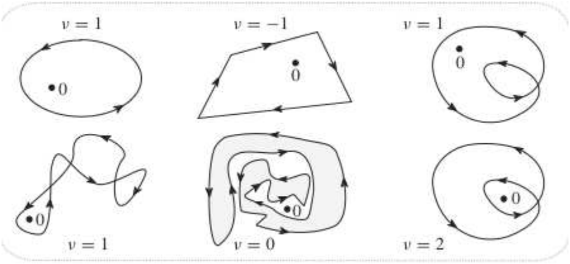
Topology is a branch of mathematics where we study those properties of a system which remain the same if small changes are made in the system

If we can define an integer which remains the same under small changes, it is called a topological invariant

Being an integer, a topological invariant does not change under small perturbations of the system

Closed curve in two dimensions

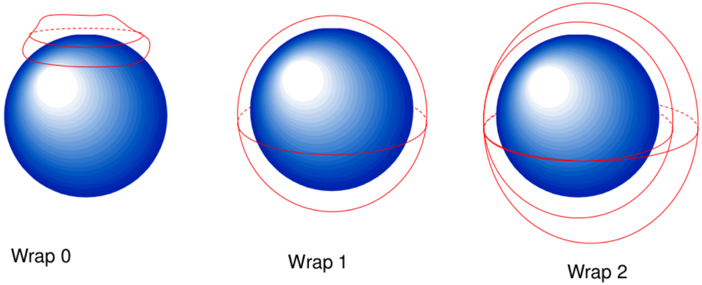
An example of a topological invariant is the number of times a closed curve in a plane winds around the origin in the anticlockwise direction. This integer is called the winding number



http://usf.usfca.edu/vca/PDF/vca-winding.pdf

Closed surface in three dimensions

Another example of a topological invariant is the number of times the surface of one sphere (red) wraps around the surface of another sphere (blue) in three dimensions



http://www3.nd.edu/ mbehren1/presentations/spheres.pdf

Berry curvature

Consider a two-dimensional system with a translation invariant Hamiltonian which has a number of bands. In band n, let $E_n(k_x, k_y)$ and $\psi_n(k_x, k_y)$ denote the energy and wave function as functions of the momentum (k_x, k_y)

We define the Berry connection

$$\begin{aligned} A_{n,x} &= i\, \psi_n^\dagger \frac{\partial \psi_n}{\partial k_x} \\ A_{n,y} &= i\, \psi_n^\dagger \frac{\partial \psi_n}{\partial k_y} \end{aligned}$$

The Berry curvature is then

$$B_n = \frac{\partial A_{n,y}}{\partial k_x} - \frac{\partial A_{n,x}}{\partial k_y} = -\, 2\, \mathrm{Im} \left[\frac{\partial \psi_n^\dagger}{\partial k_x} \frac{\partial \psi_n}{\partial k_y} \right]$$

Chern number

The Chern number is given by the integrated Berry curvature

$$C_n = \frac{1}{2\pi} \int\int dk_x dk_y \, B_n$$

If the wave functions have two components, the Chern number can be thought of as the wrapping number of one sphere around another

But we will study systems where the wave functions will have three components

Topological systems

A topological system has the following properties

- The different bulk bands are separated from each other by finite gaps

- One or more of the bulk bands are characterized by topological invariants which are non-zero integers

- There are states at the boundaries of the system which contribute to transport

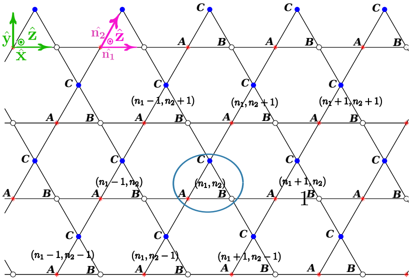
- Bulk-boundary correspondence: The number of boundary states is related to the topological invariants of the bulk bands

Spin system on a kagome lattice

Some spin systems in two dimensions have magnon (spin wave) bands which are gapped and have non-zero Chern numbers

Mook, Henk and Mertig, Phys. Rev. B 89, 134409 (2014)

We have studied a model with anisotropic ferromagnetic and Dzyaloshinskii-Moriya (DM) interactions between nearest-neighbor sites on a kagome lattice. The unit cells are triangles labeled as $\vec{n}$ with three sites $A_{\vec{n}}$, $B_{\vec{n}}$ and $C_{\vec{n}}$



Hamiltonian

The Hamiltonian is given by a sum over nearest-neighbor pairs $\langle \vec{n}\vec{n}' \rangle$

$$
\begin{aligned}
H &= H_\Delta + H_{DM} \\
H_\Delta &= -J \sum_{\langle \vec{n}\vec{n}' \rangle} [\, A^x_{\vec{n}} B^x_{\vec{n}'} + B^x_{\vec{n}} C^x_{\vec{n}'} + C^x_{\vec{n}} A^x_{\vec{n}'} \\
&\qquad + A^y_{\vec{n}} B^y_{\vec{n}'} + B^y_{\vec{n}} C^y_{\vec{n}'} + C^y_{\vec{n}} A^y_{\vec{n}'} \\
&\qquad + \Delta\, (A^z_{\vec{n}} B^z_{\vec{n}'} + B^z_{\vec{n}} C^z_{\vec{n}'} + C^z_{\vec{n}} A^z_{\vec{n}'}) \,] \\
H_{DM} &= D\, \hat{z} \cdot \sum_{\langle \vec{n}\vec{n}' \rangle} (\vec{A}_{\vec{n}} \times \vec{B}_{\vec{n}'} + \vec{B}_{\vec{n}} \times \vec{C}_{\vec{n}'} + \vec{C}_{\vec{n}} \times \vec{A}_{\vec{n}'})
\end{aligned}
$$

Setting $J = 1$, we have two parameters to vary, Δ and D

In general, we can consider a model in which all sites have spin S, namely, $\vec{S}^2_{\vec{n}} = S(S+1)\hbar^2$ for all $\vec{n}$

Spin wave theory

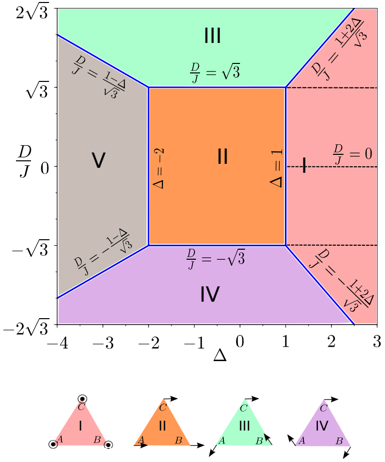
For large S , we can use spin wave theory to find the ground state phase diagram as a function of Δ and D

We first take the classical limit $S \to \infty$. We find the classical ground state spin configuration which minimizes the energy. Then we perform a Holstein-Primakoff transformation from spins to bosons. For instance, if at a site $A_{\vec{n}}$, the classical spin configuration has $A^z_{\vec{n}} = S\,\hat{z}$, we write

$$
\begin{aligned}
A^z_{\vec{n}} &= S\,\hat{z} - a^\dagger_{\vec{n}} a_{\vec{n}} \\
A^+_{\vec{n}} &\equiv A^x_{\vec{n}} + iA^y_{\vec{n}} \simeq \sqrt{2S}\, a_{\vec{n}} \\
A^-_{\vec{n}} &\equiv A^x_{\vec{n}} - iA^y_{\vec{n}} \simeq \sqrt{2S}\, a^\dagger_{\vec{n}}
\end{aligned}
$$

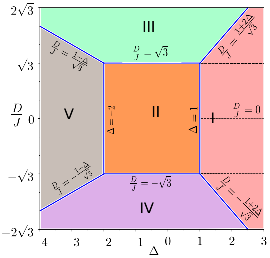
where $a^\dagger_{\vec{n}}$, $a_{\vec{n}}$ are bosonic creation and annihilation operators. We then write the Hamiltonian up to second order in the bosonic operators and diagonalize it to find the spectrum of excitations which are called magnons

Classical phase diagram



Classical phase diagram as a function of Δ and D

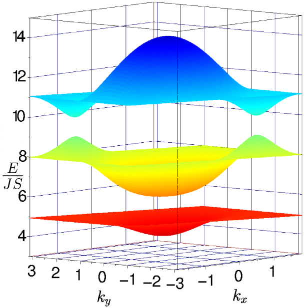
A topological phase

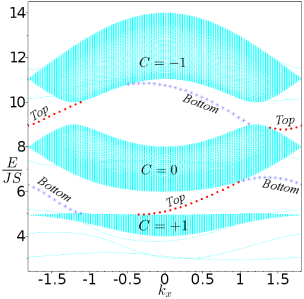


It turns out that phase I is topological. In fact, it has four topological sub-phases where the different magnon bands i are separated from each other and have different Chern numbers C_i. The transitions between these sub-phases occur at the lines $D/J = -\sqrt{3},\ 0,\ \sqrt{3}$

A strip of the system which is infinitely long in the $\hat{x}$ direction and a finite width in the $\hat{y}$ direction has edge states. The number of edge states lying at either the top or bottom edge within the $i^{\rm th}$ band gap is related to the Chern number as $\nu_i = |\sum_{j\leq i} C_j|$

Magnon dispersion and Chern numbers





Left: Magnon energy dispersion in the bulk for $\Delta = 2$ and $D/J = 2/\sqrt{3}$

Right: Dispersion as a function of k_x for bulk (shaded) and edge (dotted) states, and the Chern numbers of the different bands

Thermal Hall effect

The Chern number is obtained by integrating the Berry curvature $B_i(\vec{k})$. The Berry curvature leads to the thermal Hall effect: a temperature difference along some direction generates an energy current in the perpendicular direction

The thermal Hall conductivity κ^{xy} at a temperature T is given by a sum over all the magnon bands i

$$\kappa^{xy} = -\frac{k_B^2 T}{4\pi^2\hbar}\sum_i \int\int d^2\vec{k}\; c_2(\rho_i(\vec{k}))\, B_i(\vec{k})$$

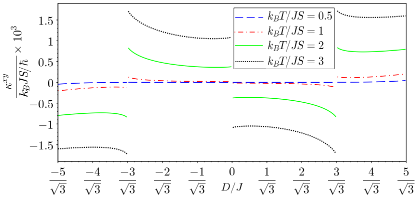
$$c_2(z) = (1+z)\left(\ln\frac{1+z}{z}\right)^2 - (\ln z)^2 - 2\mathrm{Li}_2(-z)$$

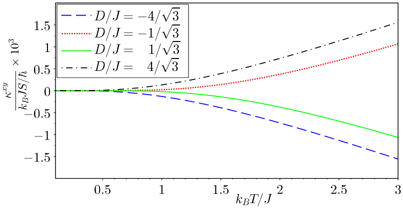
$$\mathrm{Li}_2(z) = -\int_0^z du\,\frac{\ln(1-u)}{u}$$

$$\rho_i(\vec{k}) = \frac{1}{e^{E_i(\vec{k})/(k_B T)} - 1}$$

Matsumoto and Murakami, Phys. Rev. B 84, 184406 (2011)

Thermal Hall conductivity





Variation of κ^{xy} with temperature T (top) and DM interaction strength D (bottom) for $\Delta = 2$

Summary

We have studied a spin system on a Kagome lattice with anisotropic ferromagnetic and Dzyaloshinskii-Moriya interactions between nearest-neighbor sites

The system has five different phases. One of these phases has magnon bands which are topologically non-trivial

Two of the magnon bands have non-zero Chern numbers. An infinitely long strip of the system has states which are localized at the top and bottom edges

The edge states give rise to a thermal Hall effect. (Unlike an electronic system, the magnon Hall conductivity is not quantized)

R. Seshadri and D. Sen, Phys. Rev. B 97, 134411 (2018)

Josephson junction of multiple superconducting wires

A junction of three or more superconducting wires may have Andreev bound states which have a non-zero Berry curvature as a function of the superconducting phases

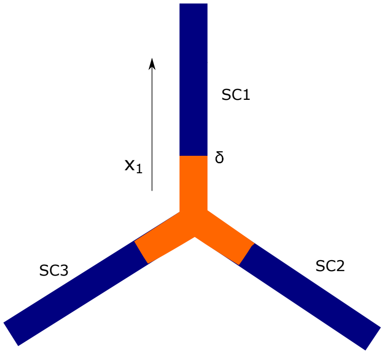
Riwar, Houzet, Meyer and Nazarov, Nature Communications 7, 11167 (2016)

The role of momentum in usual topological systems is played by the superconducting phases in the different wires

Junctions of four superconducting wires can be interesting because the Andreev bounds states may have Weyl points as a function of the superconducting phases

H.-Y. Xie, M. G. Vavilov, and A. Levchenko, Phys. Rev. B 97, 035443 (2018)

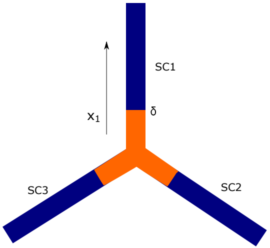
Josephson junction of three wires



Three superconducting wires meeting at a junction which is a normal region of size δ which is much smaller than the coherence length.

The junction is characterized by a scattering matrix S

Josephson junction of three wires



At the junction, there is normal reflection and transmission (electrons to electrons and holes to holes) governed by S, while in the $i^{\rm th}$ superconductor at $x_i = \delta$, there is Andreev reflection (electrons to holes with and holes to electrons with amplitudes proportional to $a\, e^{-i\phi_i}$ and $a\, e^{i\phi_i}$ respectively, where $a = (E - i\sqrt{\Delta^2 - E^2})/\Delta$ and $\Delta e^{i\phi_i}$ is the superconducting pairing amplitude). Some of the amplitudes have additional factors of $\pm$ depending on the spin of the electron and whether the superconductor has *s*-wave or *p*-wave pairing

Andreev bound state energies

For a system of 3 superconducting wires, we introduce an 3×3 diagonal matrix $e^{i\phi}$ whose diagonal entries are given by $e^{i\phi_i}$. We will assume that the magnitude of the pairing amplitude, Δ, is the same on all the wires

Recalling that $a = (E - i\sqrt{\Delta^2 - E^2})/\Delta$, we find that the Andreev bound state energies E and the corresponding electron wave functions ψ are given by the eigenvalue equation

$$S e^{i\phi} S^* e^{-i\phi} \psi \; = \; \frac{1}{a^2} \; \psi$$

if all the wires have s-wave pairing (we have suppressed the spin label σ)

Berry curvature and Chern number

Solving the eigenvalue equation, we obtain 3 bands of Andreev bound states for spin-up and 3 bands for spin-down quasiparticles. We label these bands as (n, σ)

If we vary only two of the phases, say, ϕ_1 and ϕ_2, and hold the third phase fixed, we have a two-dimensional system in which (ϕ_1, ϕ_2) play the same role as the momenta (k_x, k_y)

Hence, if all the bands are gapped, we can define the Berry curvature $B_{n,\sigma,ij}$ and the corresponding Chern number $C_{n,\sigma,ij}$ in each band

Berry curvature matrix

In a given band (n, σ), the Berry curvature matrix is given by

$$B_{n,\sigma,ij}(\phi_1, \phi_2, \cdots, \phi_N) \;=\; -\,2\;\mathrm{Im}\left[\frac{\partial\psi^\dagger_{n,\sigma}}{\partial\phi_i}\frac{\partial\psi_{n,\sigma}}{\partial\phi_j}\right]$$

This a real antisymmetric matrix

We find that for both spin-up and spin-down, $\sigma = \pm 1$,

$$\sum_{i=1}^{3} B_{n,\sigma,ij} \;=\; \sum_{j=1}^{3} B_{n,\sigma,ij} \;=\; 0$$

Hence the Berry curvature matrix has only one real parameter

Choices of scattering matrix at the junction

We consider a junction of three superconducting wires with *s*-wave pairing and two examples of the scattering matrix S at the junction

(i) A matrix which satisfies time-reversal symmetry, namely, S is symmetric

$$S = \begin{pmatrix} r & t & t \\ t & r & t \\ t & t & r \end{pmatrix}$$

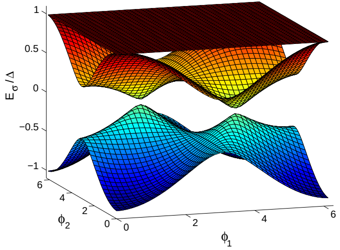
$$\text{where} \quad r = -\frac{1 + i\lambda}{3 + i\lambda} \quad \text{and} \quad t = \frac{2}{3 + i\lambda}$$

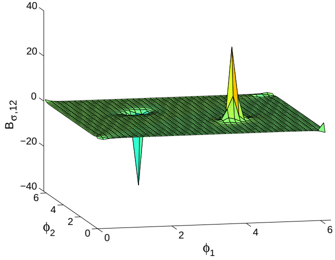
where λ is the strength of a barrier at the junction

(ii) A randomly generated unitary matrix which is asymmetric and therefore does not satisfy time-reversal symmetry

$$S = \begin{pmatrix} 0.8389 - 0.0346i & -0.2399 + 0.3146i & -0.3542 + 0.1138i \\ -0.0163 + 0.3446i & 0.7254 - 0.0341i & -0.4820 + 0.3483i \\ 0.2591 + 0.3299i & 0.3589 + 0.4328i & 0.7119 - 0.0341i \end{pmatrix}$$

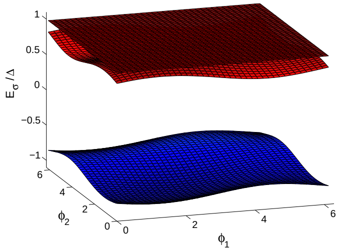
Energy bands and Berry curvature

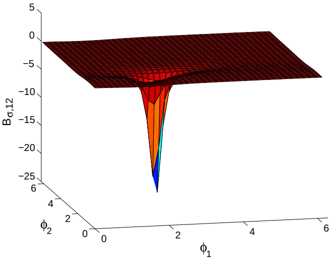




(Left) Andreev bound state energies for the three bands and (right) Berry curvature for the negative energy band $(C = 0)$, for the S scattering matrix with $\lambda = 0.1$ and $\phi_3 = 0$. One band lies at $E = \Delta$, and the other two have energies in $\pm$ pairs. These results do not depend on the spin. (For a symmetric matrix, the Chern number must be zero in each band)

Energy bands and Berry curvature





(Left) Andreev bound state energies for the three bands and (right) Berry curvature for the negative energy band $(C = -1)$, for the asymmetric S matrix and $\phi_3 = 0$. One band lies at $E = \Delta$, and the other two have energies in $\pm$ pairs. These results do not depend on the spin. (For an asymmetric matrix, the Chern number may be non-zero in some bands)

Josephson current

When a voltage V_i is applied to the superconducting wire i, the phase in that superconductor changes in time according to

$$\dot{\phi}_i = \frac{2e}{\hbar} V_i$$

Next, the Josephson current in wire i is given by

$$\begin{aligned} I_i &= \frac{1}{2} \sum_{\sigma=\pm 1} \sum_{n=1}^{N} [f(E_{n,\sigma}) - \frac{1}{2}] \\ &\times \left[\frac{2e}{\hbar} \frac{\partial E_{n,\sigma}}{\partial \phi_i} - 2e \sum_{j=1}^{N} B_{n,\sigma,ij} \dot{\phi}_j \right] \\ f(E_{n,\sigma}) &= \frac{1}{e^{\beta E_{n,\sigma}} + 1} \end{aligned}$$

The last term, proportional to the Berry curvature, is the new thing that can appear for a system with three or more wires

AC Josephson effect

Suppose that we apply constant voltages V_i. The first term in the Josephson current oscillates in time and averages to zero, but the second term (Berry curvature) has a non-zero time average

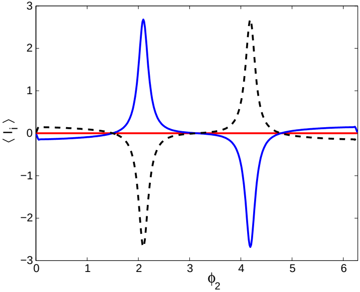
Let V_1 be non-zero while $V_2 = V_3 = 0$. Then $\phi_1 = (2e/\hbar)V_1 t$ and ϕ_2, ϕ_3 are constant. Taking $\phi_3 = 0$, we find, at zero temperature, that the time averaged currents are given by $\langle I_1 \rangle = 0$ and

$$\langle I_2 \rangle = -\langle I_3 \rangle = \frac{2e^2 V_1}{\hbar} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \left[B_{-,+1,12}(\phi_1, \phi_2) - B_{+,+1,12}(\phi_1, \phi_2) \right]$$

where the first subscript $\pm$ in $B_{n,\sigma,ij}$ denotes positive (negative) energy bands. We thus have a transconductance given by $\langle I_2 \rangle / V_1$. Further,

$$\int_0^{2\pi} \frac{d\phi_2}{2\pi} \frac{\langle I_2 \rangle}{V_1} = \frac{4e^2}{h} C_{-,+1,12}$$

AC Josephson effect

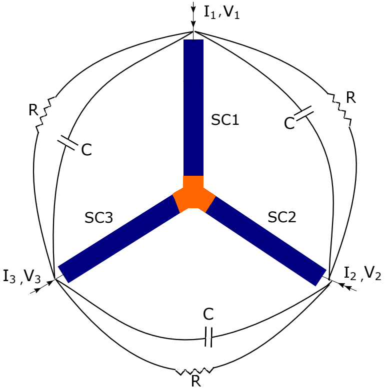


Plot of average currents $\langle I_1 \rangle$ (red solid line), $\langle I_2 \rangle$ (blue solid line), and $\langle I_3 \rangle$ (black dashed line) versus ϕ_2, for $V_1 = 5 \times 10^{-5}$ V, $V_2 = V_3 = 0$, $\phi_3 = 0$, $\Delta = 10^{-6}$ eV, and the symmetric S matrix with $\lambda = 0.1$. Currents are in units of $10^{-4}\,(e/\hbar)(\mathrm{eV}) \simeq 24$ nA

The peaks in $\langle I_2 \rangle$ coincide with the peaks in the Berry curvature

RC circuit with superconducting wires

We consider an RC circuit with three superconducting wires



Each of the superconducting wires is in parallel to a resistance R and a capacitance C. The voltages V_i and incoming currents I_i are shown

Circuit equations

The earlier equations for the current now become

$$I_i = \frac{1}{2}\sum_{n,\sigma}[f(E_{n,\sigma}) - \frac{1}{2}]\left[\frac{2e}{\hbar}\frac{\partial E_{n,\sigma}}{\partial \phi_i} - 2e\sum_{j\neq i} B_{n,\sigma,ij}\dot{\phi}_j\right] + \frac{\hbar}{2e}\sum_{j\neq i}[C(\ddot{\phi}_i - \ddot{\phi}_j) + \frac{\dot{\phi}_i - \dot{\phi}_j}{R}]$$

We now consider what happens if $I_1 = I + A\sin(\omega t)$ and $V_2 = V_3 = 0$. Then ϕ_2, ϕ_3 will remain constant, but ϕ_1 will vary with time following the equation

$$\frac{1}{2}\sum_{n,\sigma}[f(E_{n,\sigma}) - \frac{1}{2}]\left[\frac{2e}{\hbar}\frac{\partial E_{n,\sigma}}{\partial \phi_1}\right] + \frac{(N-1)\hbar}{2e}[C\ddot{\phi}_1 + \frac{\dot{\phi}_1}{R}] = I + A\sin(\omega t)$$

This is a non-linear equation and has to be solved numerically. But we can understand the solution qualitatively

Shapiro plateaus

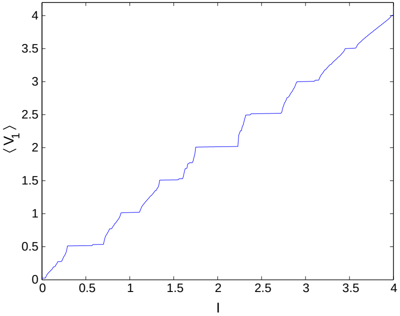
Since $E_{n,\sigma}$ is a function of ϕ_1 with period $2\pi,$ we can write

$$\frac{\partial E_{n,\sigma}}{\partial \phi_1} = \sum_{n=-\infty}^{\infty} c_n \, e^{in\phi_1}$$

We then find that the time-averaged value of $(2e/\hbar)V_1 = \dot{\phi}_1$ can have plateaus as a function of I at the values $(m/n)\,\omega,$ where m is an integer

For a two-wire junction with $E_{n,\sigma}$ proportional to $\cos\phi_1,$ only terms with $n = \pm 1$ appear in the Fourier series. Hence we can only get plateaus at integer multiples of $\omega.$ But more complicated functions $E_{n,\sigma}$ vs ϕ_1 can give plateaus at any rational multiple of $\omega,$ and we can have a devil's staircase of plateaus

Shapiro plateaus



Plot of $\langle V_1 \rangle$ vs I, for $\hbar\omega = 10^{-6}$ eV, $\hbar\omega C/e^2 = \hbar/(e^2 R) = 0.5$, $\phi_2 = 2\pi/3$, $\phi_3 = 0$, $\Delta = 10^{-6}$ eV, $A = 4$, and the symmetric S matrix with $\lambda = 0.1$. I, A are in units of $10^{-6} \times (e/\hbar)(\mathrm{eV}) \simeq 0.24$ nA, and V_1 is in units of 10^{-6} V.

We see prominent plateaus at $\langle V_1 \rangle$ at 0.5, 1, 1.5, 2, 2.5, 3, 3.5 times 10^{-6} eV (integer multiples of $\hbar\omega/(2e)$), and narrow plateaus at 0.25, 0.75, 1.75, 2.25 (half-odd-integer multiples of $\hbar\omega/(2e)$)

Summary

- A Josephson junction of three superconducting wires is a topological system in which the Andreev bound state bands have Berry curvature and Chern numbers

- The system has a transconductance; applying a constant voltage to one wire can produce a current in the other two wires. The integral of the transconductance over one of the superconducting phases is equal to the Chern number multiplied by $4e^2/h$

- Shapiro plateaus can appear in a plot of the time-averaged voltage versus current when the current in one of the wires varies periodically with time. The plateaus can occur at any rational multiple of the driving frequency

O. Deb, K. Sengupta and D. Sen, Phys. Rev. B 97, 174518 (2018)