# Kelvin circulation theorem, dynamic metric, and FQHE

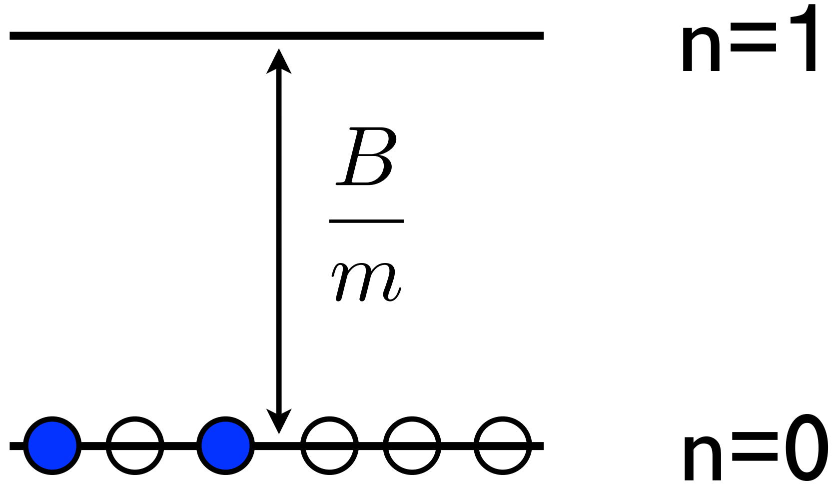
Dam Thanh Son (University of Chicago)
Novel Phases of Quantum Matter
ICTS, 27 December 2019

# Broad view

- Fractional quantum Hall effect, rich physics
  - wave function, CFT
  - TQFT, anyons
  - flux attachment
  - LLL approaches (Shankar-Murthy...)
  - duality

# Fractional quantum Hall effect

$$H = \sum_a \frac{(\mathbf{p}_a + e\mathbf{A}_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$



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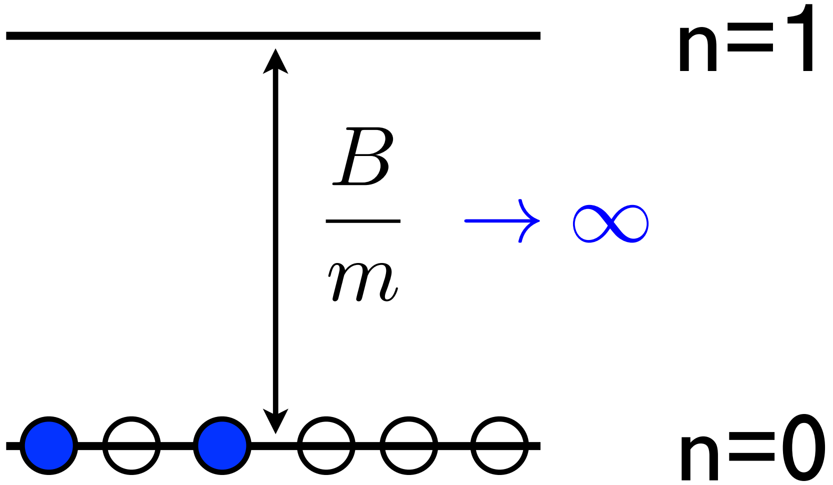
$m \to 0$



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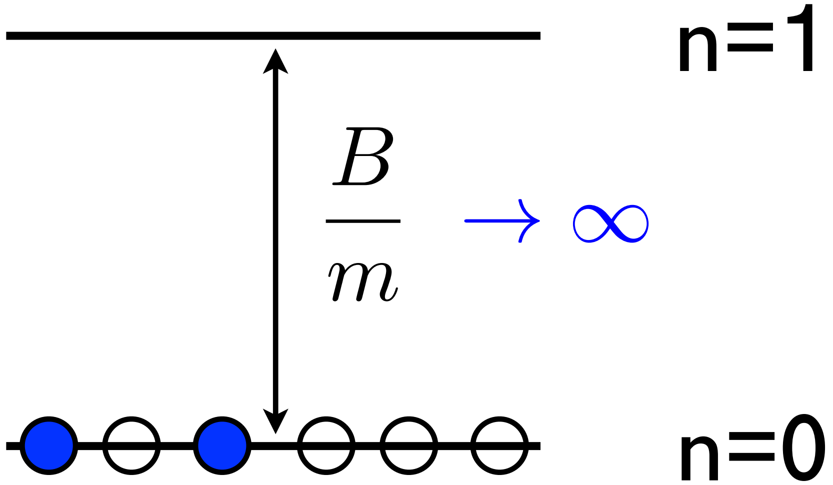
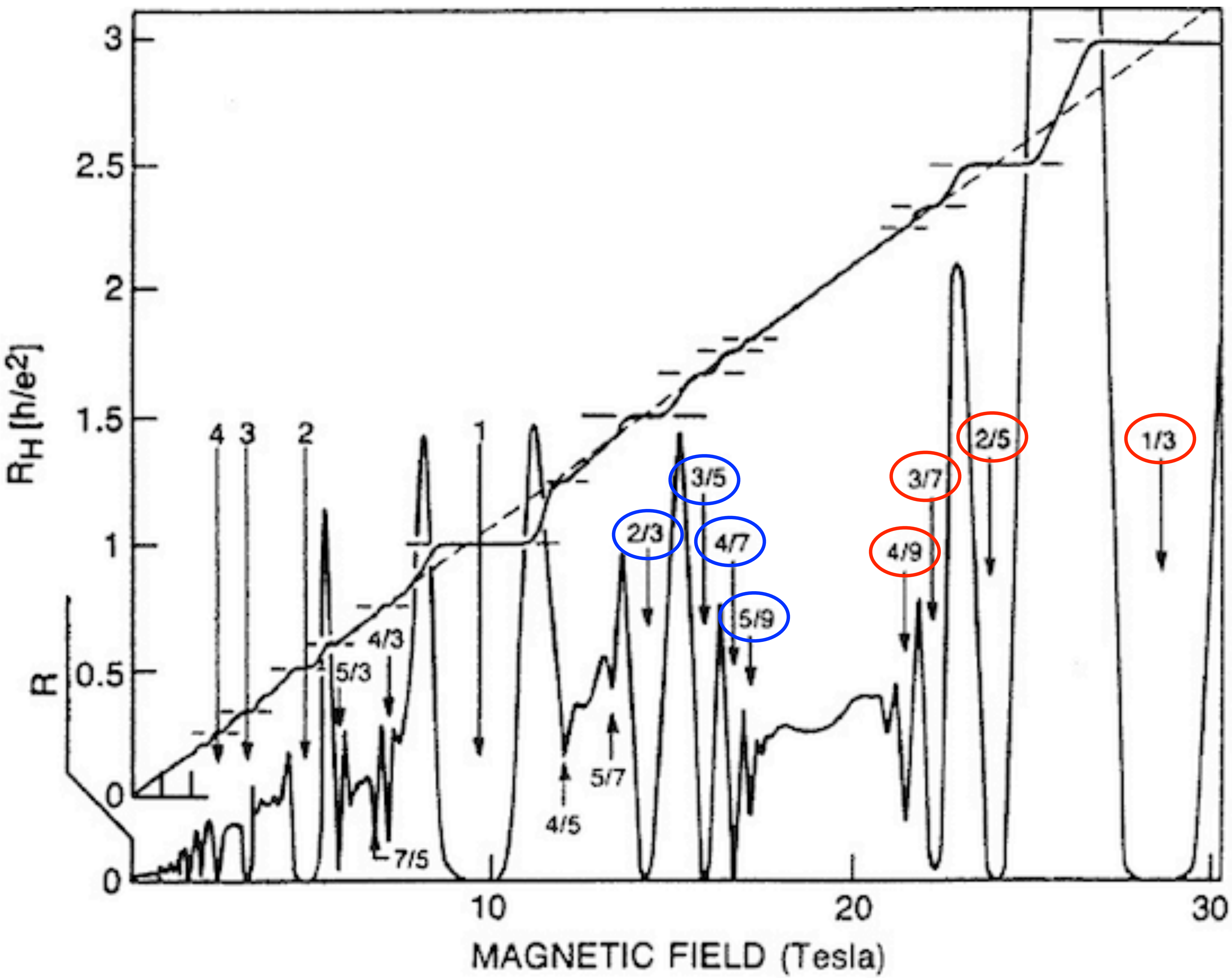



$$H = P_{\rm LLL} \sum_{a,b} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$

Projection to
lowest Landau level

# Rich phenomenology




gapped QH states at $\nu = \dfrac{N+1}{2N+1}$ $\nu = \dfrac{N}{2N+1}$

# Composite fermion

- Near half filling: a new quasiparticle
- Traditional view "composite fermion" = electron + 2 flux quanta
- That picture however is not particle-hole symmetric
- New view: composite fermion as a "dual fermion"

# Dirac composite fermion

- Composite fermions are “massless Dirac fermions”: have Berry phase pi around any loop around p=0
- Number of CFs = 1/2 number of magnetic flux quanta
- CFs live in a magnetic field $b = B - 4 \pi n_e$
- Number of electron is NOT  number of CFs

# LLL projection

- Some fundamental questions remain :
  - is there projection to LLL in the composite fermion theory?
  - in particular, does the theory realize the algebra of the projected electron density operator (GMP algebra)
  - does density-density correlator ~ q^4 ?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\rho \sim \frac{qj}{\omega}$$

$$\langle \rho\rho \rangle \sim q^2$$

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$$j \sim \frac{qT}{B}$$

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$$\langle \rho\rho \rangle \sim \cancel{q^2}\; q^4$$

$$\cancel{\frac{\partial}{\partial t}(m\mathbf{j})} + \nabla \cdot \mathbf{T} = \mathbf{j} \times \mathbf{B}$$

$$j \sim \frac{qT}{B}$$

$$m \to 0$$

# Spin of magnetoroton

- Lowest neutral excitation: magnetoroton (Girvin, MacDonald, Platzman 1986)
- near q=0: excitations can be classified by orbital angular momentum (spin)

$$\langle 0|\rho|0\rangle \sim \underbrace{\langle 0|\rho|\mathrm{MR}, q\rangle}_{q^2} \underbrace{\langle \mathrm{MR}, q|\rho|0\rangle}_{q^2} \sim q^4$$

$$\langle \mathrm{spin} \pm n, q|\rho|0\rangle \sim (q_x \pm i q_y)^n$$

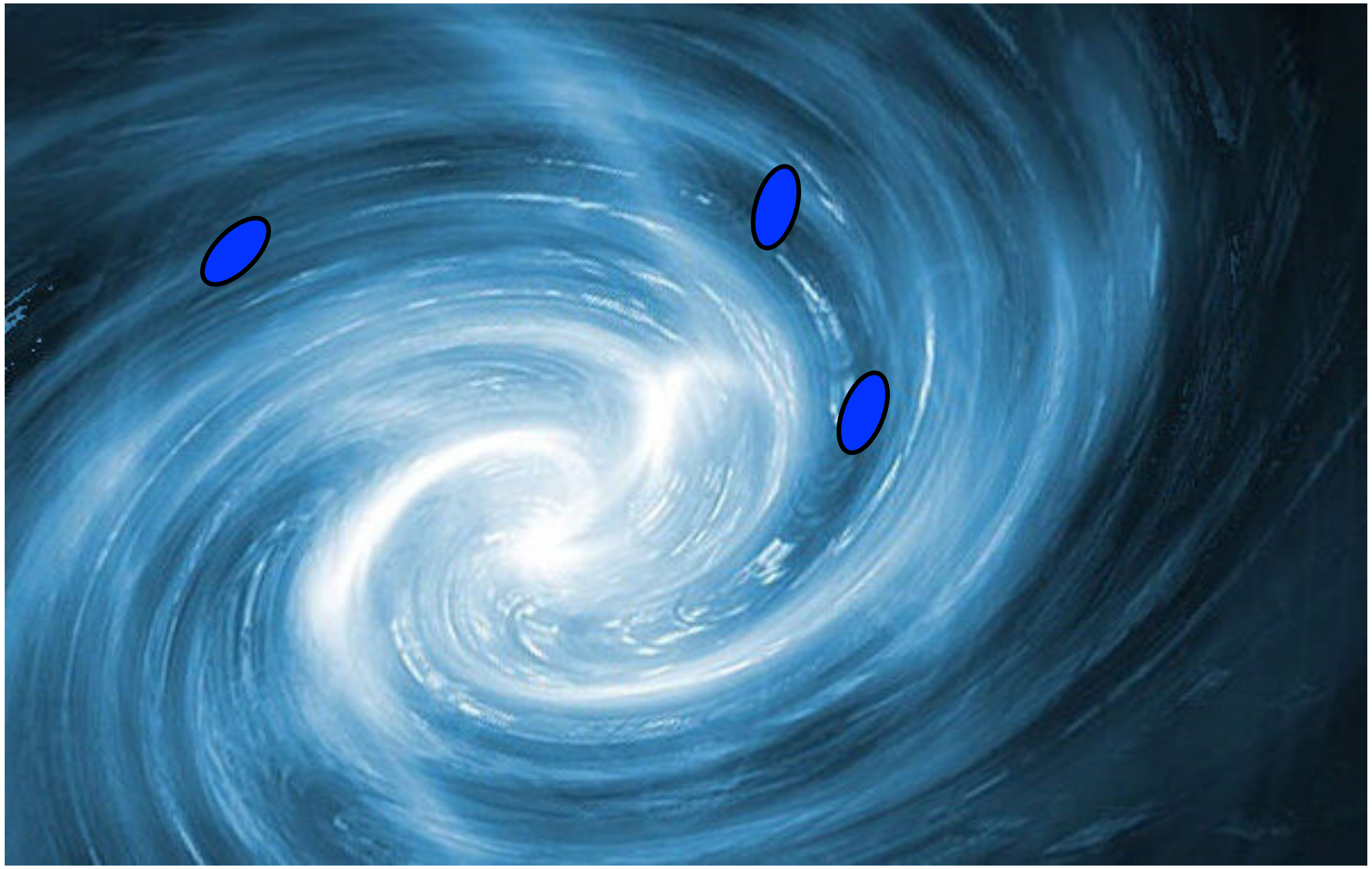
Magnetoroton at q=0 has spin 2
a dynamical graviton (Haldane)?

# A fluid view on FQHE

- Quantum Hall fluid is a fluid, described by a hydrodynamic theory
- The fluid has an internal tensor degree of freedom: a dynamic "metric"
- Particles of the fluid are dipoles
- Provide a consistent and predictive picture of the long-distance behavior of QH fluids

Ref.: DTS 1907.07187
works with A. Gromov and D.X. Nguyen

# A crash course in hydrodynamics



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# Hydrodynamics

Landau 1941

- can be formulated as a dynamical system with the Poisson brackets

$$\{\pi_i(\mathbf{x}),\, n(\mathbf{y})\} = n(\mathbf{x})\partial_i\delta(\mathbf{x}-\mathbf{y})$$

$$\{\pi_i(\mathbf{x}),\, \pi_j(\mathbf{y})\} = [\pi_j(\mathbf{x})\partial_i + \pi_i(\mathbf{y})\partial_j]\delta(\mathbf{x}-\mathbf{y})$$

- and Hamiltonian

$$H = \int d\mathbf{x}\left[\frac{1}{2m}\frac{\vec{\pi}^2(\mathbf{x})}{n(\mathbf{x})} + \epsilon(n(\mathbf{x}))\right]$$

$$\dot{n} = \{H,\, n\}$$

$$\dot{\pi}_i = \{H,\, \pi_i\}$$

# Extending Poisson algebra

- Let's introduce a "dynamical metric" $G_{ij}(x)$
- The fact that $G_{ij}$ transforms like a tensor fixes the Poisson bracket

$$\{G_{ij}(\mathbf{x}),\, \pi_k(\mathbf{y})\} = (G_{ik}(\mathbf{x})\partial_j + G_{jk}(\mathbf{x})\partial_i + \partial_k G_{ij})\delta(\mathbf{x}-\mathbf{y})$$

- {G, G} = ?

# Chiral metric hydro

In 2 spatial dimensions

$$\{G_{ij}(\mathbf{x}),\, G_{kl}(\mathbf{y})\} = -\frac{1}{s}(\varepsilon_{ik}G_{jk} + \varepsilon_{il}G_{jk} + \varepsilon_{jk}G_{il} + \varepsilon_{jl}G_{ik})\delta(\mathbf{x}-\mathbf{y})$$

- We can consistently impose $(\det G)^{1/2} = n$
- Hydrodynamics equations

$$\dot{A} = \{H,\, A\}$$

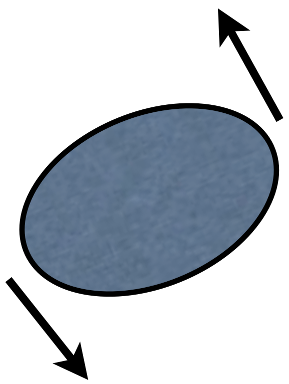
# Qualitative discussion

- "Metric" perturbation: a gapped spin-2 mode

$$G_{ij} = n(\delta_{ij} + Q_{ij})$$

$$Q_{xx} = -Q_{yy} \sim \cos\omega t$$

$$Q_{xy} \sim \sin\omega t$$



$$\omega = \frac{2\mu}{ns}$$

"Lamé constant"

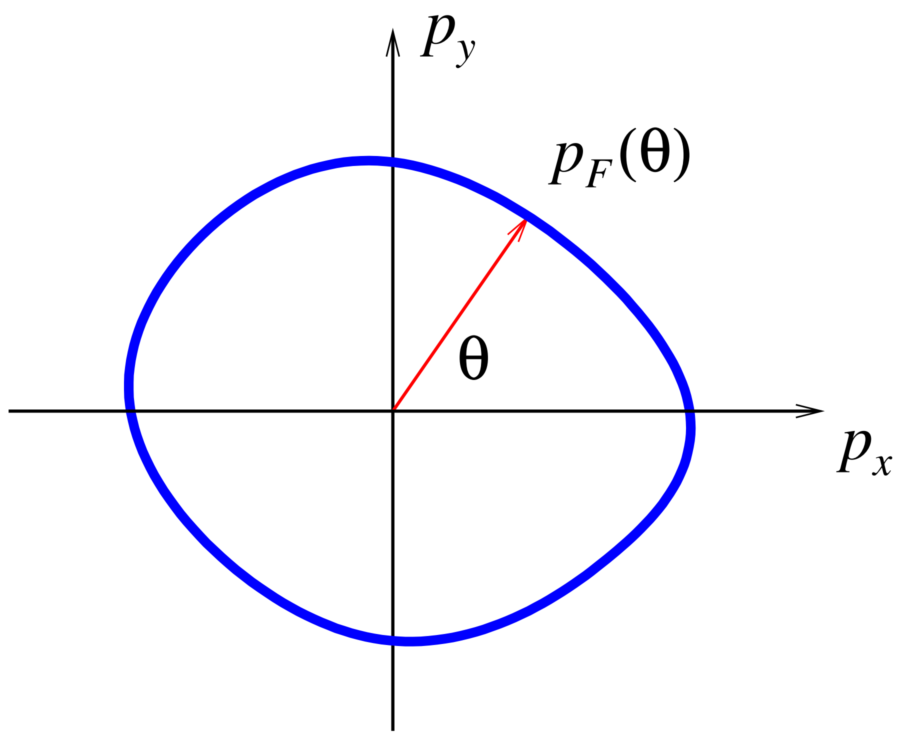
At small frequencies, a fluid with Hall viscosity

$$\eta^H = \frac{sn}{2}$$

s = average "orbital spin"

# What is the relevance to FQHE?

# CF Fermi surface




Low-energy, long-wavelength excitations: fluctuations of the shape of the Fermi surface

$$p_F(t, \mathbf{x}, \theta) = p_F^0 + \sum_{n=-\infty}^{\infty} u_n(t, \mathbf{x})\, e^{-in\theta}.$$

One scalar field per spin

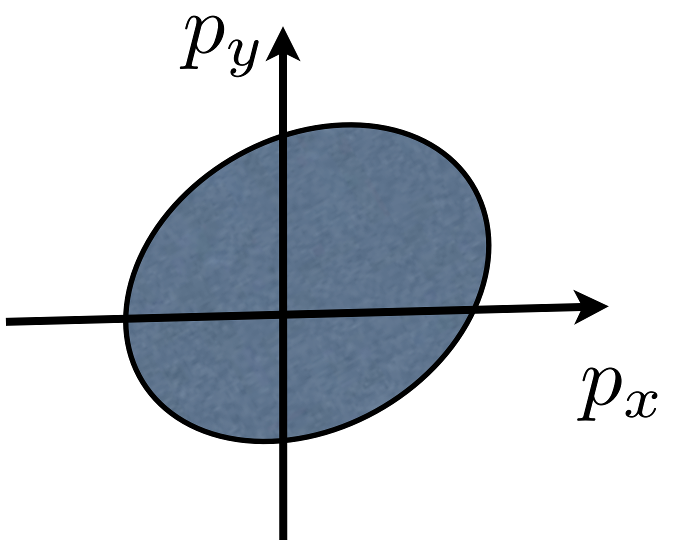
At low momenta we can limit ourselves to a few lowest modes

$$v_F q \ll \omega$$

$G_{ij}$ parametrizes an elliptical Fermi surface

# "Nematic" hydrodynamics

- Degrees of freedom:




  - density

$$n(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^2} f(\mathbf{x}, \mathbf{p})$$

  - momentum density

$$\pi_i(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^2} p_i f(\mathbf{x}, \mathbf{p})$$

  - effective metric

$$\int \frac{d\mathbf{p}}{(2\pi)^2} p_i p_j f(\mathbf{x}, \mathbf{p}) = \frac{\pi_i \pi_j}{n} + \pi n(\mathbf{x}) G_{ij}(\mathbf{x})$$

$$\sqrt{\det G} = n$$

# Chiral metric hydro

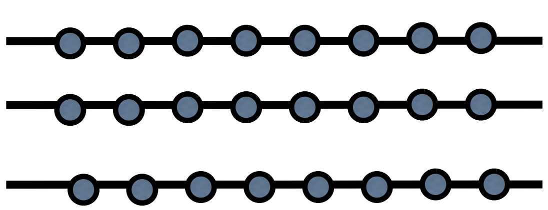
- The commutation relations can be "derived" from semiclassical arguments

s related from the Hall viscosity

= average "orbital spin" of composite fermion

$$\nu = \frac{N}{2N+1}$$

$$s = \frac{1}{N+\frac{1}{2}}\left(\frac{1}{2}\cdot 0 + 1 + 2 + \cdots + N\right) = \frac{N(N+1)}{2N+1}\,.$$



$n = 1$

# Dipoles

- On the LLL, the CF are electric dipole
- dipole moment proportional to and perpendicular to momentum
- dipole density

$$\frac{\epsilon^{ij}\pi_j}{B}$$

# Electron density

$$\rho = \frac{B-b}{4\pi} - \epsilon^{ij}\partial_i\left(\frac{\pi_j}{B}\right)$$

dipole contribution

$$= n - \frac{b+\omega}{4\pi}$$

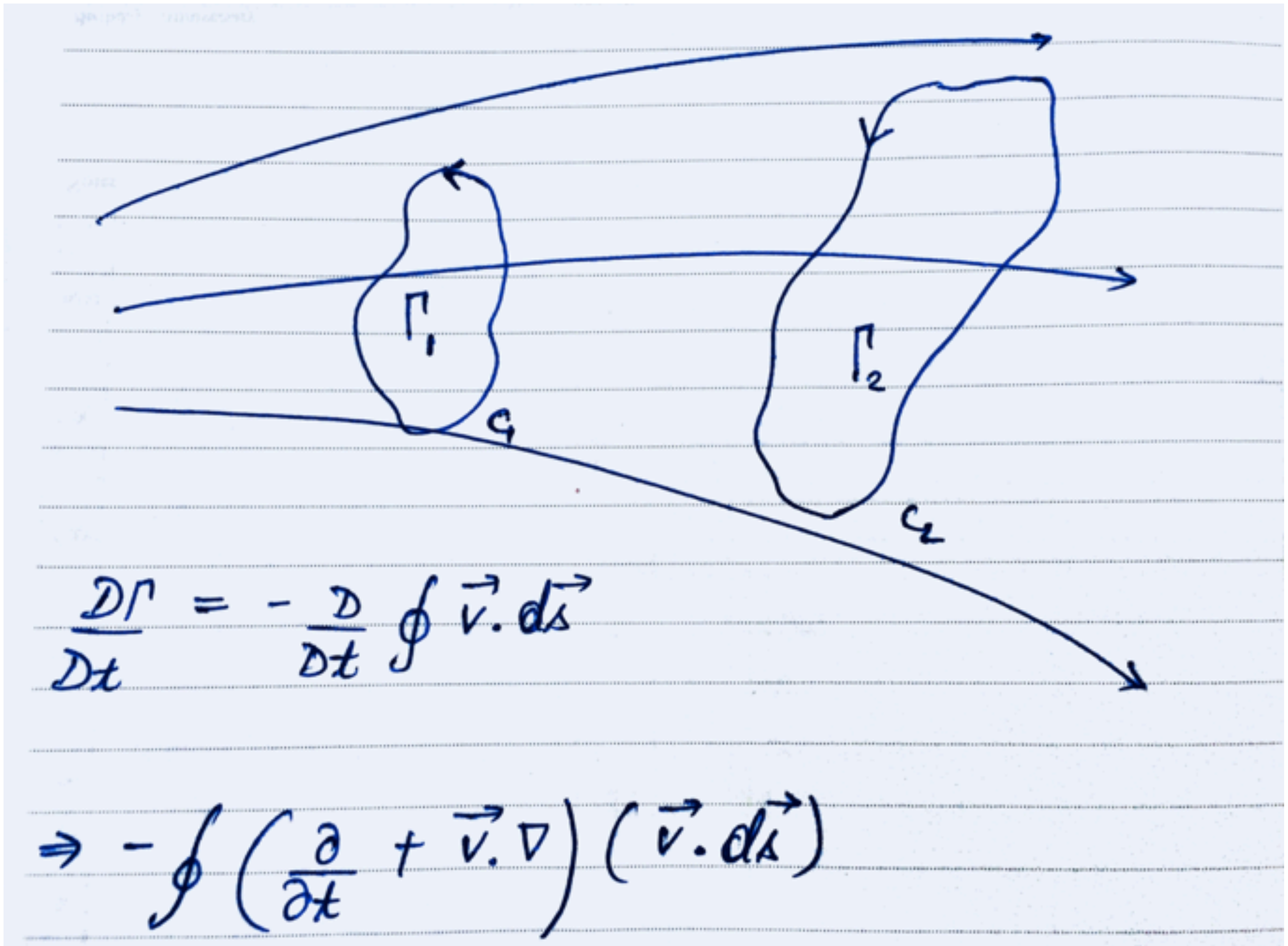
$$\omega = \vec{\nabla} \times \left(\frac{\vec{\pi}}{n}\right)$$

"vorticity"

$$H = H_0[n, \pi_i, G_{ij}] + \int d\mathbf{x}\left(-a_0 n + \frac{\varepsilon^{ij}E_j}{B}\pi_i\right)$$

# Kelvin’s circulation theorem

1869

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- In ideal hydrodynamics vorticity is carried with the flow

vorticity

$$\omega = \vec{\nabla} \times \left( \frac{\vec{\pi}}{n} \right)$$

$$\dot{\omega} + \vec{\nabla} \cdot (\omega \vec{v}) = 0$$

- Leads to an infinite number of conserved quantities (Casimirs of the Poisson algebra)

# Kelvin's circulation theorem

- In the presence of magnetic field and metric degree of freedom, Kelvin's theorem is modified

$$\Omega = b + \omega + \frac{s}{2}\sqrt{G}R[G] \qquad \dot{\Omega} + \vec{\nabla}\cdot(\Omega\vec{v}) = 0$$

$$\rho_e = \frac{B}{4\pi} - \frac{b+\omega}{4\pi}$$

- $\Omega$ = constant

$$\boxed{\delta\rho_{\mathrm{e}} = \frac{s}{8\pi}\sqrt{G}\,R[G].}$$

# An immediate consequence

$$\delta\rho_e = \frac{s}{8\pi}\sqrt{G}R[G] \quad \sim \partial_i\partial_j G_{ij}$$

$$\to \langle \delta\rho_e \delta\rho_e \rangle_{\omega,q} \sim q^4$$

Property of the lowest Landau level

In fact, numerical coefficient can be found for $\nu = \frac{N}{2N+1}$

$$\langle \delta\rho_{\rm e} \delta\rho_{\rm e} \rangle_q = \frac{N(N+1)}{2N+1}\frac{q^4}{16\pi B}\,.$$

*N*=1: matches exactly with the Laughlin wave function

# Conclusion

- Low-q regime of FQH liquid: described by a fluid with internal metric degree of freedom, coupled to a gauge field
- Electron density ~ curvature of dynamic metric
- Static structure factor: algebraic calculation

# Thank you