International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

The Maldacena Duality Conjecture and Applications

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Subramanyan Chandrasekhar Lecture
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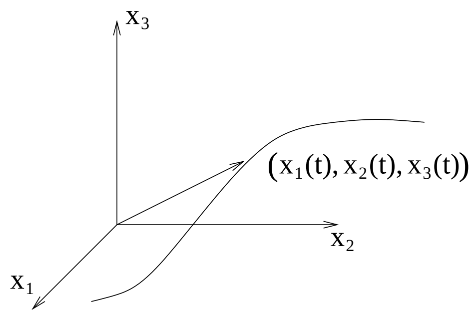
Main message of this lecture is that:

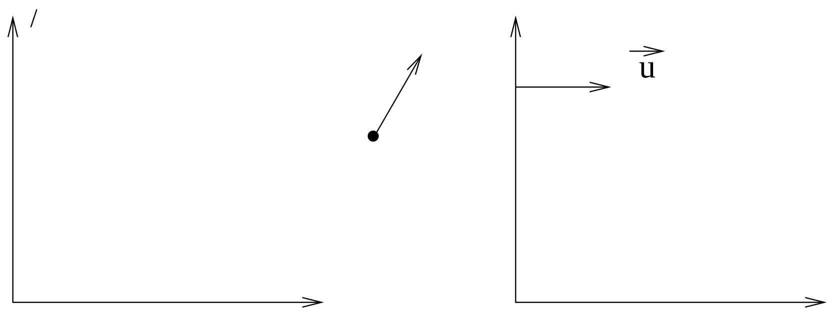
- Strongly coupled QFT is dual to a Einstein type theory of gravity
- Under certain circumstances under which Newton's coupling $G_N^{-1} \sim$ (effective degrees of freedom) is small the strong coupling quantum field theory can be solved by semi-classical gravity

Space Time-1 (Galileo, Newton)



t is 'universal time' the same for all observers traveling at a constant velocity relative to each other.

Simultaneity of events is independent of the 'state of motion'.



$$\vec{v}' = \vec{u} + \vec{v}$$

Transformation properties of space and time are reflected in the covariance of the equations of motion.

e.g.

Newton's laws: $m\ddot{\vec{x}}_\alpha = \vec{F}_\alpha(\vec{x}_1, \cdots, \vec{x}_n)$,

$\alpha = 1, \cdots, n$

Navier-Stokes eqns.:

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) \vec{v} = \nu \nabla^2 \vec{v} - \vec{\nabla} P$$

$\nabla \cdot \vec{v} = 0, \quad \vec{v}(\vec{x}, t) =$ fluid velocity

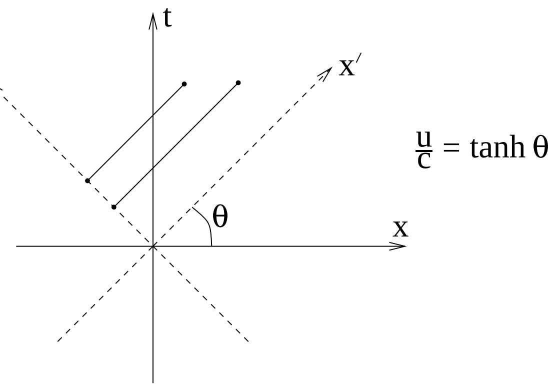
$P(\vec{x}, t)$ is the pressure

Space Time-2 (Einstein Special Relativity)

Speed of light is finite!

Simultancity of events is no more independent of the state of relative motion.

Space and time become space-time as they transform into each other.



Once again transformation properties of space time are reflected in the covariance of the equations of electro-magnetism

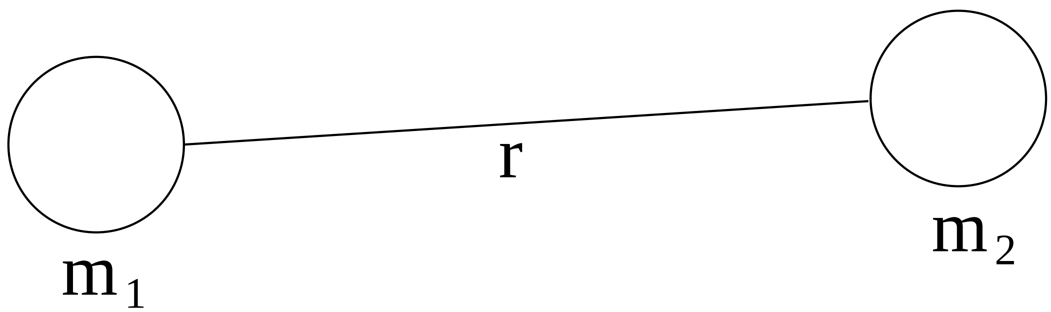
$$\frac{\partial}{\partial x^\mu} F^{\mu\nu}(x) = J^\nu(x) \text{ (Maxwell's equations)}$$

(Historically Maxwell's equations came before special relativity.)

Space Time-3 (Einstein's General Relativity)

Newton's law of gravitation

$$\vec{F}_{12} = -G_N \frac{m_1 m_2}{r^2} \hat{r}$$



Gravitational force is instantaneous

Inconsistent with special relativity

Einstein (1907): Principle of equivalence

"The effect of a constant gravitational field is equivalent to a uniformly accellerated frame."

One important consequence of this is that a clock slows down as the strength of the gravitational field increases or equivalently a photon looses energy as it ‘climbs’ in a gravitational field (ν decreases, λ increases)

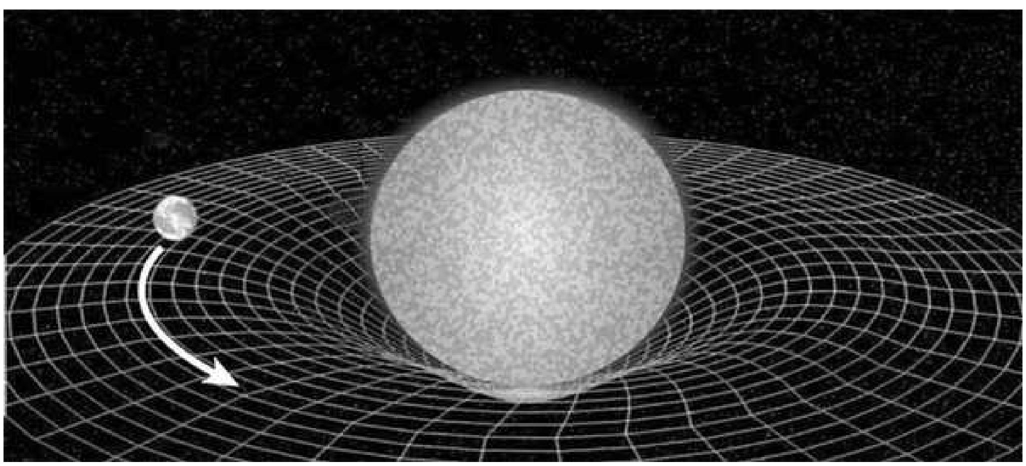
$$\Delta\tau = \sqrt{1 - \frac{2MG_N}{c^2 r}}\Delta t$$

General Relativity (1915):

Explains 'gravity' as a local warping of space-time.

The metric $G_{\mu\nu}(x,t)$ of space-time responds to matter according to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R - \wedge G_{\mu\nu} = -8\pi G_N T_{\mu\nu}$$



Black Holes:

Einstein's equations predict black holes.

(Schwarzschild, Oppenheimer and Snyder, Chandrasekhar)

$M > 3M_\odot$ gravitational collapse to form a black hole

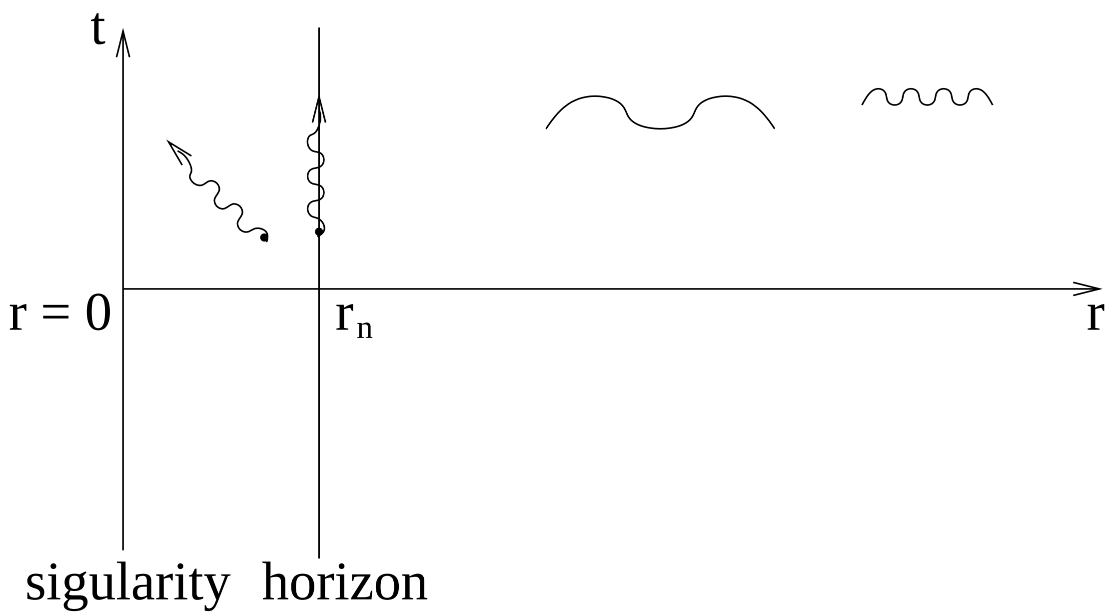
$$
\begin{aligned}
ds^2 &= G_{\mu\nu}dx^\mu dx^\nu \\
&= -\left(1-\frac{2GM}{c^2 r}\right)dt^2 + \frac{1}{\left(1-\frac{2GM}{c^2 r}\right)}dr^2 + r^2 d\Omega^2
\end{aligned}
$$

The most important feature of a black hole space-time is the fact that it has a 'horizon'.

$\left(1-\frac{2GM}{c^2 r}\right) = 0$

$r_H = \frac{2GM}{c^2}$

Black Hole Horizon



$$\Delta\tau = \sqrt{1 - \frac{2GM}{c^2 r}}\Delta t$$

classical general relativity $\Rightarrow$ black hole is black. No light can escape from $r < r_h$. Matter falling into a black hole cannot come out. As you approach $r \to r_h+$, clocks become infinitly slow: $\Delta\tau \to 0$.

Black hole horizon, Quantum Mechanics and Thermodynamics

In quantum mechanics if a black hole absorbs it has to emit

$$\langle i|H_{\text{int}}|f\rangle = \langle f|H_{\text{int}}|i\rangle^\star$$

Black holes are characterized by a temperature

$$T = \frac{\hbar c}{8\pi GM} = \frac{\hbar}{4\pi c r_h}$$

(Schwarzschild black hole)

and an entropy consitent with the first law of thermodynamics

$$S = \frac{A_h c^3}{4\hbar G_N} = \frac{A_h}{4\ell_p^2}, \qquad \ell_p^2 = \frac{\hbar G_N}{c^3}$$

(Bekenstein-Hawking)

The notion of entropy makes sense only in a finite quantum theory of gravity.

The black hole behaves like a thermodynamic object and satisfies the laws of thermodynamics.

1st Law : $dM = TdS + \Omega dJ + \phi dQ$

2nd Law : $(S_{bh} + S_{rad.}) \geq 0$

Black Hole Microstates

$$S_{BH} = \frac{A}{4\hbar G} \qquad\qquad S_B = \ell n \ \Omega$$

Geometry   Statistical mechanics

$$\text{Is} \quad S_{BH} \quad = \quad S_B \quad ?$$

Yes.

In many tractable cases this is indeed true (first example due to Strominger and Vafa). The black hole can be understood as a bound state of 'elementary objects' called D-branes.

D-branes are solitonic domain walls of string theory. A Dp brane is a p-dim. domain wall. p can take values between 0 and 9.

e.g. a D0 brane is a point like object

D1 brane is a string like object

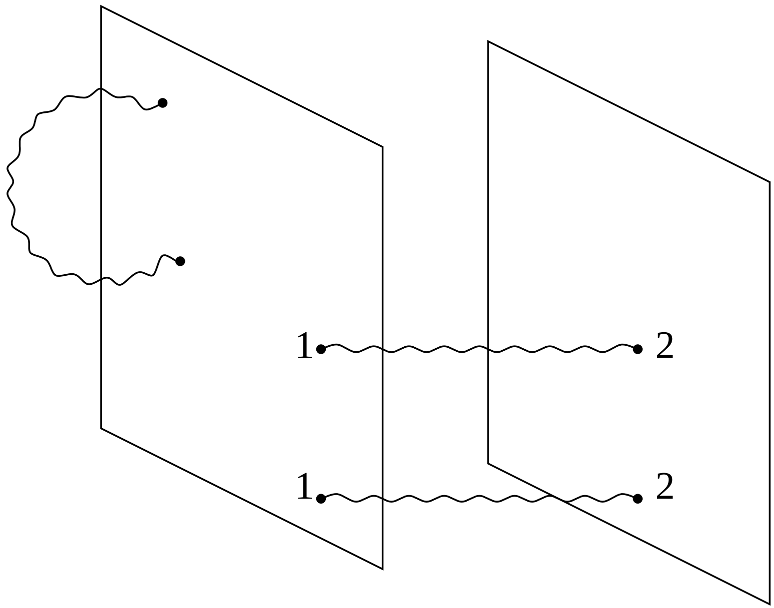
D2 brane is a 2-dim. membrane

etc.

Properties of D-branes:

(i) finite mass per unit volume $\propto \frac{1}{g_{\text{str}}}$. Gravitational potential $\propto \frac{1}{g_{\text{str}}}$.

(ii) interact by the emission and absorption of open strings. The massless mode of an open string is a 'gauge field' or a scalar.



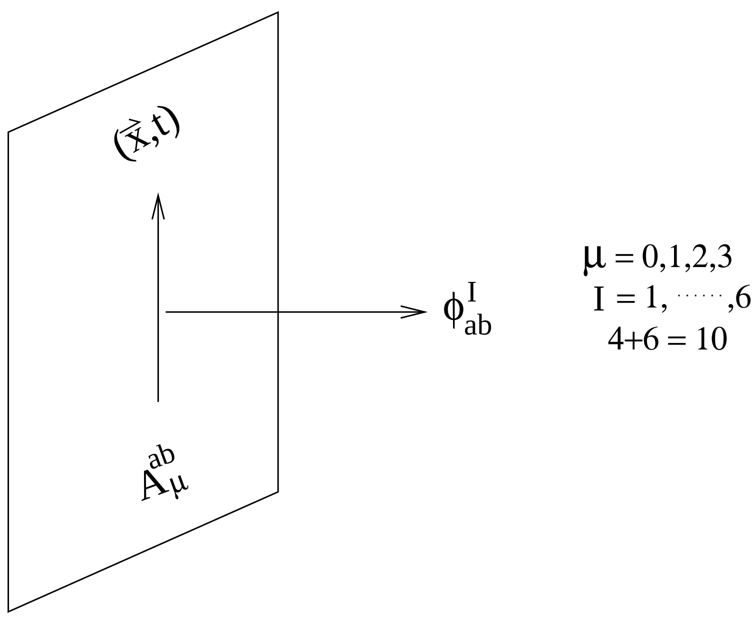
Hence there is a possibility of an abundant variety of bound states whose excitation spectrum in the long wavelength ($\gg \ell_s$) limit is given by a 'gauge theory'.

We now turn to a more detailed exposition of these ideas in the case N D3 branes.

D3-branes and the AdS/CFT correspondence (Maldacena Duality) Gauge theory:

N coincident $D3$ branes.

In the long wavelength limit $(\gg \ell_s)$, massless modes of the open strings are gauge fields



(i) in the 3+1 dim. space-time occupied by the $D3$ branes there is a 3+1 dim. $SU(N)$ gauge field

(ii) in the dim. transverve to the brane they behave like scalars in 3+1 dim.

The interactions of these low lying modes are governed by a $SU(N)$ non-abelian gauge theory containing the fields: A_μ, Φ^I, ψ^I (required by supersymmetry).

$$\mathcal{L} = -\frac{1}{4g_{YM}^2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{2}\mathrm{tr}\left(\nabla_\mu \Phi_I \nabla^\mu \Phi^I\right) - \sum_{I<J} \mathrm{tr}\left([\Phi_I, \Phi_J]^2\right) + \mathrm{O.T.}$$

The trace is over $SU(N)$ indices

It turns out that this theory is super symmetric and conformally invariant in 3+1 dim. for all values of the coupling constant g_{YM}.

The space-time symmetry group is the conformal group in 3+1 dim. $SO(2,4)$. The other symmetry is $SO(6)$, because we have 6-scalars.

Hence the 'global' symmetry of the CFT is $SO(2,4) \times SO(6)$. (This symmetry is enhanced to a super-conformal symmetry).

Note that $SO(2,4)$ contains the simple scale transformation

$$\vec{x} \to \alpha \vec{x}, \; t \to \alpha t, \; \alpha > 0.$$

Gravity theory:

N $D3$ branes act as a source of fields of a (super) gravity theory in 9+1 dim. It turns out that the space time thus created is

$$\mathrm{AdS}_5 \times S^5$$

AdS_5 is a 'hyperbolic space' of -ve constant curvature in 5-dim. S^5 is the 5-dim. sphere.

AdS_5:

$$x_0^2 + x_5^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2$$

S^5:

$$y_1^2 + y_2^2 + y_3^3 + y_4^2 + y_5^2 = R^2$$

The isometry of $\mathrm{AdS}_5 \times S^5$ is precisely $SO(2,4) \times SO(6)$. Geometrical symmetry of the space-time is the same as the global symmetry of the CFT.

AdS$_5$:

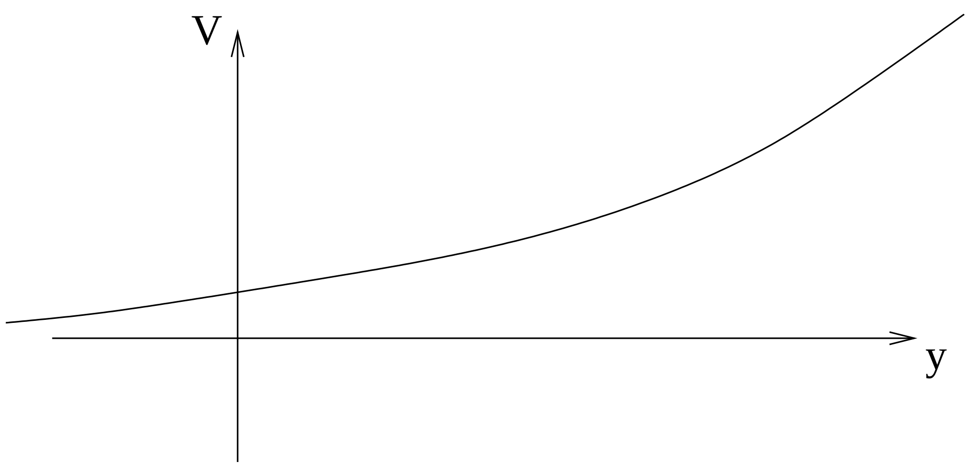
$$ds^2 = R^2 \left[e^{2y} \left(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + dy^2 \right]$$

$y \to \infty$ is the 4-dim. boundary of the space time

$$ds^2 = R^2 e^{2y} \left(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right)$$

Gravitational potential energy of a mass m

$$V = mc^2 \sqrt{-g_{00}} \propto e^y$$



$y \to \infty$ is the boundary of Minkowski space-time in 3+1 dim.

$$dS^2_{\text{boundary}} \approx R^2 e^{2y} \left(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 \right)$$

Scale invariance of the metric:

$$x_\mu \to \alpha x_\mu, \; y \to y - \ell n \; \alpha$$

The extra dim. can be interpreted as the scale of the QFT. Scaling x_μ to larger values corresponds to going deeper into the interior of AdS_5 space.

Short distance physics is coded near the boundary.

Long distance physics is coded deep in the interior of AdS_5.

One can imagine generalizing the 'warp factor' e^{2y} to a more general function $e^{A(y)}$.

Precise statement of AdS/CFT correspondence (Maldacena)

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dim. is dual to type IIB string theory in a space-time with $\mathrm{AdS}_5 \times S^5$ boundary conditions.

Correspondence of gauge theory and string theory parameters:

(YM coupling) $g^2_{YM} = g_s$ (string coupling)

(t'Hooft coupling) $\lambda = g_s N = \left(\frac{R}{\ell_s}\right)^4$

R is the AdS radius and ℓ_s is the string length

(Newton's coupling) $G_{10} = R^5 G_5 = g_s^2 \ell_s^8$

$$G_{10} = \frac{1}{N^2}\left(\frac{R}{\ell_s}\right)^8 \ell_s^8$$

AdS/CFT is a precise formulation of 'holography':

"Quantum Field Theory in d dim. is a hologram of a theory with gravity in $d+1$ dim."

Qualitative argument follows from the Bekenstein-Hawking area law for the entropy of a black hole and the second law of thermodynamics.

The entropy of a physical system in a volume V with surface area A is not greater than the entropy of a black hole that can be formed in the same region.

$$S_V \leq S_{BH} = \frac{A}{4\hbar G_N}$$

String theory $\rightarrow$ supergravity

$\frac{R}{\ell_s} \gg 1$ $\quad$ ℓ_s is the fundamental length of string theory

- String theory can be approximated by a theory of gravity in an expansion in $\frac{\ell_s}{R} \ll 1$.

- $\frac{R}{\ell_s} \gg 1$ implies t'Hooft coupling $\lambda \gg 1$ t'Hooft coupling is 'strong'.

- However if $N^2 \gg 1$ such that
$$G_{10} = \left(\frac{R}{\ell_s}\right)^8 \frac{\ell_s^8}{N^2} = \frac{\lambda^2 \ell_s^8}{N^2} \ll 1$$
Then semi-classical treatment of gravity is valid!

Strong coupling $(\lambda \gg 1)$ theory can be analysed by semi-classical gravity $(G_{10} \ll 1)$. This requirement can be somewhat relaxed by choosing to work in the long wavelength limit.

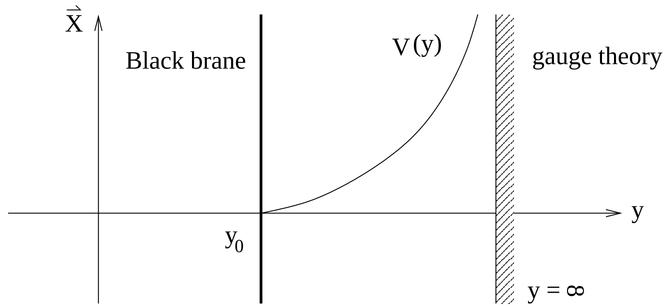
Tests of AdS/CFT

The AdS/CFT correspondence has been tested for exact agreement using the large amount of supersymmetry of the supersymmetric gauge theory.

e.g. masses of supersymmetric states do not depend on the gauge coupling. These are calculated at weak coupling with exact agreement with gravity calculations at strong coupling.

Applications of the AdS/CFT correspondence

Near extremal $D3$ branes in AdS_5 at $y = y_0$



$$ds^2 = R^2 \left[e^{2y} \left(-h dx_0^2 + d\vec{x}^2 \right) + \frac{dy^2}{h} \right] + R^2 d\Omega_5^2$$

$$h = 1 - e^{-4(y - y_0)}$$

$y = y_0$ is the horizon where $h = 0$. The $D3$ brane near the boundary $y \to \infty$ is AdS_5. It is a solution of the 5-dim. Einstein eqns. with a -ve cosmological constant

$$R_{\mu\nu} = -\frac{4}{R^2} G_{\mu\nu}$$

All solutions have constant scalar curvature

$$G^{\mu\nu} R_{\mu\nu} = -\frac{20}{R^2} < 0$$

Thermodynamics of the $D3$ black brane

Horizon implies a

Hawking temperature: $T = \frac{e^{y_0}}{\pi R}$

The Bekenstein-Hawking formula for the entropy

$$s = \frac{S}{V_3} = \frac{\pi^6 R^8 T^3}{4G_{10}}$$

The holographic dual of a translationally invariant black brane in AdS_5 with a horizon at $y = y_0$ is the $\mathcal{N} = 4$ gauge theory in 3+1 dim. at finite temperature T (Witten).

$$\text{AdS/CFT} \Rightarrow R^4 = \frac{\sqrt{8\pi G_{10}}}{2\pi^{5/2}} N$$

$$\Rightarrow s = \frac{S}{V_3} = \frac{\pi^2}{2} N^2 T^3$$

Conformal invariance $\Rightarrow$

$$\epsilon = 3P$$

$$\epsilon + P = Ts$$

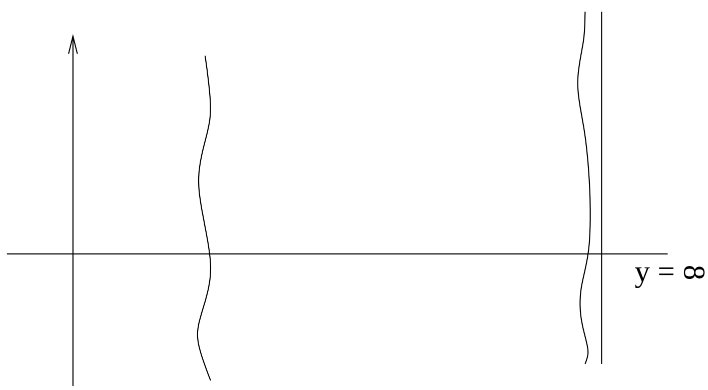
$$P = \frac{1}{4} Ts = \frac{\pi^2}{8} N^2 T^4$$

Fluctuating horizons and Fluid dynamics

Generalize black brane (hole) thermodynamics to fluid dynamics

Local thermal equilibrium

Parameters of the metric, e.g. the horizon become slowly varying functions of the boundary coordinates $(\vec{x}, t)$. $r_h = r_h(\vec{x}, t)$



A ripple on the horizon of a black hole is absorbed in a characteristic time by the black brane. Black holes (branes) have no hair in 3+1 dim.

Ripples can be analyzed in terms of quazi normal modes which have a complex frequency:

$$w = w_R + iw_I$$

$w_I \propto \frac{1}{T}$

In the gauge theory this corresponds to dissipation due to viscous effects.

Kubo formula:

$$\eta = \lim_{w \to 0} \frac{1}{2w} \int d^4x e^{iwt} \langle [T_{xy}(t, \vec{x}), T_{xy}(0)] \rangle \theta(t)$$

The stress tensor 'sources' the graviton in the gravity theory

$$S = \int d^4x \tilde{g}_{\mu\nu} T^{\mu\nu} + S_{QFT}$$

where $\tilde{g}_{\mu\nu}(\bar{x}, t) = \lim_{y \to \infty} g_{\mu\nu}(\vec{x}, t, y)$

Cross-section for absorption of graviton g_{xy}

$$\sigma(w) = V_3 \frac{8\pi G_N}{w} \int d^4 x e^{iwt} \langle [T_{x,y}(\vec{x},t), T_{xy}(0,0)] \rangle \theta(t)$$

V_3 is the 3-brane volume in the $\vec{x}$ direction.

On the gravity side $\sigma(w)$ is the absorption cross-section to 'fall into the horizon'. This can be calculated by solving a linear equation for $g_{xy}(\vec{x},t,y)$ in the background of the 3-brane geometry.

$$\sigma(w \to 0) = A_\hbar = 4 G_N S \quad (c = \hbar = 1)$$

Dhar, Mandal, Wadia, Das and Mathur

$$\eta = \frac{\sigma(0)}{16\pi G_N V_3} = \frac{s}{4\pi}$$

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}$$ Son, Policastro, Starinets

$$\frac{\eta}{s} \ll 1$$ as opposed to

the perturbative estimate $\frac{\eta}{s} \sim \frac{1}{\lambda^2 \ell n \frac{1}{\lambda}} \gg 1$ seems favored by RHIC.

AdS/CFT and non-linear Fluid Dynamics

$$\begin{aligned} r_0 &\Rightarrow \delta r_0(\vec{x},t) + r_0 \\ T_0 &\Rightarrow T_0 + \delta T_0(\bar{x},t) \end{aligned}$$

$r = r_0$

$$\frac{\partial}{\partial x^\mu}\frac{\delta T_0}{T_0} \sim \frac{1}{LT_0} \ll 1$$

Hydrodynamic description of the gauge theory is valid for largest time and length scales.

$$u^\mu(\vec{x},t),\ u^\mu u_\mu = -1,$$

$$T(\vec{x},t)$$

4 independent variables, the 4-velocity $u^\mu(\vec{x},t)$ and the local temperature $T(\vec{x},t)$. AdS/CFT gives a precise meaning to these variables at strong coupling.

Bhattacharya et. al. (BHMR):

$G_{10} \sim N^{-2} \to 0$

$D3$ brane in terms of "infalling"

Edington-Finklestein coordinates

$$ds^2 = 2dtdv - r^2 f(r)dt^2 + r^2 d\vec{x}^2$$

$\downarrow$ boosted metric
well behaved solution

$$ds^2 = -2u_\mu dx^\mu dv - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu$$

$$+r^2 P_{\mu\nu} dx^\mu dx^\nu$$

$P_{\mu\nu} = (\eta_{\mu\nu} + u_\mu u_\nu)$.

(u_μ, b) 4 parameters of

$$\frac{SO(3,1)}{SO(3)} \times D.$$

slowly varying $u^\mu(\bar{x},t),\ b(\bar{x},t)$ (Nambu-Goldstone modes)

solve Einstein eqn. in AdS_5

provided : (Relativistic fluid-dynamics).

$\partial_\mu T^{\mu\nu} = 0$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu}$$

$$-2\eta\left(P^{\mu\alpha}P^{\mu\beta}\left[\partial_\alpha u_\beta + \partial_\beta u_\alpha\right]\frac{1}{2} - \frac{1}{3}P^{\mu\nu}\partial_\alpha u^\alpha\right)$$

$\epsilon = 3P,\ P = \frac{\pi^2}{8}N^2T^4,\ \eta = \frac{\pi}{8}N^2T^3$

Since $s = \frac{\pi^2}{2}N^2T^3$

$\frac{\eta}{s} = \frac{1}{4}$ (Policastro-Son-Starinets)

Hence solutions of relativistic fluid dynamics give rise to solutions of Einstein equations that describe ripples on the horizon of D3-brane.

Systematic higher order corrections can be calculated in the derivative expansion.

Non-Relativistic Limit

(Bhattacharya, Minwalla, SRW)

Speed of sound in fluid:

$$v_s^2 = \frac{\partial P}{\partial \epsilon} = \frac{1}{3}$$

$$\frac{|\vec{V}|}{v_s} \ll 1, \quad \vec{V}(\vec{x}, t)$$

$$u^0 = \frac{1}{\sqrt{1 - \vec{v}^2}}, \; u^i = \frac{V^i}{\sqrt{1 - \vec{v}^2}}$$

Scaling limit to project out sound mode:

$$\vec{V}(\bar{x}, t) = \epsilon \vec{v}(\epsilon \vec{x}, \epsilon^2 t)$$

$$P(\bar{x}, t) = P_0 + \epsilon^2 p(\epsilon \vec{x}, \epsilon^2 t) \rho_e$$

$$\rho_e = 4 P_0$$

$\epsilon \to 0$ in relativistic eqn. $\Rightarrow$

$$\partial_\mu T^{\mu 0} = \epsilon^2 \left[4 P_o \partial_i v^i \right] + o(\epsilon^4)$$

$$\Rightarrow \qquad \partial_i v^i = 0$$

$$\partial_\mu T^{\mu i} = \epsilon^3 4 P_0 \left[\partial_t v^i + v^j \partial_j v^i - v \partial^2 v^i + \partial^i p \right]$$

$$+ \; o(\epsilon^5)$$

$$\nu = \frac{\eta}{\epsilon_0 + P_0} = \frac{\eta}{4 P_0}$$

$$\Rightarrow \; \partial_t v^i + v^j \partial_j v^i = \nu \partial^2 v^i - \partial^i p$$

Navier-Stokes eqns. for incompressible fluid.

Applications:

Quantum Gravity

- The AdS/CFT correspondence provides us with a well defined quantum theory of gravity in terms of the dual gauge theory.

- An immediate consequence is the resolution of the black hole information paradox.

- Gauge theory provides a framework to resolve the singularities of black holes.

- Cosmological singularity.

- Cosmological models (KKLT ....).

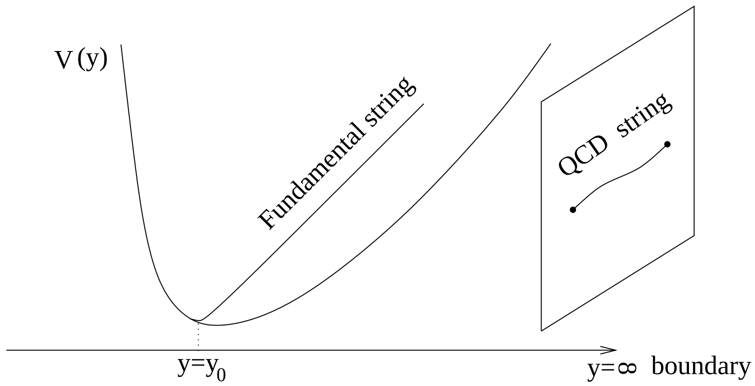
- Black holes in the sky: GRS 1915 + 105

Extreme Kerr $\frac{J}{GM} > .98$ (McClintock et. al.)

AdS/CFT methods to compute properties of this black hole using the near horizon geometry NHEK (Strominger et. al.)

Quantum Chromodynamics (QCD)

Why are 'quarks confined'?



- QCD string connecting quarks in the boundary conformal field theory is a hologram of a fundamental string in a warped geometry in the extra dimension (Maldacena, Klebanov Strassler and others)

- Physics of strongly coupled plasmas in heavy ion collisions (Son, Gubser, Liu and others)

- Gluon scattering amplitudes (Maldacena, Alday ....)

Strong coupling calculations in condensed matter systems

Boundary condensed matter system (QFT)

Fluid dynamics limit

Conservation laws

$\partial_\mu T^{\mu\nu} = 0,\ \partial_\mu J^\mu = 0$ ect.

$S = S_{QFT} + \int dx g_{\mu\nu} T^{\mu\nu} + \int A_\mu J^\mu + \cdots$

Conserved currents source the 'gauge' fields $g_{\mu\nu}, A_\mu, \cdots$ in the gravity theory in 1 higher dimension. Tranport coefficients like viscosity, charge and heat conductivity can be calculated using AdS/CFT by solving linear equations in an appropriate black hole/brane background

- Quantum Critical phenomena

- Anomalous Nernst effect
- Conductivity calculations
- de Haas-van Alphen oscillations

etc.

(Sachdev, Muller, Herzog, Hartnoll, Gubser, Mukherjee, Basu, Polchinski, Kachru, Trivedi, Kovtun, Son, David and many others).