Path Components in the Space of Polynomial Knots

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We have produced the polynomial representations of all the knots up to 6 crossings in degree at most 7 and determined the minimal polynomial degree for some knots.

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For a fixed positive integer n, the set $\mathfrak{P}_n$ of all polynomial knots $\phi = (f, g, h)$ with $deg(f) < deg(g) < deg(h) = n$ can be thought of as a subset of $\mathbb{R}^{3n}$ and it is equipped with the subspace topology induced from $\mathbb{R}^{3n}$.

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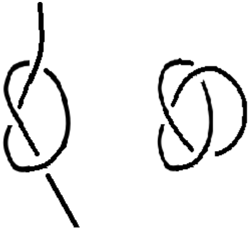
In this talk we discuss about determining a lower bound on the number of path components of $\mathfrak{P}_5, \mathfrak{P}_6$ and $\mathfrak{P}_7$.

We define a path equivalence in the space $\mathfrak{P}_n$ and show that it is stronger than the topological equivalence.

Polynomial Knot

Definition (1)

A **long knot** *is a smooth embedding* $\phi : \mathbb{R}^1 \to \mathbb{R}^3$ *such that* $t \mapsto \| \phi(t) \|$ *is strictly monotone outside a closed interval and* $\| \phi(t) \| \longrightarrow \infty$ *as* $| t | \longrightarrow \infty$.



A long & a classical trefoil

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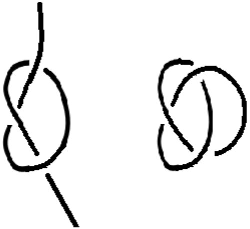

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Definition (2)

A long knot $(f, g, h) : \mathbb{R}^1 \to \mathbb{R}^3$, where f, g and h are real polynomials, is called as a **polynomial knot**.

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Degree of a Polynomial Knot

Definition (3)

A **degree** *of a polynomial knot* $\phi := (f, g, h)$ *is defined as* $deg(\phi) = \max\{deg(f), deg(g), deg(h)\}$.

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Proposition (4)

Any polynomial knot ϕ *of degree* n *is ambient isotopic to a polynomial knot* $\psi := (f, g, h)$ *with* $deg(f) < deg(g) < deg(h) = n$.

Polynomial Representation

Theorem (5)

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So each knot $\mathcal{K} : \mathbf{S}^1 \to \mathbf{S}^3$ is ambient isotopic to a one point compactification of some polynomial knot $\mathcal{P} : \mathbb{R}^1 \to \mathbb{R}^3$ via an embedding $F : \mathbb{R}^3 \to \mathbf{S}^3$.

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This $\mathcal{P}$ is called as a **polynomial representation** of the knot type $[\mathcal{K}]$.

Polynomial Degree of a Knot Type

Definition (6)

A ***polynomial degree*** $p[\mathcal{K}]$ *of a knot type* $[\mathcal{K}]$ *is the least positive integer* n *having a polynomial representation of* $[\mathcal{K}]$ *in degree* n.

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- If a knot $[\mathcal{K}]$ is represented by a polynomial knot (f, g, h), then $(f, g, -h)$ represents it's mirror image.
- Thus a knot $[\mathcal{K}]$ and it's mirror image have same polynomial degree.
- Hence the polynomial degree can not detect the **chirality** of a knot.

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Representations of Some Knots

The minimal polynomial representation and polynomial degree was known for the knots $3_1, 4_1, 5_1$ and 8_{19}.

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We have produced polynomial representations of the following knots:

Knot Types	Degree
3_1	5
4_1	6
$5_1, 5_2, 6_1, 6_2, 6_3, 3_1\#3_1, 3_1\#3_1^*$ & 8_{19}	7

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The representations of these knots are given below:

Polynomial Representation of 3_1

$$x(t) := 4\,t\,(-25 + t^2)\,,$$
$$y(t) := (-25 + t^2)\,(-6 + t^2)\,,$$
$$z(t) := -\,0.2\,t\,(-26.8 + t^2)\,(0.04 + t^2)$$

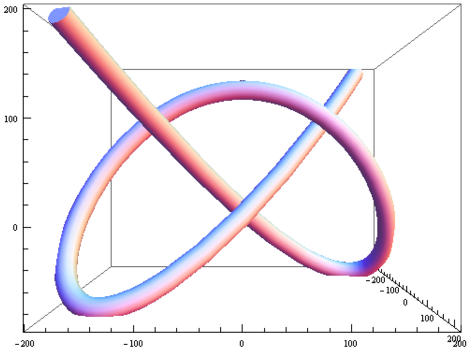

Figure : 3_1 with degree sequence $(3, 4, 5)$

Polynomial Representation of 4_1

$x(t) := (-4.8 + t)\ (-0.3 + t)\ (3.6 + t)\ (10 + t)$,
$y(t) := (-4.8 + t)\ (-3.3 + t)\ (-0.3 + t)\ (2.3 + t)\ (4.6 + t)$,
$z(t) := 0.5\ t\ (-0.19 + t)\ (21.22 - 9.19\ t + t^2)\ (17.78 + 8.42\ t + t^2)$

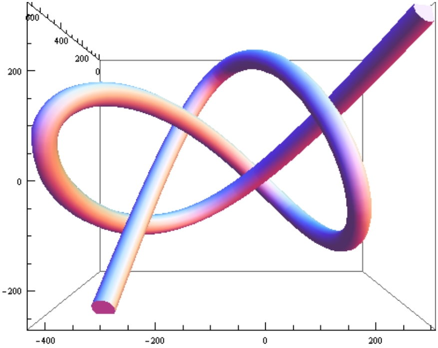

Figure : 4_1 with degree sequence $(4, 5, 6)$

Polynomial Representation of 5_1

$x(t) := 4\ (-24.01 + t^2)\ (-4 + t^2)\ ,$
$y(t) := t\ (-30.25 + t^2)\ (-12.25 + t^2)\ ,$
$z(t) := -\ 0.1\ t\ (-26.8328 + t^2)\ (-13.6702 + t^2)\ (0.1135 + t^2)$

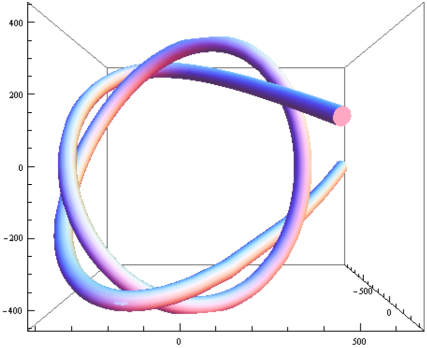

Figure : 5_1 with degree sequence $(4, 5, 7)$

Polynomial Representation of 5_2

$x(t) := 20\ (-17+t)\ (-10+t)\ (15+t)\ (21+t)$,
$y(t) := t\ (-400+t^2)\ (-121+t^2)$,
$z(t) := -\ 0.005\ t\ (-20.1133216+t)\ (-14.260128+t)\ (12.2430449+t)$
$(20.5785825+t)\ (0.0107598-0.0343124\ t+t^2)$

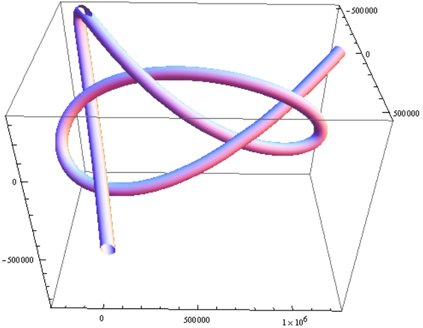

Figure : 5_2 with degree sequence $(4,5,7)$

Polynomial Representation of 6_1

$$x(t) := 60\ (-43.4 + t)\ (-28 + t)\ (5 + t)\ (31.4 + t)\ (47.6 + t)\ ,$$
$$y(t) := (-49 + t)\ (-38 + t)\ (-8 + t)\ (-6 + t)\ (28 + t)\ (43.6 + t)\ ,$$
$$z(t) := -\ 0.07\ (-45.995024874 + t)\ (5.231021635 + t)\ (19.036560084 + t)$$
$$(758.763745443 - 54.4650519227\ t + t^2)\ (2059.948386689 + 90.4819595699\ t + t^2)$$

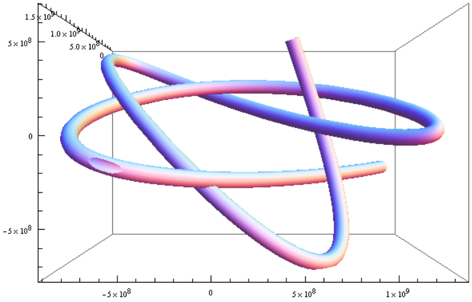

Figure : 6_1 with degree sequence $(5, 6, 7)$

Polynomial Representation of 6_2

$x(t) := 4\ (-39+t)\ (-5+t)\ (35+t)\ (-625+t^2)$,
$y(t) := 0.1\ (-39+t)\ (-30+t)\ (-10+t)\ (20+t)\ (25+t)\ (41+t)$,
$z(t) := 0.005\ t\ (-39.8753791+t)\ (-27.4156408+t)\ (28.436878+t)$
$(37.25572585+t)\ (0.002423881-0.005429486\ t+t^2)$

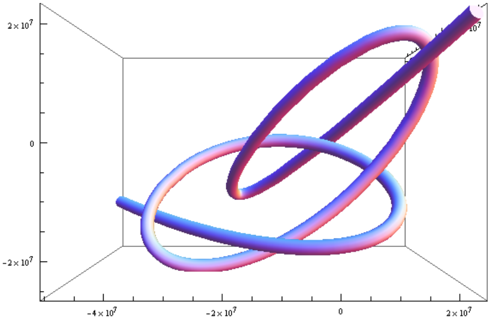

Figure : 6_2 with degree sequence $(5, 6, 7)$

Polynomial Representation of 6_3

$$x(t) := 15\,(-29+t)\,(-20+t)\,(10+t)\,(30+t)^2\;,$$
$$y(t) := (-32+t)\,(-6+t)\,(4+t)\,(30+t)\,(-400+t^2)\;,$$
$$z(t) := -\;0.06\,(-33.329044815+t)\,(376.737563885-37.8892469397\,t+t^2)$$
$$(144.275534095+21.404400212\,t+t^2)\,(955.985733648+61.56649851\,t+t^2)$$

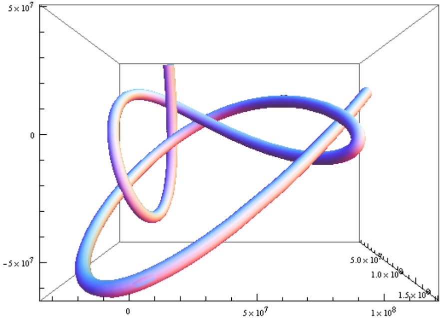

Figure : 6_3 with degree sequence $(5, 6, 7)$

Polynomial Representation of $3_1\#3_1$

$$x(t) := 5\,t\,(77.3 - 17.5\,t + t^2)(77.3 + 17.5\,t + t^2)\,,$$
$$y(t) := (-102.01 + t^2)\,(-53.29 + t^2)\,(-4.84 + t^2)\,,$$
$$z(t) := -\,0.15\,t\,(-99.695462027 + t^2)\,(-68.11720396 + t^2)\,(0.025367747 + t^2)$$

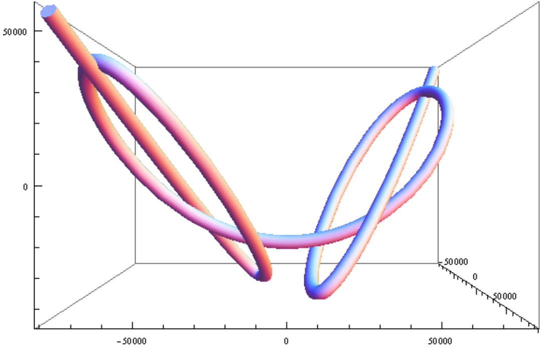

Figure : $3_1\#3_1$ with degree sequence $(5, 6, 7)$

Polynomial Representation of $3_1\#3_1^*$

$x(t) := 30\,(-32.5+t)\,(-21.3+t)\,(-3.3+t)\,(16.2+t)\,(28+t)$,
$y(t) := (-34+t)\,(-23+t)\,(-6.8+t)\,(12+t)\,(21.7+t)\,(33.1+t)$,
$z(t) := -\,0.03\,t\,(-32.807367+t)\,(-24.209735+t)\,(15.257278+t)$
$(28.289226+t)\,(0.0043718-0.0082068\,t+t^2)$

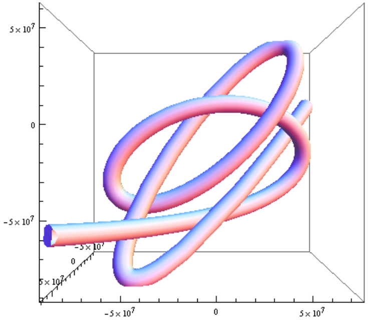

Figure : $3_1\#3_1^*$ with degree sequence $(5, 6, 7)$

Polynomial Representation of 8_{19}

$x(t) := t^5 - 5.5\ t^3 + 4.5\ t$,
$y(t) := t^6 - 7.35\ t^4 + 14\ t^2$,
$z(t) := t^7 - 8.13297\ t^5 + 18.5762\ t^3 - 10.4337\ t$

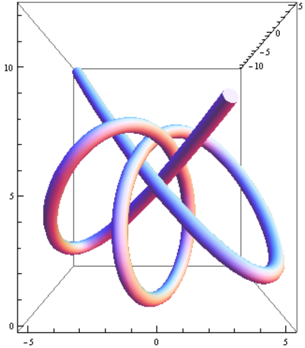

Figure : 8_{19} with degree sequence $(5, 6, 7)$

There are some inequalities between polynomial degree and other knot invariants like crossing number, bridge index and super bridge index.

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Definitions of the bridge index and the super bridge index are given below:

Bridge Index and Super Bridge Index

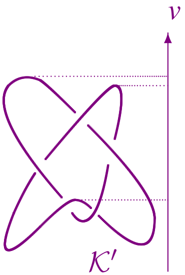

Figure : $m_v(\mathcal{K}') = 3$

Given a knot $\mathcal{K}'$ and a vector $v \in \mathbf{S}^2$.

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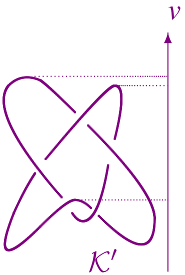

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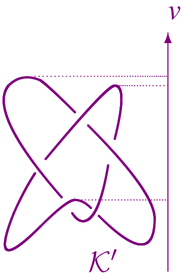

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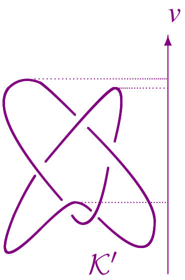

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A **bridge index** of knot type $[\mathcal{K}]$ is,

$$b[\mathcal{K}] := \min_{\mathcal{K}' \in [\mathcal{K}]} \min_{v \in \mathcal{S}_{\mathcal{K}'}} m_v(\mathcal{K}')$$

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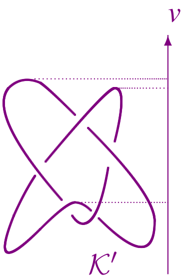

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A **super bridge index** of a knot type $[\mathcal{K}]$ is,

$$sb[\mathcal{K}] := \min_{\mathcal{K}' \in [\mathcal{K}]} \max_{v \in \mathcal{S}_{\mathcal{K}'}} m_v(\mathcal{K}')$$

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Polynomial Degree and Other Knot Invariants

Proposition (7)

For a nontrivial knot $[\mathcal{K}]$ *:*

1. $2.c[\mathcal{K}] \leq (p[\mathcal{K}] - 2)(p[\mathcal{K}] - 3)$
2. $2.b[\mathcal{K}] \leq p[\mathcal{K}] - 1$
3. $2.sb[\mathcal{K}] \leq p[\mathcal{K}] + 1$

Where $c[\mathcal{K}], b[\mathcal{K}], sb[\mathcal{K}]$ *and* $p[\mathcal{K}]$ *denote the crossing number, bridge index, super bridge index and polynomial degree of* $[\mathcal{K}]$ *respectively.*

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Where $c[\mathcal{K}], b[\mathcal{K}], sb[\mathcal{K}]$ *and* $p[\mathcal{K}]$ *denote the crossing number, bridge index, super bridge index and polynomial degree of* $[\mathcal{K}]$ *respectively.*

The polynomial representations of the knots $3_1, 4_1, 5_1, 3_1\#3_1$, $3_1\#3_1^*$ & 8_{19} are minimal, but the representations of the knots $5_2, 6_1, 6_2$ & 6_3 may be reduced further.

Polynomial Degree

We have proved the following theorem.

Theorem (8)

If a polynomial knot ϕ has a regular projection (f, g) with n transversal double points and the crossing data of the knot is such that there are m changes from under crossing to over crossing or vice-versa, then there is a polynomial h with $deg(h) \leq \min\{n+2, m\}$ such that the polynomial knots ϕ and $\psi := (f, g, h)$ are topologically equivalent.

Polynomial Degree

For an alternating knot $\mathcal{K}$ with minimal number of crossings, we have $c[\mathcal{K}]$ number of transversal double points and $2.c[\mathcal{K}] - 1$ number of crossing changes. Hence the following corollary follows immediately from the previous theorem.

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Corollary (8.1)

If a knot type $[\mathcal{K}]$ is represented by an alternating knot $\mathcal{K}$, then $p[\mathcal{K}] \leq c[\mathcal{K}] + 2$.

Where $c[\mathcal{K}]$ and $p[\mathcal{K}]$ denote the crossing number and polynomial degree of $[\mathcal{K}]$ respectively.

Spaces of Polynomial Knots

- For a fixed positive integer n, the set $\mathfrak{P}_n$ of all polynomial knots $\phi = (f, g, h)$ with $deg(f) < deg(g) < deg(h) = n$ can be thought of as a subset of $\mathbb{R}^{3n}$ and it is equipped with the subspace topology induced from $\mathbb{R}^{3n}$.

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- The set $\mathfrak{P} = \cup_n \mathfrak{P}_n$ of all polynomial knots can be given the inductive limit topology.
- So $\mathfrak{P}_n$ and $\mathfrak{P}$ are topological spaces.

Polynomial Isotopy

Definition (9)

Two polynomial knots ϕ and ψ are said to be **polynomially isotopic** *if there exists a one parameter family of polynomial knots $\{\mathcal{P}_t | t \in [0,1]\}$ such that $\mathcal{P}_0 = \phi$ and $\mathcal{P}_1 = \psi$.*

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- It was proved that, two polynomial knots are ambient isotopic (topologically equivalent) as long knots if and only if they are polynomially isotopic. [Rama Mishra, 1994]

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Two polynomial knots ϕ and ψ are said to be ***polynomially isotopic*** *if there exists a one parameter family of polynomial knots $\{\mathcal{P}_t | t \in [0,1]\}$ such that $\mathcal{P}_0 = \phi$ and $\mathcal{P}_1 = \psi$.*

- Being polynomially isotopic is an equivalence relation in $\mathfrak{P}$ for which it is easy to note that the equivalence classes are nothing but the path components of the space $\mathfrak{P}$.
- It was proved that, two polynomial knots are ambient isotopic (topologically equivalent) as long knots if and only if they are polynomially isotopic. [Rama Mishra, 1994]
- Thus two knots lie in the same path component of $\mathfrak{P}$ if and only if they are ambient isotopic.

Path Equivalence in $\mathfrak{P}_n$

Two polynomial knots of different degree may represent equivalent long knots and the polynomial isotopy may pass through polynomial knots of various degrees. For the spaces $\mathfrak{P}_n$, there is another equivalence defined as:

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It is obvious that if two polynomial knots in $\mathfrak{P}_n$ are path equivalent then they are topologically equivalent. However the converse is not true.

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We have proved the following theorem.

Theorem (11)

Suppose (f, g, h) is a minimal degree polynomial representation of a knot $[\mathcal{K}]$ with $deg(f) < deg(g) < deg(h) = n$. Then (f, g, h) and it's mirror image given by $(f, g, -h)$ belong to the distinct path components of $\mathfrak{P}_n$.

Remarks (12)

If (f, g, h) is a minimal degree polynomial representation of a knot $[\mathcal{K}]$ with $deg(f) < deg(g) < deg(h) = n$, then the following hold :

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1. *If the degree of f is minimal in the sense that, by reducing the degree of f results in a knot with less than $c[\mathcal{K}]$ number of crossings, then (f,g,h), $(-f,g,-h)$, $(-f,g,h)$ and $(f,g,-h)$ are lie in 4 distinct path components of $\mathfrak{P}_n$.*

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2. *Similarly, if the degree of g is minimal in the above sense, then there are at least 4 distinct path components of $\mathfrak{P}_n$ corresponding to $[\mathcal{K}]$.*

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2. *Similarly, if the degree of g is minimal in the above sense, then there are at least 4 distinct path components of $\mathfrak{P}_n$ corresponding to $[\mathcal{K}]$.*
3. *If the degree of each of f and g is minimal in the sense that, by reducing the degree of any one of them results in a knot with less than $c[\mathcal{K}]$ number of crossings, then there are at least 8 distinct path components of $\mathfrak{P}_n$ corresponding to $[\mathcal{K}]$.*

Main Questions

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1. Given a knot, what is its polynomial degree?
2. Given a positive integer n, what are the knots which have a polynomial representation in $\mathfrak{P}_n$?

Both questions are equally interesting and are not answered completely, and answer to each question helps in answering the other question.

- We have partially answered the Question 2 for the spaces $\mathfrak{P}_6$ & $\mathfrak{P}_7$, and estimated some lower bounds on the number of path components of each of the spaces $\mathfrak{P}_5, \mathfrak{P}_6$ & $\mathfrak{P}_7$.

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- The number of topologically distinct knots in $\mathfrak{P}_n$ together with Theorem 11 and Remarks 12.1, 12.2 & 12.3 provide us a lower bound on the number of path components of $\mathfrak{P}_n$.

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- The number of topologically distinct knots in $\mathfrak{P}_n$ together with Theorem 11 and Remarks 12.1, 12.2 & 12.3 provide us a lower bound on the number of path components of $\mathfrak{P}_n$.
- All the knots that are realized in degree n are also realized in degree $n+1$.

The Spaces $\mathfrak{P}_n$ for $n \leq 4$

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The trivial knot is the only knot that can be realized in $\mathfrak{P}_n$ for $n \leq 4$.

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In fact for $n \leq 4$ there is a stronger result:

Theorem (14)

The space $\mathfrak{P}_n$ for $n \leq 4$ is path connected.

The Space $\mathfrak{P}_5$

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- The polynomial degree of 3_1 is 5.

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The Space $\mathfrak{P}_5$

Lower bound on the number of path components of $\mathfrak{P}_5$:

s.n.	knot type	# of path components corresponding to the knot type
1.	0_1	at least 1
2.	3_1	at least 4
3.	3_1^*	at least 4
	# of path components of $\mathfrak{P}_5$	at least 9

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Thus, the space $\mathfrak{P}_5$ has at least 9 path components.

The Space $\mathfrak{P}_6$

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- The knots $0_1, 3_1, 3_1^*$ & 4_1 can be realized in $\mathfrak{P}_6$.
- The polynomial degree of 4_1 is 6.

The Space $\mathfrak{P}_6$

Lower bound on the number of path components of $\mathfrak{P}_6$:

s.n.	knot type	# of path components corresponding to the knot type
1.	0_1	at least 1
2.	3_1	at least 1
3.	3_1^*	at least 1
3.	4_1	at least 8
	# of path components of $\mathfrak{P}_6$	at least 11

The Space $\mathfrak{P}_6$

Lower bound on the number of path components of $\mathfrak{P}_6$:

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3.	4_1	at least 8
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Thus, the space $\mathfrak{P}_6$ has at least 11 path components.

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- Any knot with polynomial degree 7 has at most 10 crossings.
- All the knots up to 6 crossings (including 8_{19} and 8_{19}^*) can be realized in $\mathfrak{P}_7$.
- The polynomial degree of each of the knot $5_1, 3_1\#3_1, 3_1\#3_1^*$ and 8_{19} is 7.
- The polynomial degree of each of the knot $5_2, 6_1, 6_2$ and 6_3 is either 6 or 7.

The Space $\mathfrak{P}_7$

Lower bound on the number of path components of $\mathfrak{P}_7$:

s.n.	knot type	# of path components corresponding to the knot type
1.	0_1	at least 1
2.	3_1	at least 1
3.	3_1^*	at least 1
4.	4_1	at least 1
5.	5_1	at least 2
6.	5_1^*	at least 2
7.	5_2	at least 1
8.	5_2^*	at least 1
9.	6_1	at least 1
10.	6_1^*	at least 1

The Space $\mathfrak{P}_7$

11.	6_2	at least 1
12.	6_2^*	at least 1
13.	6_3	at least 1
14.	$3_1 \# 3_1$	at least 2
15.	$3_1^* \# 3_1^*$	at least 2
16.	$3_1 \# 3_1^*$	at least 2
17.	8_{19}	at least 2
18.	8_{19}^*	at least 2
	# of path components of $\mathfrak{P}_7$	at least 25

The Space $\mathfrak{P}_7$

11.	6_2	at least 1
12.	6_2^*	at least 1
13.	6_3	at least 1
14.	$3_1 \# 3_1$	at least 2
15.	$3_1^* \# 3_1^*$	at least 2
16.	$3_1 \# 3_1^*$	at least 2
17.	8_{19}	at least 2
18.	8_{19}^*	at least 2
	# of path components of $\mathfrak{P}_7$	at least 25

Thus, the space $\mathfrak{P}_7$ has at least 25 path components.

Conjecture

We have conjectured the following:

Conjecture (15)

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- However it is conjectured that, the only three super bridge knots are 3_1 and 4_1. If this is proved, then it will imply the above conjecture.
- Once conjecture 15 is proved, it will bring at least 7 more path components in $\mathfrak{P}_7$.
- On the contrary, if the conjecture 15 is disproved, then it will produce example of a three super bridge knot other than 3_1 & 4_1 and will bring more path components in $\mathfrak{P}_6$.

Thank You !