On Hempel distance of bridge splittings of links

Yeonhee Jang
(joint work with Ayako Ido and Tsuyoshi Kobayashi)

Nara Women's University

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Outline

Curve complex

S : orientable surface of genus g with p punctures s.t. $3g + p - 4 > 0$
(i.e., $(g, p) \neq (0,0), (0,1), (0,2), (0,3), (0,4), (1,0), (1,1)$).

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The (1-skeleton of the) curve complex $C(S)$ is a simplicial complex defined as follows:

- **0-simplex** $\leftrightarrow$ (isotopy class of) an essential s. c. curve on S,
- two $\mathbf{0}$-simplexes are joined by a **1-simplex** if they can be realized by disjoint curves.

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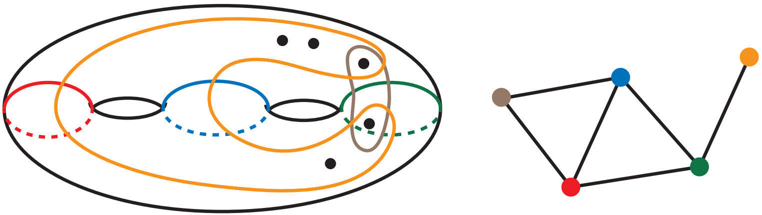

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Distance in the curve complex

- For vertices a and b ($\in C^0(S)$),
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Fact (Harvey '81, Hempel '01)

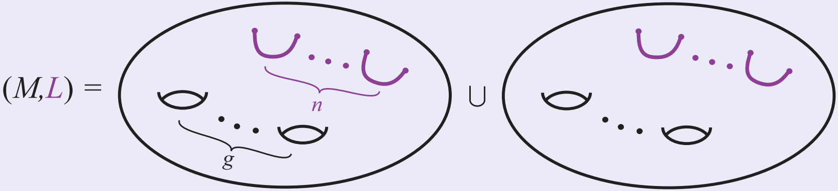
$C(S)$: connected
(i.e., $\exists$ a path connecting a and b $\forall$ a, b : essential curves on S).
In fact, $d(a,b) \leq 2 + 2\log_2 \iota(a,b)$,
where $\iota(a,b)$: geometric intersection number of a and b.

Bridge surfaces

Bridge surfaces

Let L be a link in a closed orientable 3-manifold M. We call $B_+ \cup_S B_-$ a (g, n)-bridge splitting of L and S a (g, n)-bridge surface if

- S is a genus-g Heegaard surface of M,
- $L \cap B_\pm$ is "trivial" in $B_\pm$.



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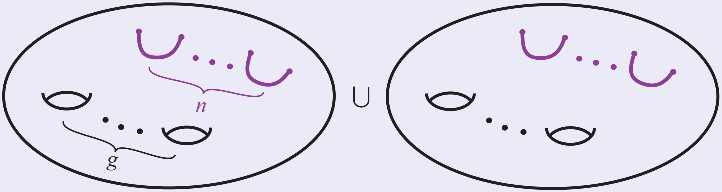
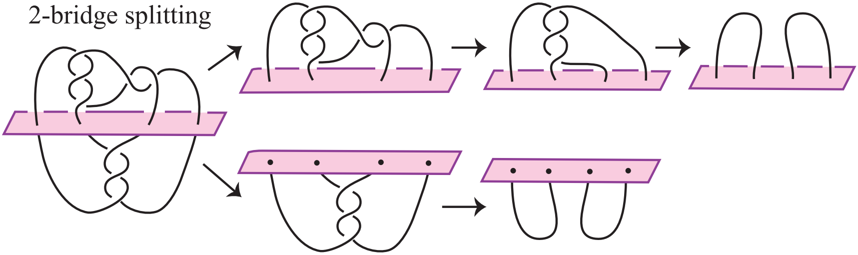
$(M, L) =$ 

When $g = 0$, we call $B_+ \cup_S B_-$ and S, respectively, an n-bridge splitting and an n-bridge sphere of L.

Bridge surfaces



Hempel Distance of bridge surfaces

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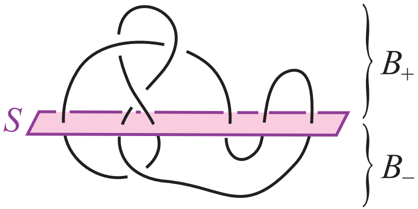
$B_+ \cup_S B_-$: (g,n)-bridge splitting of L,

$\mathcal{C}(S \setminus L)$: curve complex of $S \setminus L$

$\mathcal{D}(B_+ \setminus L)$ ($\mathcal{D}(B_- \setminus L)$) : the set of curves ($\in \mathcal{C}(S \setminus L)$),
which bound disks in $B_+ \setminus L$ ($B_- \setminus L$)

: upper/lower disk set (in this talk).

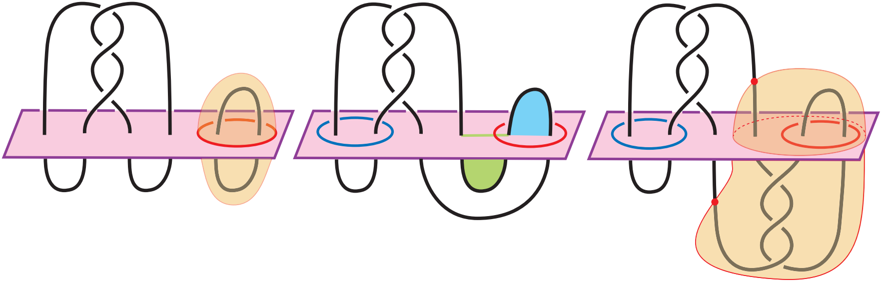
$\rightsquigarrow d(L,S) := d_{\mathcal{C}(S \setminus L)}(\mathcal{D}(B_+ \setminus L), \mathcal{D}(B_- \setminus L))$.



Examples

Fact

- $\exists$ essential sphere in $S^3 \setminus L \Leftrightarrow d(L, S) = 0$
- S : "stabilized" $\Rightarrow d(L, S) \leq 1$
- $L(\subset S^3)$: "composite" link $\Rightarrow d(L, S) \leq 1$



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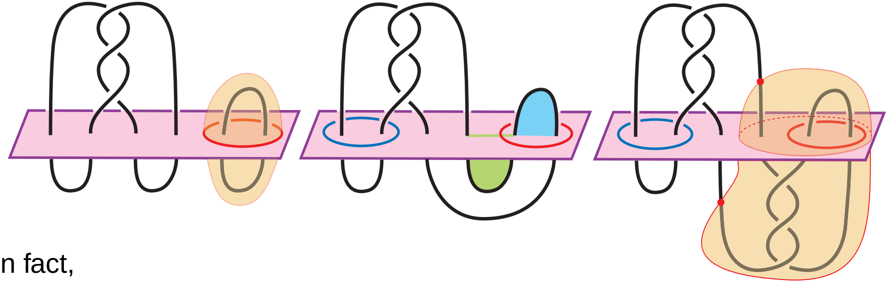

In fact,

$d(L, S) = 0$ $\qquad$ $d(L, S) = 1$ $\qquad$ $d(L, S) = 1$

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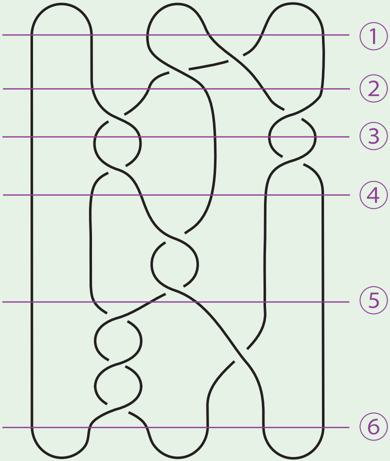

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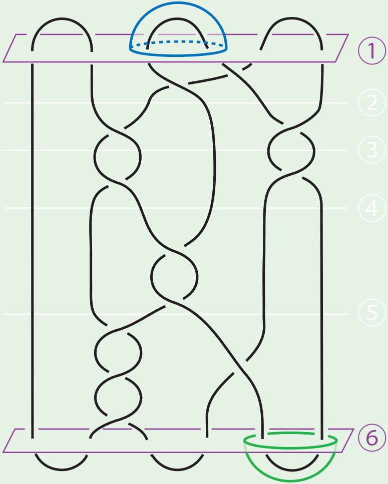

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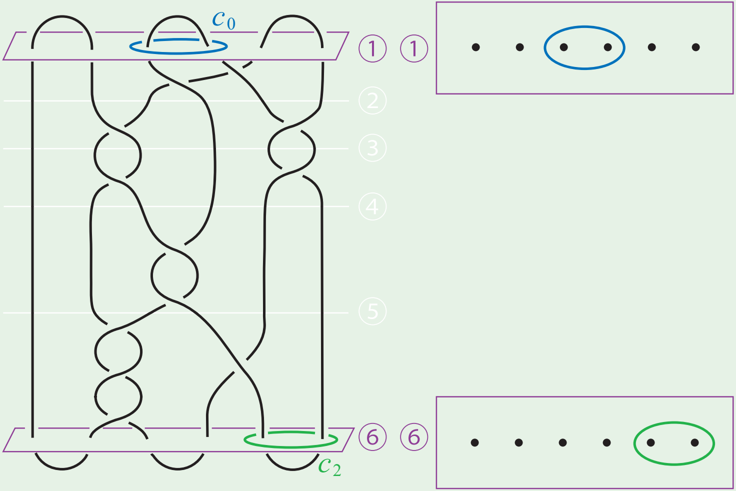

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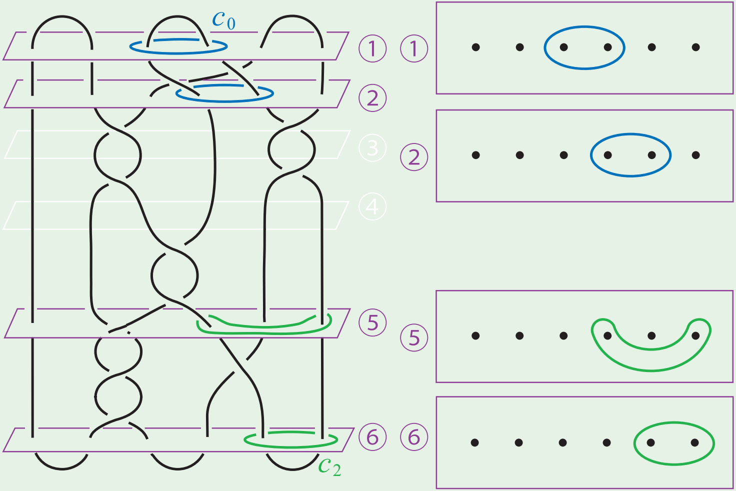

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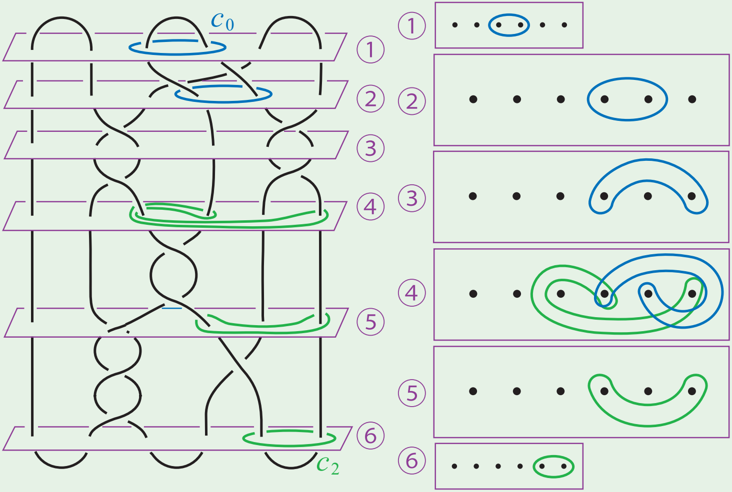

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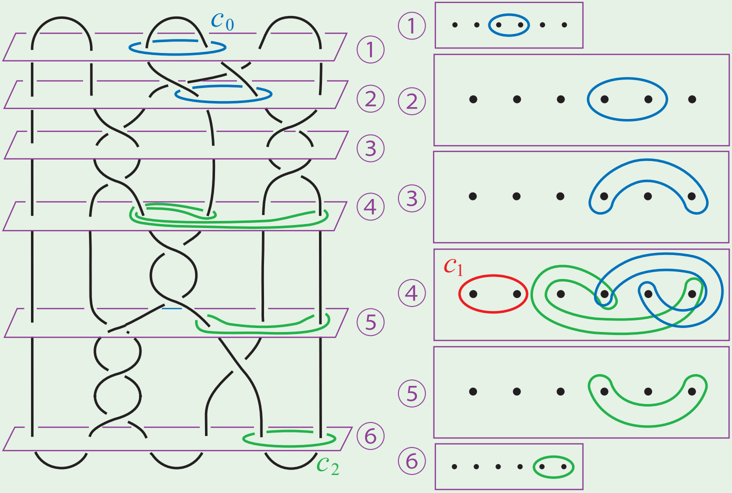

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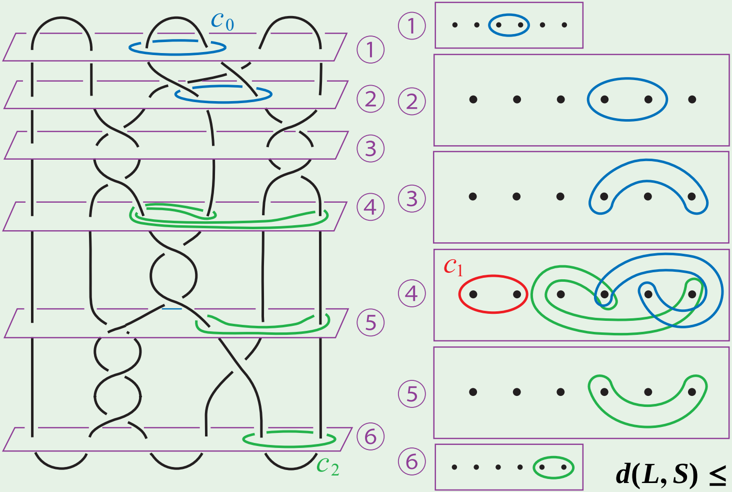

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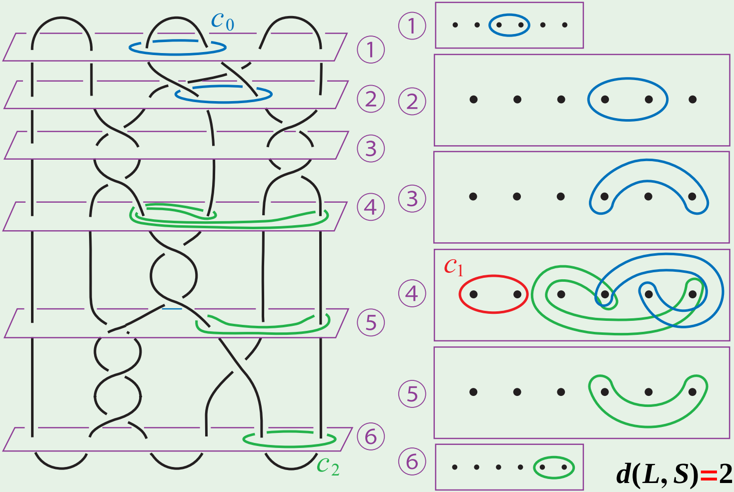

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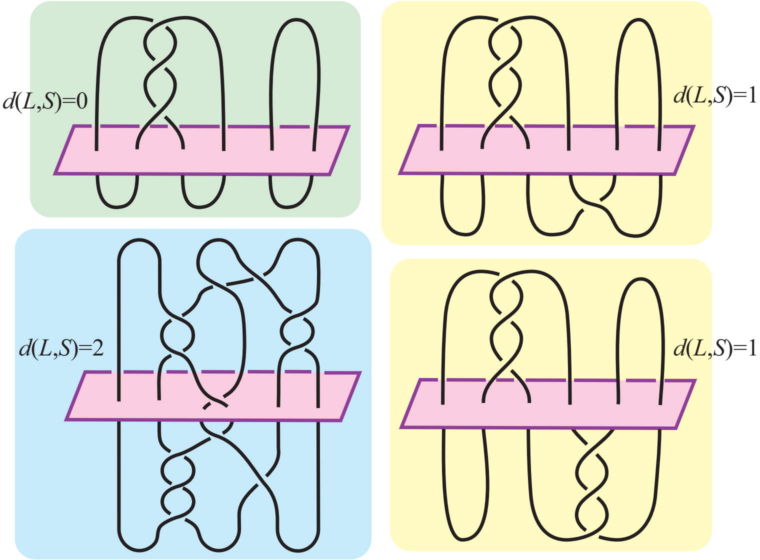

Summary of the first half

surface → curve complex

bridge splitting → Hempel distance

measures complexity of bridge splittings

Upper bounds

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- (Bachman-Schleimer '05) $\exists\, F$: essential surface ($\subset E(L)$)
  $\Rightarrow d(L, S) \leq -\chi(F) + 2$,
- (J. to appear) $\exists\, F$: essential $n(\geq 4)$-punctured sphere ($\subset S^3 \setminus L$)
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($M_2(L)$: double branched cover of S^3 branched over L)

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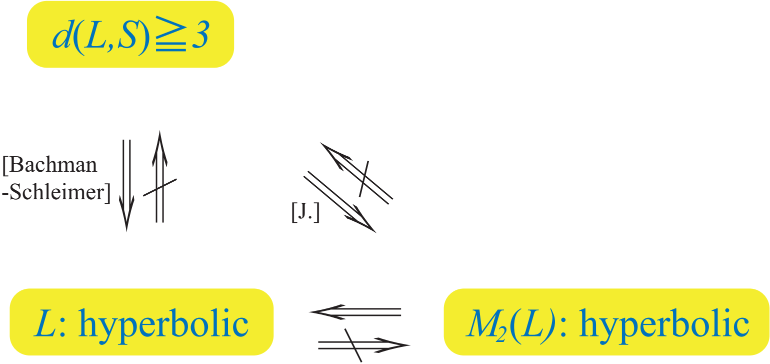

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Existence of high distance knots (Saito '04, Campisi-Rathbun '12, Blair-Tomova-Yoshizawa '13, Ichihara-Saito '13)

For any integer n, there exists a knot K in some 3-manifold and a bridge surface S of K such that $d(K, S) > n$.

Main Theorem

Main Theorem (Ido-J.-Kobayashi)

For any integers $n \geq 2$, $g \geq 0$ and $b \geq 1$ (except for $(g, b) = (0, 1), (0, 2)$),
$\exists$ a (g, b)-bridge splitting of some link with distance exactly n.

Subsurface projection

S : surface, X : essential non-simple subsurface of S,
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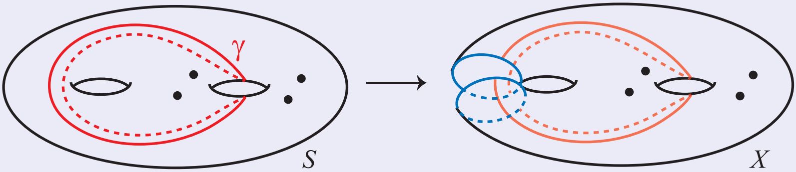

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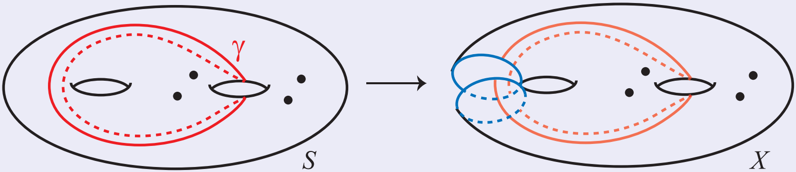

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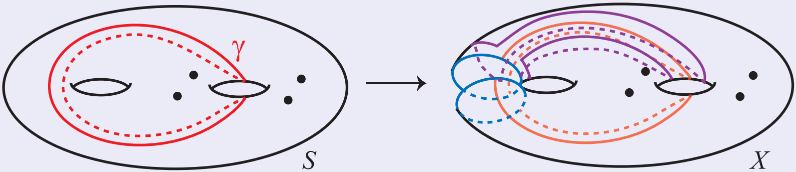

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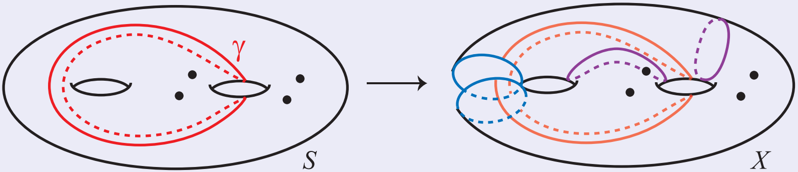

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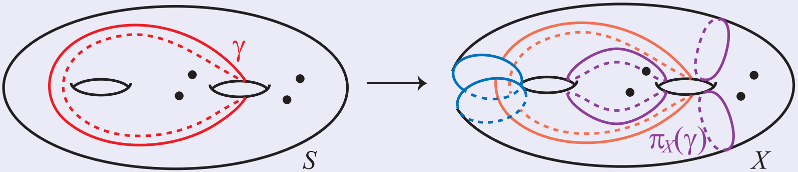

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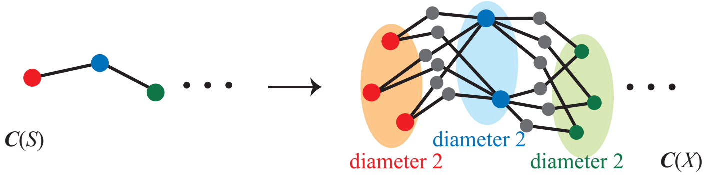

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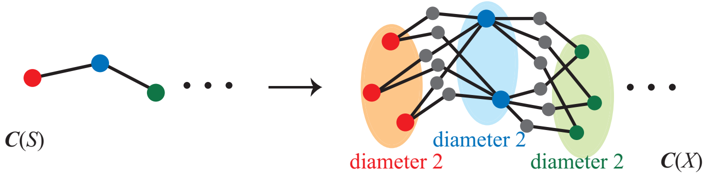

Lemma 2

Let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$: a path in $C(S)$ $(n \geq 3)$ s.t.

- $[\alpha_0, \ldots, \alpha_i]$, $[\alpha_i, \ldots, \alpha_n]$: geodesics,
- α_i cuts S into a twice-punctured disk and the other component X_i,
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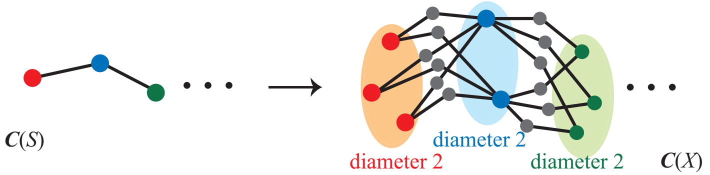

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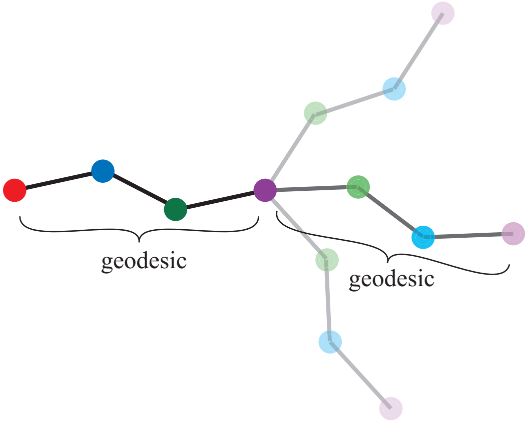
Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

Extending geodesics

By using Lemma 2, we can “construct” a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ $\forall n$.



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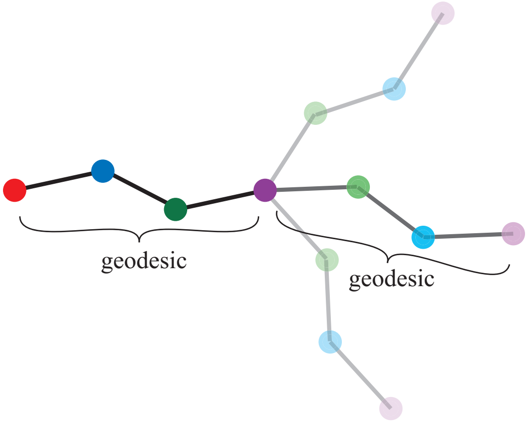
By using Lemma 2, we can “construct” a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ $\forall n$. Moreover, we can choose a geodesic so that each of α_0 and α_n cuts off a twice-punctured disk from S.



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Let S be a surface with genus g and $2b$ punctures.
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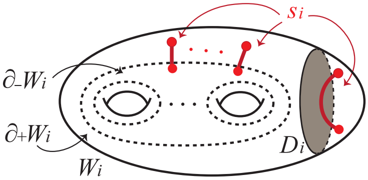

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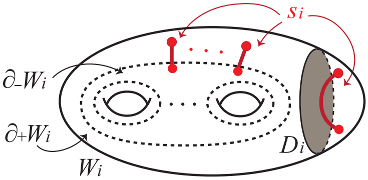
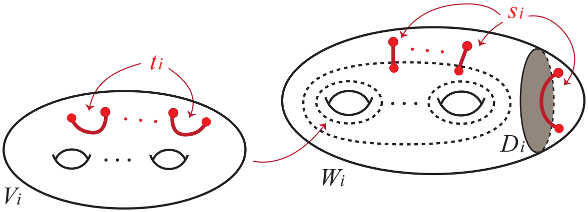


Identify $\partial_+ W_1$ and $\partial_+ W_2$ with S so that $\partial D_1 = \alpha_0$ and $\partial D_2 = \alpha_n$.

Proof of Main Theorem

For $i = 1, 2$, let V_i be a handlebody, t_i the union of trivial arcs in V_i.
Glue (V_i, t_i) to $\partial_- W_i$, and let $(V_i^*, t_i^*) := (W_i, s_i) \cup (V_i, t_i)$.



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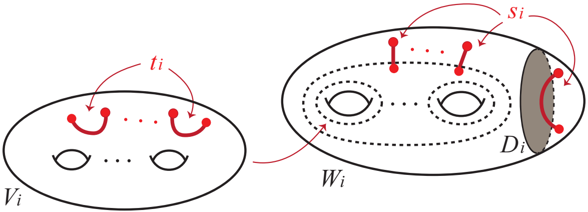

Then $(V_1^*, t_1^*) \cup_S (V_2^*, t_2^*)$ is a (g, b)-bridge splitting.

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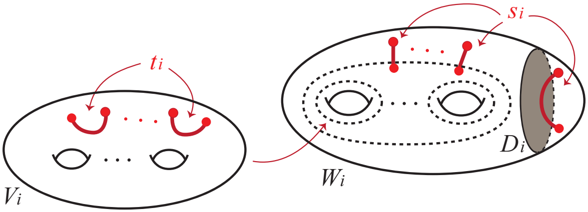
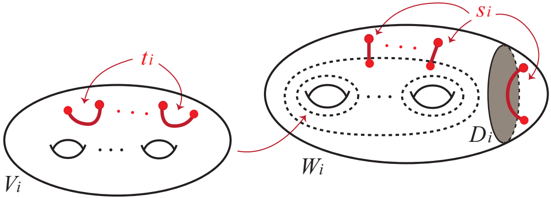


Then $(V_1^*, t_1^*) \cup_S (V_2^*, t_2^*)$ is a (g, b)-bridge splitting.

If we choose the gluing homeomorphisms “complicated enough”, then we can see that the distance of the bridge splitting $(V_1^*, t_1^*) \cup_S (V_2^*, t_2^*)$ is n, which is realized by $d(\partial D_1, \partial D_2)$.

How complicated must the gluing map be?

S_i : component of $S \setminus \partial D_i$ that is not a twice-punctured disk,
$\pi_{S_i} : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{C}^0(S_i))$: the subsurface projection,
$P_i : S_i \to S_i \cup D_i \to \partial_- W_i$: the natural map,
$\mathcal{D}(V_i \setminus t_i)$, $\mathcal{D}(V_i^* \setminus t_i^*)$: the disk complexes



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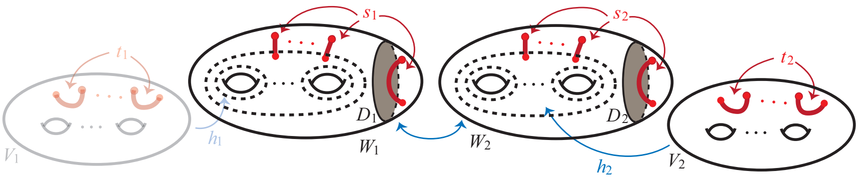

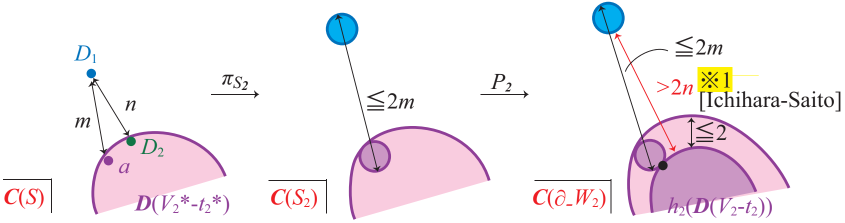

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