Discussion Meeting

Component-conservative invertibility of links and Samsara 4-manifolds on 3-manifolds

The preprint in:

http://www.sci.osaka.cu.ac.jp/~kawauchi/index.html

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1. <u>A meaning of an invertible knot</u>

An oriented knot K in S^3 is invertible if $\exists$ an orientation-preserving self-homeomorphism f of S^3 sending K to -K.

A topological meaning of an invertible knot has not been enough observed until now. We find here a meaning in constructing a 4-manifold.

The self-homeomorphism $f: S^3 \to S^3$ induces an orientation-preserving self-homeomorphism $h: M \to M$ for the Dehn surgery 3-manifold $M = \chi(K;r)$ for any $r \in Q$ such that

$$h_* = -1: H_1(M;Z) \to H_1(M;Z).$$

Let Σ be the mapping torus of h:

$$\Sigma = M \times [0,1]/\{h(x,0)=(x,1) \mid x \in M\}.$$

Since $H_1(M;Z)$ is cyclic,

$$H_1(\Sigma;Z) = Z \oplus H_1(M;Z_2) = Z \oplus Z_2^s$$

for s=0 or 1.

By Poincare duality and $\chi(\Sigma)=0$, we have

$$H_d(\Sigma;Z)=\begin{cases} Z & (d=0,3,4) \\ Z_2^{\,s} & (d=2) \\ Z\oplus Z_2^{\,s} & (d=1) \\ 0 & (\text{others}), \end{cases}$$

where $s = \beta_1(M;Z_2) = 0$ or 1.

Thus, for the subring

$$Z_{/2}=Z[1/2]\subset Q,$$

Σ is a $Z_{/2}$-homology $S^1\times S^3$ and $\exists$ an embedding $k\colon M \to \Sigma$ such that $k_*[M]\in H_3(\Sigma;Z)$ is a generator.

Let M^0 be a punctured 3-manifold of M, i.e., $M^0 = cl(M - B^3)$ for a 3-ball B in M. Since Σ is an M-bundle over S^1, let Σ* be a closed 4-manifold obtained from Σ by a surgery killing a section S^1:

$$\Sigma\text{^} = cl(\Sigma - S^1 \times D^3) \cup (D^2 \times \partial D^3).$$

Then

$$H_d(\Sigma\text{^};Z) = \begin{cases} Z & (d=0,4) \\ Z_2^s & (d=1,2) \\ 0 & (\text{others}), \end{cases}$$

where $s = \beta_1(M;Z_2) = 0$ or 1.

Thus, M^0 is embedded in Σ^, a $Z_{/2}$-homology 4-sphere.

Let X be a connected oriented 4-manifold, and M a closed connected oriented 3-manifold.

Then ∃ two types of (smooth) embeddings

$$M \to X.$$

<u>Definition</u>. An embedding f: M → X is <u>of type 1</u> if X-f(M) is connected, and <u>of type 2</u> if X-f(M) is disconnected.

Type 1

Type 2

Note: If ∃ type 1 embedding $f:M \to X$, then $H_1(X;Z)$ has a direct summand Z, because ∃ $[C] \in H_1(X;Z)$ with $Int_X(C,fM)=\pm 1$.

For an abelian group G,

let $G^{(2)} = \{x \in G \mid 2x = 0\}$.

For a connected oriented 4-manifold X,

let $\beta^{(2)}{}_d(X;Z) = H_d(X;Z)^{(2)}$.

Then we have

$$s = \beta_1(M;Z_2) = \beta^{(2)}{}_2(\Sigma;Z) = \beta^{(2)}{}_2(\Sigma\hat{};Z) = 0 \text{ or } 1.$$

Observation 1.

Every r-surgery manifold $M=\chi(K;r)$ of an invertible knot K is type 1 embedded in Σ, a $Z_{/2}$-homology $S^1 \times S^3$ with $\beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma;Z)=0$ or 1.

Further, M^0 is embeddable in Σ^, a $Z_{/2}$-homology 4-sphere with $\beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma\text{^};Z)=0$ or 1.

<u>Remark.</u> The $Z_{/2}$-homology 4-sphere Σ^ cannot be replaced by S^4 in general.

(1) For the lens space L(p,q)=χ(O;p/q) (p>0, even) for the trivial knot O which is invertible, $L(p,q)^0$ is NOT embeddable in S^4.

D. B. A. Epstein, Embedding punctured manifolds, Proc. Amer. Math. Soc. 16(1965), 175-176.

(2) For the 0-surgery manifold M=χ(K;0) of the trefoil knot K (known to be invertible), M^0 is NOT embeddable in S^4.

A. Kawauchi, On n-manifolds whose punctured manifolds are imbeddable in (n+1)-sphere and spherical manifolds, Hiroshima Math. J. 9(1979),47-57.

2. A generalization to a component-conservatively invertible link

Definition.

An oriented link L with components K_i (i=1,2,…,n) in S^3 is component-conservatively invertible if ∃ an orientation-preserving self-homeomorphism f of S^3 sending K_i to $-K_i$ for every i.

The self-homeomorphism f induces an orientation-preserving self-homeomorphism h of the r-surgery 3-manifold M=χ(L;r) for any $r \in Q^n$ such that $h_* = -1: H_1(M;Z) \to H_1(M;Z)$.

Let Σ be the mapping torus of h:

$$\Sigma = M \times [0,1]/\{h(x,0)=(x,1) \mid x \in M\}.$$

Then $H_1(\Sigma;Z)= Z \oplus H_1(M;Z_2)=Z \oplus Z_2^s$.

By Poincare duality and the Euler characteristic $\chi(\Sigma)=0$, we have

$$H_d(\Sigma;Z)= \begin{cases} Z & (d=0,3,4) \\ Z_2^s & (d=2) \\ Z \oplus Z_2^s & (d=1) \\ 0 & (\text{others}), \end{cases}$$

where $s= \beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma;Z)$.

Σ is a $Z_{/2}$-homology $S^1 \times S^3$ and $\exists$ a type 1 embedding $k: M \to \Sigma$.

Since Σ is an M-bundle over S^1, let Σ be a closed 4-manifold obtained from Σ by a surgery killing a section S^1:

$$\Sigma^\wedge = cl(\Sigma - S^1 \times D^3) \cup (D^2 \times \partial D^3).$$

Then

$$H_d(\Sigma^\wedge;Z) = \begin{cases} Z & (d=0, 4) \\ Z_2{}^s & (d=1, 2) \\ 0 & (\text{others}), \end{cases}$$

where $s = \beta_1(M;Z_2) = \beta^{(2)}{}_2(\Sigma^\wedge;Z)$.

Thus, M^0 is embeddable in Σ^, a $Z_{/2}$-homology 4-sphere.

Observation 2.

Every r-surgery manifold M=χ(L;r) of every component-conservatively invertible link L is type 1 embedded in Σ, a $Z_{/2}$-homology $S^1 \times S^3$ with $\beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma;Z)$.

Further, M^0 is embeddable in a $Z_{/2}$-homology 4-sphere Σ^ with $\beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma\hat{};Z)$.

3. Invertible 3-manifolds

Definition.

A closed connected oriented 3-manifold M is invertible if ∃ an orientation-preserving self-homeomorphism h of M such that

$$h_*=-1: H_1(M)\to H_1(M).$$

Observation 3.

Cf. A. Kawauchi, The imbedding problem of 3-manifolds into 4-manifolds, Osaka J. Math. 25 (1988), 171-183.

Every invertible 3-manifold M is embedded in Σ, a $Z_{/2}$-homology $S^1 \times S^3$ with $\beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma;Z)$. Further, M^0 is embeddable in a $Z_{/2}$-homology 4-sphere Σ^ with $\beta_1(M;Z_2)=\beta^{(2)}{}_2(\Sigma\hat{};Z)$.

Examples of invertible 3-manifolds

(1) Every Dehn surgery 3-manifold obtained from S^3 along every component-conservatively invertible link is an invertible 3-manifold.

(2) The double branched cover of S^3 branched along every link is an invertible 3-manifold.

(3) Every closed connected orientable 3-manifold of Heegaard genus $\leqq 2$ is an invertible 3-manifold.

Examples of non-invertible 3-manifolds

(1) A closed connected oriented hyperbolic 3-manifold with no symmetry or with only odd symmetries.

(2) A closed connected oriented 3-manifold M such that $\exists\, u_1, u_2, u_3 \in H^1(M;Z_p)$ (p odd prime) with $u_1 \cup u_2 \cup u_3 \neq 0$ in $H^3(M;Z_p) = Z_p$.

(e. g., $M = T^3 \# M'$).

Proof of (1). If M is invertible, then M has an even order isometry by Mostwo rigidity.

Proof of (2). Suppose $\exists$ an orientation-preserving self-homeomorphism h of M such that $h_*=-1: H_1(M)\to H_1(M)$.

Then $h^*=-1: H^1(M;Z_p)\to H^1(M;Z_p)$, so that

$$h^*(u_1\cup u_2 \cup u_3)=(h^*u_1\cup h^*u_2 \cup h^*u_3)$$

$$= -(u_1\cup u_2 \cup u_3).$$

Thus, $h^*=-1: H^3(M;Z_p)\to H^3(M;Z_p)$ and h must be orientation-reversing, a contradiction.//

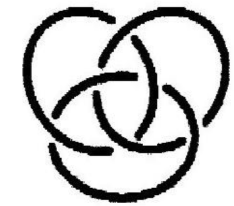
4. Samsara 4-manifold

Let M be a closed connected oriented 3-manifold.

Definition.

A closed Samsara 4-manifold on M is a 4-manifold Σ with $Z_{/2}$-homology of $S^1 \times S^3$ such that ∃ a type 1 embedding k:M → Σ.

Let $T^3=S^1 \times S^1 \times S^1$.

Let L_B be the Borromean rings in S^3: 

Let $D(T^3) = D^4 \cup$ 0-framed three 2-handles on L_B be the 0-surgery trace on B with $\partial D(T^3)=\chi(L_B,0)=T^3$.

Let $D(sT^3)$ be the disk sum of s copies of $D(T^3)$ with $\partial D(sT^3)=\#sT^3$. Note that

$$H_d(D(sT^3);Z) = \begin{cases} Z^{3s} & (d=2) \\ Z & (d=0) \\ 0 & (\text{others}). \end{cases}$$

The intersection form on $H_2(D(sT^3);Z)$ is 0-form.

Definition.

A **bounded Samsara 4-manifold** on M is a compact oriented 4-manifold Σ with $\partial\Sigma = \#sT^3$ such that

(1) Σ has the $Z_{/2}$-homology of $S^1 \times S^3 \# D(sT^3)$ for some $s>0$, and

(2) $\exists$ a type 1 embedding $k: M \to \Sigma$ with

$$k_* = 0: H_2(M;Z_{/2}) \to H_2(\Sigma;Z_{/2}).$$

Definition.

A reduced closed Samsara 4-manifold on M^0 is a $Z_{/2}$-homology 4-sphere Σ^ with M^0 embedded.

Definition.

A reduced bounded Samsara 4-manifold on M^0 is a 4-manifold Σ^ with ∂Σ^= $\#sT^3$ such that

(1) $H_*(\Sigma\hat{};Z_{/2}) = H_*(S^4\#D(sT^3);Z_{/2}) = H_*(D(sT^3);Z_{/2})$,

(2) ∃ an embedding $k^0: M^0 \to \Sigma\hat{}$ such that $k^0{}_*=0: H_2(M^0;Z_{/2}) \to H_2(\Sigma\hat{};Z_{/2})$.

Observation 4.

(1) Given a reduced Samsara 4-manifold Σ^ on M^0, ∃ a Samsara 4-manifold Σ on M with $H_2(\Sigma;Z)=H_2(\Sigma\hat{};Z)$.

Conversely, given a Samsara 4-manifold Σ on M, ∃ a reduced Samsara 4-manifold Σ^ on M^0 with $H_2(\Sigma\hat{};Z)=H_2(\Sigma;Z)$ by a surgery killing a generator of $H_1(\Sigma;Z)/(2\text{-torsion}) = Z$.

(2) For a Samsara 4-manifold Σ on M and every integer n>0, ∃ a Samsara 4-manifold Σ' on M with

$$\beta^{(2)}_2(\Sigma';Z)=\beta^{(2)}_2(\Sigma;Z)+n.$$

For a reduced Samsara 4-manifold Σ^ on M^0 and every integer n>0, ∃ a reduced Samsara 4-manifold Σ^' on M^0 with

$$\beta^{(2)}_2(\Sigma\hat{}';Z)=\beta^{(2)}_2(\Sigma\hat{};Z)+n.$$

(3) ∃ Samsara 4-manifolds Σ on M and reduced Samsara 4-manifolds Σ^ on M^0 for some M, such that

$$\beta^{(2)}_2(\Sigma;Z)<\beta_1(M;Z_2) \text{ and } \beta^{(2)}_2(\Sigma\hat{};Z)<\beta_1(M;Z_2) .$$

For example, take M with $\beta_1(M;Z_2)>0$ such that M^0 is embedded in S^4. Then S^4 is a reduced closed Samsara 4-manifold on M^0 with

$$\beta^{(2)}_2(S^4;Z)=0<\beta_1(M;Z_2).$$

By a surgery of S^4 along the 2-knot $S^2=\partial M^0$, ∃ a closed Samsara 4-manifold Σ with Z-homology of $S^1\times S^3$ on M with $\beta^{(2)}_2(\Sigma;Z)=0<\beta_1(M;Z_2)$.

<u>Theorem</u>.

(1) For every closed connected oriented 3-manifold M, ∃ a (closed or bounded) Samsara 4-manifold Σ on M with $\beta^{(2)}_2(\Sigma;Z)=\beta_1(M;Z_2)$.

(2) For every integer n>0, ∃ ∞-many M such that every (closed or bounded) Samsara 4-manifold Σ on M has $\beta^{(2)}_2(\Sigma;Z) \geqq \beta_1(M;Z_2)=n$.

Corollary. For every closed connected oriented 3-manifold M, ∃ a reduced (closed or bounded) Samsara 4-manifold Σ^ on M^0 with

$$\beta^{(2)}_2(\Sigma\text{^};Z)=\beta_1(M;Z_2).$$

Further, for every integer n>0, ∃ ∞-many M such that every reduced (closed or bounded) Samsara 4-manifold Σ^ on M^0 has

$$\beta^{(2)}_2(\Sigma\text{^};Z)\geqq\beta_1(M;Z_2)=n.$$

Proof of (1) of Theorem.

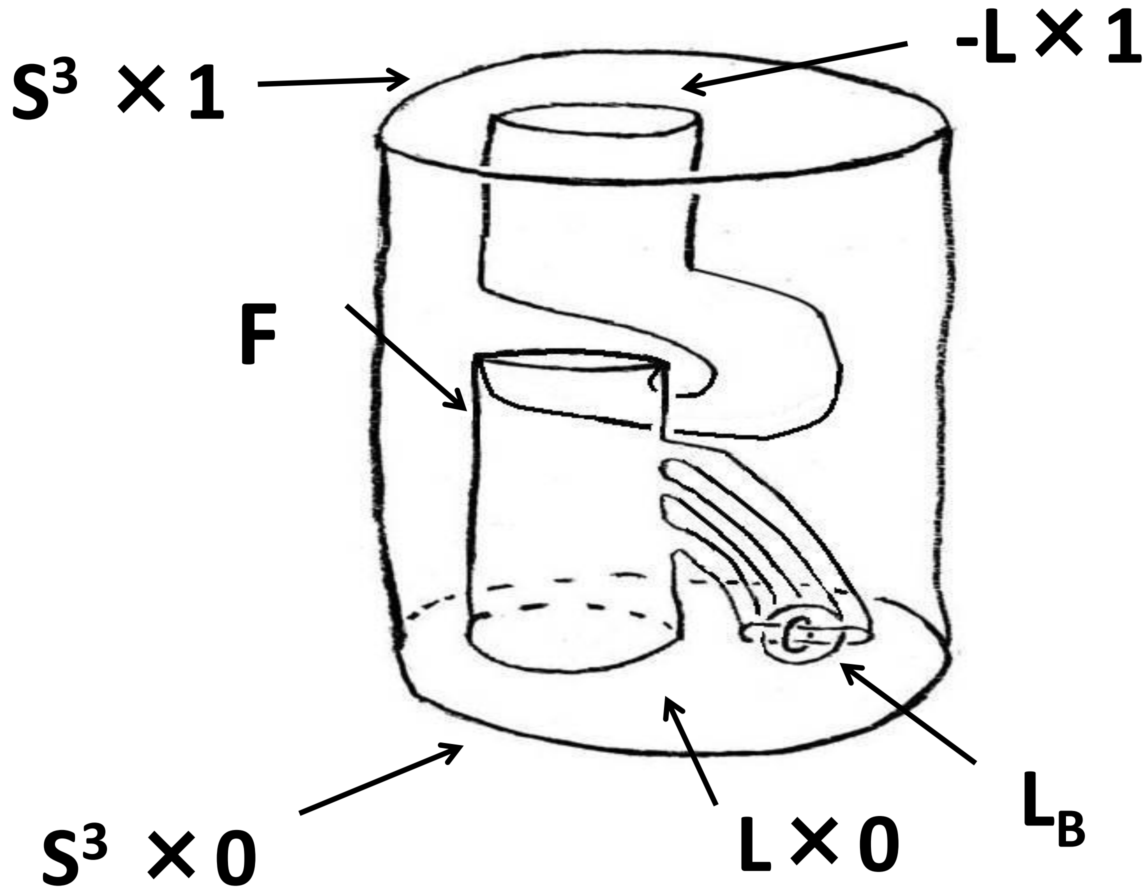
Let $M=\chi(L,0)$, the 0-surgery of S^3 along a link L with r components.

By the following paper:

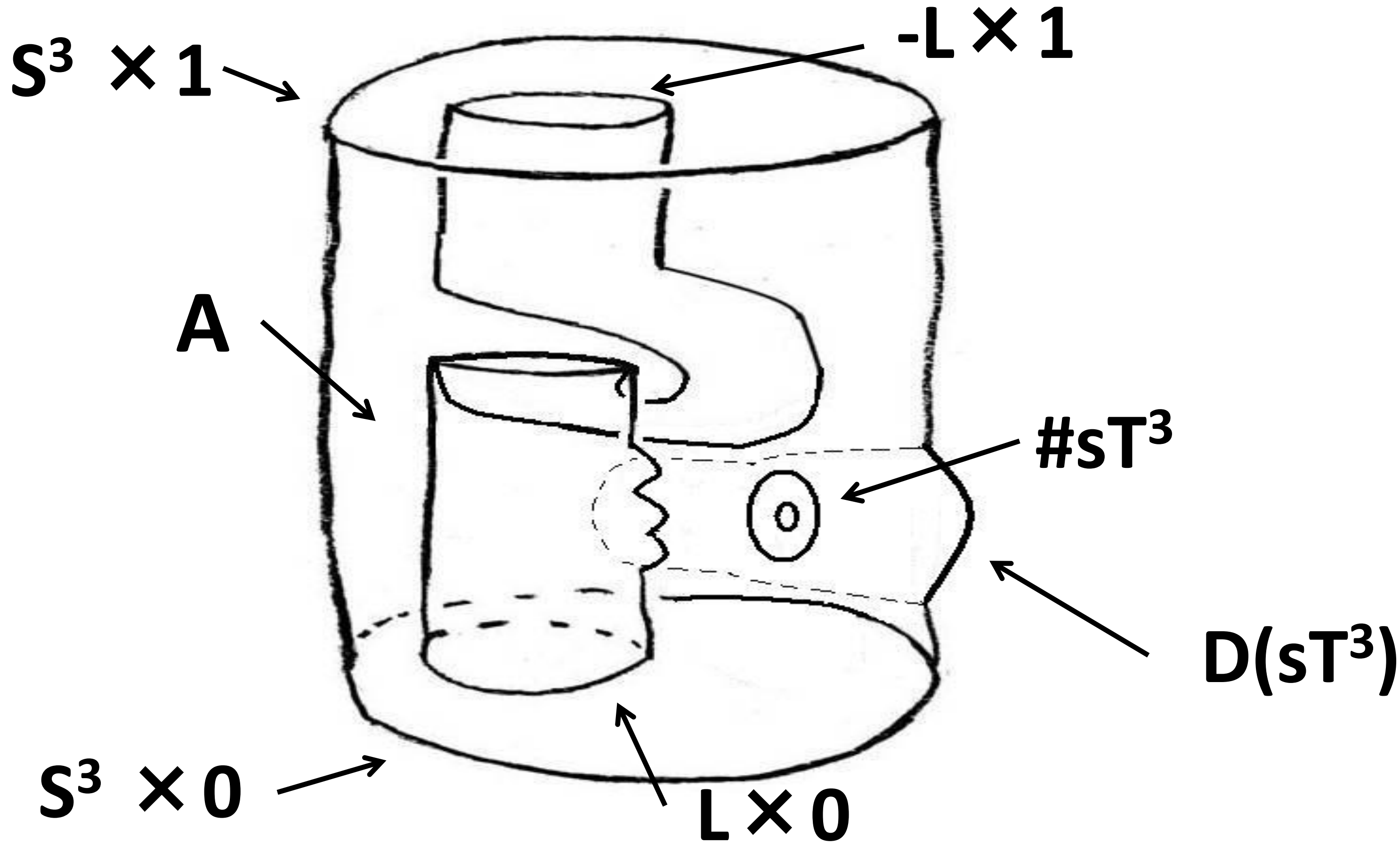
H. Murakami and Y. Nakanishi, On a certain move generating link-homology, Math. Ann. 284(1989), 75-89

the link –L is a fusion of a split union of L and some copies of the Borromean rings L_{B_i} (i=1,2,...,s).

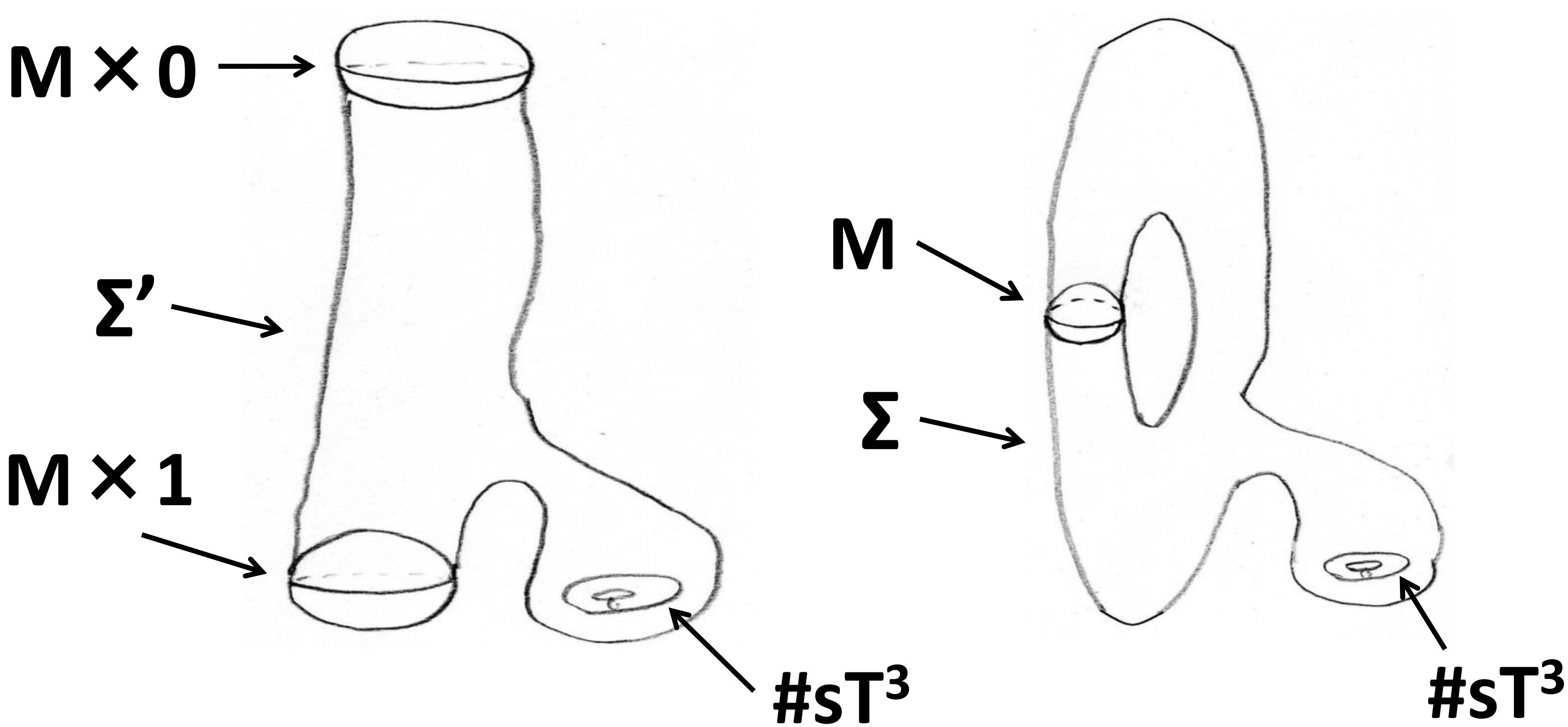
∃ a proper oriented surface F consisting of punctured annuli in $S^3 \times [0,1]$ such that $\partial F = (L \cup L_B) \times 0 \cup (-L) \times 1$, where L_B is the union of Borromean rings L_{B_i} (i=1,2,...,s).



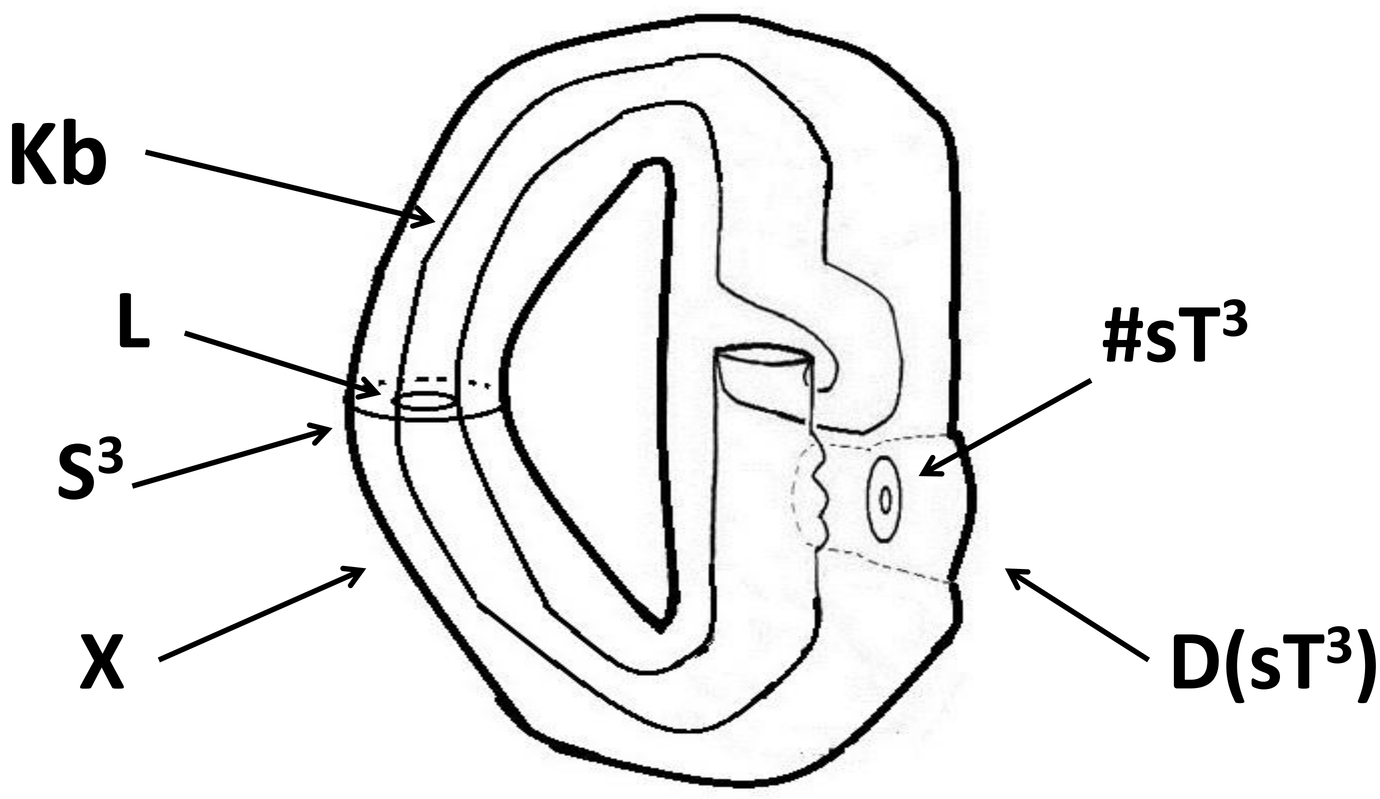
Attach 0-framed $D^2 \times D^2_i$ ($i=1,2,\ldots,3s$) to $S^3 \times 0$ along L_B. Then F extends the union A of r proper annuli with $\partial A = L \times 0 \cup (-L) \times 1$ in the connected sum $X' = S^3 \times [0,1] \# D(sT^3)$.



Let Σ' be the "0-surgery" of X' along A, so that $\partial Y = \#sT^3 \cup M \times 0 \cup -M \times 1$. A desired bounded Samsara 4-manifold Σ on M with $\partial\Sigma = \#sT^3$ is obtained from X' by identifying $M \times 0$ with $-M \times 1$.



Let $X = S^1 \times S^3 \# D(sT^3)$ be the manifold obtained from X' by identifying $S^3 \times 0$ with $S^3 \times 1$, and Kb the union of r Klein bottles obtained from A by identifying the boundaries.



(0) Σ is also the "0-surgery" of X along Kb .

(1) Since X' is simply connected, every element of H_1(X'-A) is generated by meridians of A in X'. Hence the natural map

$$H_1(S^3 \times 0 - L \times 0;Z) \rightarrow H_1(X'-A;Z)$$

is onto, so that the natural map

$$H_1(M \times 0;Z) \rightarrow H_1(\Sigma';Z)$$

is onto.

(2) For the inclusion $k: M \subset \Sigma$, $\exists$ a natural exact sequence

$$H_1(M;Z) \xrightarrow{k_*} H_1(\Sigma;Z) \to Z \to 0$$

and the image $Im(k_*) \subset H_1(\Sigma)$ is generated by order 2 elements. Hence

$$H_1(\Sigma;Z_{/2})=Z_{/2}$$

and

$$k_*=0: H_1(M; Z_{/2}) \to H_1(\Sigma; Z_{/2}).$$

For a generator $[C] \in H_1(\Sigma;Z)/(2\text{-torsion})=Z$,

$$Int(M,C)=1.$$

(3) $\exists$ a Z-basis $x_i \in H_2(D(sT^3);Z)$ (i=1,2,…,3s) with Int $(x_i,x_j)=0(\forall i,j)$ such that each x_i is represented by an embedded surface S_i disjoint from A.

(4) $\exists$ a Z-basis $y_i \in H_2(D(sT^3), \#sT^3;Z)$ (i=1,2,…,3s) with Int $(x_i,y_j)=\delta_{ij}(\forall i,j)$ such that each y_i is represented by an embedded proper 2-disk D_i transversely meeting S_i with one point and $D_i \cap D_j = \phi$ and $D_i \cap S_j = \phi$ for i≠j. Also, every D_i transversely meets A with one point in X'.

(5) ∂D_i (i=1,2,…,3s) forms a Z-basis of $H_1(\#sT^3;Z)$.

(6) From D_i and its parallel D'_i, we can construct an annulus $A(D_i)$ in X disjoint from K with

$$[\partial A(D_i)]=[\partial D_i + \partial D_i']=2[\partial D_i] \in H_1(\#sT^3;Z)$$

by piping them along K.

Let $y_i^{(2)} = (1/2)[A(D_i)] \in H_2(\Sigma,\partial\Sigma;Z_{/2})$.

Then $\partial_*: H_2(\Sigma,\partial\Sigma;Z_{/2}) \to H_1(\partial\Sigma;Z_{/2})$ is onto.

Hence the natural map $H_1(\Sigma; Z_{/2}) \to H_1(\Sigma,\partial\Sigma; Z_{/2})$ is an isomorphism and

$$H_1(\Sigma; Z_{/2})=H_1(\Sigma,\partial\Sigma; Z_{/2})= Z_{/2}.$$

(7) By Poincare duality, $H_3(\Sigma,\partial\Sigma;Z)=Z$ and M represents a generator. Since $H_2(\partial\Sigma;Z)$ is Z-free, the natural map $H_3(\Sigma;Z) \rightarrow H_3(\Sigma,\partial\Sigma;Z)=Z$ is an isomorphism. Hence $[M] \in H_3(\Sigma;Z)$ is a generator.

(8) $\chi(\Sigma) = \chi(X) = 3s-1$ implies $\dim_Q H_3(\Sigma;Q)=3s$.
By Poincare duality, $H_2(\Sigma;Z_{/2})$ is $Z_{/2}$-free.
Regarding $x_i \in H_2(\Sigma; Z_{/2})$ $(i=1,2,\ldots,3s)$,

$$\mathrm{Int}\,(x_i, y_i^{(2)})=\delta_{ij}(\forall\, i,j).$$

Hence $x_i(i=1,2,\ldots,3s)$ form a $Z_{(2)}$-basis of $H_2(\Sigma; Z_{/2})$ and by Poincare duality $y_i^{(2)}$ $(i=1,2,\ldots,3s)$ are a $Z_{(2)}$-basis of $H_2(\Sigma,\partial\Sigma; Z_{/2})$.
In particular, $H_*(\Sigma; Z_{/2}) = H_*(X; Z_{/2})$.

(9) Since M is disjoint from a 2-cycle representing $y_i^{(2)} \in H_2(\Sigma,\partial\Sigma; Z_{/2})$ and $x_i (i=1,2,\ldots,3s)$ form a $Z_{/2}$-basis of $H_2(\Sigma; Z_{/2})$, we see from Int $(x_i, y_i^{(2)}) = \delta_{ij}$ $(\forall i,j)$ that $k_*=0: H_2(M;Z_{/2}) \to H_2(\Sigma;Z_{/2})$. //

Proof of (2) of Theorem.

We use the signature theorem in:

(1) A. Kawauchi, On the signature invariants of infinite cyclic coverings of even dimensional manifolds, Advanced Studies in Pure Math. 9(1986), 177-188.
http://www.sci.osaka-cu.ac.jp/~kawauchi/index.html

(2) A. Kawauchi, The imbedding problem of 3-manifolds into 4-manifolds, Osaka J. Math. 25 (1988), 171-183.

Let Y be a compact oriented 4-manifold with boundary a closed 3-manifold B.
Let (Y_∞, B_∞) be the infinite cyclic covering of (Y,B) associated with a homomorphism $\gamma : H_1(Y;Z) \to Z$.
Let $\dot{\gamma} : H_1(Y;Z) \to Z$ be the restriction of γ.
Consider the Γ-intersection form

$$\mathrm{Int}_\Gamma : H_2(Y_\infty;Q) \times H_2(Y_\infty;Q) \to \Gamma.$$

Let $A(t)$ be a Γ-Hermitian matrix representing the Γ-intersection form Int_Γ on $H_2(Y_\infty;Q)/(\Gamma\text{-torsions})$.

Let $a, x \in (-1, 1)$.

Define

$$\tau_{a\pm 0}(Y_\infty) = \lim_{x\to a\pm 0} \mathrm{sign}\, A(x+(1-x^2)^{1/2} i).$$

The signature invariants $\sigma_x(B_\infty)$, $x \in (-1, 1)$, of B_∞ is defined on the quadratic form

$$b: Tor_\Gamma H_1(B_\infty;Q) \times Tor_\Gamma H_1(B_\infty;Q) \to Q$$

so that

$$\sigma_{(a,1]}(B_\infty) = \sum_{a<x<1} \sigma_x(B_\infty).$$

Signature Theorem.

$$\tau_{a-0}(Y_\infty)\text{-sign } Y = \sigma_{[a,1]}(B_\infty),$$

$$\tau_{a+0}(Y_\infty)\text{-sign } Y = \sigma_{(a,1]}(B_\infty).$$

Corollary. For $\forall a \in (-1,1)$,

$$|\sigma_{[a,1]}(B_\infty)| - \kappa_1(B_\infty) \leqq 2\hat{\beta}_2(Y;Z)$$

where

$\kappa_1(B_\infty)$ = Z-rank of $\mathrm{Ker}(t-1: H_1(B_\infty) \to H_1(B_\infty))$.

$\hat{\beta}_2(Y;Z)$ = rank of Int: $H_2(Y;Z) \times H_2(Y;Z) \to Z$.

For every integer n>0, take n knots K_i ($1 \leqq i \leqq n$) whose signatures $\sigma(K_i)$($1 \leqq i \leqq n$) have:

$|\sigma(K_1)| > 0$ and $|\sigma(K_i)| > |\sum_{j=1}^{i-1} \sigma(K_j)|$ (i=2,3,...,n).

Let $M_i = \chi(K_i,0)$ and $M=M_1\#M_2\#...\#M_n$.

Call M an <u>efficient 3-manifold of rank</u> n.

<u>Fact:</u> For every integer n>0, ∃∞-many efficient 3-manifolds of rank n.

Claim: For every efficient M of any rank $n \geqq 1$ and every reduced Samsara 4-manifold Σ^ on M^0, we have $\beta^{(2)}_2(\Sigma\hat{};Z) \geqq \beta_1(M;Z_2)=n$.

(∵) If $\beta^{(2)}_2(\Sigma\hat{};Z) < \beta_1(M;Z_2)$, then apply Signature Theorem for $Y=cl(\Sigma\hat{} \setminus M^0 \times I)$ and $B=\partial Y \supset DM^0$ to obtain a Z_2-asymmetric homomorphism $\dot{\gamma}: H_1(DM^0;Z) \to Z$ with $\sigma_{(-1,1]}(DM^0_\infty)=0$, contradicting that M is efficient. //

5. A meaning of a Samsara 4-manifold

cf. A. Kawauchi, On 4-dimensional universe for every 3-dimensional manifold, preprint.

Definition.

A (4-dimensional) universe = a boundary-less orientable 4-manifold U with every M embedded.

Problem. Characterize the topological shape of a universe.

<u>Fact</u>. **For ∀ closed orientable 4-manifold X, ∃ M such that M^0 is not embeddable in X, so that M is not embeddable in X.**

Cf. A. Kawauchi, The imbedding problem of 3-manifolds into 4-manifolds, Osaka J. Math. 25 (1988), 171-183.

Thus, a universe must be an open 4-manifold.

Let Σ_i (i=1,2,3,…) be bounded Samsara 4-manifolds on all M.

<u>A Samsara universe</u> is: U_{SM}= int $(R^4_+ \bigcup_{i=1}^{+\infty} \Sigma_i)$,

where $\bigcup$ denotes the boundary disk sums.

(1) Int=0: $H_2(U_{SM}; Z) \times H_2(U_{SM}; Z) \to Z$.

(2) For $\forall$ M, $\exists$ a type 1 embedding k:M $\to U_{SM}$

with k_*=0: $H_d(M;Z_{/2}) \to H_d(U_{SM};Z_{/2})$, d=1, 2.