On Coxeter links associated to cycle graphs

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Abstract: Through study of zeros of the Alexander polynomials of links, we came to study the class of "Coxeter links".
Coxeter links are fibered links associated to labeled graphs with certain conditions.
In this talk, we classify those links arising from cycle graphs, and determine which of them are Coxeter links. We calculate the multivariate Alexander polynomials and see that all the zeros of the reduced Alexander polynomials of these links are on the unit circle.

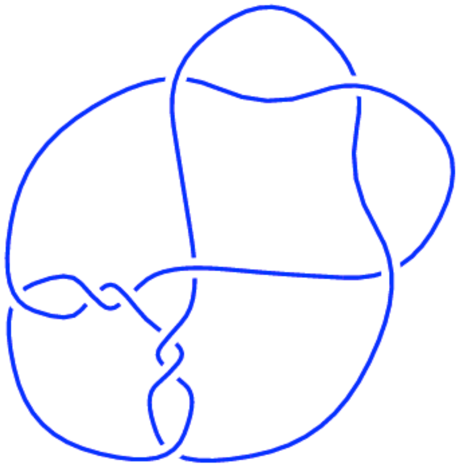
Motivation

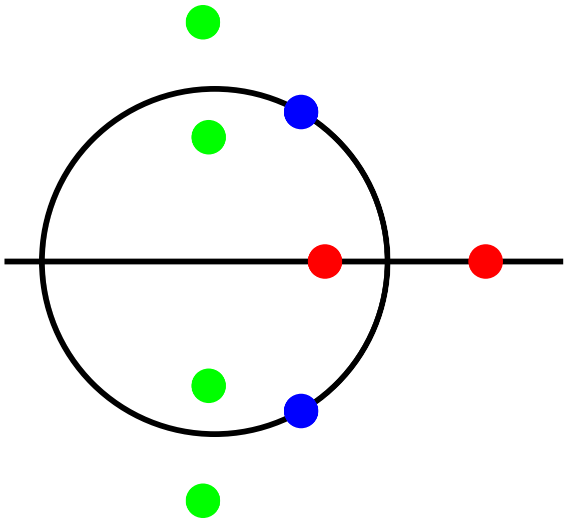
- Study of zeros of Alexander pol. of links.

$$\Delta_L(\alpha) = 0, \ \alpha \in \mathbb{C}$$

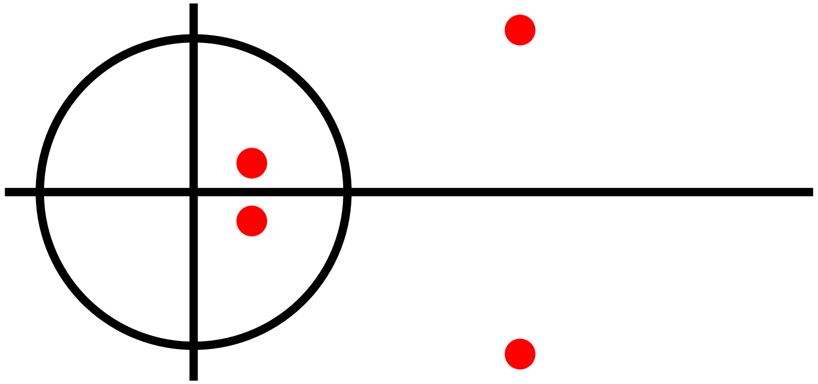
For a given link, how they are distribute on $\mathbb{C}$?

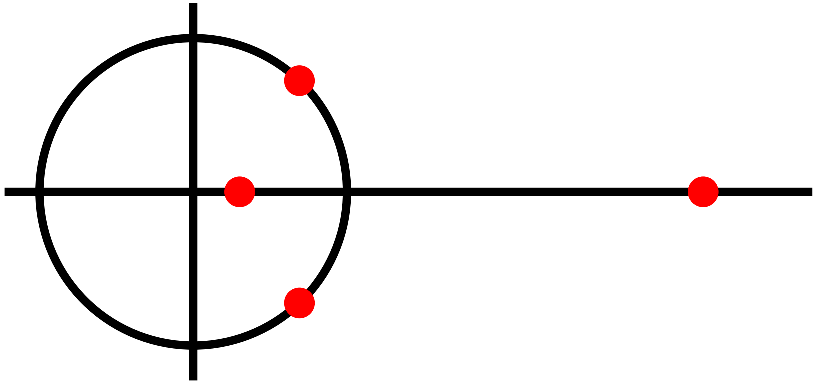
e.g. 10_{64}

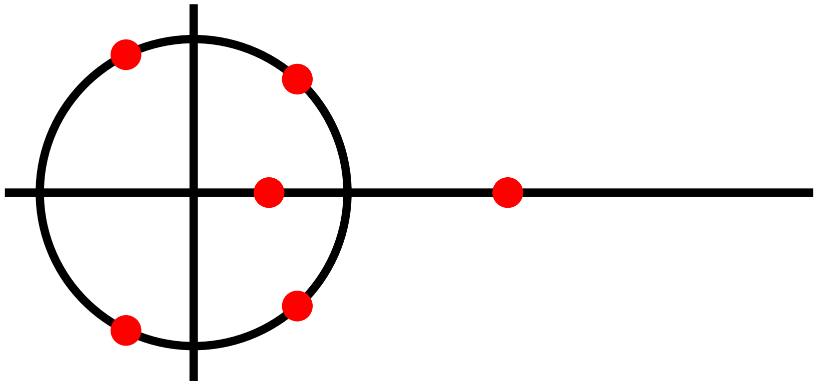




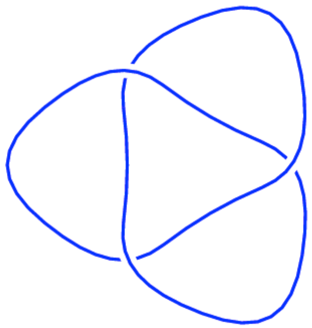
$$1 - 3t + 6t^2 - 10t^3 + 11t^4 - 10t^5 + 6t^6 - 3t^7 + t^8$$

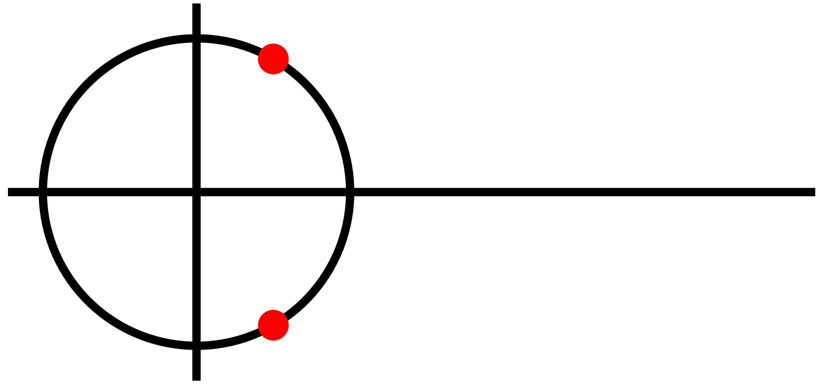






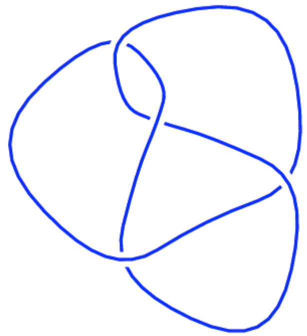
$\mathbf{3_1}$:

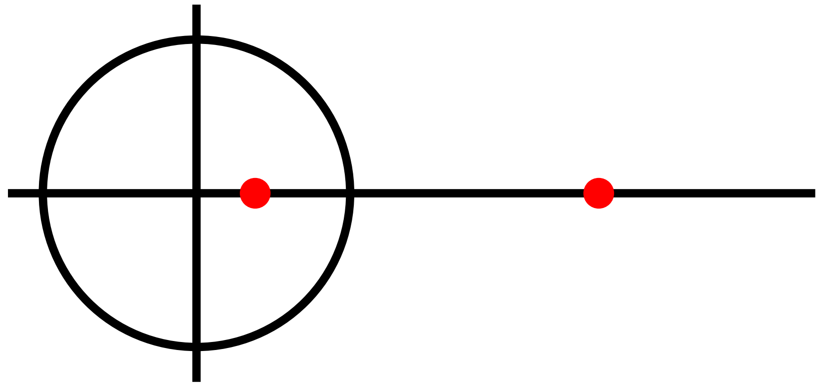




$$t^2 - t + 1 = 0 \quad \Rightarrow \quad t = \frac{1 \pm \sqrt{3}\, i}{2}$$

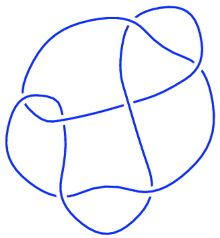
$\mathbf{4_1}$:

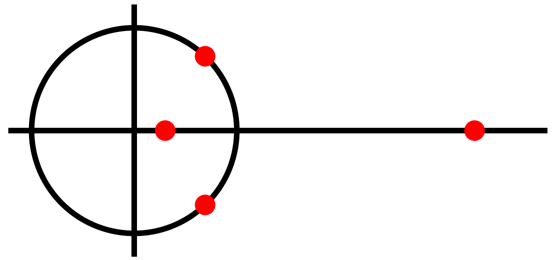




$$t^2 - 3t + 1 = 0 \quad \Rightarrow \quad t = \frac{3 \pm \sqrt{5}}{2}$$

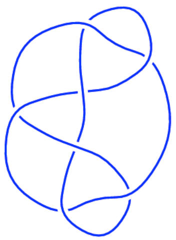
$\mathbf{7_6}$:

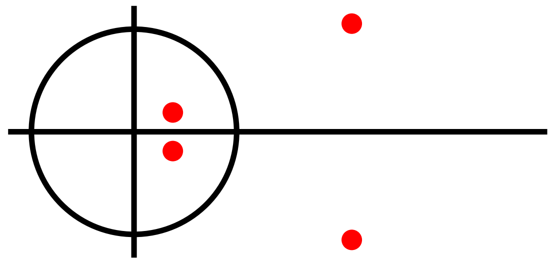




$$1 - 5t + 7t^2 - 5t^3 - t^4$$

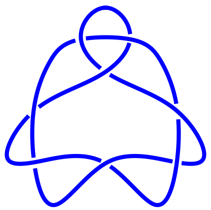
$\mathbf{7_7}$:



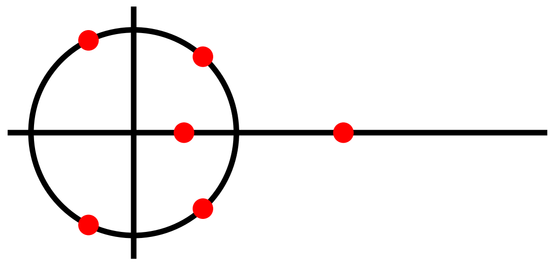


$$1 - 5t + 9t^2 - 5t^3 + t^4$$

$\mathbf{8_2}$:



8_2



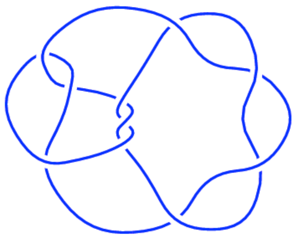
$$1 - 3t + 3t^2 - 3t^3 + 3t^4 - 3t^5 + t^6$$

Note that for any alternating knot K,

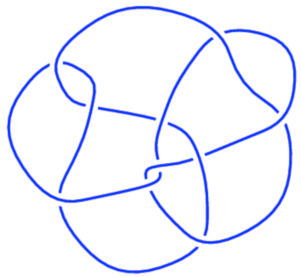
$$\boxed{\Delta_K(\alpha) = 0, \alpha \in \mathbb{R} \Rightarrow \alpha > 0}.$$

This shows that the following are non-alt'g.
(All knots up to 10c with ”real zero < 0”.)

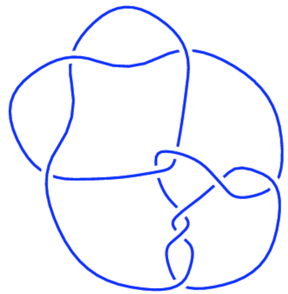
10_{139}



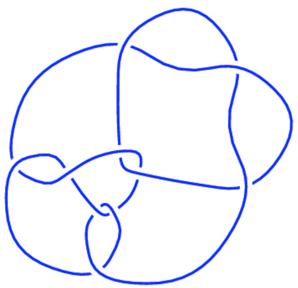
10_{145}



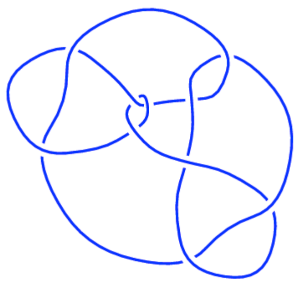
10_{152}



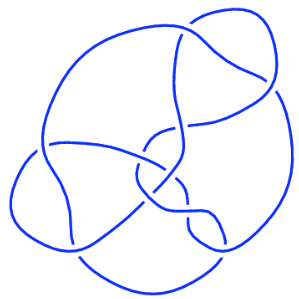
10_{153}



10_{154}



10_{161}



The following are non-alt'g.
(All knots up to 10c with "real zero $< \mathbf{0}$".)

$\mathbf{10_{139}, 1 - t + 2t^3 - 3t^4 + 2t^5 - t^7 + t^8}$

$\mathbf{10_{145}, 1 + t - 3t^2 + t^3 + t^4}$

$\mathbf{10_{152}, 1 - t - t^2 + 4t^3 - 5t^4 + 4t^5 - t^6 - t^7 + t^8}$

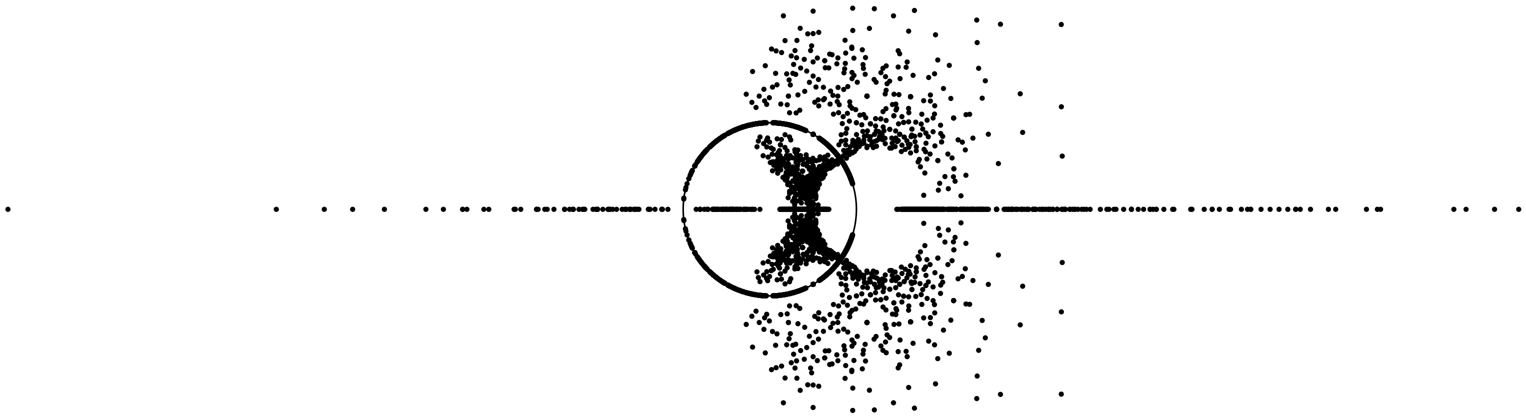
$\mathbf{10_{153}, 1 - t - t^2 + 3t^3 - t^4 - t^5 + t^6}$

$\mathbf{10_{154}, 1 - 4t^2 + 7t^3 - 4t^4 + t^6}$

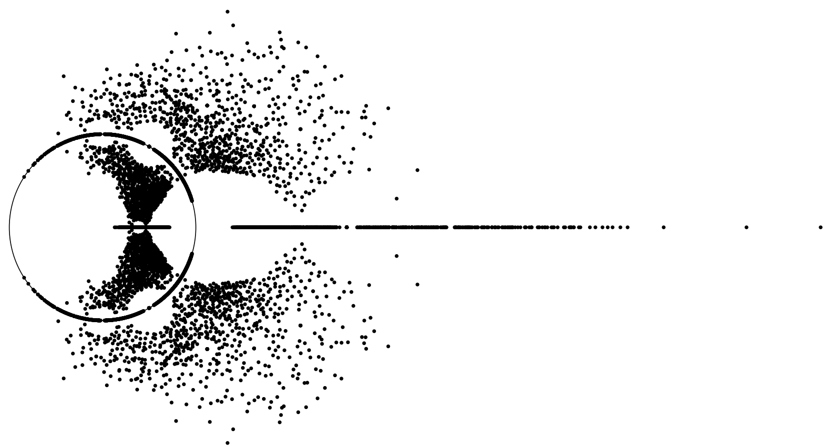
$\mathbf{10_{161}, 1 - 2t^2 + 3t^3 - 2t^4 + t^6}$

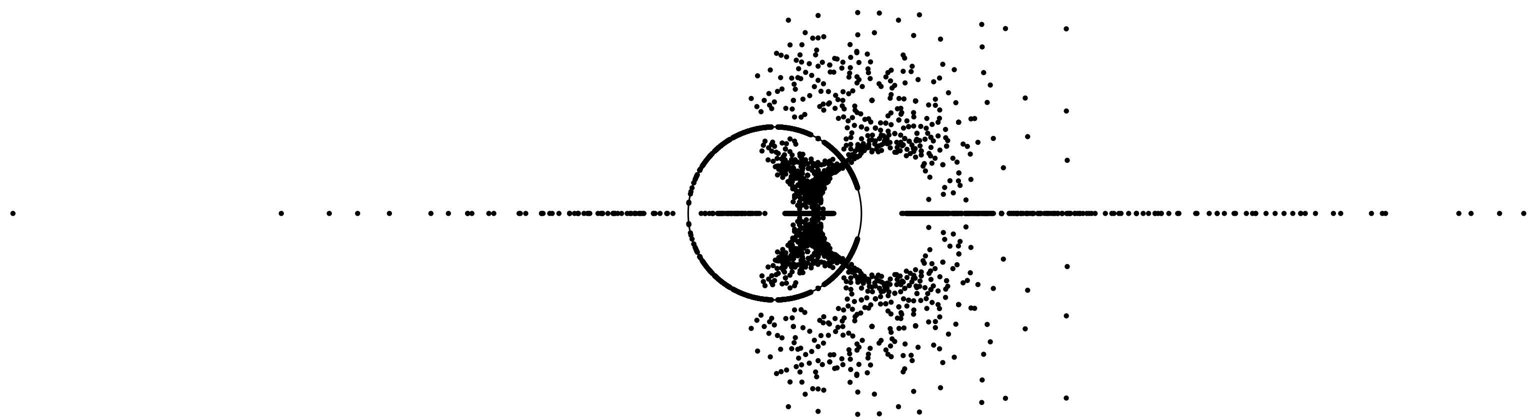
For alt'g knots, coefficients of Alex.pol. alternate in sign.
Note the famous "trapezoid conjecture".

What do you see in this ?

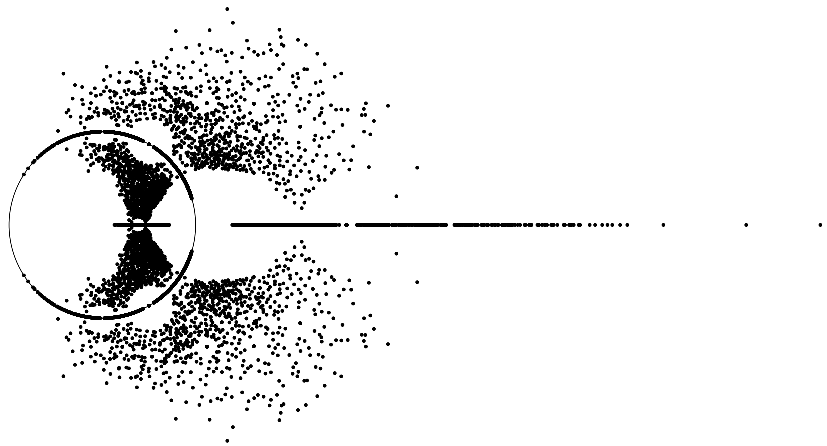


How about this ?



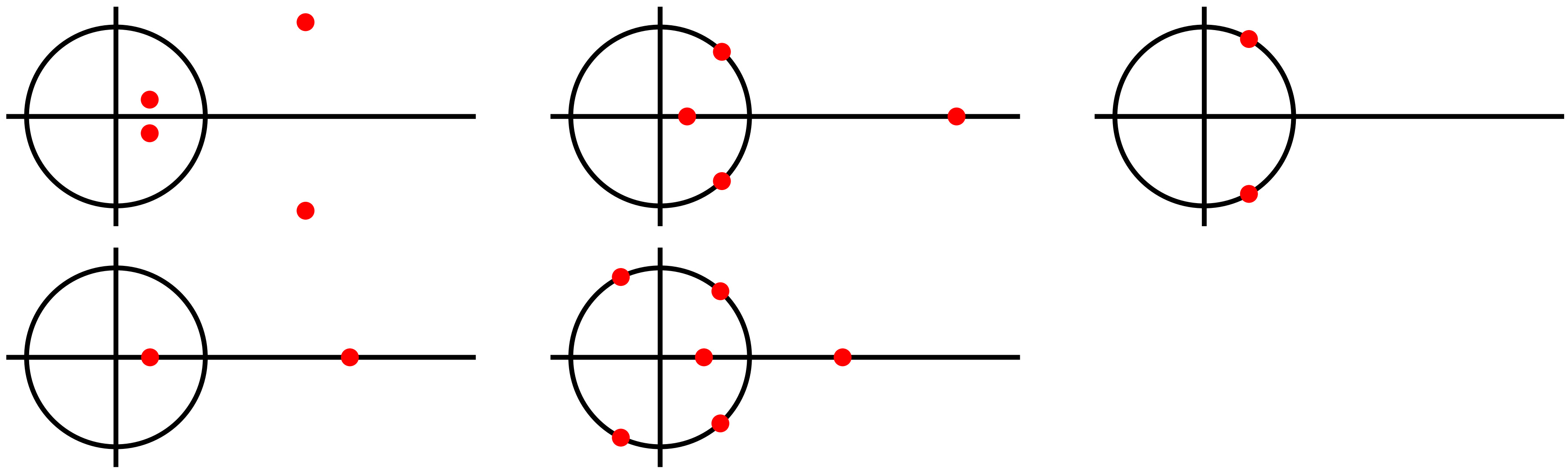


zeros for non-alt'g knots w/ crossings 11 and 12.

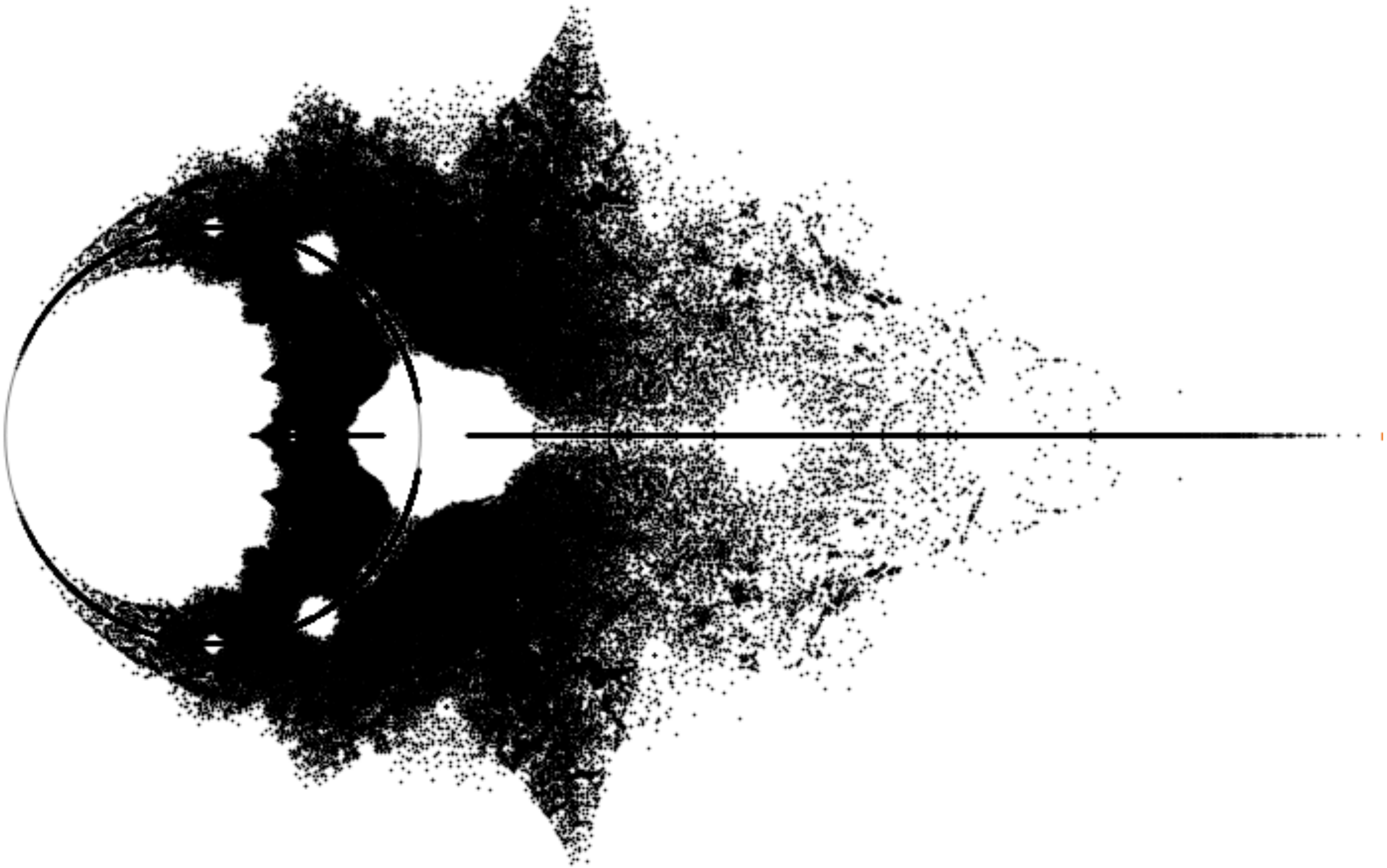


zeros for alt'g knots w/ crossings 11 and 12.

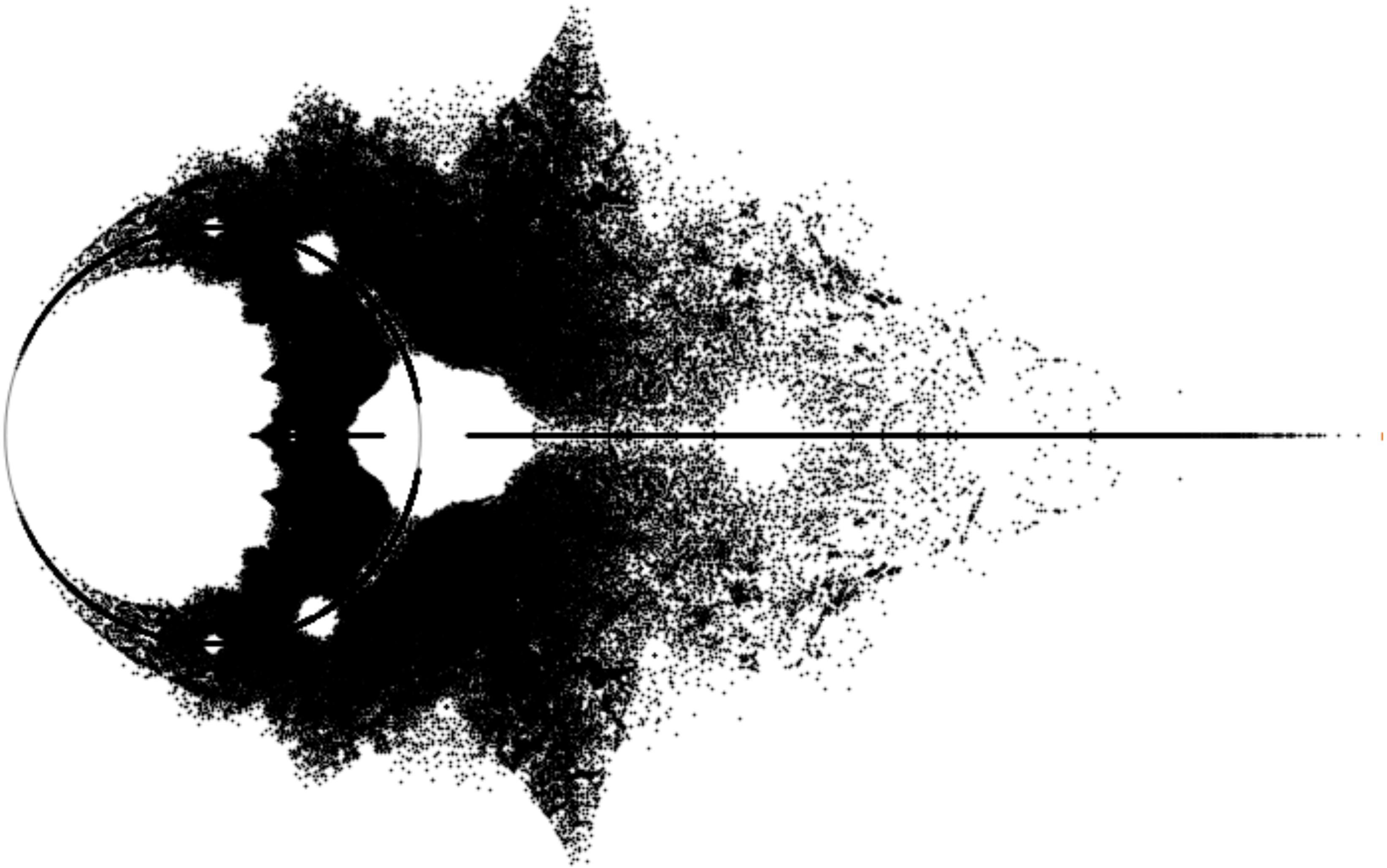
Conjecture 1.1. *(J. Hoste 2002)*
For any alternating knot K, we have $\boxed{Re(\alpha) > -1}$*, where* $\Delta_K(\alpha) = 0$.



It is unknown: What $\Delta_K(t)$ can be that of "alternating knot" or even "2-bridge knot".

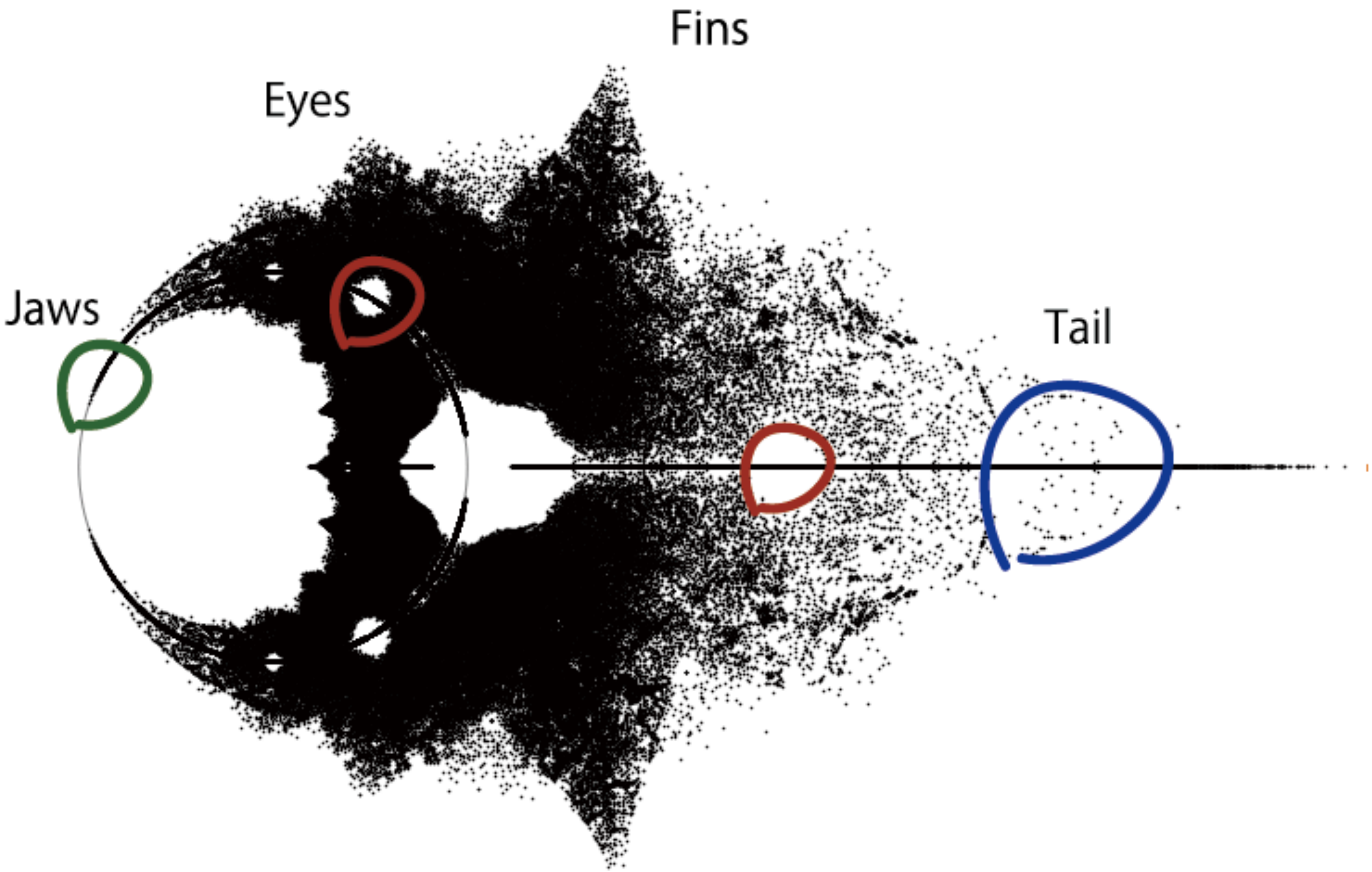


What is this?



zeros for rational knots up to 20 crossings.
Murasugi-Lyubich proved $-\mathbf{3} < \mathbf{Re}(\boldsymbol{\alpha}) < \mathbf{6}$. (Top. Appl. 2012)

Beside Hoste's conjecture, we have many conjectures for the zeros for 2-bridge knots.



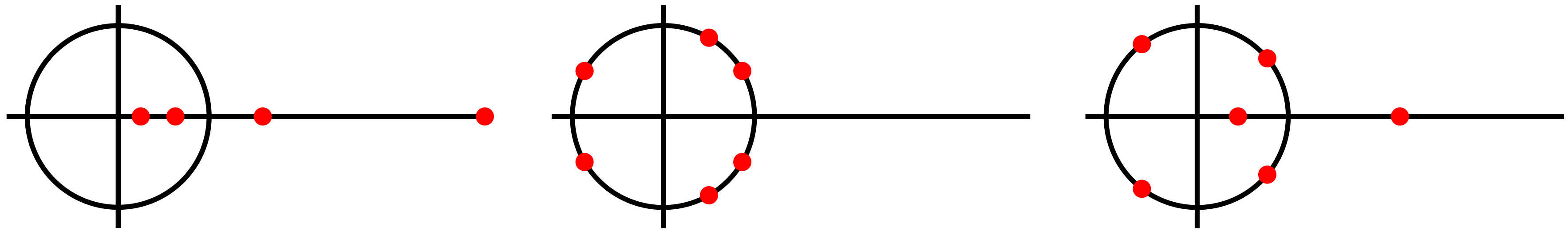
Posisiton of zeros of Alex.pol. not only distinguishes links but also indicates some are similar.

Definition 1.2. α: zero of $\Delta_L(t)$.

L is **real stable** if $\forall \alpha \in \mathbb{R}$.

L is **circular cable** if $\forall |\alpha| = 1$.

L is **bi-stable** if $\forall \alpha \in \mathbb{R}$ or $|\alpha| = 1$.



Note.

Proposition 1.3. *Trapezoidal conjecture holds for $\boldsymbol{r}$-stable alternating knots.*

H.S.Wilf, Generatingfunctionolgy,
Academic press (1990) See p127.

Rational knots

Rational knots (a.k.a 2-bridge knots) are boundary of linear plumbing of annuli with twists $[2a_1, 2b_1, \ldots, 2a_n, 2b_n]$. e.g. $3_1 = [2, 2], 4_1 = [2, -2]$.

Theorem 1.4.
(i) $\forall a_i, b_i > 0 \Rightarrow K$ *is* c*-stable.*
(ii) $\forall a_i > 0, b_i < 0 \Rightarrow K$ *is* r*-stable.*

• exceptionally stable rational knots.

Proposition 1.5.
(i) For $[2a, -2, -2b, 2c], a, b, c > 0$*,*

$\quad bc > 2a(c+1) \Rightarrow K$ *is* r*-stable.*

(ii) For $[2a, 2b, -2b, -2a], a, b > 0$*,*

$\quad a \geq 4b \Leftrightarrow K$ *is* r*-stable.*

• systematic construction of c-stable knots.

Theorem 1.6. $\forall$ *Seifert surface* F*, we can twist some bands of* F *into* F'*, where* $\partial F'$ *is* c*-stale.*

Sideways: flat plumbing basket

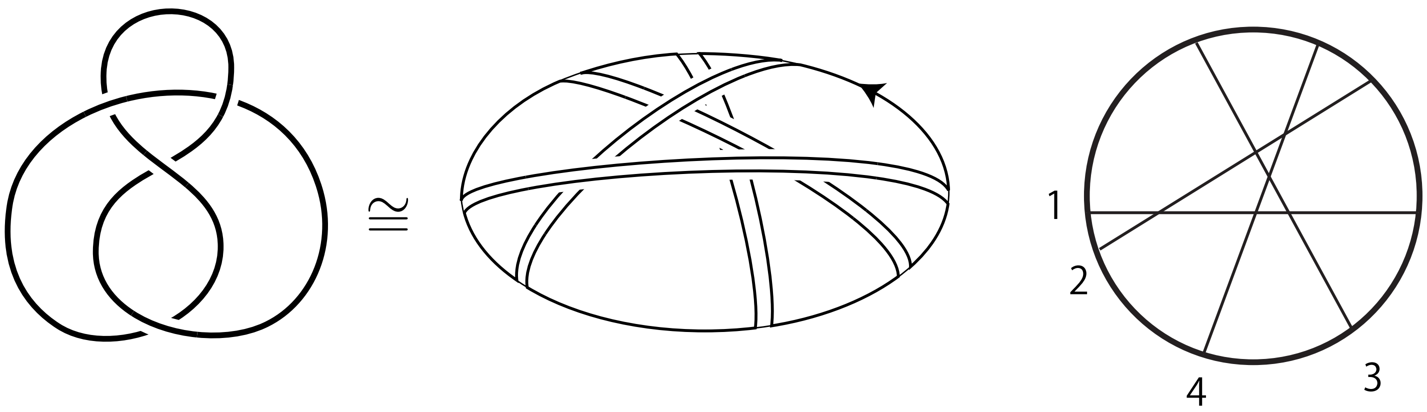
∀ link has a Seifert surf. $\boldsymbol{F}$ obtained by attaching flat bands to a disk. $\boldsymbol{F}$ is called *flat plumbing basket*.

We gave an **algorithm to obtain f.p.b.**

R.Furihata-H-T.Kobayashi

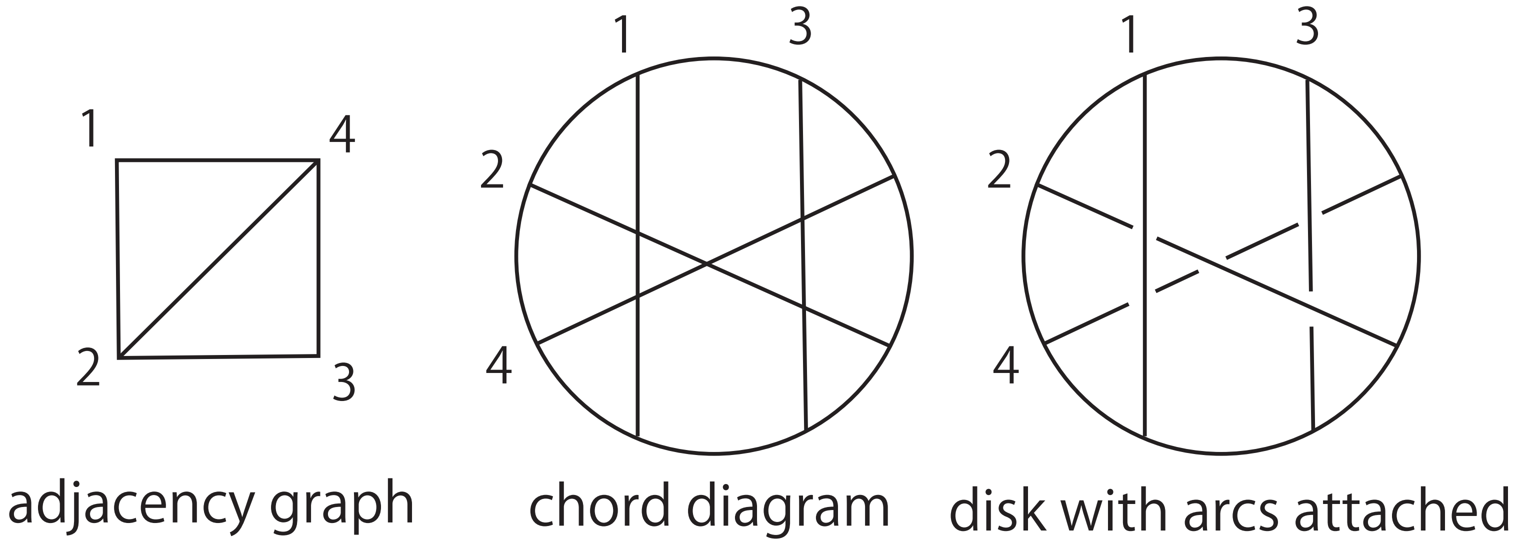
Bull. L.M.S. 40 (2008) 405-414.

Seifert surfaces in open books, and a new coding algorithm for links.

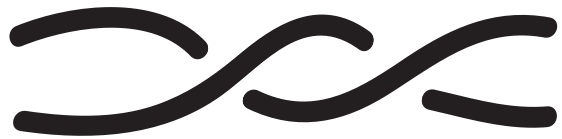


links associated to a labelled graph

Adjancency graph, labeled chord diag. $\&$ accisiated link L.



A right-full-twisted bands are attached to a disk to form a Seifert surface F. Then ∂F is the link associated to Γ.



Note that these links are fibered, i.e.,
$cl(S^3 - N(L)) \cong (F \times I)/\sim$.
F: a Seifert surface for L.

i.e., the link complements is a surface bundle over F.

Therefore, not all links are associated to chord diagram.
And fibered knot 4_1 does not arise, since it is a plumbing of $+$ and $-$ Hopf bands.

Background of Coxeter links

c.f E.Hironaka, J.London Math. Soc 69 (2004) 243-257.

Chord diagrams and Coxeter links

$\boldsymbol{\Gamma}$: finite graph w/o loops and multi-edge.

$\boldsymbol{S = \{s_1, \cdots, s_n\}}$: ordering of vertices of $\boldsymbol{\Gamma}$.

$\boldsymbol{A}$: adjacency mtx of $\boldsymbol{\Gamma}$, i.e., $\boldsymbol{a_{ij} = 1}$ or $\boldsymbol{0}$.

$$\boldsymbol{W = \langle S | (s_i s_j)^{m_{ij}} = 1 \rangle, m_{ij} = \begin{cases} 1 & (i + j) \\ a_{ij} + 2 & (i \neq j) \end{cases}}$$

$\boldsymbol{(W, S)}$ is called a simply-laced Coxeter system,

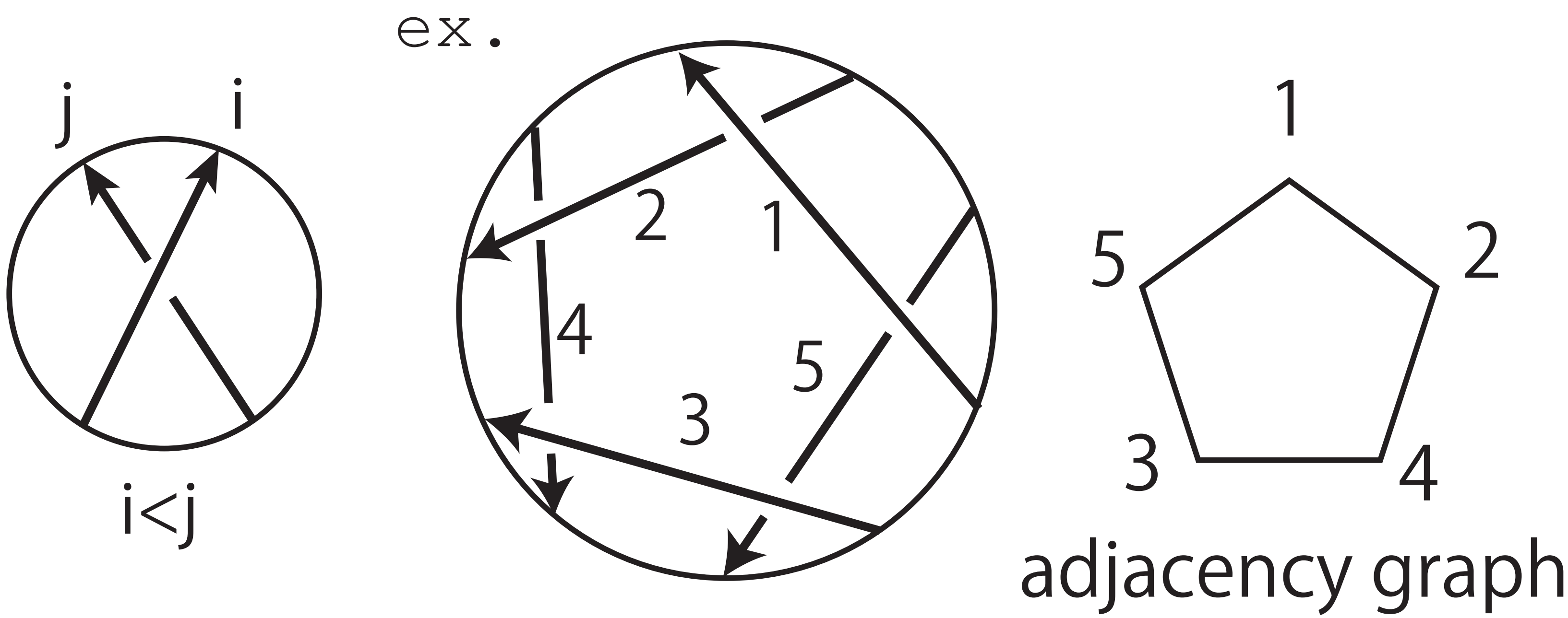
with Coxeter system $\boldsymbol{c = s_1 s_2 \ldots s_n}$.

Coxeter system has a natural repre. as an action on $\mathbb{R}^{\boldsymbol{n}}$.
$(\boldsymbol{W}, \boldsymbol{S})$: *spherical* if $\boldsymbol{W}$ is a finite reflection group.
$(\boldsymbol{W}, \boldsymbol{S})$: *affine* if $\boldsymbol{W}$ is isom to a group of affine reflection.

Theorem 1.7. *(Howlett 1982)* $(\boldsymbol{W}, \boldsymbol{S})$ *is*
(i) spherical $\Leftrightarrow$ *all e-val of* $\boldsymbol{c}$ *are roots of unity* $(\neq \mathbf{1})$.
(ii) affine $\Leftrightarrow$ $\boldsymbol{c}$ *has e-val* $\mathbf{1}$ *and other e-val's are modulus 1.*

E.Hironaka formulated Coxeter links so that their Alex.pols are the Characteristic pols of $\boldsymbol{c}$.

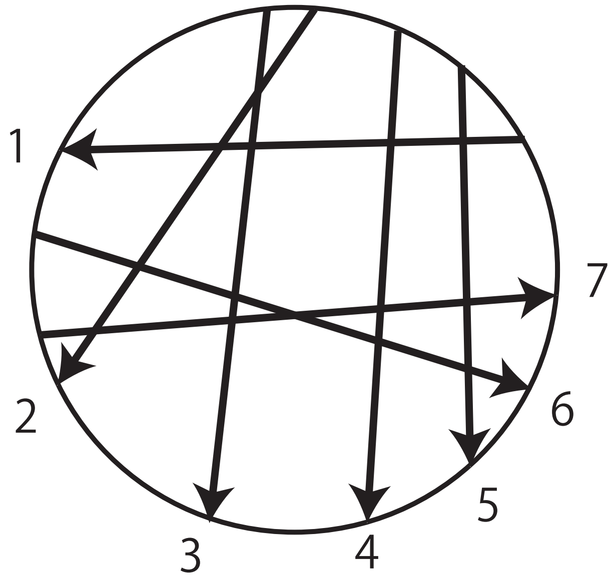
Definition 1.8. A Coxeter link is a link obtained from an ordered chord diagram such that for some orientation of chords, the following is satisfied.

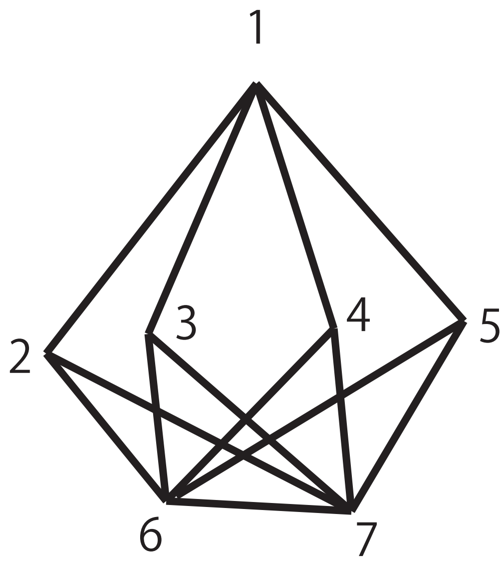


An ordered chord diagram is of Coxeter type if it yields a Coxeter links.

Proposition 1.9. *For any chord diagram, we can give some ordering and orientation so that it yields a coveter link.*

e.g.

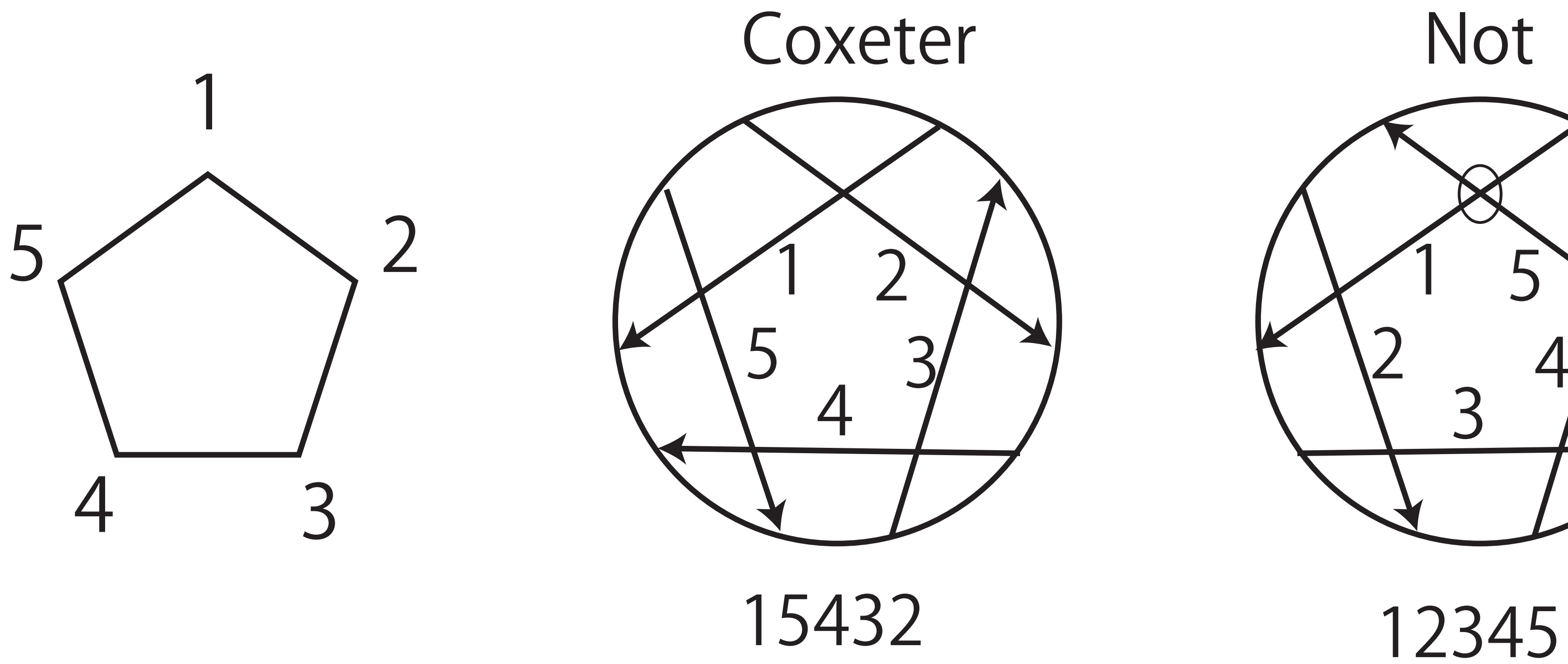




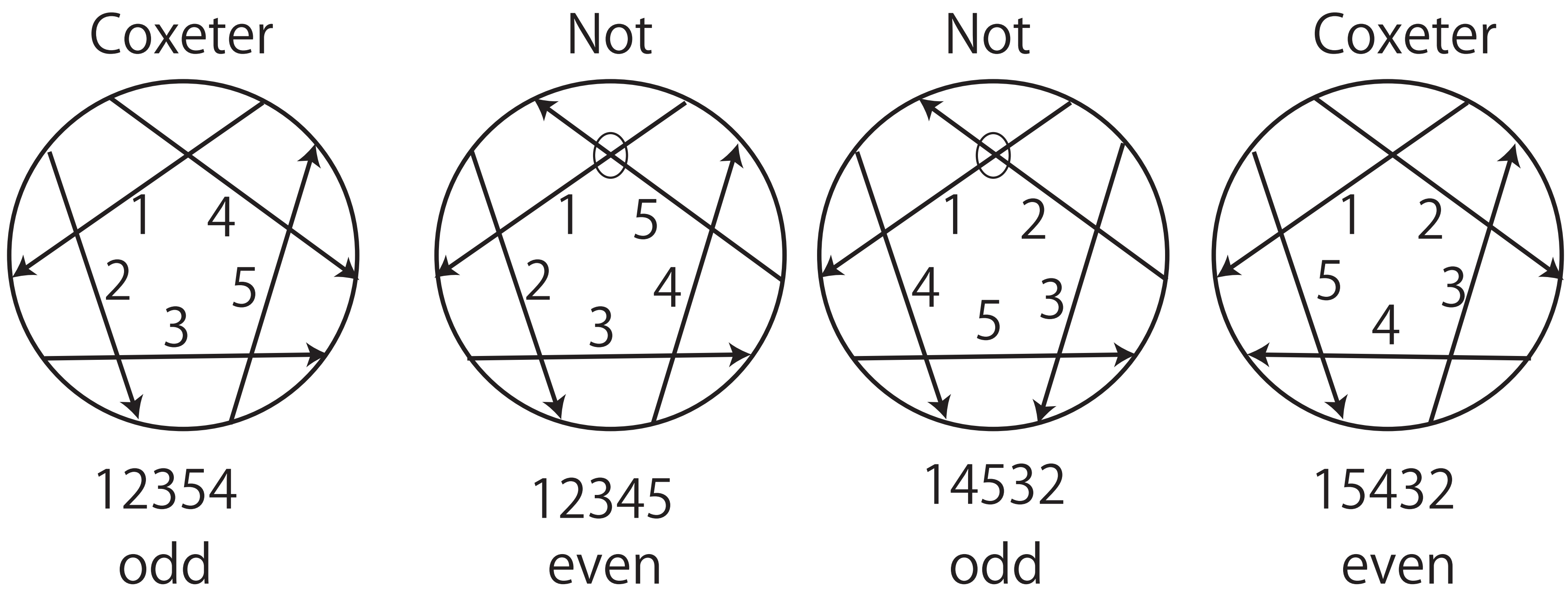
Not all labelled graphs are realizable.
(i.e, Some can not yield a ordered chord diagram.)

In this talk, we deal with links associated to chord diagrams **whose adjacency graph is a cycle**.

Note: For a given ordered cycle graph, there are exactly two ways to realize it by ordered chord diagram.



To denote the link, read the chords counter-clockwise.



Parity of permutations does not work to detect whether Coxeter type or not.

To determine which chord diag (of length n) is of Coxeter-type, we introduce the swirl move.

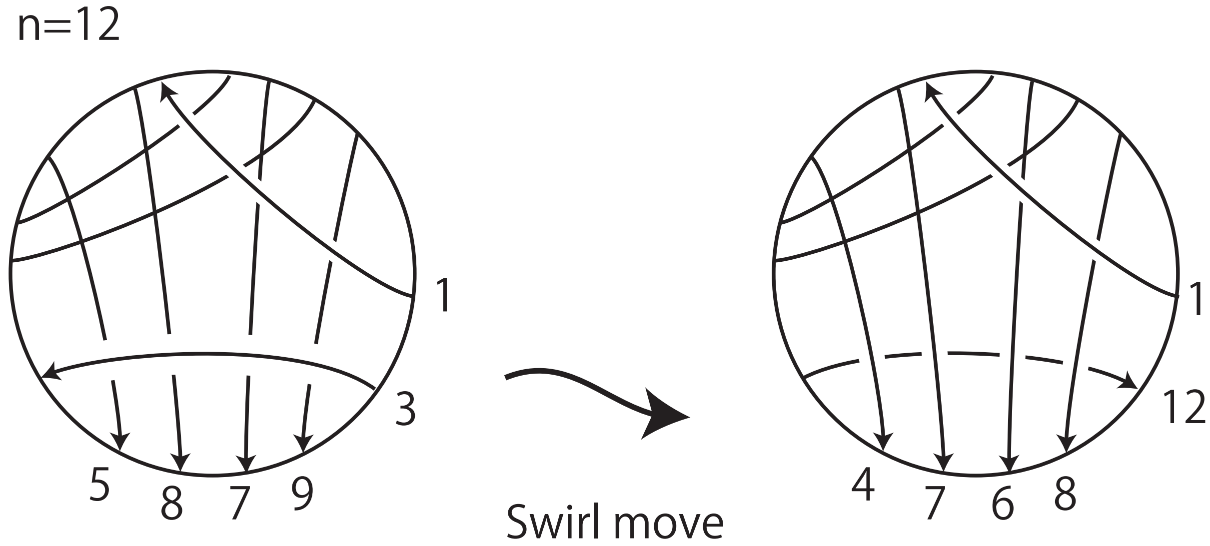
Suppose chord k is locally highest (i.e, k is locally minimal).

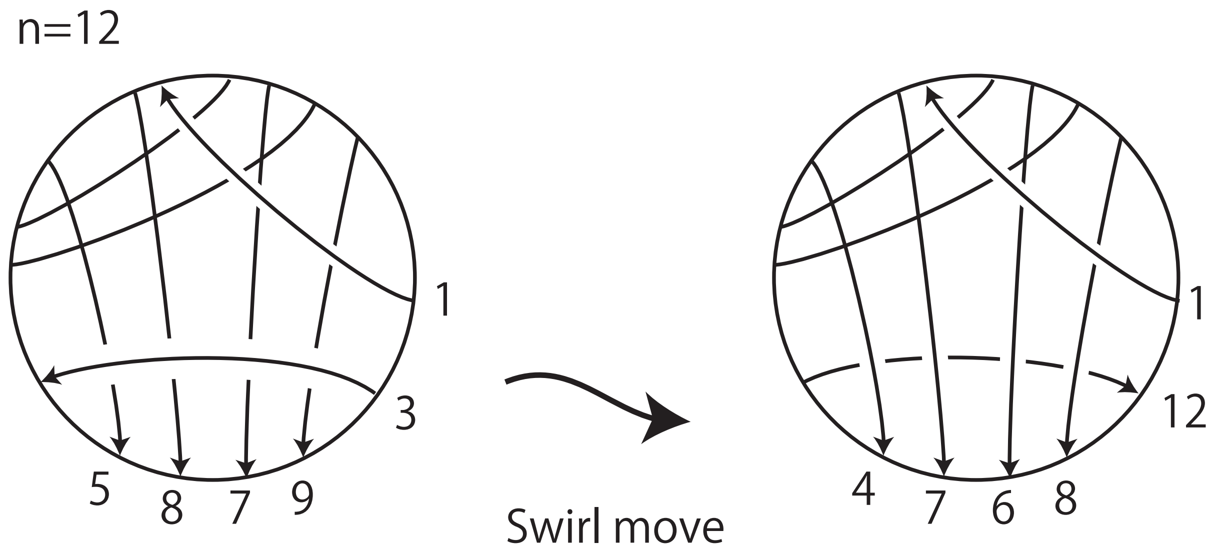
(i) Swirl the chord to be locally lowest.

(ii) re-label it as $n+1$.

(iii) reverse the orientation of the swirled chord.

(iv) slide the labels $k+1, k+2, \cdots, n+1$ down by 1.

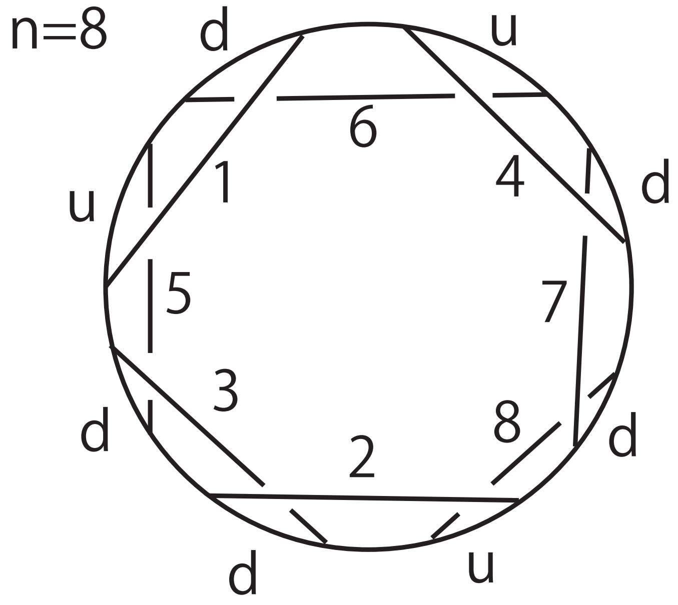




Note that the swirl move does not change the accosted link, but change the bases of $H_1(F)$.
However, by base change, we have the following.

Theorem 1.10. *The swirl move preserves whether or not the chord diagrams are of Coxeter-type.*

To give a standard form, we introduce the **$\boldsymbol{u}$-$\boldsymbol{d}$ notation** of labelled chord diagrams.

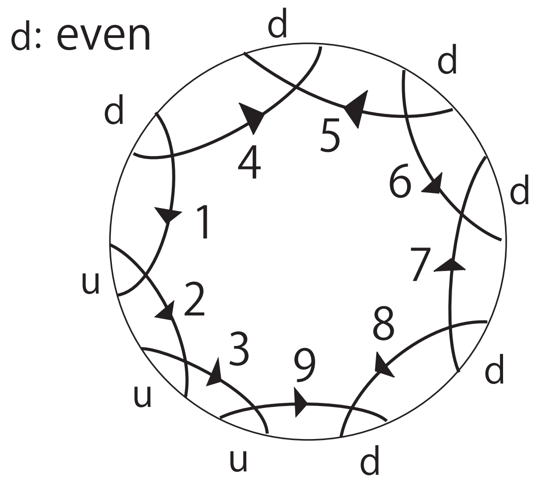


${}^{1}u^{5}d^{3}d^{2}u^{8}d^{7}d^{4}u^{6}d^{1}$

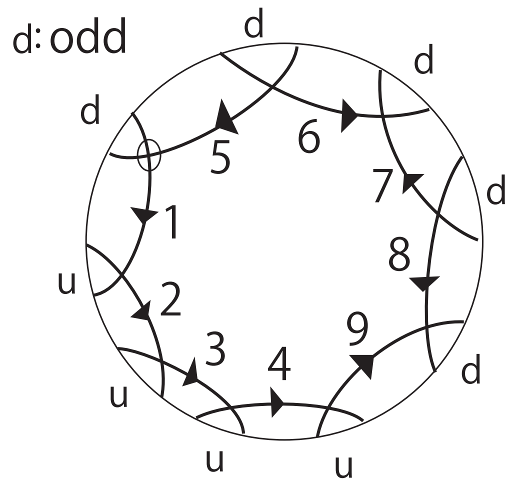
Proposition 1.11. *The swirl move makes adjacent $\boldsymbol{d}$ and $\boldsymbol{u}$ changes to adjacent $\boldsymbol{u}$ and $\boldsymbol{d}$.*
Therefore, we can change the chord diagram so that the $\boldsymbol{u}$-$\boldsymbol{d}$ notation is $\boldsymbol{uu\cdots udd\cdots d}$.

In chord diagram whose $\boldsymbol{u}$-$\boldsymbol{d}$ notation is $\boldsymbol{uu\cdots udd\cdots d}$, it is of Coxeter-type $\Leftrightarrow \#(\boldsymbol{d})$ is even.
Therefore, we have;

Theorem 1.12. *Let $\boldsymbol{D}$ be a relatively ordered chord diagram associated to a cycle graph. Then, $\boldsymbol{D}$ is of Coxeter-type $\Leftrightarrow \#(\boldsymbol{d})$ is even.*



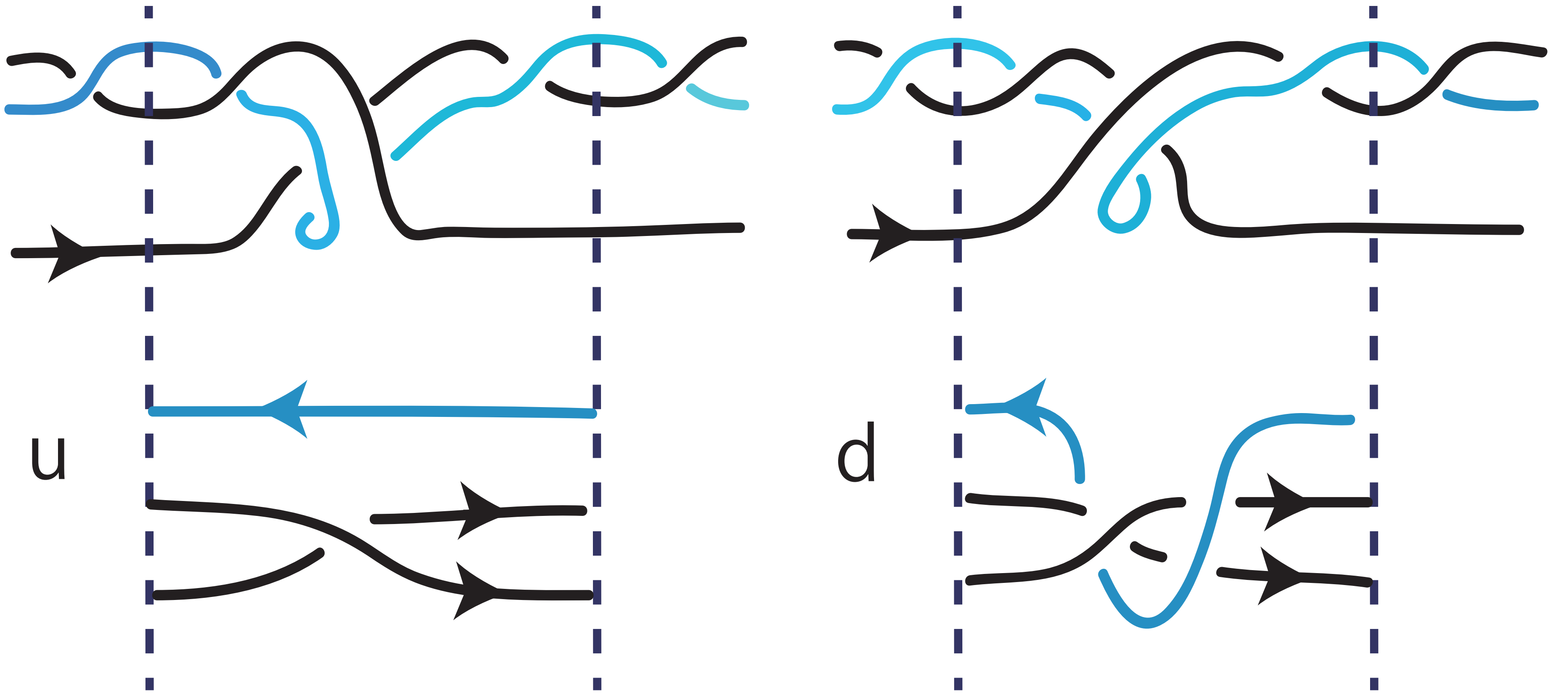
Coxeter

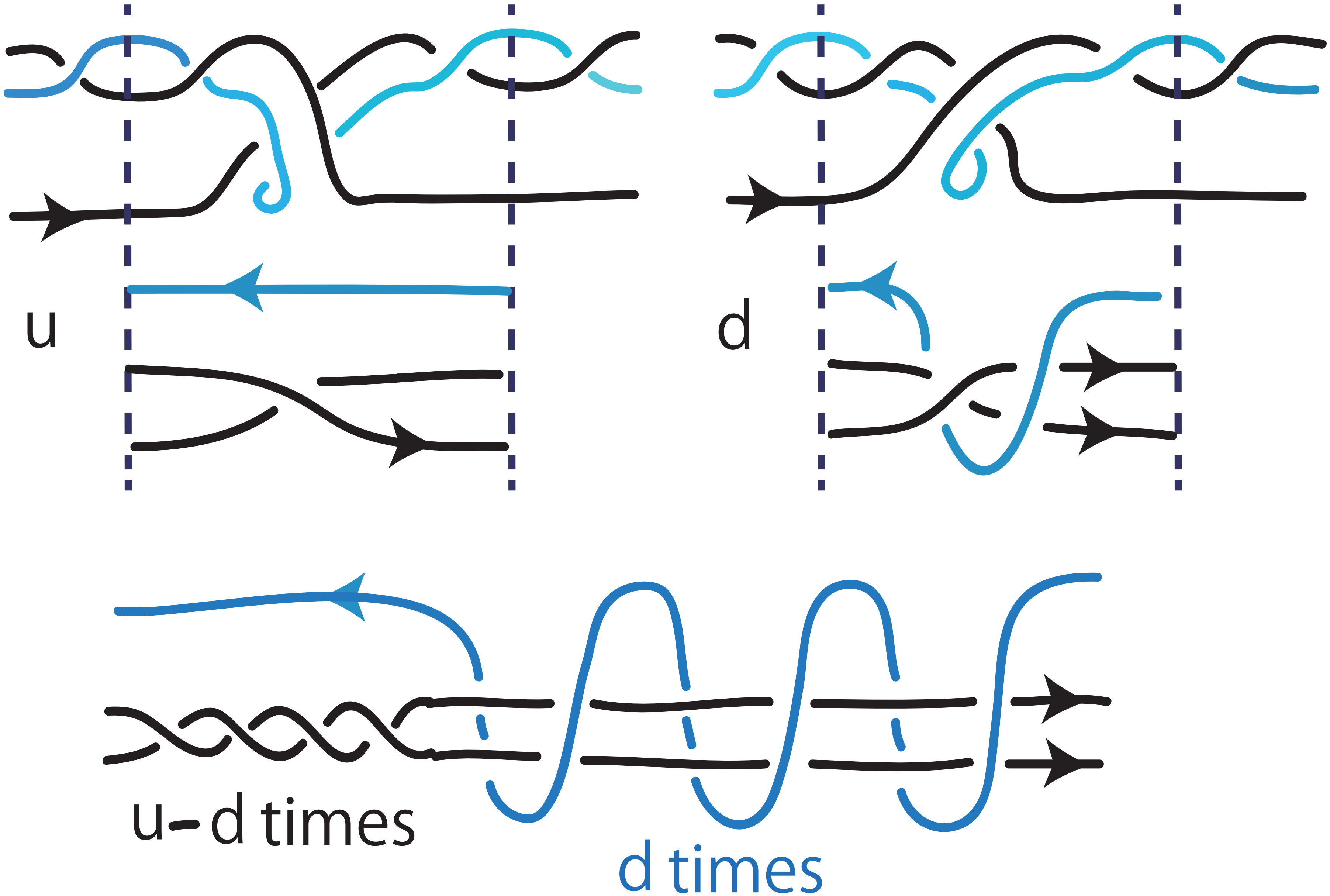


Not

Now we classify the links associated to cycle graph !

Local twisting of strings:





For Alex. pol, we used Summers' cabling formula.

Case 1: n is odd. L has 2 components.

- Reduced Alexander polynomial (coming from Seifert mtx).

$$\Delta_L(t) = (t^u + 1)(t^d - 1)$$

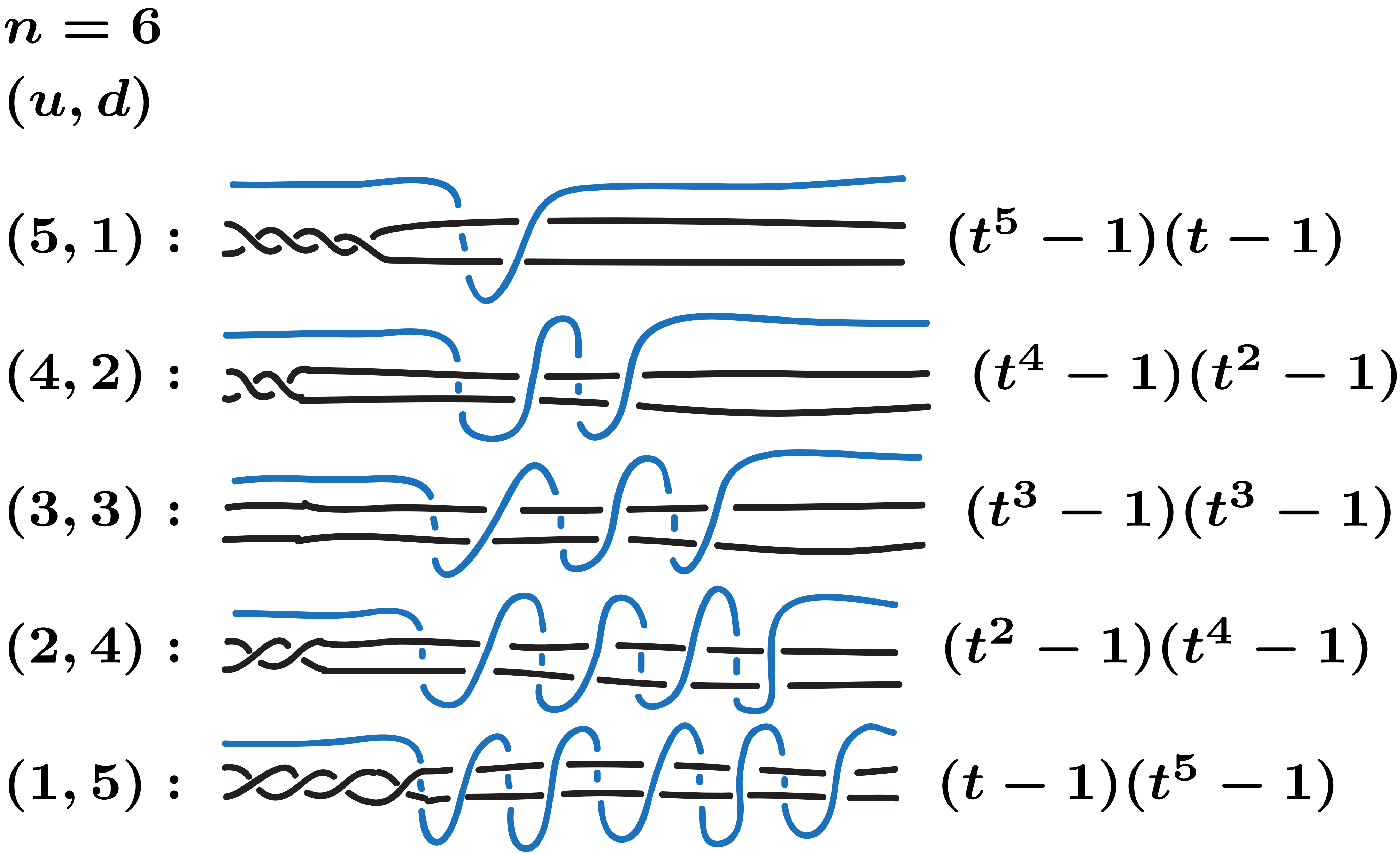
- Multivariate Alexander polynomial

$$D_L(t_1, t_2) = \left(t_1^d \, t_2^{u-d} + 1\right) \frac{(t_1^{-1} t_2^2)^d - 1}{t_1^{-1} t_2^2 - 1}$$

In this case, links associated to cycle graphs are determined completely by their Alexander polynomial alone.

Their zeros are on the unit circle,

more precisely, 1 and roots of ± 1.



n = 6
(u, d)
(5, 1) : (t^5 − 1)(t − 1)
(4, 2) : (t^4 − 1)(t^2 − 1)
(3, 3) : (t^3 − 1)(t^3 − 1)
(2, 4) : (t^2 − 1)(t^4 − 1)
(1, 5) : (t − 1)(t^5 − 1)

Case 2. n is even. L has 3 components.

- Reduced Alexander polynomial (coming from Seifert mtx).

$$\Delta_L(t) = (t^u - 1)(t^d - 1)$$

- Multivariate Alexander polynomial

$$D_L(t_1, t_2, t_3) =$$
$$\left(t_1^d\, t_2^{\frac{u-d}{2}} t_3^{\frac{u-d}{2}} - 1\right) \frac{(t_1^{-1} t_2 t_3)^d - 1}{t_1^{-1} t_2 t_3 - 1}$$

In this case, the links are not determined by their Alex. pol. alone, but 3-variate Alex. pol. classifies then.

In stead of Multi-variate Alex. pol., reduced Alex. pol and linking number among components also can classify them.

Their zeros are roots of 1.

Conclusion for links accosted to n-cycle graphs.

There are $(n-1)$ such links.
We can tell when a Coxeter link arises.

When n is odd: They all have different Alex. pol. and all the zeros are roots of ± 1.
$\frac{n-1}{2}$ of them are Coxeter links.

When n is even: Some of them have the same Alex. pol. but has different linking numbers.
All the zeros are roots of 1.
$\frac{n}{2}-1$ of them are Coxeter links.

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