Geometrically maximal knots

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Discussion Meeting on Knot theory and its Applications
IISER Mohali, India
Dec 2013
Outline

Hyperbolic geometry

Hyperbolic knots & 3-manifolds

Ideal triangulations

Simplest hyperbolic knots

Knot diagrams and Volume bounds

Weaving knots

Main Theorem & Proof

Motivation & Conjectures
In 1980s, William Thurston’s seminal work established a strong connection between hyperbolic geometry and knot theory, namely that most knot complements are hyperbolic. Thurston introduced tools from hyperbolic geometry to study knots that led to new geometric invariants, especially hyperbolic volume.
The upper half-plane model of hyperbolic plane \( \mathbb{H}^2 = \{(x, t)|t > 0\} \) with metric \( ds^2 = \frac{dx^2 + dt^2}{t^2} \). The boundary of \( \mathbb{H}^2 \) is \( \mathbb{R} \cup \infty \) called the circle at infinity.

- Geodesic lines are vertical lines or semicircles orthogonal to the \( x \)-axis (with centers on the \( x \)-axis).
- Hyperbolic lines either intersect in \( \mathbb{H}^2 \) or intersect at infinity, or are parallel.
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- Other models include Poincare ball model, Klein model etc.
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The upper half-space model of hyperbolic 3-space \( \mathbb{H}^3 = \{(x, y, t)| t > 0\} \) with metric \( ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2} \). The boundary of \( \mathbb{H}^3 \) is \( \mathbb{C} \cup \infty \) called the sphere at infinity.
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- $\text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$ which acts as Mobius transforms on $\mathbb{C} \cup \infty$. 
A 3-manifold $M$ is said to be *hyperbolic* if it has a complete, finite volume hyperbolic metric i.e. small balls in $M$ look like small balls in $\mathbb{H}^3$. 

(Margulis 1978) If $M$ is orientable and noncompact then $M = \partial M'$ where $\partial M' = \bigcup T_2$. Each end is of the form $T_2 \times [0, \infty)$ with each section is scaled Euclidean metric, called a cusp. 

(Mostow-Prasad Rigidity, 1968) Hyperbolic structure on a 3-manifold is unique. This implies geometric invariants, e.g. hyperbolic volume, are topological invariants.
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Hyperbolic knots

A knot or link $L$ in $S^3$ is hyperbolic if $S^3 - L$ is a hyperbolic 3-manifold.
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**Theorem (Thurston)** Every knot in $S^3$ is either a torus knot, a satellite knot or a hyperbolic knot.

**Theorem (Menasco)** If $L$ has a connected prime alternating diagram, except the standard $(2, q)$-torus link diagram, then $L$ is hyperbolic.

Reduced alternating diagram of $L \leftrightarrow\rightarrow$ decomposition of $S^3 - L$ into two ideal hyperbolic polyhedra with faces identified, according to the checkerboard coloring of the diagram.
Hyperbolic 3-manifolds are formed by gluing hyperbolic polyhedra.

The basic building block is an ideal tetrahedra which is a geodesic tetrahedra in \( \mathbb{H}^3 \) with all vertices on \( \mathbb{C} \cup \infty \).
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$$\text{Vol}(\triangle(z)) = \text{Im}(\text{Li}_2(z)) + \log |z| \text{arg}(1 - z)$$

where $\text{Li}_2(z)$ is the dilogarithm function. $\text{Vol}(\triangle(z)) \leq v_3 \approx 1.01494$, $v_3$ is the volume of the regular ideal tetrahedron (all dihedral angles $\pi/3$).
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$\text{Vol}(M)$ is a sum of volumes of ideal tetrahedra.
Example: Figure-8 knot
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Computing hyperbolic structures

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SnapPy also includes census of hyperbolic manifolds triangulated using at most 8 tetrahedra (≈ 17000 manifolds) and census of low volume closed hyperbolic 3-manifolds.
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Hyperbolic knots with geometric complexity up to 6 tetrahedra were found by Callahan-Dean-Weeks (1999), extended to 7 tetrahedra by C-Kofman-Paterson (2004) and to 8 tetrahedra by C-Kofman-Mullen (2012).
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Simplest hyperbolic knots
D. Thurston gave upper bound by decomposing $S^3 - L$ into octahedra, placing one octahedron at each crossing, and pulling remaining vertices to $\pm \infty$:

Any hyperbolic octahedron has volume $\leq v_8 \approx 3.66386 = \text{Vol}(\text{regular ideal octahedron})$.

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Improved by C. Adams: If $c(K) \geq 5$ then

$$\text{Vol}(S^3 - K) \leq v_8 (c(K) - 5) + 4v_3$$
Let $D$ be a prime alternating diagram of $L$. Two crossings $p, q$ are geometric complexity if they form a bigon (clasp).
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**Twist number** $t(D) :=$ number of twist-equivalence classes of $D$.

**Example:** $t(D) = 3$. 
Let $D$ = prime alternating (twist-reduced) diagram of hyperbolic link $L$.

**Thm. (Lackenby + Agol-D.Thurston + Agol-Storm-W.Thurston)**

$$\frac{v_8}{2} (t(D) - 2) \leq Vol(S^3 - L) < 10v_3 (t(D) - 1)$$

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where $v_3 = Vol(\text{regular ideal tetrahedron}) \approx 1.01494$ and $v_8 = Vol(\text{regular ideal octahedron}) \approx 3.66386$.

If $D$ also has no bigons then $t(D) = c(D)$, so

$$\frac{v_8}{2} (c(D) - 2) \leq Vol(S^3 - L) < 10v_3 (c(D) - 1)$$
Geometrically maximal knots

Agol-Storm-W. Thurston + Adams upper bound give the best current volume bounds per crossing number for a knot $K$ with a prime alternating (and twist-reduced) diagram with no bigons:

$$\frac{v_8}{2} + c_1 \leq \frac{Vol(S^3 - K)}{c(K)} \leq v_8 + c_2$$

where $c_1, c_2 \to 0$ as $c(K) \to \infty$. 

We say a sequence of knots $K_n$ is geometrically maximal if $\lim_{n \to \infty} Vol(S^3 - K_n) c(K_n) = v_8$. 

Question: Which knot families are geometrically maximal?
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Weaving knots

X.-S. Lin suggested that *weaving knots* \( W(p, q) \) asymptotically maximize the volume per crossing number.

\( W(p, q) \) is the alternating knot with the same projection as the torus knot \( T(p, q) \). For example, \( W(5, 4) \) is the closure of this braid:

\[
c(W(p, q)) = q(p - 1)
\]

**Conjecture (Lin)**

\[
\lim_{|p| + |q| \to \infty} \frac{\text{Vol}(W(p, q))}{c(W(p, q))} = v_8
\]
Theorem (C-Kofman-Purcell 2013) If \( p \geq 3 \) and \( q \geq 2 \), then

\[
v_8 (p - 2) q \leq \text{Vol}(W(p, q) \cup \text{axis}) \leq (v_8 (p - 3) + 4v_3) q
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Corollary If \( p \geq 3 \) and \( q \geq 6 \), then

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v_8(p - 2)q \left(1 - \frac{(2\pi)^2}{q^2}\right)^{3/2} \leq \text{Vol}(W(p, q)) \leq (v_8(p - 3) + 4v_3)q
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**Corollary**  \[ \lim_{|p|+|q| \to \infty} \frac{Vol(W(p, q))}{c(W(p, q))} = v_8 \]
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Outline of the proof

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6. Casson & Rivin $\implies$ $\text{Vol}(W(p, 1) \cup \text{axis}) \geq \text{Vol}(\mathcal{P})$. 
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1. \( W(p, q) \cup \text{axis} \) is a \( q \)-fold cover of \( W(p, 1) \cup \text{axis} \).
2. \( S^3 \setminus (W(p, 1) \cup \text{axis}) \) has an ideal polyhedral decomposition \( \mathcal{P} \) with 4 ideal tetrahedra and \( p - 3 \) ideal octahedra.
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6. Casson & Rivin \( \implies \) \( Vol(W(p, 1) \cup \text{axis}) \geq Vol(\mathcal{P}) \).
7. The meridian of the braid axis of \( W(p, q) \) has length \( \geq q \).
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8. By Futer-Kalfagianni-Purcell, the lower bound for \( \text{Vol}(W(p, q)) \) follows.
The 3–strand case

Theorem (C-Kofman-Purcell 2013) If \( p \geq 3 \) and \( q \geq 2 \), then

\[
v_8(p - 2)q \leq Vol(W(p, q) \cup axis) \leq (v_8(p - 3) + 4v_3)q
\]

If \( p = 3 \) the upper bound in the above Theorem is achieved. This case is special because all edges of \( \mathcal{P} \) are 6–valent, so all dihedral angles are \( \pi/3 \). Thus \( \mathcal{P} \) has only regular ideal tetrahedra, which is the geometric triangulation.

\[
Vol(W(3, q) \cup axis) = 4q v_3
\]
Commensurability in the 3–strand case

\[ \text{Vol}(W(3, q) \cup \text{axis}) = 4q \nu_3 \]
Motivation: The Infinite Weave

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Conjectures

- **Conjecture** Family of alternating knots obtained by closing up parts of the infinite weave, with number of squares in both directions $\to \infty$ is (a) geometrically maximal (b) geometrically converges to $\mathbb{R}^3 - W$. 

- **Theorem (C-Kofman-Purcell)** For a sequence $K_n$ of knots as above, $\lim_{n \to \infty} 2\pi \log \det(K_n) = v_8$. 

- **Conjecture** If $K$ is any knot, $2\pi \log \det(K) \leq v_8$. 

- **Conjecture (Kenyon)** If $G$ is any finite planar graph, $\tau(G) = \# \text{spanning trees of } G$, $C \approx 0.916$ is Catalan's constant, $\log \tau(G) e(G) \leq 2\pi = v_8/2\pi \approx 0.58312$. 

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- **Theorem (C-Kofman-Purcell)** For a sequence $K_n$ of knots as above, \( \lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8. \)

- **Conjecture** If $K$ is any knot, \( \frac{2\pi \log \det(K)}{c(K)} \leq v_8. \)

- **Conjecture (Kenyon)** If $G$ is any finite planar graph, $\tau(G) = \#$ spanning trees of $G$, $C \approx 0.916$ is Catalan’s constant, \( \frac{\log \tau(G)}{e(G)} \leq \frac{2C}{\pi} = \frac{v_8}{2\pi} \approx 0.58312. \)
Questions
Questions

Thank You

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