Geometrically maximal knots

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Discussion Meeting on Knot theory and its Applications
IISER Mohali, India
Dec 2013

Outline

Hyperbolic geometry

Hyperbolic knots & 3-manifolds

Ideal triangulations

Simplest hyperbolic knots

Knot diagrams and Volume bounds

Weaving knots

Main Theorem & Proof

Motivation & Conjectures

Now enters geometry ....

In 1980s, William Thurstons seminal work established a strong connection between hyperbolic geometry and knot theory, namely that most knot complements are hyperbolic. Thurston introduced tools from hyperbolic geometry to study knots that led to new geometric invariants, especially hyperbolic volume.

Hyperbolic plane

- The upper half-plane model of hyperbolic plane $\mathbb{H}^2 = \{(x,t) | t > 0\}$ with metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}t^2}{t^2}$. The boundary of $\mathbb{H}^2$ is $\mathbb{R} \cup \infty$ called the circle at infinity.
- Geodesic lines are vertical lines or semicircles orthogonal to the x-axis (with centers on the x-axis).
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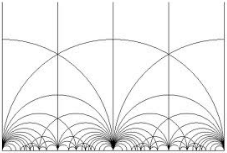
- $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2, \mathbb{R})$.
- Other models include Poincare ball model, Klein model etc.



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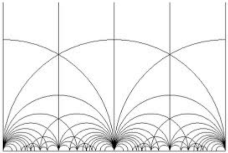

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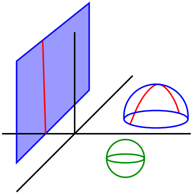
- $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$ which acts as Mobius transforms on $\mathbb{C} \cup \infty$.



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- (Margulis 1978) If M is orientable and noncompact then $M = \overset{\circ}{M'}$ where $\partial M' = \cup\, T^2$. Each end is of the form $T^2 \times [0, \infty)$ with each section is scaled Euclidean metric, called a cusp.

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- (Mostow-Prasad Rigidity, 1968) Hyperbolic structure on a 3-manifold is unique. This implies geometric invariants, e.g. hyperbolic volume, are topological invariants !

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Theorem (Thurston) Every knot in S^3 is either a torus knot, a satellite knot or a hyperbolic knot.

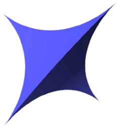
Theorem (Menasco) If L has a connected prime alternating diagram, except the standard (2, q)-torus link diagram, then L is hyperbolic.

Reduced alternating diagram of $L \longleftrightarrow$ decomposition of $S^3 - L$ into two ideal hyperbolic polyhedra with faces identified, according to the checkerboard coloring of the diagram.

Ideal tetrahedra in $\mathbb{H}^3$

Hyperbolic 3-manifolds are formed by gluing hyperbolic polyhedra.

The basic building block is an ideal tetrahedra which is a geodesic tetrahedra in $\mathbb{H}^3$ with all vertices on $\mathbb{C} \cup \infty$.



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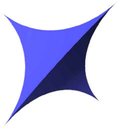

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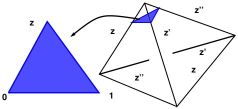

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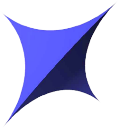

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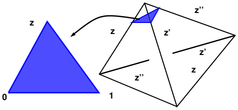

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$\text{Vol}(\triangle(z)) = \text{Im}(\text{Li}_2(z)) + \log|z| arg(1 - z)$ where $\text{Li}_2(z)$ is the dilogarithm function. $\text{Vol}(\triangle(z)) \leq v_3 \approx 1.01494$, v_3 is the volume of the regular ideal tetrahedron (all dihedral angles $\pi/3$).

Ideal triangulations

An ideal triangulation of a cusped (non-compact) hyperbolic 3-manifold M is a decompostion into ideal tetrahedra glued along the faces with the vertices deleted.

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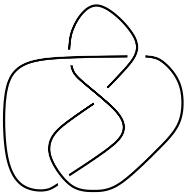
Vol(M) is a sum of volumes of ideal tetrahedra.

Example: Figure-8 knot



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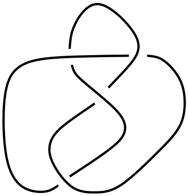

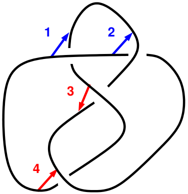

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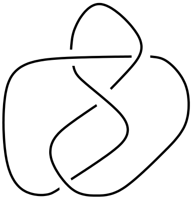
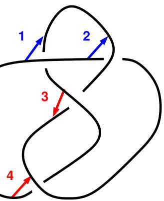
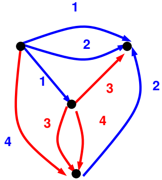
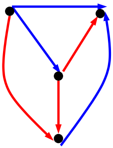
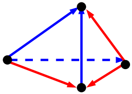




B+







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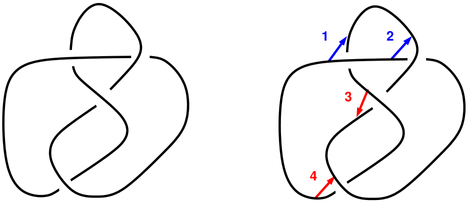

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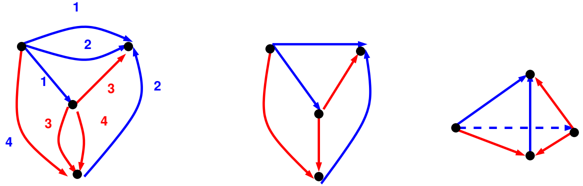

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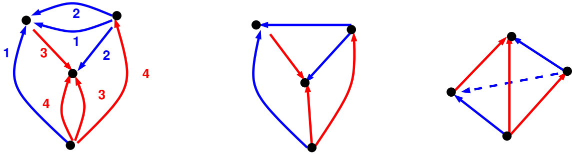

Computing hyperbolic structures

The program SnapPea by Weeks (1999) and its recent modification SnapPy by Culler and Dunfield computes hyperbolic structures and invariants on 3-manifolds and knots by triangulating and solving gluing equations.

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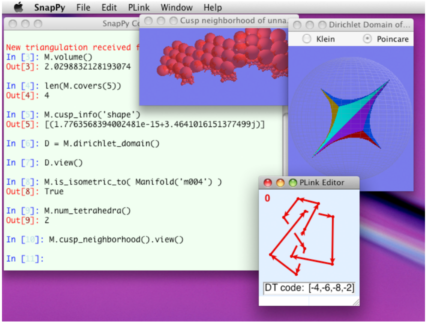
SnapPy also includes census of hyperbolic manifolds triangulated using at most 8 tetrahedra ($\approx$ 17000 manifolds) and census of low volume closed hyperbolic 3-manifolds.



Simplest hyperbolic knots

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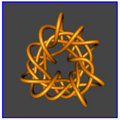
Tetrahedra	1	2	3	4	5	6	7	8	≤ 8
Manifolds	0	2	9	52	223	913	3388	12241	16828
Knots	0	1	2	4	22	43	129	299	500

Simplest hyperbolic knots


82

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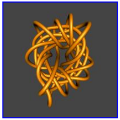
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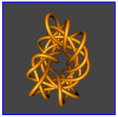
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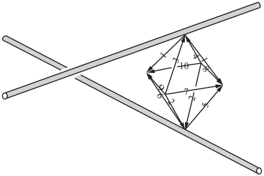
Upper Volume bounds

D. Thurston gave upper bound by decomposing $S^3 - L$ into octahedra, placing one octahedron at each crossing, and pulling remaining vertices to $\pm\infty$:



Any hyperbolic octahedron has volume $\leq$ v_8 $\approx$ 3.66386 $=$ Vol(regular ideal octahedron).

$\Longrightarrow$ $Vol(S^3 - L) < v_8\, c(L)$.

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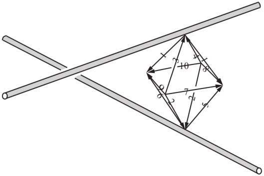

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$\implies Vol(S^3 - L) < v_8\, c(L)$.

Improved by C. Adams: If $c(K) \geq 5$ then

$$Vol(S^3 - K) \leq v_8\,(c(K) - 5) + 4v_3$$

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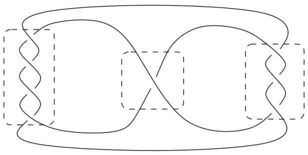
Twist number $t(D) :=$ number of twist-equivalence classes of D.

Example: $t(D) = 3$.



Volume bounds from twist number

Let $D =$ prime alternating (twist-reduced) diagram of hyperbolic link L.

Thm. (Lackenby + Agol-D.Thurston + Agol-Storm-W.Thurston)

$$\frac{v_8}{2}(t(D)-2) \leq Vol(S^3 - L) < 10v_3\,(t(D)-1)$$

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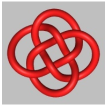
If D also has no bigons then $t(D) = c(D)$, so



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Geometrically maximal knots

Agol-Storm-W.Thurston + Adams upper bound give the best current volume bounds per crossing number for a knot K with a prime alternating (and twist-reduced) diagram with no bigons:

$$\frac{v_8}{2} + c_1 \leq \frac{Vol(S^3 - K)}{c(K)} \leq v_8 + c_2$$

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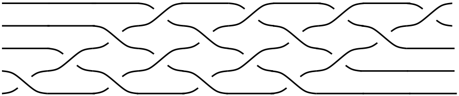
Question: Which knot families are geometrically maximal ?

Weaving knots

X.-S. Lin suggested that *weaving knots* $W(p, q)$ asymptotically maximize the volume per crossing number.

$W(p, q)$ is the alternating knot with the same projection as the torus knot $T(p, q)$. For example, $W(5, 4)$ is the closure of this braid:



$$c(W(p, q)) = q(p - 1)$$

Conjecture (Lin)

$$\lim_{|p|+|q|\to\infty} \frac{Vol(W(p, q))}{c(W(p, q))} = v_8$$

Main Theorem

Theorem(C-Kofman-Purcell 2013) If $p \geq 3$ and $q \geq 2$, then

$$v_8\,(p-2)q \leq Vol(W(p,q) \cup \mathrm{axis}) \leq (v_8\,(p-3) + 4v_3)q$$

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Corollary If $p \geq 3$ and $q \geq 6$, then

$$v_8\,(p-2)q\,\left(1 - \frac{(2\pi)^2}{q^2}\right)^{3/2} \leq Vol(W(p,q)) \leq (v_8\,(p-3) + 4v_3)q$$

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Corollary

$$\lim_{|p|+|q|\to\infty} \frac{Vol(W(p,q))}{c(W(p,q))} = v_8$$

Outline of the proof

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2. $S^3 - (W(p,1) \cup \mathrm{axis})$ has an ideal polyhedral decomposition $\mathcal{P}$ with 4 ideal tetrahedra and $p-3$ ideal octahedra.

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8. By Futer-Kalfagianni-Purcell, the lower bound for $Vol(W(p,q))$ follows.

The 3–strand case

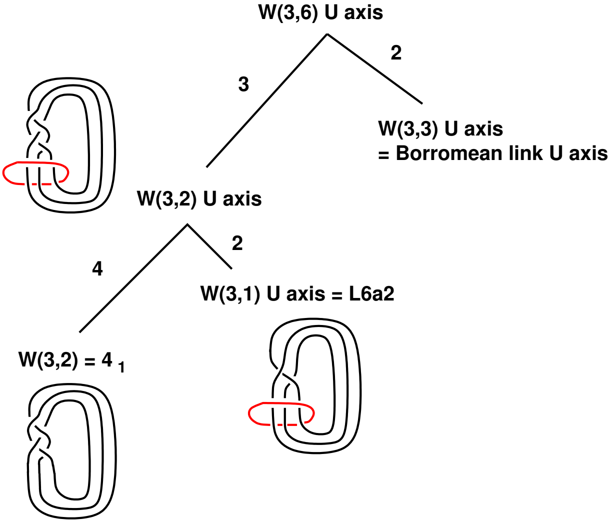
Theorem(C-Kofman-Purcell 2013) If $p \geq 3$ and $q \geq 2$, then

$$v_8\,(p-2)q \leq Vol(W(p,q) \cup \mathrm{axis}) \leq (v_8\,(p-3) + 4v_3)q$$

If $p = 3$ the upper bound in the above Theorem is achieved. This case is special because all edges of $\mathcal{P}$ are 6–valent, so all dihedral angles are $\pi/3$. Thus $\mathcal{P}$ has only regular ideal tetrahedra, which is the geometric triangulation.

$$Vol(W(3,q) \cup \mathrm{axis}) = 4q\,v_3$$

Commensurability in the 3–strand case



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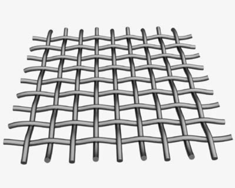
Motivation: The Infinite Weave

Menasco's polyhedral decomposition for $S^3 - W(p, q)$ approaches that of the infinite weave W as $|p| + |q| \to \infty$



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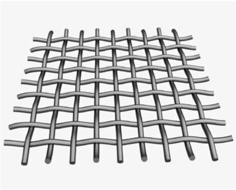

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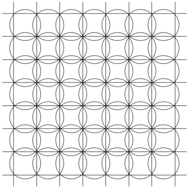

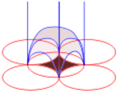

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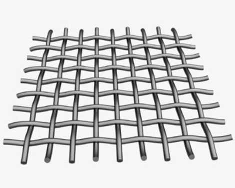

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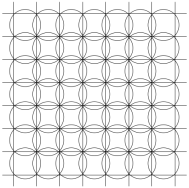

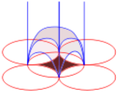

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Conjectures

- Conjecture Family of alternating knots obtained by closing up parts of the infinite weave, with number of squares in both directions $\to \infty$ is (a) geometrically maximal (b) geometrically converges to $\mathbb{R}^3 - W$.

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- Conjecture(Kenyon) If G is any finite planar graph, $\tau(G) = \#$ spanning trees of G, $C \approx 0.916$ is Catalan's constant,
$$\frac{\log \tau(G)}{e(G)} \leq \frac{2\mathrm{C}}{\pi} = \frac{v_8}{2\pi} \approx 0.58312.$$

Questions

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Thank You

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http://www.math.csi.cuny.edu/abhijit/