Lengths of closed geodesics in a hyperbolic knot complement in S^3

Sreekrishna Palaparthi

Department of Mathematics

Indian Institute of Technology Guwahati

December 17, 2013

Outline

1. Introduction
   - Definitions

2. Geodesic Length Bounds
   - Theorem of Adams and Reid
   - $\mathscr{L}_n(S^3 - L)$ is bounded above

Outline

1. Introduction
   - Definitions
2. Geodesic Length Bounds
   - Theorem of Adams and Reid
   - $\mathscr{L}_n(S^3 - L)$ is bounded above

Upper half space model

- Upper half-space model of $\mathbb{H}^3$: the space

$$\left\{(x, y, t) \| (x, y, t) \in \mathbb{R}^3, t > 0\right\}$$

equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$$

- Geodesics in this model are vertical lines or semicircles perpendicular to the $x - y$ plane in $\mathbb{R}^3$.
- Geodesic planes are vertical planes or hemispheres perpendicular to the $x - y$ plane.

Upper half space model

- Upper half-space model of $\mathbb{H}^3$: the space

$$\left\{(x, y, t) \| (x, y, t) \in \mathbb{R}^3, t > 0\right\}$$

equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$$

- Geodesics in this model are vertical lines or semicircles perpendicular to the $x - y$ plane in $\mathbb{R}^3$.
- Geodesic planes are vertical planes or hemispheres perpendicular to the $x - y$ plane.

Upper half space model

- Upper half-space model of $\mathbb{H}^3$: the space

$$\left\{(x, y, t) \| (x, y, t) \in \mathbb{R}^3, t > 0\right\}$$

equipped with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2 + dt^2}}{t}$$

- Geodesics in this model are vertical lines or semicircles perpendicular to the $x - y$ plane in $\mathbb{R}^3$.
- Geodesic planes are vertical planes or hemispheres perpendicular to the $x - y$ plane.

Boundary at ∞

- The set of points infinitely far away from a point in $\mathbb{H}^3$
- In the upper half-space model of $\mathbb{H}^3$, $\partial_\infty \mathbb{H}^3$ is the x-y plane in $\mathbb{R}^3$ union the point at infinity.

Boundary at ∞

- The set of points infinitely far away from a point in $\mathbb{H}^3$
- In the upper half-space model of $\mathbb{H}^3$, $\partial_\infty \mathbb{H}^3$ is the x-y plane in $\mathbb{R}^3$ union the point at infinity.

Isometries of $\mathbb{H}^3$

- The group of orientation preserving isometries of $\mathbb{H}^3 \cong PSL(2, \mathbb{C})$.
- Classification of isometries: elliptic, parabolic and loxodromic.

Isometries of $\mathbb{H}^3$

- The group of orientation preserving isometries of $\mathbb{H}^3 \cong PSL(2, \mathbb{C})$.
- Classification of isometries: elliptic, parabolic and loxodromic.

Hyperbolic 3-Manifolds

- Hyperbolic 3-manifolds: 3-manifolds equipped with a complete Riemannian metric of constant sectional curvature -1.
- Can be obtained as $\mathbb{H}^3/\Gamma$, where Γ is a torsion free Kleinian group.
- Will restrict attention to finite volume hyperbolic 3-manifolds.
- Mostow's Rigidity Theorem: unique finite volume structure.
- Parabolic elements in Γ give non-compact manifolds.

Hyperbolic 3-Manifolds

- Hyperbolic 3-manifolds: 3-manifolds equipped with a complete Riemannian metric of constant sectional curvature -1.
- Can be obtained as $\mathbb{H}^3/\Gamma$, where Γ is a torsion free Kleinian group.
- Will restrict attention to finite volume hyperbolic 3-manifolds.
- Mostow's Rigidity Theorem: unique finite volume structure.
- Parabolic elements in Γ give non-compact manifolds.

Hyperbolic 3-Manifolds

- Hyperbolic 3-manifolds: 3-manifolds equipped with a complete Riemannian metric of constant sectional curvature -1.
- Can be obtained as $\mathbb{H}^3/\Gamma$, where Γ is a torsion free Kleinian group.
- Will restrict attention to finite volume hyperbolic 3-manifolds.
- Mostow's Rigidity Theorem: unique finite volume structure.
- Parabolic elements in Γ give non-compact manifolds.

Hyperbolic 3-Manifolds

- Hyperbolic 3-manifolds: 3-manifolds equipped with a complete Riemannian metric of constant sectional curvature -1.
- Can be obtained as $\mathbb{H}^3/\Gamma$, where Γ is a torsion free Kleinian group.
- Will restrict attention to finite volume hyperbolic 3-manifolds.
- Mostow's Rigidity Theorem: unique finite volume structure.
- Parabolic elements in Γ give non-compact manifolds.

Hyperbolic 3-Manifolds

- Hyperbolic 3-manifolds: 3-manifolds equipped with a complete Riemannian metric of constant sectional curvature -1.
- Can be obtained as $\mathbb{H}^3/\Gamma$, where Γ is a torsion free Kleinian group.
- Will restrict attention to finite volume hyperbolic 3-manifolds.
- Mostow's Rigidity Theorem: unique finite volume structure.
- Parabolic elements in Γ give non-compact manifolds.

Hyperbolic 3-Manifolds

- The ends of a finite volume, orientable, non-compact, hyperbolic 3-manifold consists of finite number of *cusps*.
- A rank two cusp $\cong \mathbb{T}^2 \times [0, \infty)$.
- A cusped hyperbolic 3-manifold is a hyperbolic 3-manifold with at-least one cusp.

Hyperbolic 3-Manifolds

- The ends of a finite volume, orientable, non-compact, hyperbolic 3-manifold consists of finite number of *cusps*.
- A rank two cusp $\cong \mathbb{T}^2 \times [0, \infty)$.
- A cusped hyperbolic 3-manifold is a hyperbolic 3-manifold with at-least one cusp.

Hyperbolic 3-Manifolds

- The ends of a finite volume, orientable, non-compact, hyperbolic 3-manifold consists of finite number of *cusps*.
- A rank two cusp $\cong \mathbb{T}^2 \times [0, \infty)$.
- A cusped hyperbolic 3-manifold is a hyperbolic 3-manifold with at-least one cusp.

Lift of a Cusp

- A cusp lifts to a disjoint union of *horoballs*.
- A horoball is a Euclidean ball tangent to the $\partial_\infty \mathbb{H}^3$.
- A cross-sectional torus of the cusp is called a cusp torus.

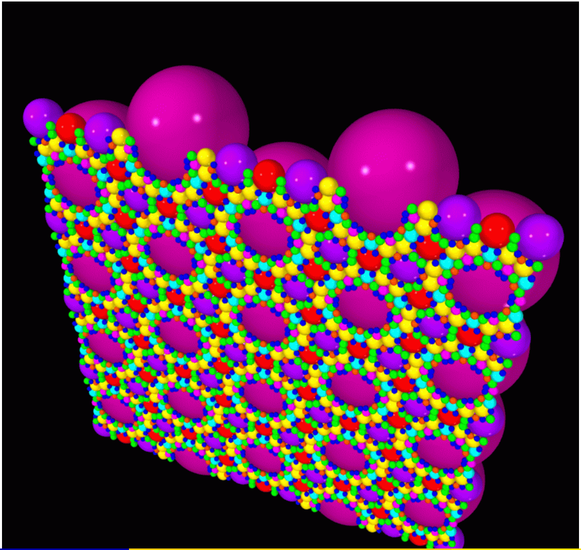
Lift of a Cusp

- A cusp lifts to a disjoint union of *horoballs*.
- A horoball is a Euclidean ball tangent to the $\partial_\infty \mathbb{H}^3$.
- A cross-sectional torus of the cusp is called a cusp torus.

Lift of a Cusp

- A cusp lifts to a disjoint union of *horoballs*.
- A horoball is a Euclidean ball tangent to the $\partial_\infty \mathbb{H}^3$.
- A cross-sectional torus of the cusp is called a cusp torus.

Horoballs in a lift of a cusped hyperbolic 3-manifold



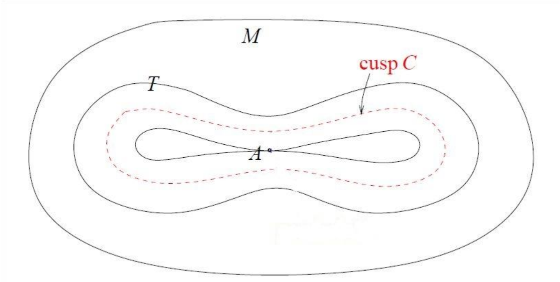
Maximal Cusp

- Expand the horoballs in the lift of a cusp to $\mathbb{H}^3$ until the first two touch.
- The projection of this configuration is called a *maximal cusp*.

Maximal Cusp

- Expand the horoballs in the lift of a cusp to $\mathbb{H}^3$ until the first two touch.
- The projection of this configuration is called a *maximal cusp*.

Maximal Cusp



Standardization Of The Maximal Cusp

- Parabolic fixed points of Γ are horoball centers.
- 0 and ∞ are parabolic fixed points.
- The horosphere centred at ∞ is the plane $t = 1$ in $\mathbb{H}^3$ (relaxable).

Standardization Of The Maximal Cusp

- Parabolic fixed points of Γ are horoball centers.
- 0 and ∞ are parabolic fixed points.
- The horosphere centred at ∞ is the plane $t = 1$ in $\mathbb{H}^3$ (relaxable).

Standardization Of The Maximal Cusp

- Parabolic fixed points of Γ are horoball centers.
- 0 and ∞ are parabolic fixed points.
- The horosphere centred at ∞ is the plane $t = 1$ in $\mathbb{H}^3$ (relaxable).

Geodesic Length Bounds

- Adams and Reid: A shortest closed geodesic in a hyperbolic link complement in S^3 is bounded above by 7.171646...
- Question: Is a second shortest closed geodesic a hyperbolic link complement in S^3 also bounded above?
- Answer: Yes.

Geodesic Length Bounds

- Adams and Reid: A shortest closed geodesic in a hyperbolic link complement in S^3 is bounded above by 7.171646...
- Question: Is a second shortest closed geodesic a hyperbolic link complement in S^3 also bounded above?
- Answer: Yes.

Geodesic Length Bounds

- Adams and Reid: A shortest closed geodesic in a hyperbolic link complement in S^3 is bounded above by 7.171646...
- Question: Is a second shortest closed geodesic a hyperbolic link complement in S^3 also bounded above?
- Answer: Yes.

Bounds on Exceptional Dehn Filling

- The 2π-Theorem of Gromov and Thurston

Theorem (Gromov and Thurston)

Let M be a cusped hyperbolic 3-manifold with n cusps. Let $T_1, .., T_n$ be disjoint cusp tori for the n cusps of M, and r_i a slope on T_i represented by a geodesic a_i whose length in the Euclidean metric on T_i is greater than 2π, for each $i = 1..n$. Then $M(r_1, .., r_n)$ admits a metric of negative curvature.

- Agol's and Lackenby's improvement

Bounds on Exceptional Dehn Filling

- The 2π-Theorem of Gromov and Thurston

Theorem (Gromov and Thurston)

Let M be a cusped hyperbolic 3-manifold with n cusps. Let $T_1, .., T_n$ be disjoint cusp tori for the n cusps of M, and r_i a slope on T_i represented by a geodesic a_i whose length in the Euclidean metric on T_i is greater than 2π, for each $i = 1..n$. Then $M(r_1, .., r_n)$ admits a metric of negative curvature.

- Agol's and Lackenby's improvement

Bounds on Exceptional Dehn Filling

- The 2π-Theorem of Gromov and Thurston

Theorem (Gromov and Thurston)

Let M be a cusped hyperbolic 3-manifold with n cusps. Let $T_1, .., T_n$ be disjoint cusp tori for the n cusps of M, and r_i a slope on T_i represented by a geodesic a_i whose length in the Euclidean metric on T_i is greater than 2π, for each $i = 1..n$. Then $M(r_1, .., r_n)$ admits a metric of negative curvature.

- Agol's and Lackenby's improvement

Hyperbolic Knot Complements

For hyperbolic knot complements in closed, orientable 3-manifolds which do not admit any Riemannian metric of negative curvature (S^3, for example):

- Dehn filling along the meridian curve gives back the manifold.
- Meridian length in the Euclidean metric on the standardized maximal cusp torus < 6.

Hyperbolic Knot Complements

For hyperbolic knot complements in closed, orientable 3-manifolds which do not admit any Riemannian metric of negative curvature (S^3, for example):

- Dehn filling along the meridian curve gives back the manifold.
- Meridian length in the Euclidean metric on the standardized maximal cusp torus < 6.

Hyperbolic Knot Complements

For hyperbolic knot complements in closed, orientable 3-manifolds which do not admit any Riemannian metric of negative curvature (S^3, for example):

- Dehn filling along the meridian curve gives back the manifold.
- Meridian length in the Euclidean metric on the standardized maximal cusp torus < 6.

Notation

- By Mostow's rigidity theorem, the length of an n^{th} shortest closed geodesic in a f.v. hyperbolic 3-manifold M is an invariant of its topological type.
- Denote this length by $\mathscr{L}_n(M)$.

Notation

- By Mostow's rigidity theorem, the length of an n^{th} shortest closed geodesic in a f.v. hyperbolic 3-manifold M is an invariant of its topological type.
- Denote this length by $\mathscr{L}_n(M)$.

Theorem for $\mathscr{L}_1(M)$

Theorem (Adams, Reid)

Let N be a finite volume hyperbolic 3-manifold with at least one cusp. Assume that in a maximal cusp torus, there is a non-trivial curve corresponding to a parabolic isometry of length equal to w. Then:

(1) $\mathscr{L}_1(N) \leq 2\mathfrak{Re}(\cosh^{-1}((2 + iw^2)/2))$ *if* $w \neq 2$

(2) $\mathscr{L}_1(N) \leq 2\ln(3 + 2\sqrt{2}) = 3.525..$ *if* $w = 2$

Hyperbolic Knot Complements in S^3

Corollary (Adams, Reid)

Let M be closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature and $K \subset M$ be a knot with hyperbolic complement. Then $\mathscr{L}_1(M - K) \leq 7.35534..$

- The bound in this Corollary can be improved to 7.171646.. by the work of Agol and Lackenby.
- Proof:

Hyperbolic Knot Complements in S^3

Corollary (Adams, Reid)

Let M be closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature and $K \subset M$ be a knot with hyperbolic complement. Then $\mathscr{L}_1(M - K) \leq 7.35534..$

- The bound in this Corollary can be improved to 7.171646.. by the work of Agol and Lackenby.
- Proof:

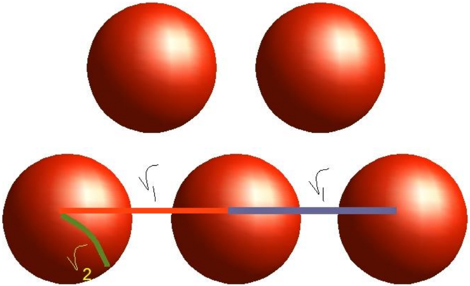
Hyperbolic Knot Complements in S^3

Corollary (Adams, Reid)

Let M be closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature and $K \subset M$ be a knot with hyperbolic complement. Then $\mathscr{L}_1(M - K) \leq 7.35534..$

- The bound in this Corollary can be improved to 7.171646.. by the work of Agol and Lackenby.
- Proof:

Lift of a Standard Maximal Cusp of N



When $w \neq 2$

case 1: Suppose $w \neq 2$, then:

- $\gamma_1^{-1}\gamma_2$ is loxodromic.
- Maximum geodesic length occurs when angle between γ_1^{-1} and γ_2 is $\pi/2$
- Can conjugate γ_1^{-1} to $\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$
- Can conjugate γ_2 to $\begin{bmatrix} 1 & 0 \\ iw & 1 \end{bmatrix}$

When $w \neq 2$

case 1: Suppose $w \neq 2$, then:

- $\gamma_1^{-1}\gamma_2$ is loxodromic.
- Maximum geodesic length occurs when angle between γ_1^{-1} and γ_2 is $\pi/2$
- Can conjugate γ_1^{-1} to $\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$
- Can conjugate γ_2 to $\begin{bmatrix} 1 & 0 \\ iw & 1 \end{bmatrix}$

When $w \neq 2$

case 1: Suppose $w \neq 2$, then:

- $\gamma_1^{-1}\gamma_2$ is loxodromic.
- Maximum geodesic length occurs when angle between γ_1^{-1} and γ_2 is $\pi/2$
- Can conjugate γ_1^{-1} to $\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$
- Can conjugate γ_2 to $\begin{bmatrix} 1 & 0 \\ iw & 1 \end{bmatrix}$

When $w \neq 2$

case 1: Suppose $w \neq 2$, then:

- $\gamma_1^{-1}\gamma_2$ is loxodromic.
- Maximum geodesic length occurs when angle between γ_1^{-1} and γ_2 is $\pi/2$
- Can conjugate γ_1^{-1} to $\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$
- Can conjugate γ_2 to $\begin{bmatrix} 1 & 0 \\ iw & 1 \end{bmatrix}$

When $w \neq 2$

case 1: Suppose $w \neq 2$, then:

- $\gamma_1^{-1}\gamma_2$ is loxodromic.
- Maximum geodesic length occurs when angle between γ_1^{-1} and γ_2 is $\pi/2$
- Can conjugate γ_1^{-1} to $\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}$
- Can conjugate γ_2 to $\begin{bmatrix} 1 & 0 \\ iw & 1 \end{bmatrix}$

When $w \neq 2$ (contd.)

- Trace of the product $\gamma_1^{-1}\gamma_2$ is $2 + iw^2$
- So, in this case we get

$$\mathscr{L}_1(N) \leq 2\mathfrak{Re}(\cosh^{-1}((2 + iw^2)/2))$$

When $w \neq 2$ (contd.)

- Trace of the product $\gamma_1^{-1}\gamma_2$ is $2 + iw^2$
- So, in this case we get

$$\mathscr{L}_1(N) \leq 2\mathfrak{Re}(\cosh^{-1}((2 + iw^2)/2))$$

When $w = 2$

case 2: Suppose $w = 2$, then

- $\angle(\gamma_1^{-1}, \gamma_2)$ could be 0 and then $\gamma_1^{-1}\gamma_2$ could be parabolic.
- Can conjugate γ_1 to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- Assume $\gamma_2 =$

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

When $w = 2$

case 2: Suppose $w = 2$, then

- $\angle(\gamma_1^{-1}, \gamma_2)$ could be 0 and then $\gamma_1^{-1}\gamma_2$ could be parabolic.
- Can conjugate γ_1 to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- Assume $\gamma_2 =$

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

When $w = 2$

case 2: Suppose $w = 2$, then

- $\angle(\gamma_1^{-1}, \gamma_2)$ could be 0 and then $\gamma_1^{-1}\gamma_2$ could be parabolic.
- Can conjugate γ_1 to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- Assume $\gamma_2 =$

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

When $w = 2$

case 2: Suppose $w = 2$, then

- $\angle(\gamma_1^{-1}, \gamma_2)$ could be 0 and then $\gamma_1^{-1}\gamma_2$ could be parabolic.
- Can conjugate γ_1 to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- Assume $\gamma_2 =$

$$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

When $w = 2$ (contd.)

- $\gamma_1^{-1}\gamma_2$ then is

$$\begin{bmatrix} 1-2r & -2 \\ r & 1 \end{bmatrix}$$

- This is parabolic $\iff$ $r = 2$.

When $w = 2$ (contd.)

- $\gamma_1^{-1}\gamma_2$ then is

$$\begin{bmatrix} 1-2r & -2 \\ r & 1 \end{bmatrix}$$

- This is parabolic $\iff r = 2$.

When $w = 2$ (contd.)

- $\gamma_1^{-1}\gamma_2^{-1}$ is loxodromic.
- The length of this geodesic is $2\ln(3 + 2\sqrt{2}) = 3.525...$
- So in this case we get:

$$\mathscr{L}_1(N) \leq 2\ln(3 + 2\sqrt{2}) = 3.525..$$

When $w = 2$ (contd.)

- $\gamma_1^{-1}\gamma_2^{-1}$ is loxodromic.
- The length of this geodesic is $2\ln(3+2\sqrt{2})$ = 3.525...
- So in this case we get:

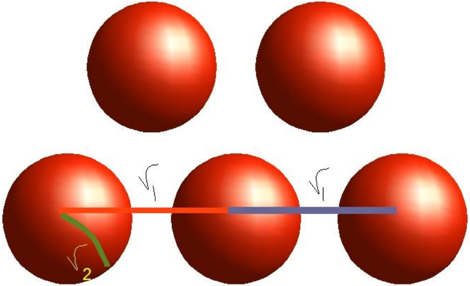
$$\mathscr{L}_1(N) \leq 2\ln(3+2\sqrt{2}) = 3.525..$$

When $w = 2$ (contd.)

- $\gamma_1^{-1}\gamma_2^{-1}$ is loxodromic.
- The length of this geodesic is $2\ln(3+2\sqrt{2})$ = 3.525...
- So in this case we get:

$$\mathscr{L}_1(N) \leq 2\ln(3+2\sqrt{2}) = 3.525..$$

$\gamma_1^{-n}\gamma_2$



Altered Standard Form

- Conjugate so that $w = 2$, then $\gamma_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- $\gamma_2 = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$
- $\gamma_1^{-n}\gamma_2 = \begin{bmatrix} 1 - 2rn & -2n \\ r & 1 \end{bmatrix}$
- Infer from the trace: $\gamma_1^{-n}\gamma_2$ *are loxodromic for* $n > 1$

Altered Standard Form

- Conjugate so that $w = 2$, then $\gamma_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- $\gamma_2 = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$
- $\gamma_1^{-n}\gamma_2 = \begin{bmatrix} 1 - 2rn & -2n \\ r & 1 \end{bmatrix}$
- Infer from the trace: $\gamma_1^{-n}\gamma_2$ *are loxodromic for* $n > 1$

Altered Standard Form

- Conjugate so that $w = 2$, then $\gamma_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- $\gamma_2 = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$
- $\gamma_1^{-n}\gamma_2 = \begin{bmatrix} 1 - 2rn & -2n \\ r & 1 \end{bmatrix}$
- Infer from the trace: $\gamma_1^{-n}\gamma_2$ *are loxodromic for* $n > 1$

Altered Standard Form

- Conjugate so that $w = 2$, then $\gamma_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- $\gamma_2 = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$
- ${\gamma_1^{-}}^{n}\gamma_2 = \begin{bmatrix} 1 - 2rn & -2n \\ r & 1 \end{bmatrix}$
- Infer from the trace: ${\gamma_1^{-}}^{n}\gamma_2$ *are loxodromic for* $n > 1$

Conjugacy

- $\gamma_1^{-k}\gamma_2$ and $\gamma_1^{-m}\gamma_2$ are not conjugate for $k \neq m$
- Proof: Is $2 - 2rk = \pm(2 - 2rm)$?

Conjugacy

- $\gamma_1^{-k}\gamma_2$ and $\gamma_1^{-m}\gamma_2$ are not conjugate for $k \neq m$
- Proof: Is $2 - 2rk = \pm(2 - 2rm)$?

A Result

Theorem

Let N be a finite volume hyperbolic 3-manifold with at least one cusp. Assume that in a maximal cusp torus, there is a non-trivial curve corresponding to a parabolic isometry of length equal to w. Then $\mathscr{L}_n(N) \leq 2\mathfrak{Re}[\cosh^{-1}((2 + iw^2(n+1))/2)]$

Statement of the Main Theorem

Theorem

Let M be a closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature. Let L be a hyperbolic link in M. Then

$$\mathscr{L}_n(M - L) \leq 2\mathfrak{Re}[\cosh^{-1}(1 + 18(n+1)i)]$$