Chart description of surface braids

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IISER MOHALI

References for my talk at the Advanced School

For a quick view (pp.2–33)

[3] S. Carter, S. Kamada, M. Saito, *Surfaces in 4-space*, Encyclopaedia of Mathematical Sciences, **142**, Low-Dimensional Topology, III, Springer-Verlag, Berlin, 2004.

For details on surface diagrams

[4] J. S. Carter, M. Saito, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs, **55**, American Mathematical Society, Providence, RI, 1998.

For details on braid forms

[7] S. Kamada, *Braid and knot theory in dimension four*, Mathematical Surveys and Monographs, **95**, American Mathematical Society, Providence, RI, 2002.

Contents

1. Surface braids

$D^2 \times I^2$: the product of two 2-disks
$pr_1 : D^2 \times I^2 \to D^2$, $pr_2 : D^2 \times I^2 \to I^2$: the projections
Q_m : a fixed set of m interior points of D^2

Def (1.1)

A compact oriented surface S properly embedded in $D^2 \times I^2$ is a surface braid (or a 2-dimensional braid) if it satisfies the following two conditions.

(1) The restriction map $pr_2|_S : S \to I^2$ to S is an m-fold branched covering of I^2. (It is denoted by $\pi_S : S \to I^2$.)

(2) For any $y \in \partial I^2$, $pr_1(S \cap pr_2^{-1}(y)) = Q_m$.

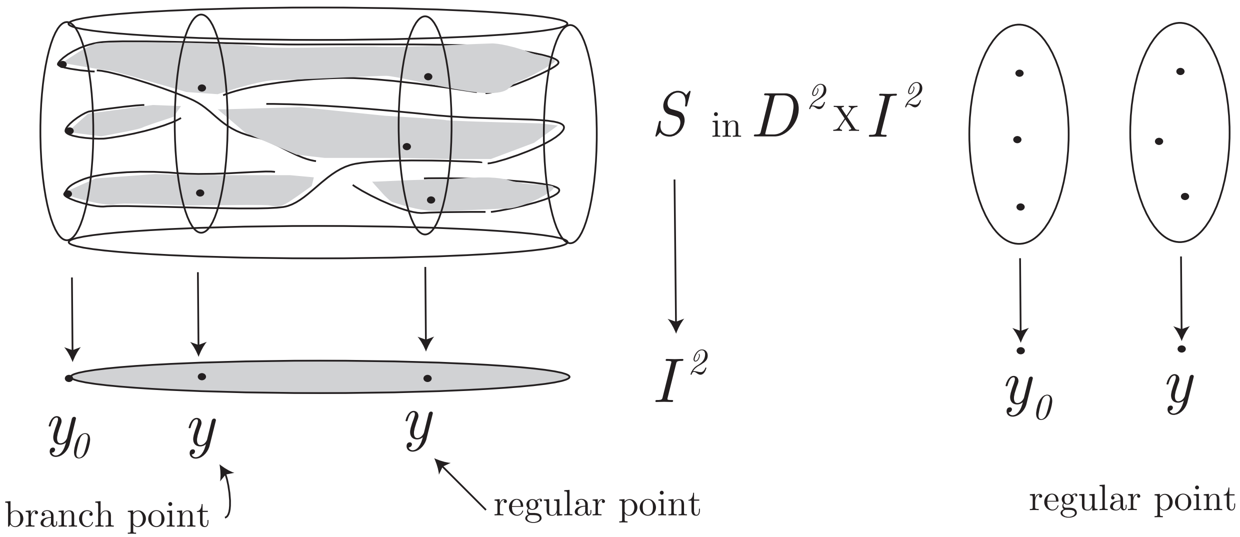
Def (1.2)

A surface braid S is regular if there exists exactly one branch point in the pre-image of each branch value of $\pi_S : S \to I^2$.

Def (1.3)

A surface braid S is simple if it is regular and the local degree at each branch point is 2.

Rem. S is simple if and only if $\#(S \cap pr_2^{-1}(y)) \geq m - 1$ for any $y \in I^2$.



We introduce two kinds of equivalence relations on surface braids: equivalent and braid ambient isotopic.

Def (1.4)

Surface braids S and S' are equivalent if there is an ambient isotopy $\{h_s\}_{s\in[0,1]}$ of $D^2 \times I^2$ satisfying the following.

(1) $h_0 = \mathrm{id}$, $h_1(S) = S'$.

(2) For each $s \in [0,1]$, h_s is fiber-preserving.
Namely, there is a homeomorphism $\underline{h}_s : I^2 \to I^2$ such that

$$\underline{h}_s \circ pr_2 = pr_2 \circ h_s.$$

(3) For each $s \in [0,1]$, the restriction map of h_s to $D^2 \times \partial I^2$ is the identity map.

Def (1.5)

Surface braids S and S' are braid ambient isotopic if there is an ambient isotopy $\{h_s\}_{s\in[0,1]}$ of $D^2 \times I^2$ satisfying the following.

(1) $h_0 = \mathrm{id}$, $h_1(S) = S'$

(2) For each $s \in [0,1]$, $h_s(S)$ is a surface braid.

(3) For each $s \in [0,1]$, the restriction map of h_s to $D^2 \times \partial I^2$ is the identity map.

equivalent $\Rightarrow$ braid ambient isotopic

The converse of "equivalent $\Rightarrow$ braid ambient isotopic "is not true.

2. Motion picture method

We regard $D^2 \times I^2 = (D^2 \times I) \times [0,1]$.
The last coordinate $[0,1]$ is considered as the time parameter, and we consider the motion picture of a surface braid S.

For each $t \in [0,1]$, let b_t be the cross section of S at the time t.
That means

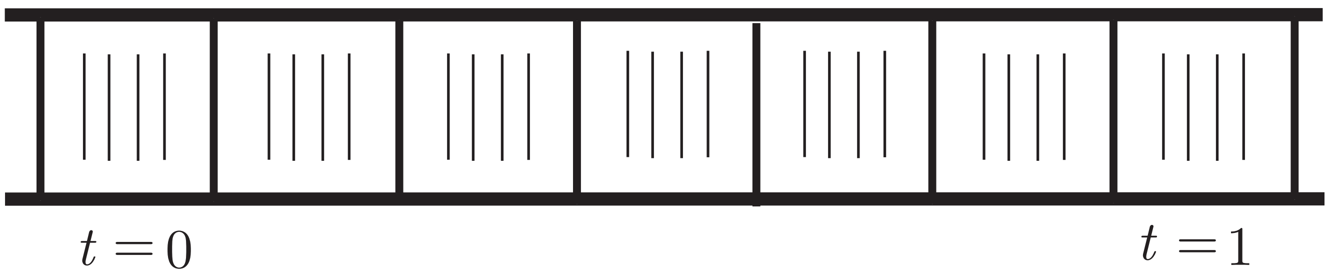
$$S \cap (D^2 \times I) \times \{t\} = b_t \times \{t\}.$$

We call the 1-parameter family $\{b_t\}$ the motion picture of S.

- For all but finitely many t, b_t is an m-braid.
- For finitely many t, b_t is a singular m-braid.
- b_0 and b_1 are the trivial m-braid $Q_m \times I$.

Eg. (2.2) Trivial surface braid

We call $S = Q_m \times D_2^2$ the trivial surface braid. The motion picture $\{b_t\}$ consists of trivial m-braids.



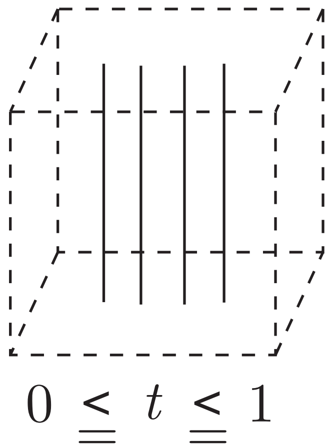


Figure: The trivial surface braid

Eg. (2.3)

See the motion picture below. At $t = 1/3$ and $t = 2/3$ we have singular braids. The singular points correspond to the branch points of the surface braid.

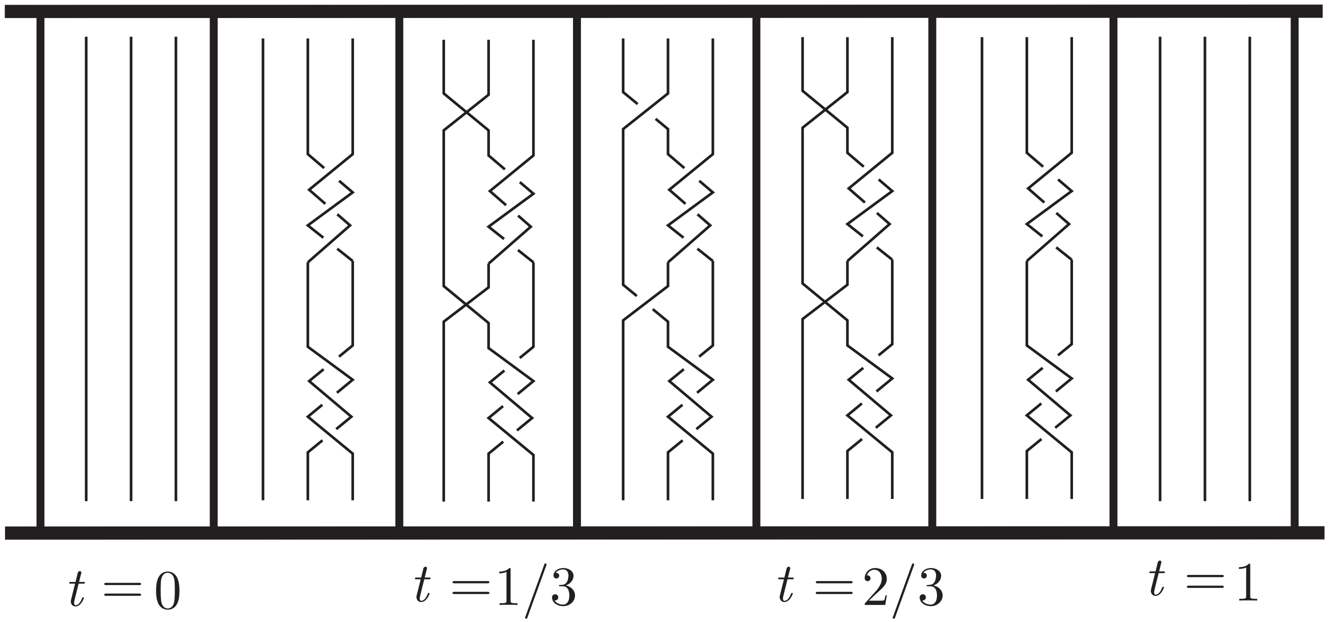


Figure: A non-trivial surface braid of degree 3

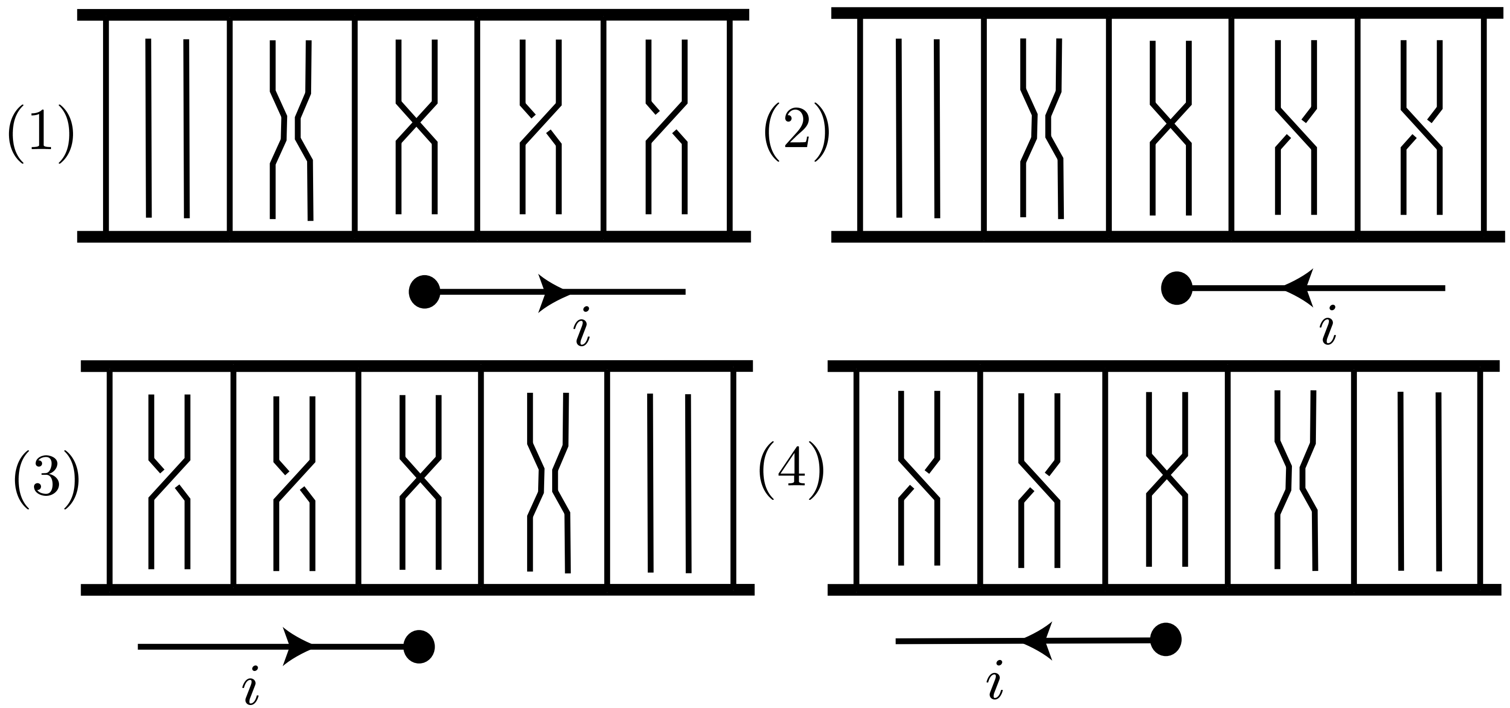
Prop (2.4) (motion picture of simple surface braids)

By a slight perturbation up to equivalence, a motion picture of a simple surface braid can be taken as follows.

(1) In a neighborhood of each singular point, it looks like the figure.

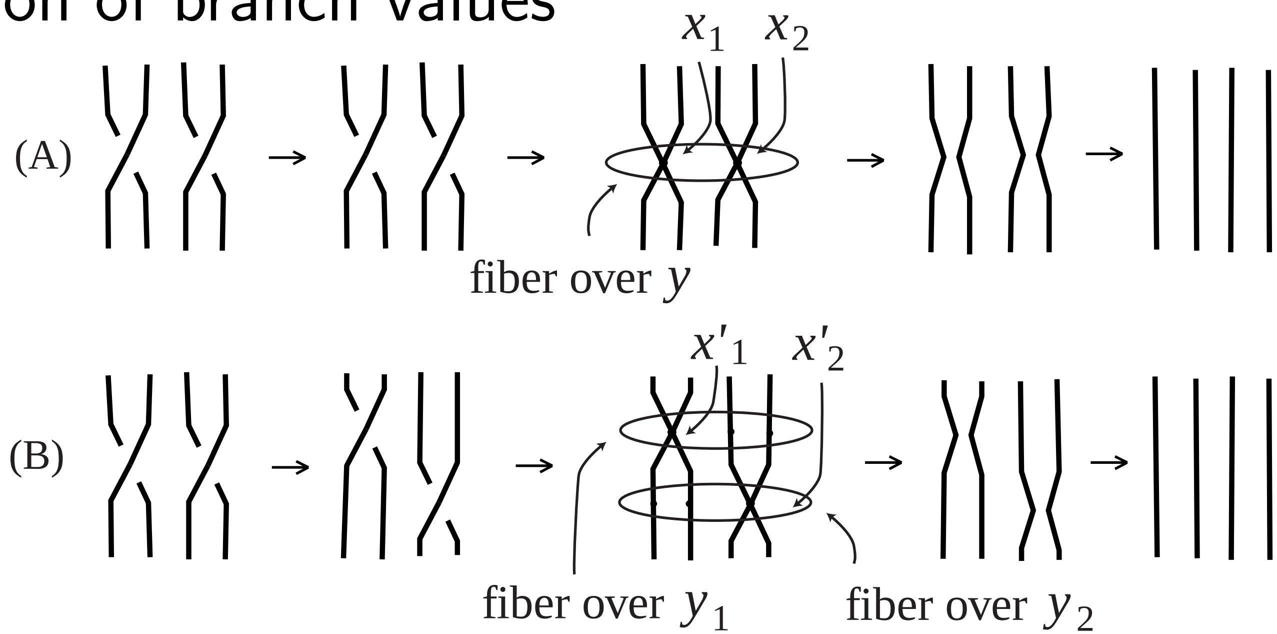
(2) b_0 and b_1 are trivial braids $Q_m \times I$.

Conversely, any motion picture satisfying these two conditions is a motion picture of a simple surface braid.

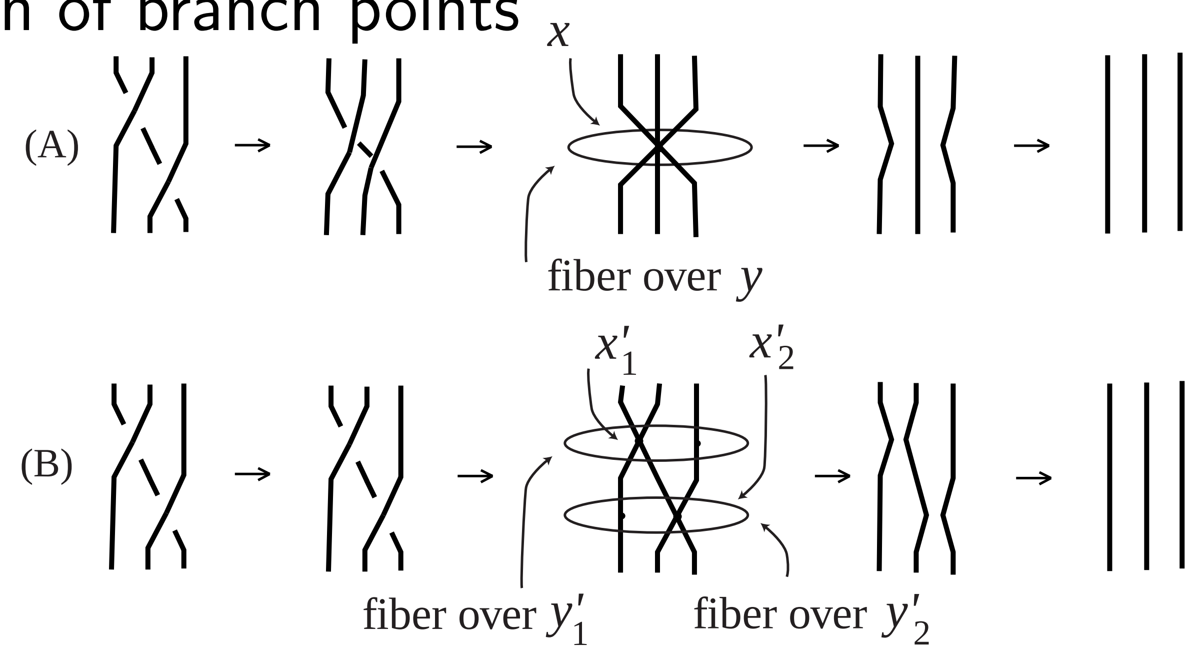


The converse of “equivalent ⇒ braid ambient isotopic ”is not true.

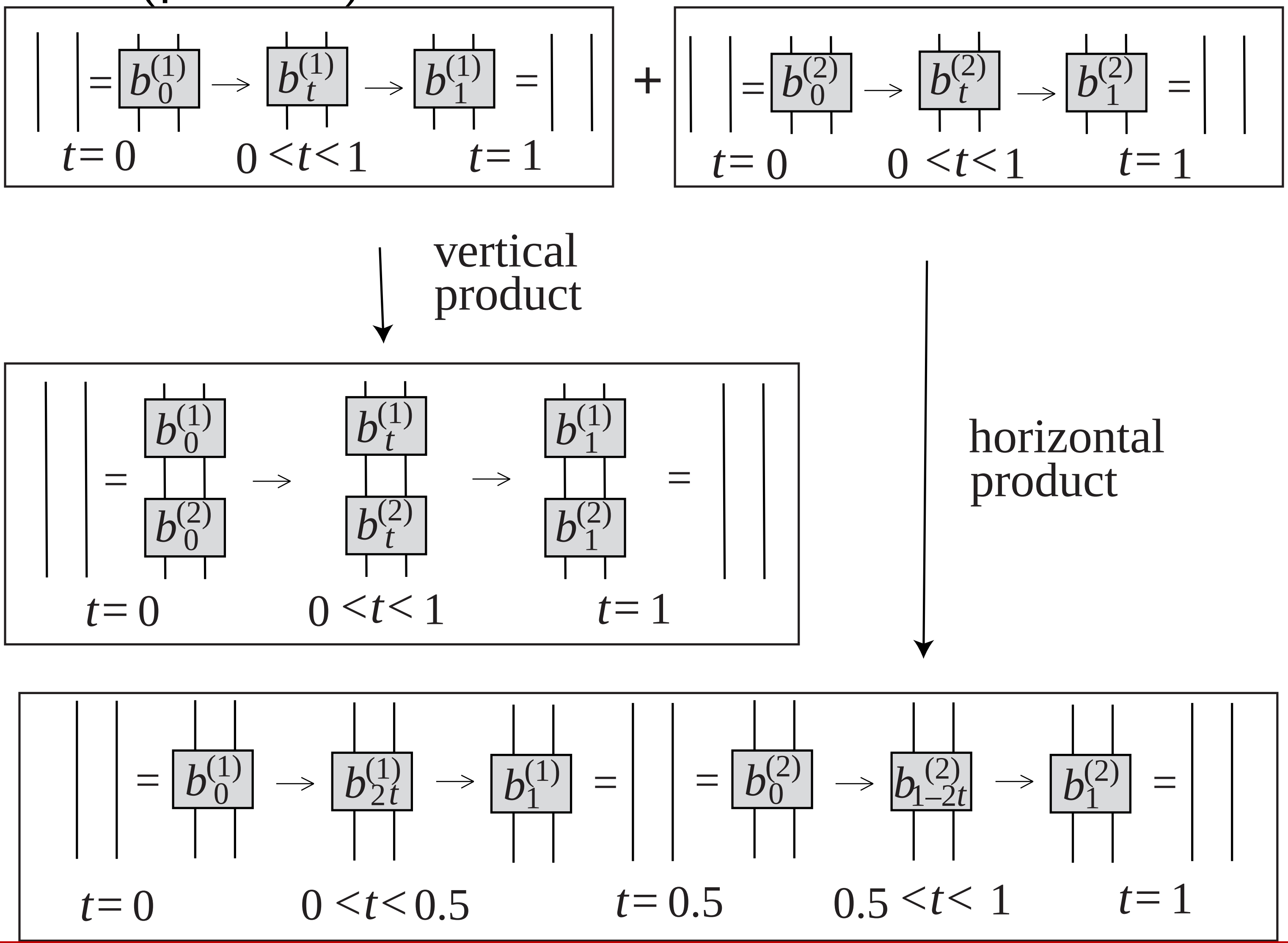
Fusion/Fission of branch values



Fusion/Fission of branch points



Composition (product) of surface braids



Prop (2.5)

The vertical product of S and S' is equivalent to the horizontal product.

3. Surface braids vs surface links

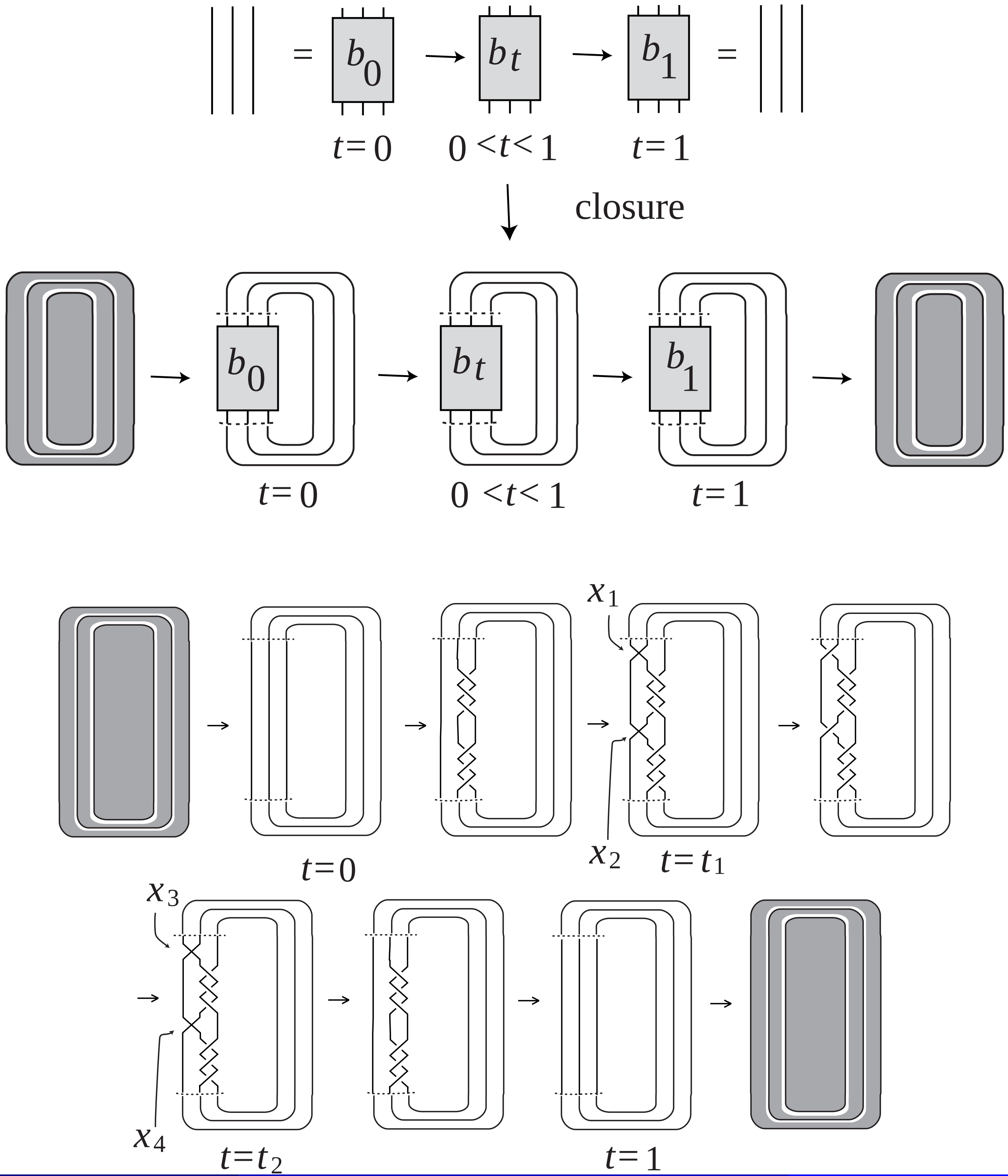
Thm (4.1)(Alexander and Markov theorem)

(1) Any oriented link can be described as the closure of a braid.
(2) Such a braid is unique up to equivalence, conjugation, stabilization and destabilization.

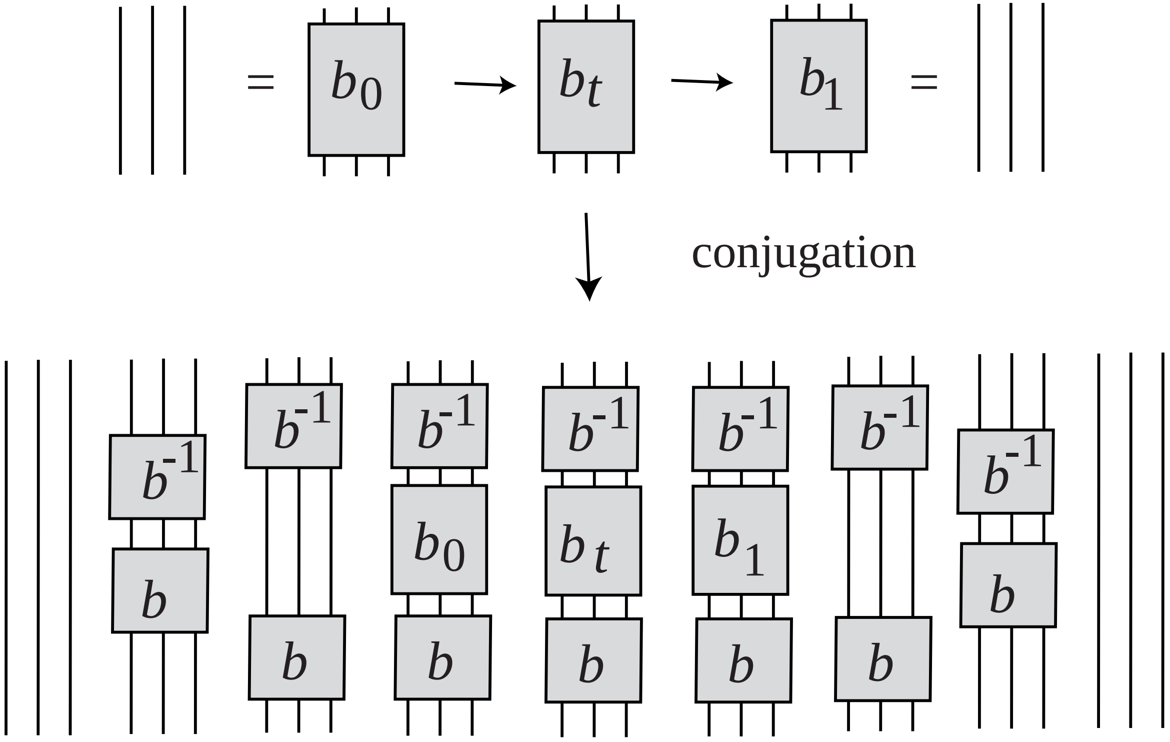
(For classical braids, equivalent $\Leftrightarrow$ braid ambient isotopic.)

Thm (4.2)

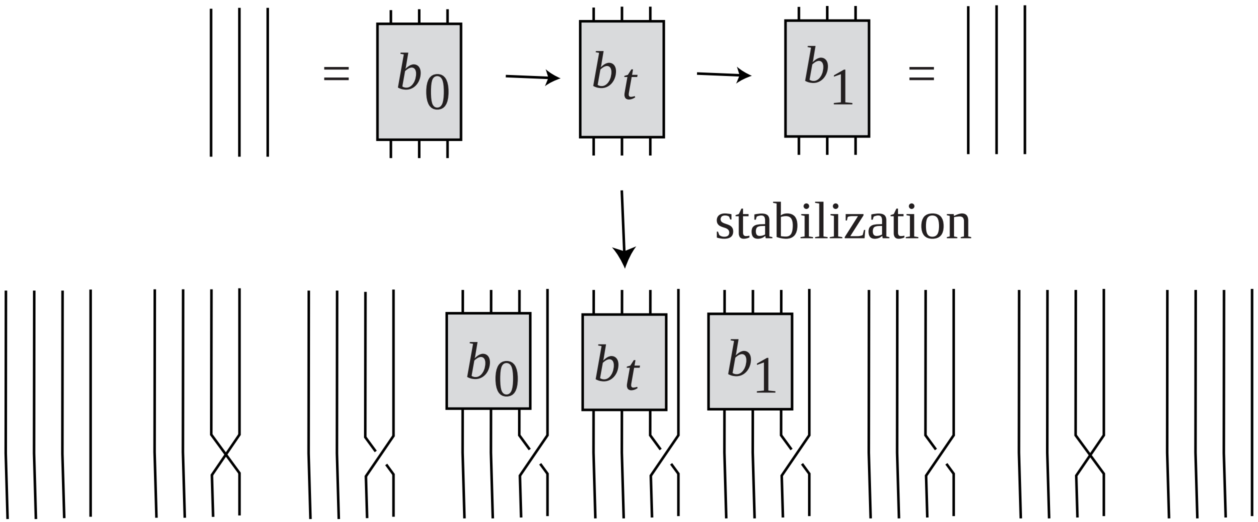
(1) Any oriented surface link can be described as the closure of a surface braid.
(2) Such a surface braid is unique up to braid ambient isotopy, conjugation, stabilization and destabilization.


b_0
b_t
b_1
$t=0$
$0<t<1$
$t=1$
closure
b_0
b_t
b_1
$t=0$
$0<t<1$
$t=1$
x_1
x_2
$t=0$
$t=t_1$
x_3
x_4
$t=t_2$
$t=1$

Conjugation



Stabilization



$$\{\text{Surface braids}\}/_\sim \overset{1:1}{\Longleftrightarrow} \{\text{Surface links}\}/_\sim$$

Def.

For a surface link F, the **braid index** of F is the minimum degree of simple surface braids describing F.

$$\begin{aligned}
\mathrm{Braid}(F) = 1 &\Leftrightarrow F \text{ is an unknotted } S^2 \\
\mathrm{Braid}(F) = 2 &\Leftrightarrow F \text{ is an unknotted } S^2 \amalg S^2 \text{ or } \Sigma_g \\
\mathrm{Braid}(F) = 3 &\Rightarrow F \text{ is a ribbon surface link}
\end{aligned}$$

Eg.

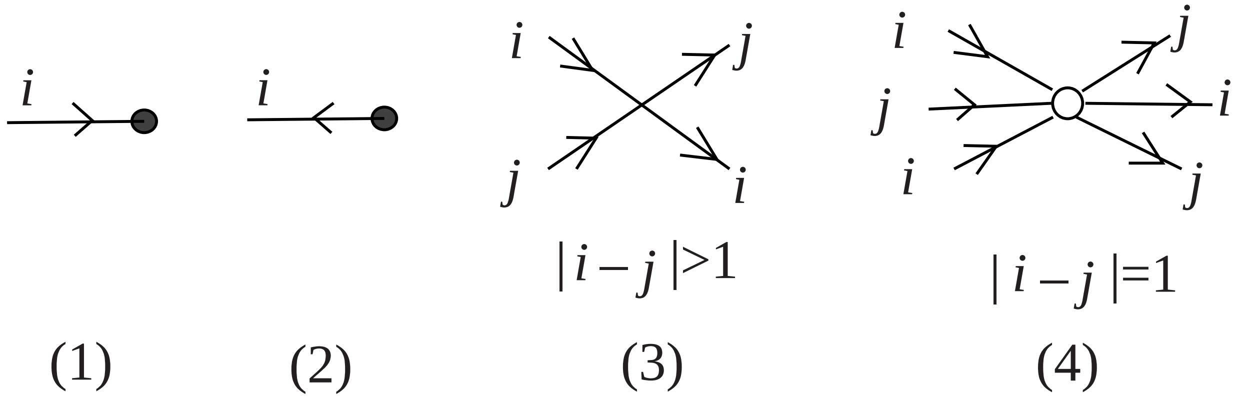
The braid index of an unknotted S^2 is 1.

The braid index of an unknotted torus is 2.

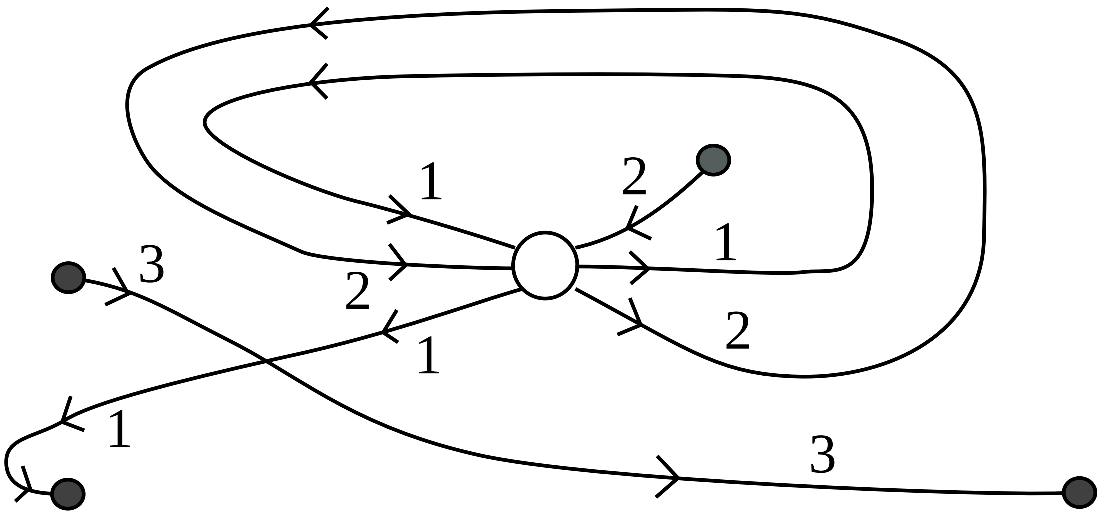
The braid index of a spun trefoil is 3.

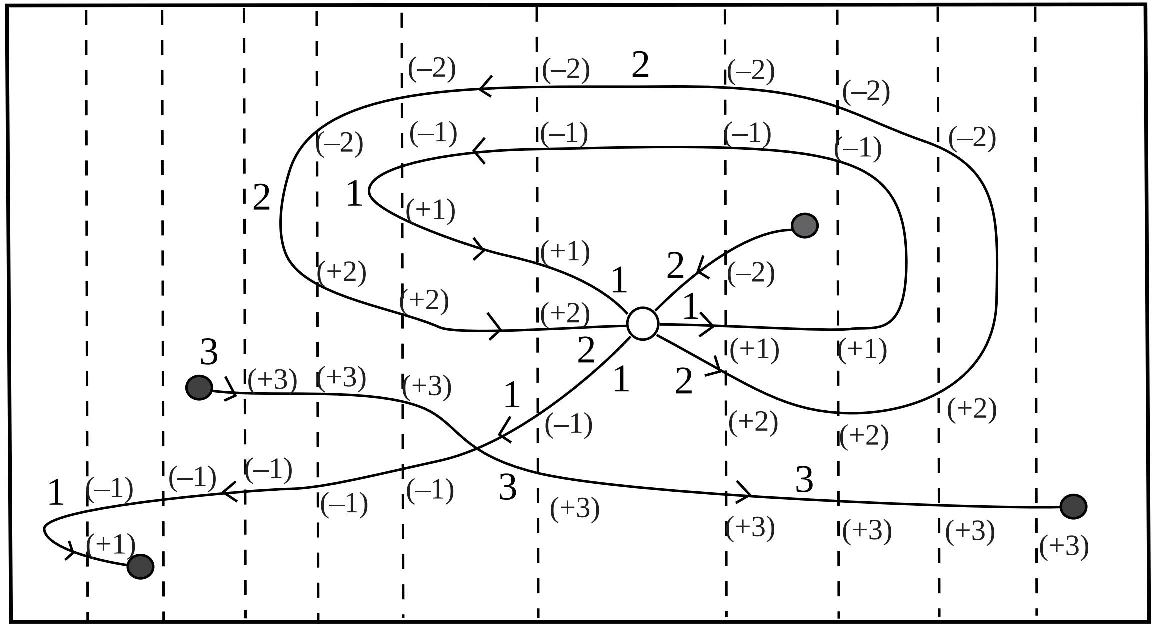
The braid index of a 2-twist spun trefoil is 4.

A simple chart or a chart of degree m is a graph in I^2 whose edges are oriented and labeled by integers from $\{1, \ldots, m-1\}$ and the vertices are as follows.



A simple chart of degree 4





(−2)
(−2)
2
(−2)
(−2)
(−2)
(−2)
(−1)
(−1)
(−1)
(−1)
2
1
(+1)
(+1)
1
2
(−2)
(+2)
(+2)
(+2)
1
3
2
(+1)
(+1)
(+3)
(+3)
(+3)
1
2
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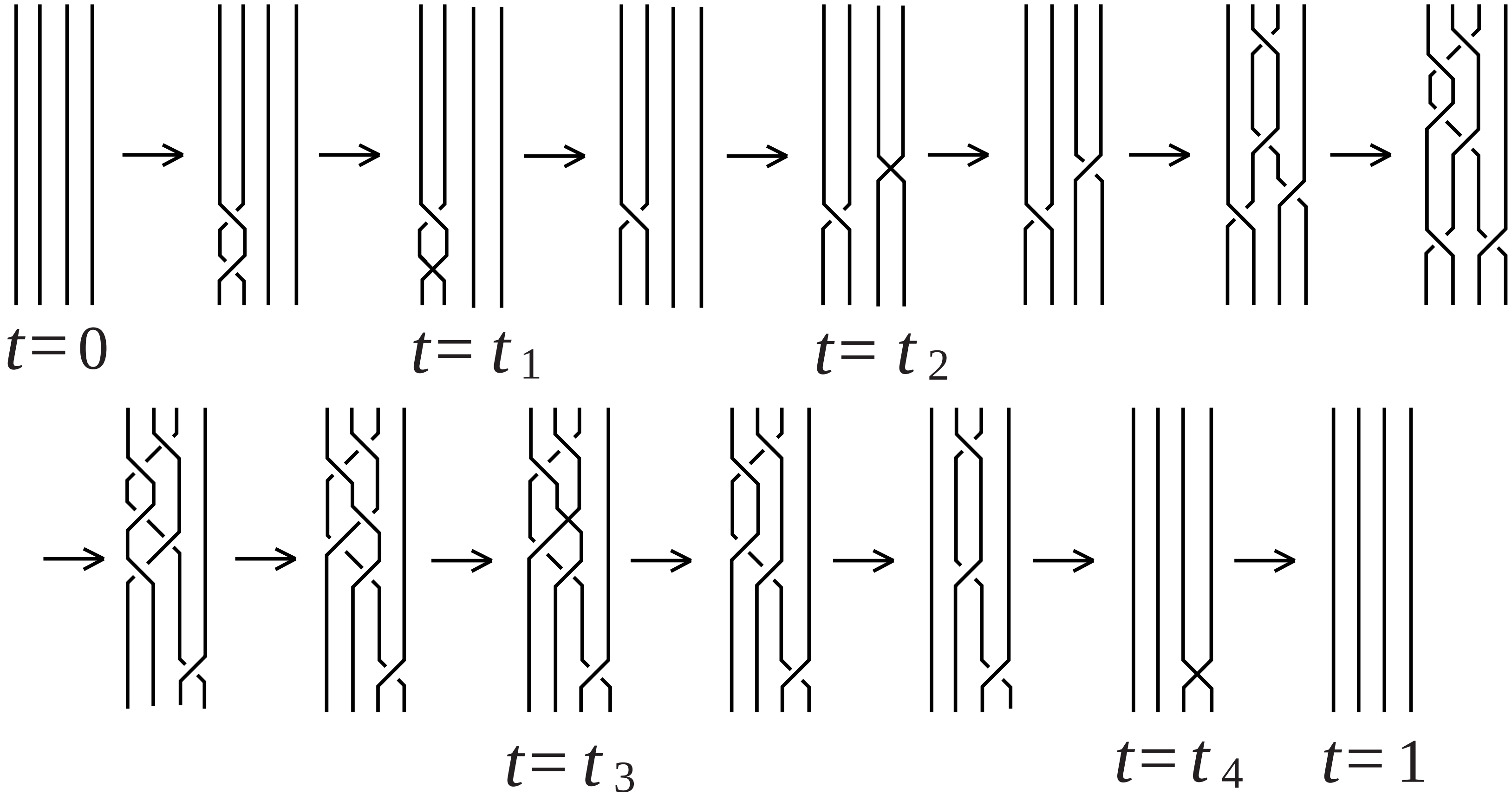
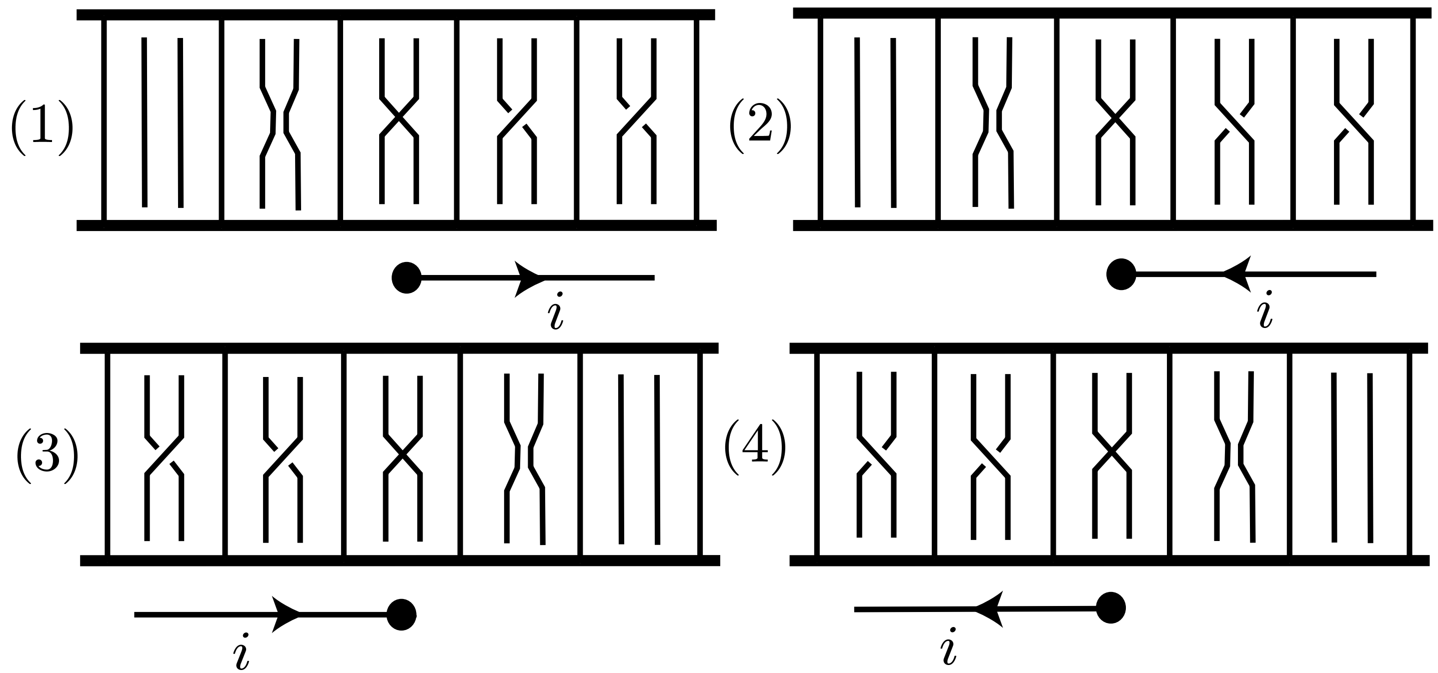
(+2)
(+2)
(+2)
(−1)
1
(−1)
(−1)
(−1)
(−1)
(−1)
3
(+3)
3
(+3)
(+3)
(+3)
(+3)
(+1)



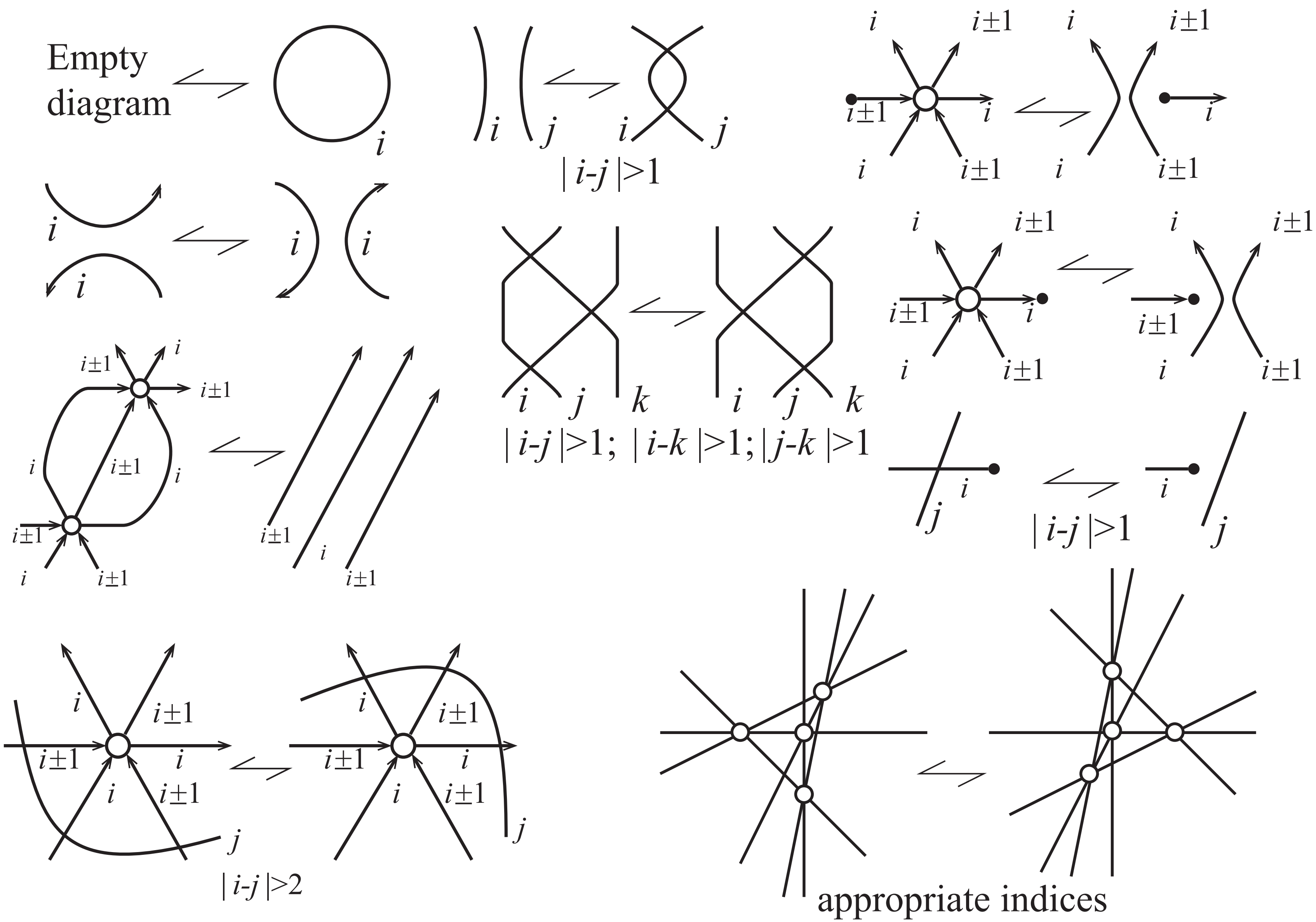
$t=0$
$t=t_1$
$t=t_2$
$t=t_3$
$t=t_4$
$t=1$



a simple chart $\Longleftrightarrow$ a simple surface braid
a black vertex $\Longleftrightarrow$ a branch point

Thm (4.2) (K. 1992, 1994)

Any simple surface braid is described by a simple chart. Such a chart is unique up to (simple) chart moves.



Empty diagram
i
| i-j |>1
i j i j
i±1
| i-j |>1; | i-k |>1; | j-k |>1
i j k
| i-j |>1
| i-j |>2
appropriate indices

5. Chart description for regular surface braids

We first define a subset of the classical m-braid group B_m.

Def (5.1)

$b \in A_m^{\mathrm{regular}} \iff b$ is conjugate to $b_1 \amalg b_2$, where b_1 is a braid of degree ≥ 2 whose closure is an unknot in S^3 and b_2 is a trivial braid.

We call the degree of b_1 the branch degree of b, and denote it by branch-deg(b).

Eg.

Let $b = \sigma_2 \sigma_3^{-1} \sigma_4 \in B_7$.
Then $b \sim b_1 \amalg b_2$, where $b_1 = \sigma_1 \sigma_2^{-1} \sigma_3 \in B_4$ whose closure in S^3 is an unknot, and $b_2 \in B_3$ is a trivial braid.
Thus $b \in A_7^{\mathrm{regular}}$ and branch-deg$(b) = \deg(b_1) = 4$.

Let $b \in A_m^{\mathrm{regular}}$ and let $w = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_q}^{\epsilon_q}$ be a braid word presenting b. If the cardinality of generators $\{\sigma_{i_1}, \ldots, \sigma_{i_q}\}$ appearing in w equals branch-deg$(b) - 1$, then we say that w is range reduced.

Eg.

Consider the above example again. Let $b = \sigma_2 \sigma_3^{-1} \sigma_4 \in B_7$. Then $b \in A_7^{\mathrm{regular}}$ and branch-deg$(b) = 4$.

Let $w = \sigma_2 \sigma_3^{-1} \sigma_4 \sigma_3 \sigma_3^{-1}$. The generator set appearing in w is $\{\sigma_2, \sigma_3, \sigma_4\}$. The cardinality is $3 = 4 - 1$. Thus w is range reduced.

Let $w' = \sigma_2 \sigma_3^{-1} \sigma_4 \sigma_5 \sigma_5^{-1}$. The generator set appearing in w' is $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5\}$. The cardinality is $4 \neq 4 - 1$. Thus w' is not range reduced.

Def (5.2)

A regular chart of degree m is a graph in I^2 whose edges are oriented and labeled by integers from $\{1, \ldots, m-1\}$ and the vertices are as in the figure, such that at each black vertex, let w be a braid word obtained by reading the labels and orientations of the edges around the vertex, then w is a range reduced word representing an element of A_m^{regular}.

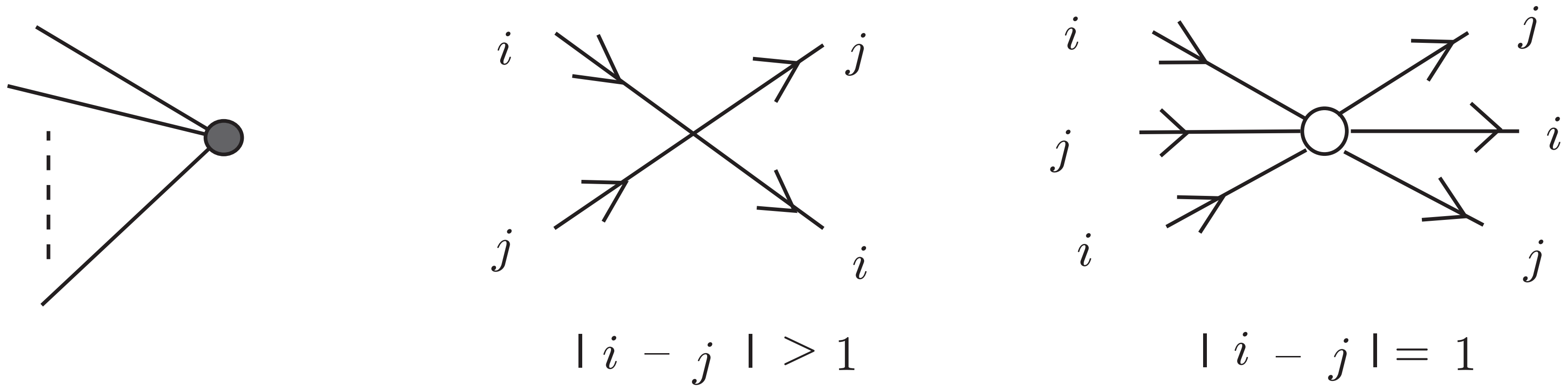
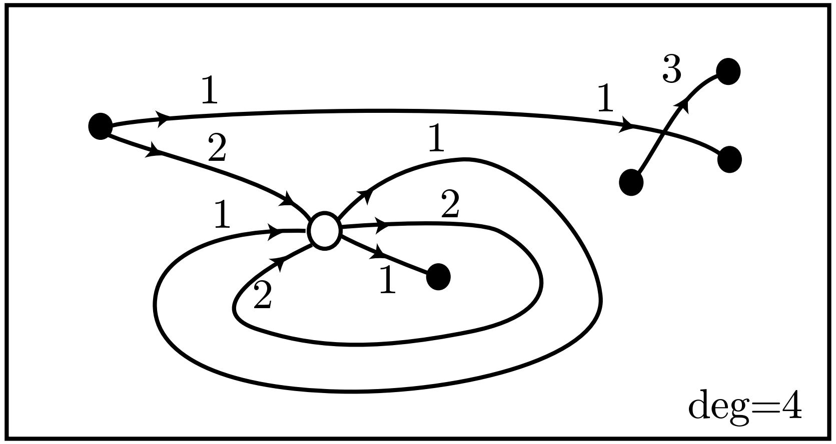
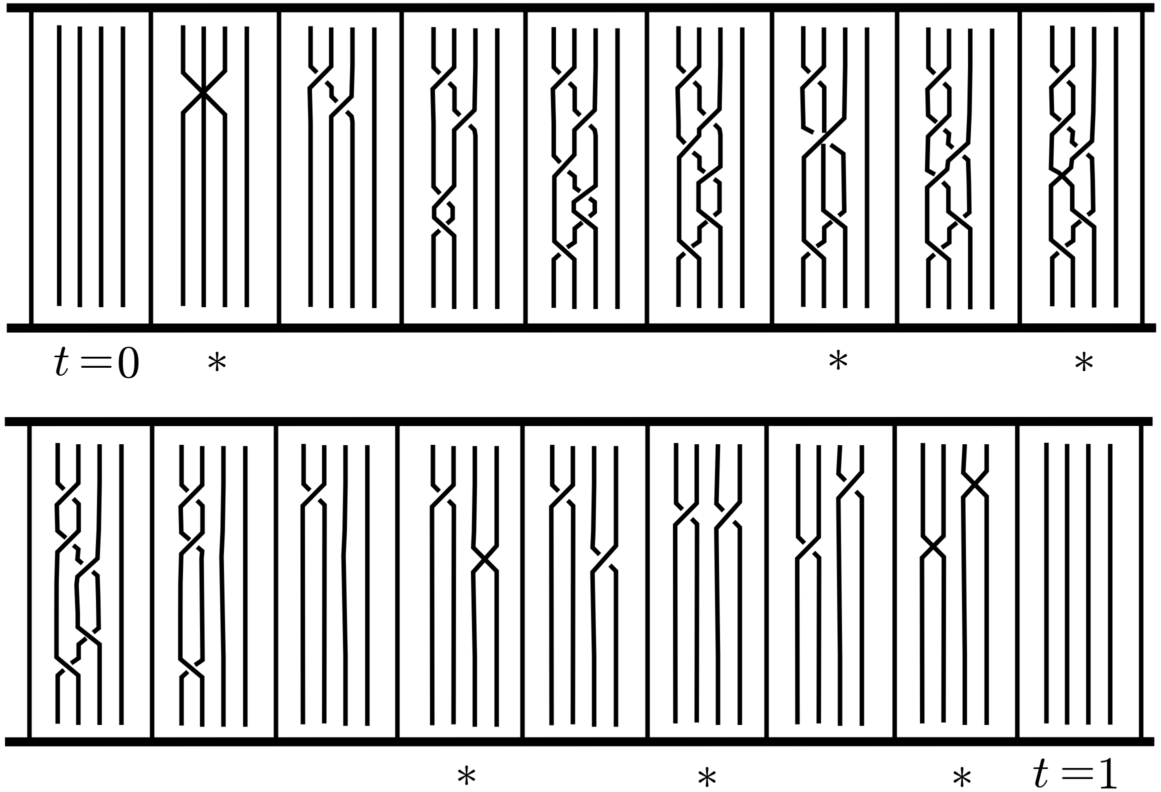
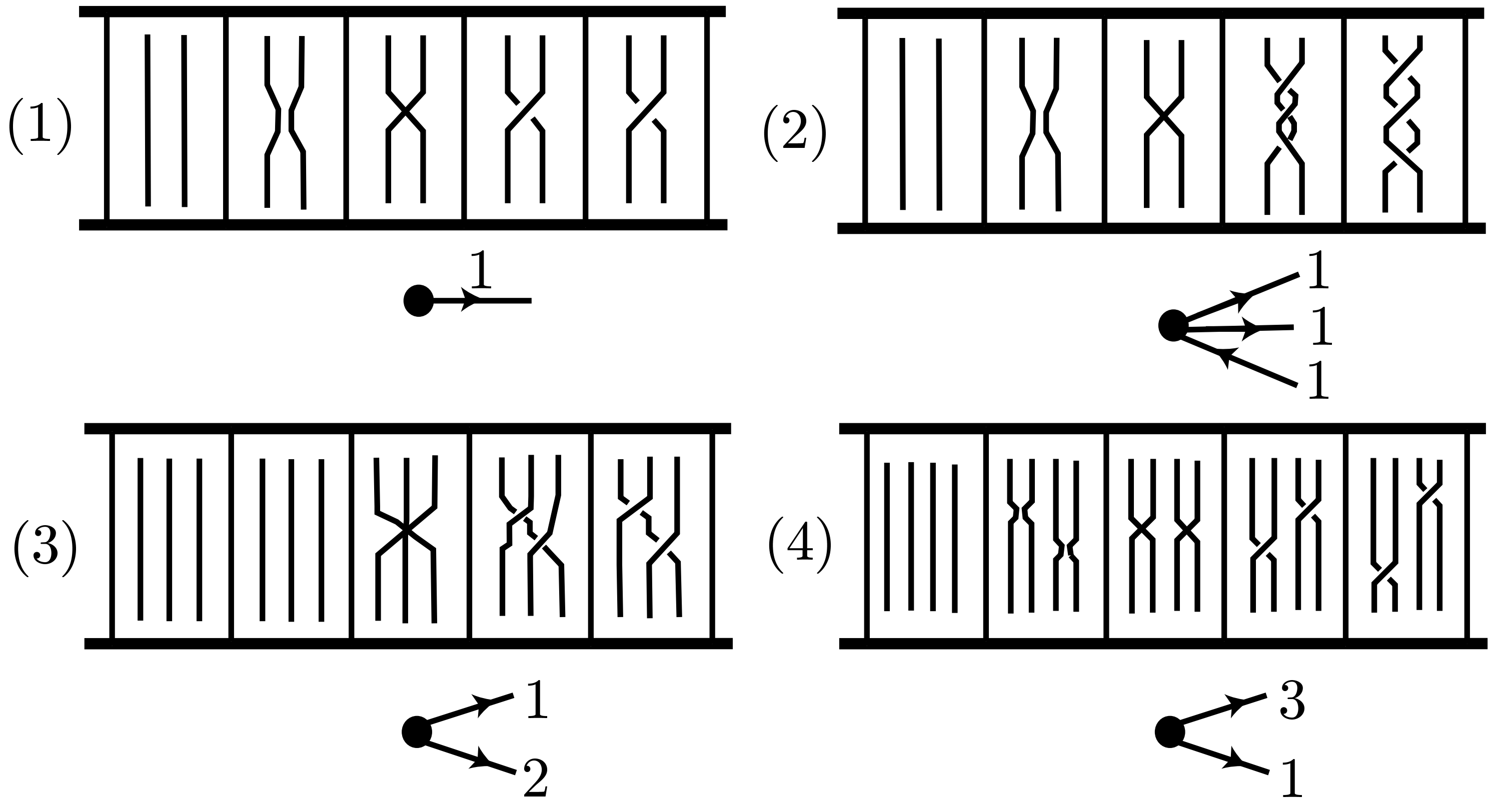


Figure: black vertex, crossing, white vertex


1
2
1
2
1
2
1
1
3
deg=4


$t=0$
$*$
$*$
$*$
$*$
$*$
$*$
$t=1$



a regular chart $\Longleftrightarrow$ a regular surface braid
a black vertex $\Longleftrightarrow$ a branch point

Thm (5.3)

Any regular surface braid is described by a regular chart. (Such a chart is unique up to regular chart moves (Thm. 5.4)).

Chart moves of type W

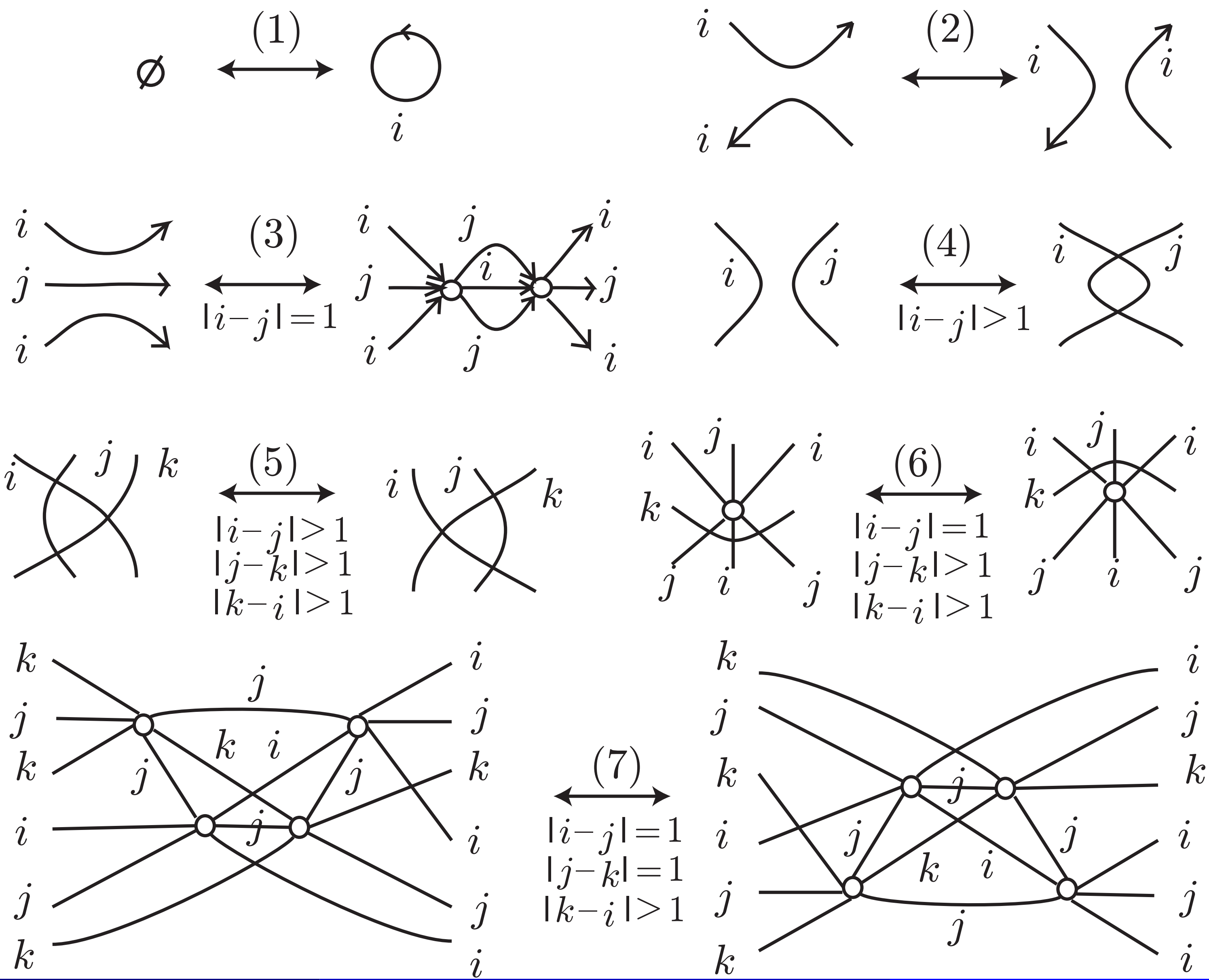
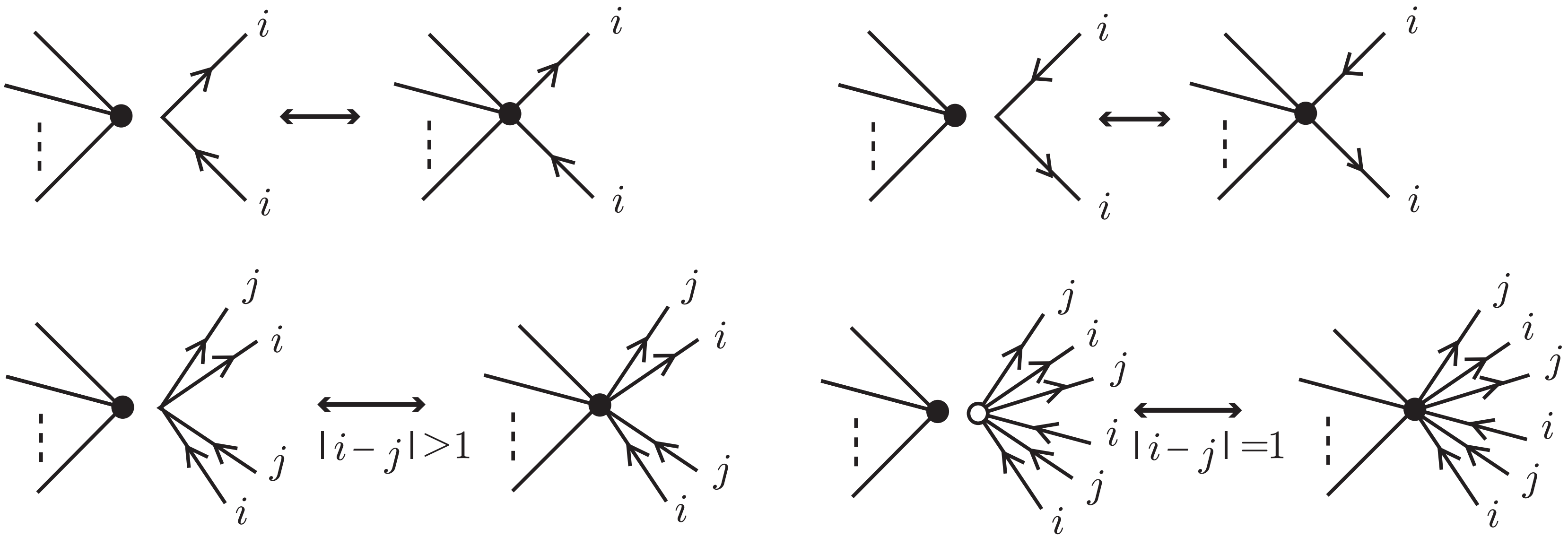
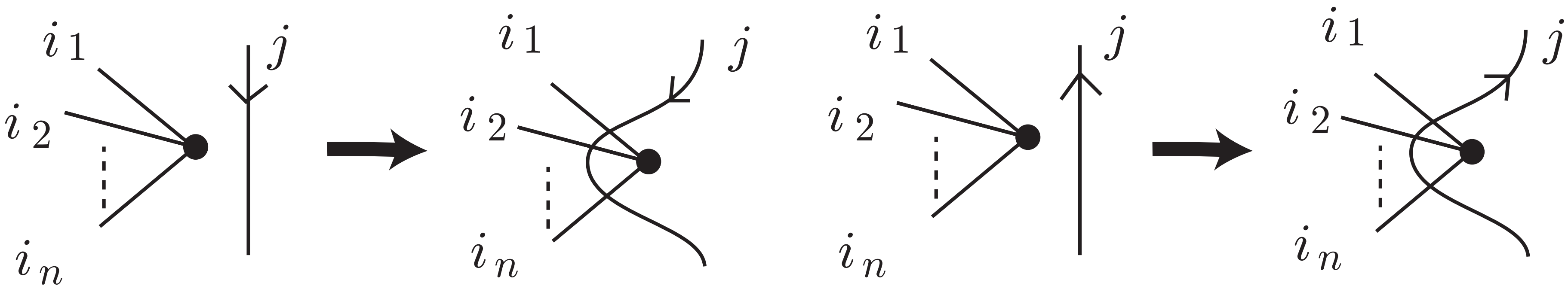


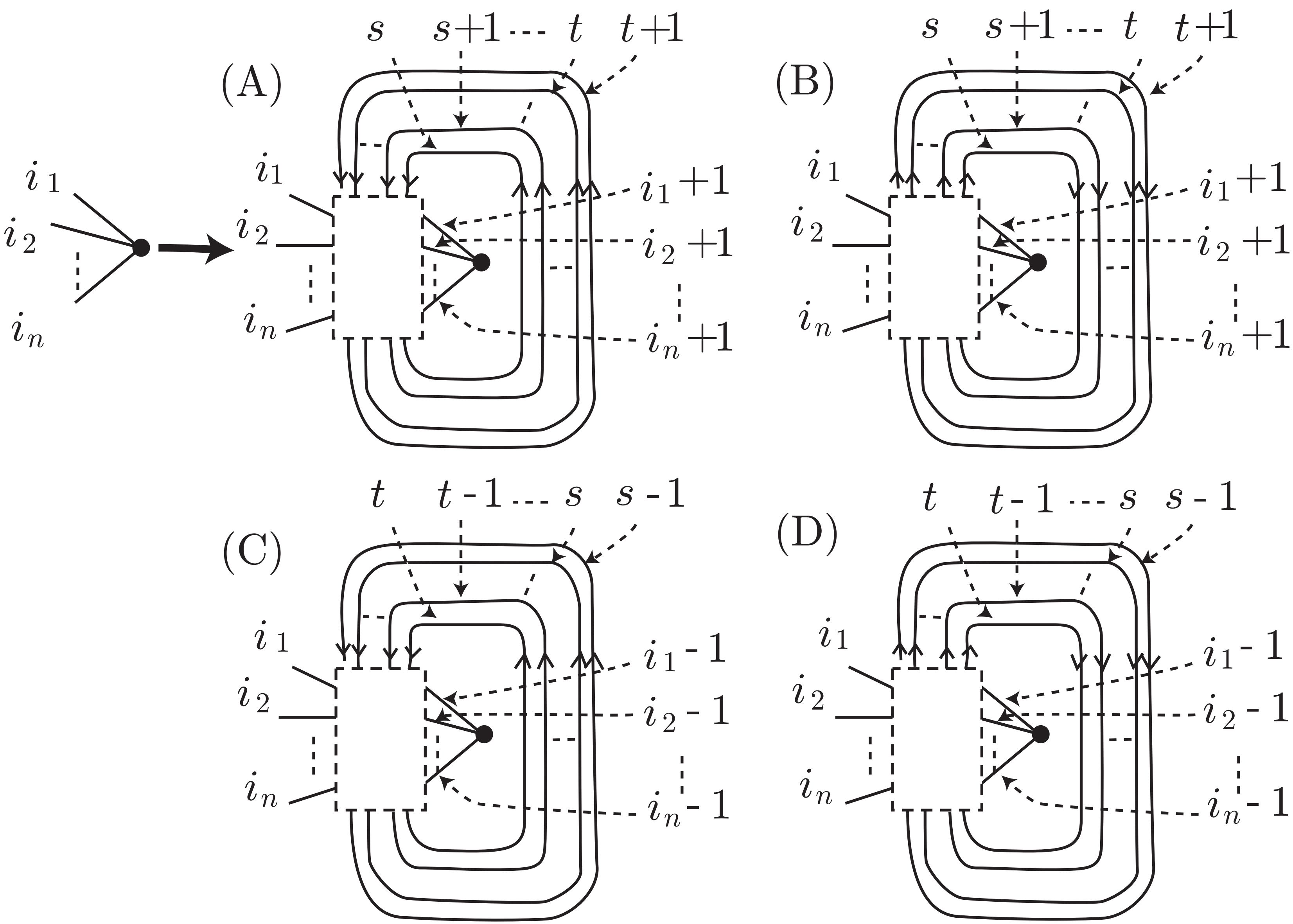
Chart moves of regular type B (Assume that i and j are elements of the labels $\{i_1, \ldots, i_n\}$ around the black vertex of the left side)



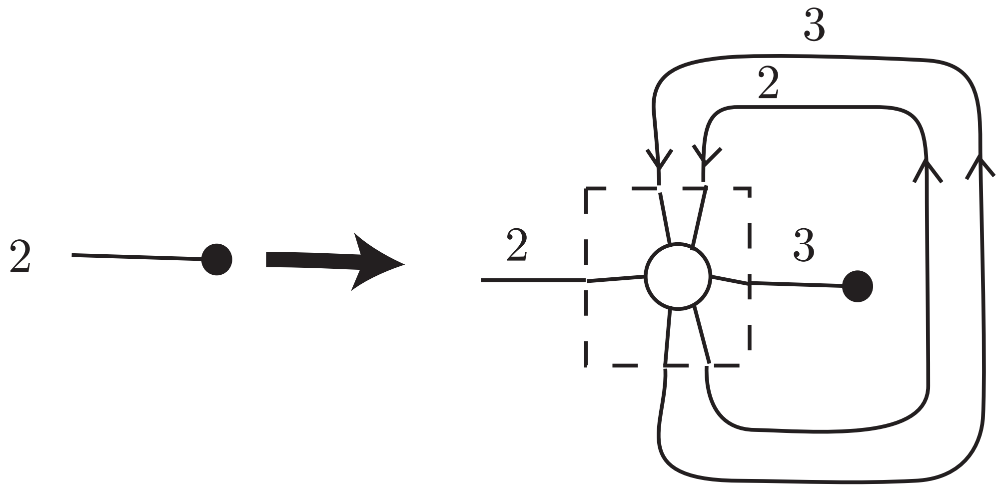
Passing moves. (Assume that j satisfies $|i_k - j| > 1$ $(k = 1, \ldots, n)$, where $\{i_1, \ldots, i_n\}$ is the labels around the black vertex)

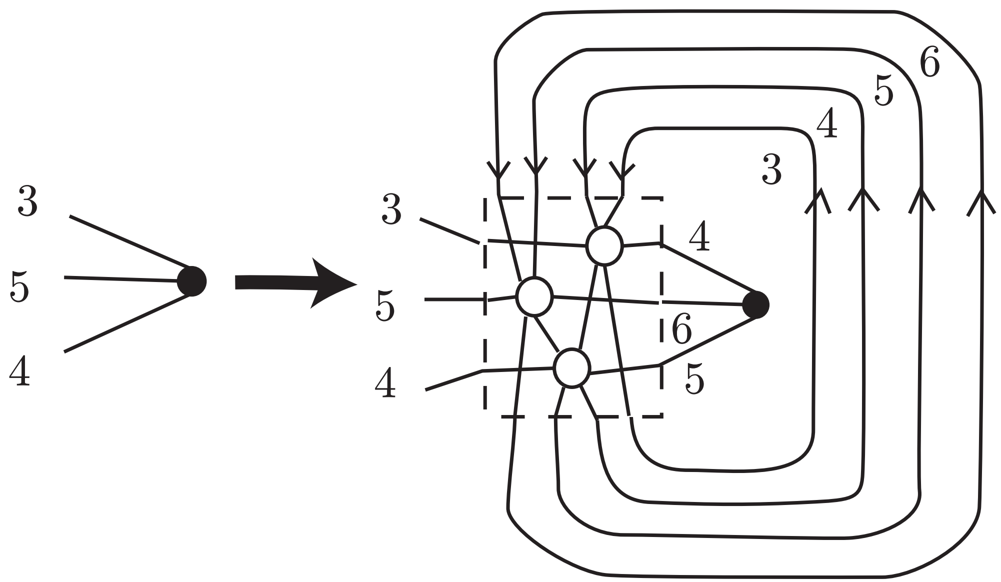


Label shift moves, which shift the labels $+1$ or -1.



Examples of label shift moves.





Thm (5.4)

A regular chart description of a regular surface braid is unique up to chart moves of type W, chart moves of regular type B, passing moves and label shift moves.

$$\{\text{simple surface braids}\}/_{\sim} \overset{1:1}{\Longleftrightarrow} \{\text{simple charts}\}/_{\sim}$$

$$\{\text{regular surface braids}\}/_{\sim} \overset{1:1}{\Longleftrightarrow} \{\text{regular charts}\}/_{\sim}$$

Thank you.