Numerical invariants of twisted knots

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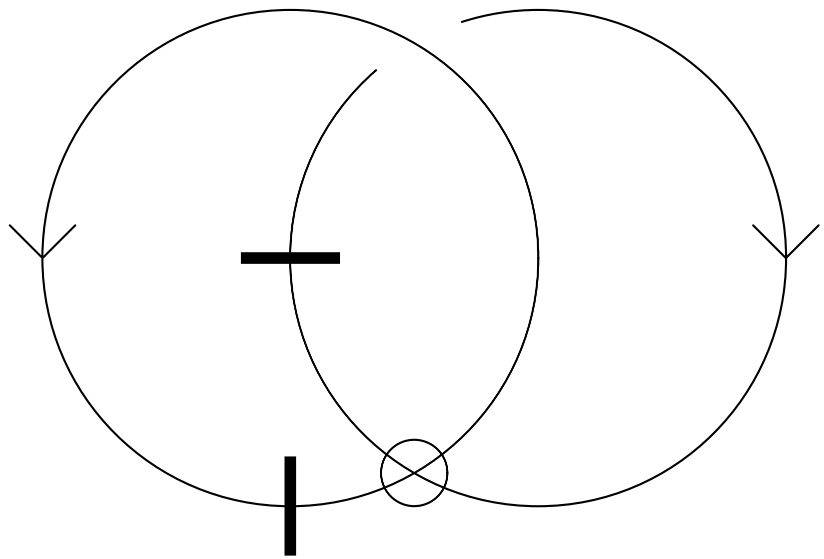
ADVANCED SCHOOL AND DISCUSSION MEETING

ON KNOT THEORY AND ITS APPLICATIONS

Twisted link diagram

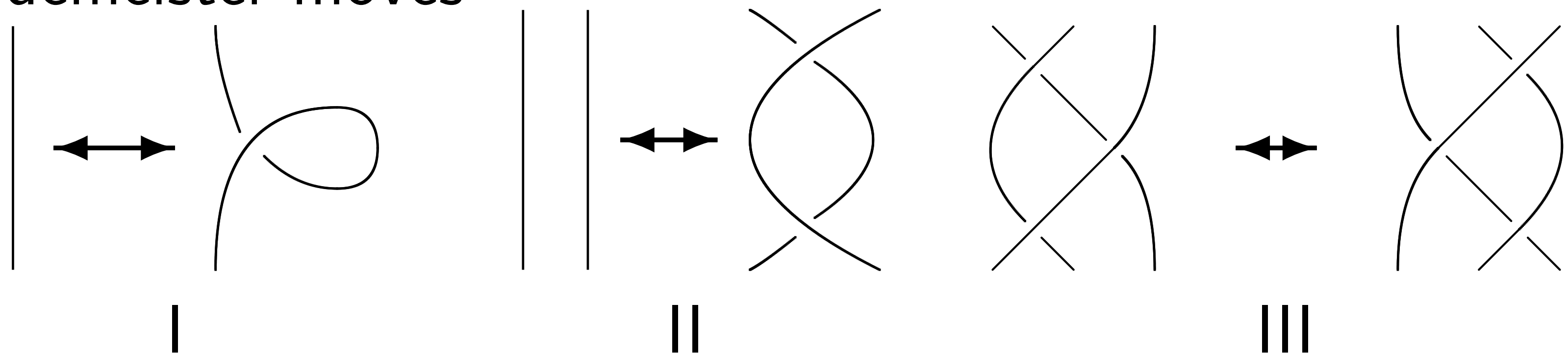
D : twisted link diagram $\Leftrightarrow$ D : a link diagram whose double points are given the informations over/under or virtual possibly with some bars on arcs



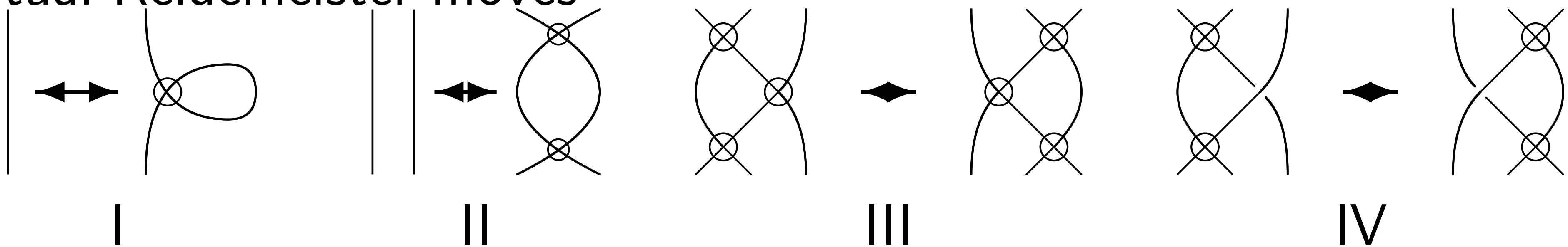
A twisted link is the equivalence class of a twisted link diagram under Reidemeister moves I, II, III, virtual Reidemeister moves I, II, III, IV and twisted Reidemeister moves I, II, III.

Generalized Reidemeister moves

Reidemeister moves

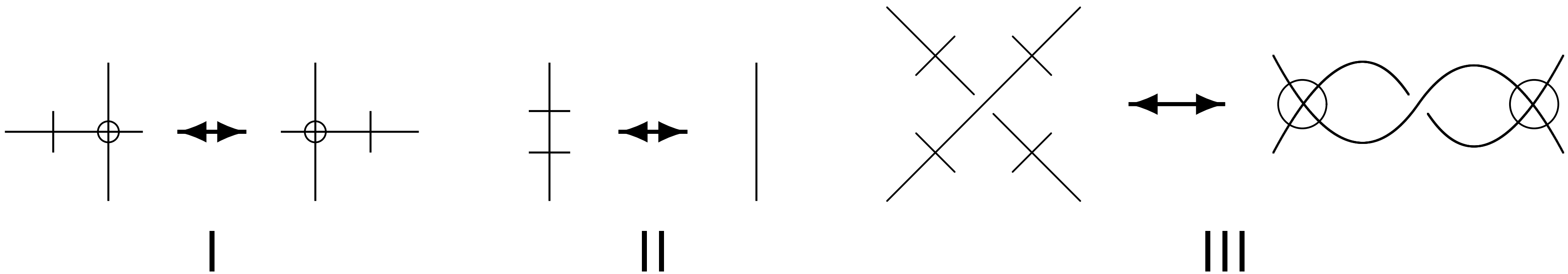


Virtual Reidemeister moves



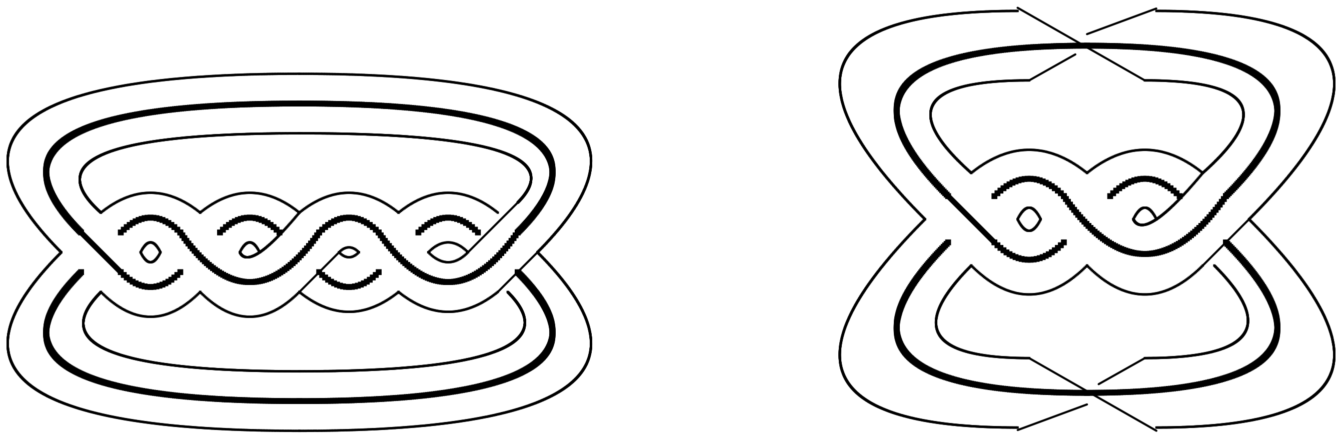
Extended Reidemeister moves

Generalized Reidemeister moves +



Abstract links

An abstract link diagram (Σ, D_Σ) : a pair of a, possibly non-orientable compact surface Σ and a link diagram D_Σ in Σ such that $|D_\Sigma|$ is a deformation retract of Σ .

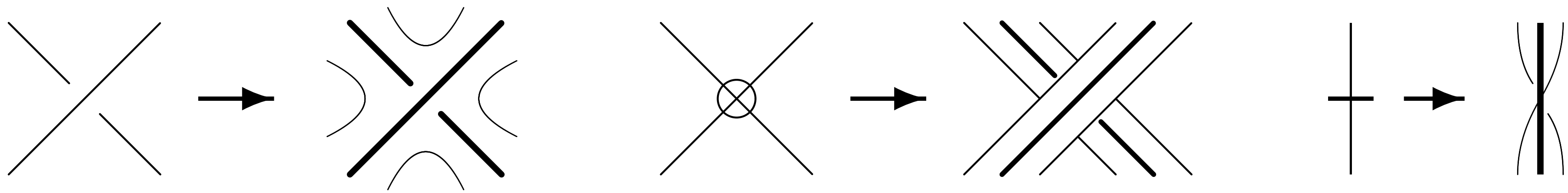


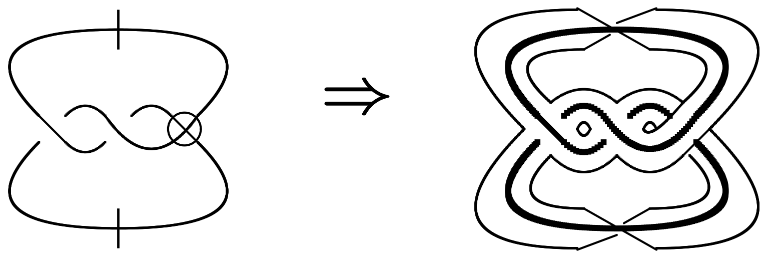
Abstract links and Twisted links

Theorem(Bourgoin)

$$\varphi : \{\text{twisted link diagrams}\} \longrightarrow \{\text{abstract link diagrams}\}$$

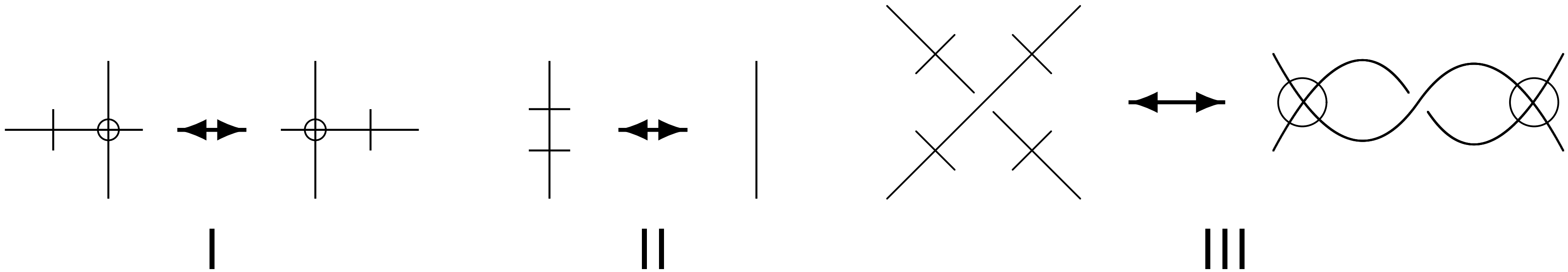
s.t. φ induce a bijection between the set of twisted links and the set of abstract links.

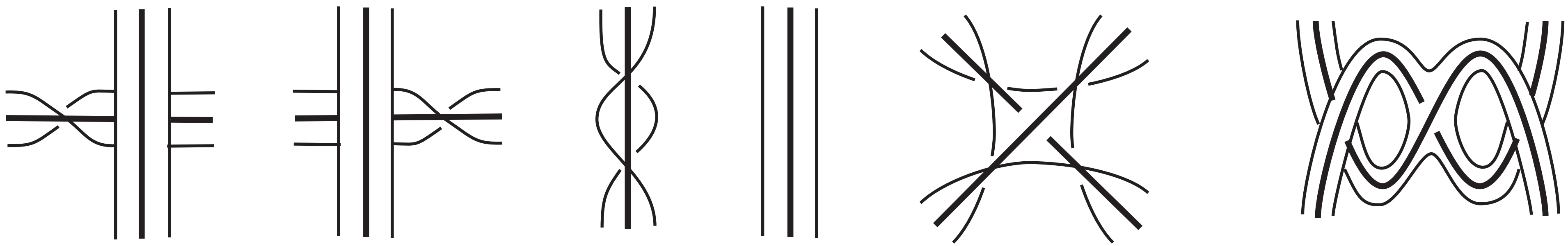




$\varphi(D) = (\Sigma, D_\Sigma)$: an abstract link diagram associated with D

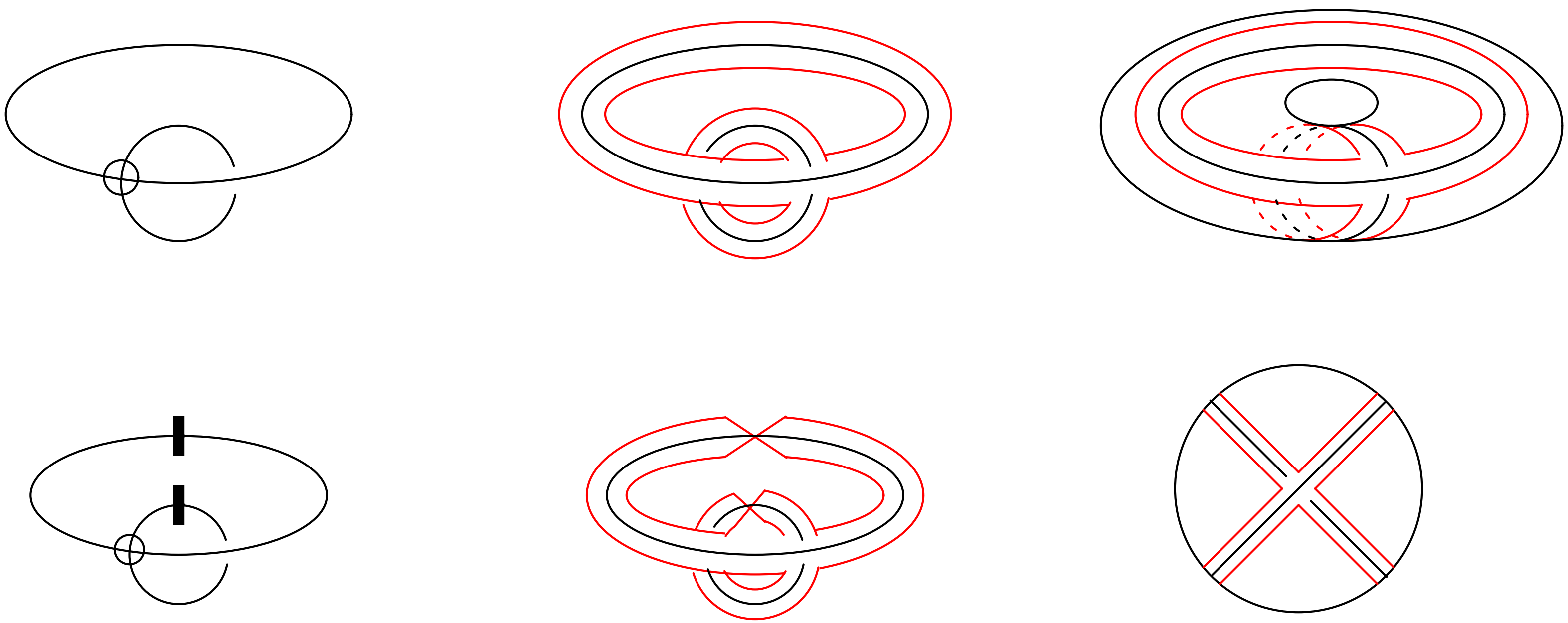
Twisted Reidemeister moves and abstract links





Link diagram realizations of twisted links

D : a twisted link diagram
(Σ, D_Σ) : an abstract link diagram associated with D
(F, D_F) : a link diagram realization of D in a closed surface F
$\Leftrightarrow$ a pair of a closed surface F and a link diagram D_F s.t. there is a embedding f from Σ to F whose image of D_Σ is D_F $(f(D_\Sigma) = D_F)$.



Twisted links and stable equivalence classes

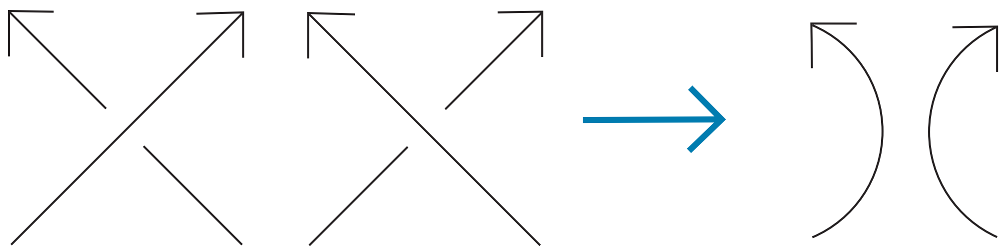
$$\{\text{twisted links}\} \Leftrightarrow \bigcup_{F \in \{\text{closed surfaces}\}} \{\text{links in } F \tilde{\times} I\} \Big/ \text{stable equivalence relation}$$

Theorem (M. Bourgoin)

Stable equivalence classes of links in oriented thickened surfaces have a unique irreducible representative.

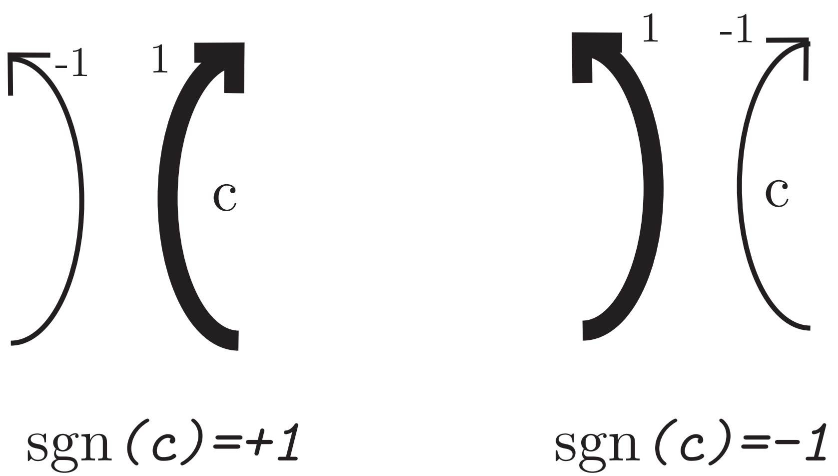
Index diagram

D : a virtual knot diagram
c : a real crossing of D
The index diagram of a real crossing c, D_c of D is a two component link diagram $d_1 \cup d_2$ which is obtained from D by smoothing at c



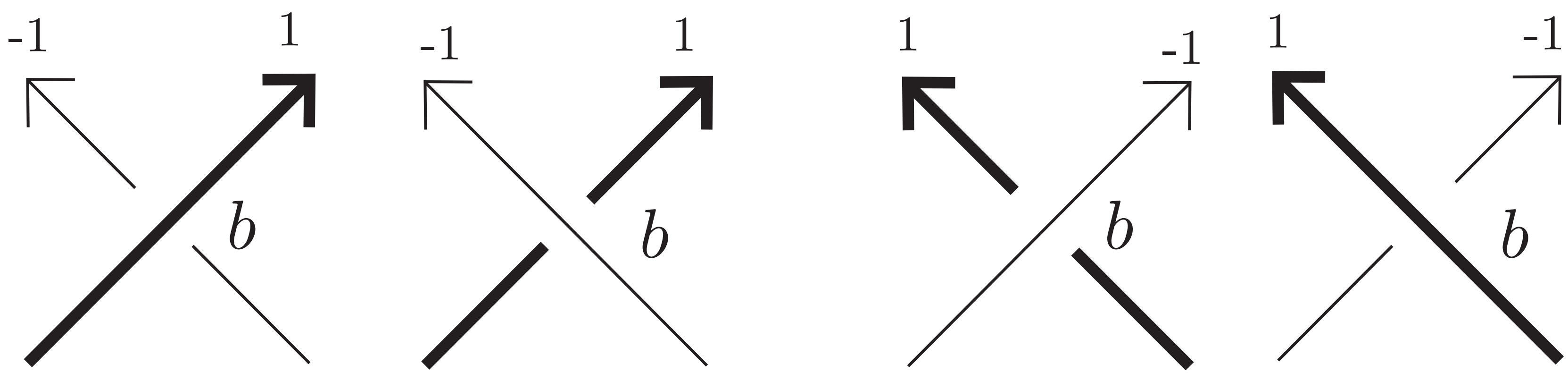
Intersection index of a real crossing

D: a virtual knot diagram
c : a real crossing of D
We label each component of D_c by $(1, -1)$ as in the figure below.



Intersection index of a real crossing

D : a virtual knot diagram c : a real crossing of D
D_c : the index diagram of c b : a non self real crossing of D_c



$\iota(b) = 1$ $\iota(b) = -1$

The intersection index $\mathrm{Ind}(c)$ of a real crossing c is defined by

$$\mathrm{Ind}(c) = \sum_{b \in d_1 \cap d_2} \iota(b)$$

The nth partial writhe

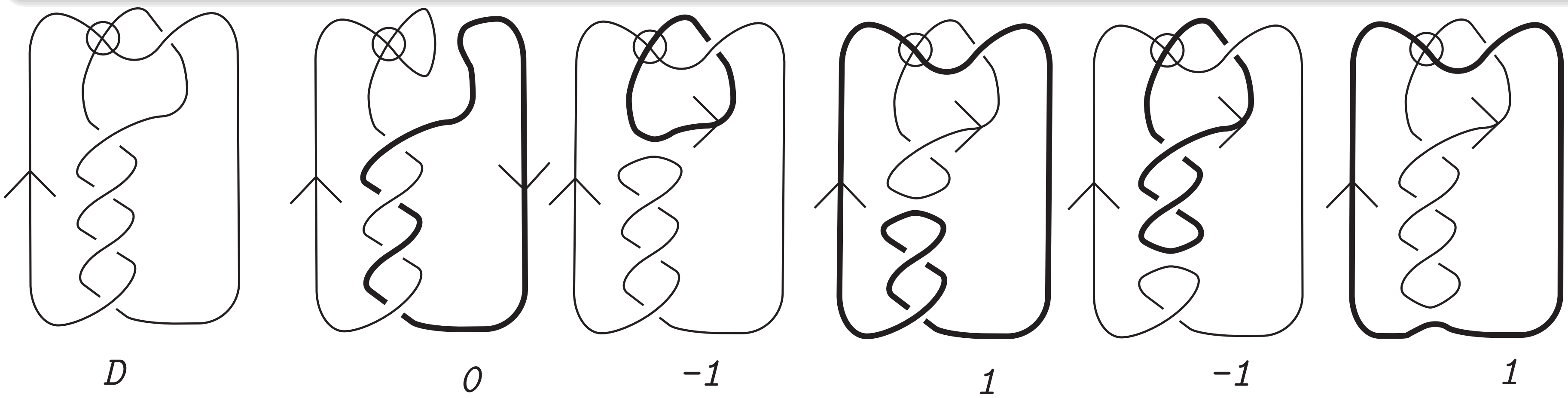
The nth partial writhe of D is defined as follows

$$J_n(D) = \sum_{\mathrm{Ind}(c)=n} \mathrm{sgn}(c)$$

where $\mathrm{sgn}(c)$ is the sign of a real crossing c.

Theorem[S. Satoh, K. Taniguchi]

$J_n(D)$ is an invariant of a virtual knot for each integer $n \neq 0$.



Odd writhe

D : a virtual knot diagram

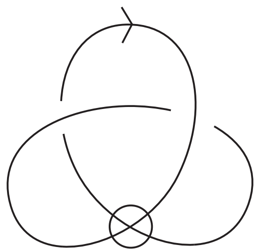
A real crossing c of D is odd if we meet an odd number of crossings when we walk along one of arcs of D whose starting point and ending point are c.

The odd writhe of D is defined as follows

$$J(D) = \sum_{c:\ \text{odd}} \text{sgn}(c)$$

Theorem [L. Kauffman]

$J(D)$ is an invariant of a virtual knot.



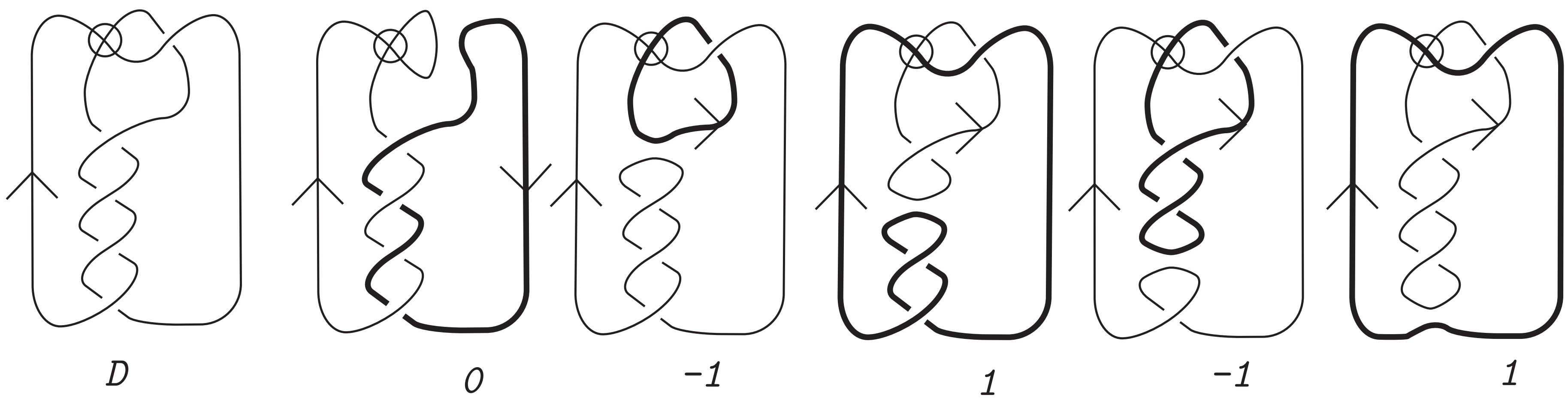
$J(D) = 2$

Odd writhe and the nth partial writhe

D : a virtual knot diagram

Corollary[S. Satoh and K. Taniguchi]

$$J(D) = \sum_{n:\text{odd}} J_n(K)$$



$$J_1(D) = -2,\ J_{-1}(D) = -2,\ J(D) = -4$$

Definition of Index polynomial

The index polynomial of a virtual knot D is defined by as follows

$$Q_D(t) = \sum_{c} \mathrm{sgn}(c)(t^{|\mathrm{Ind}(c)|} - 1)$$

where c runs over all real crossings of D.

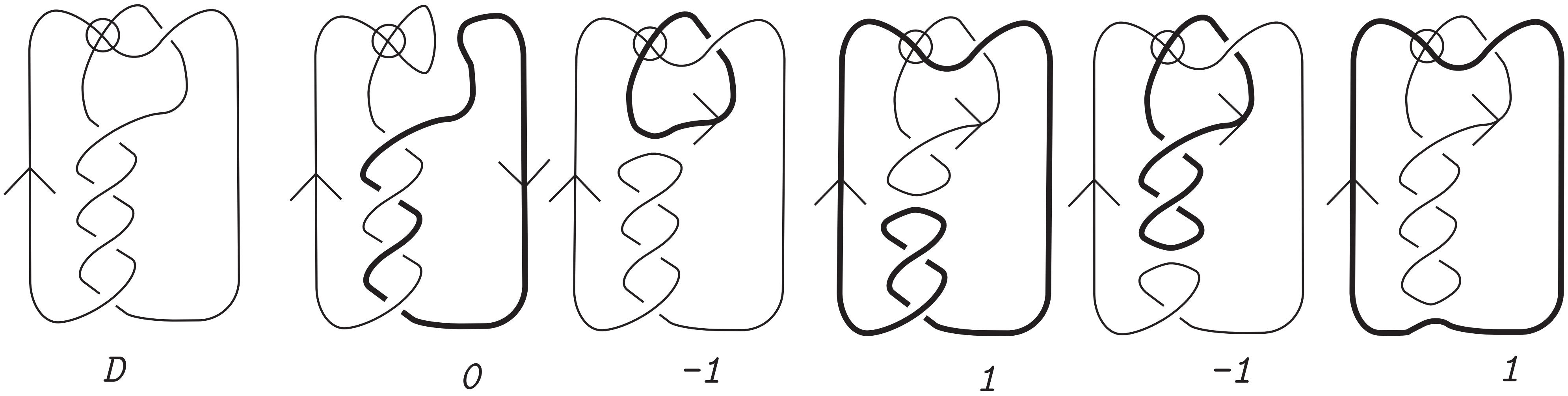
Theorem [A. Henrich, Y. H. Im, K. Lee, S. Y. Lee]

$Q_D(t)$ is an invariant of a virtual knot.

Corollary [S. Satoh and K. Taniguchi]

$$Q_D(t) = \sum_{n \neq 0} (J_n(D) + J_{-n}(D))(t^n - 1)$$

Example

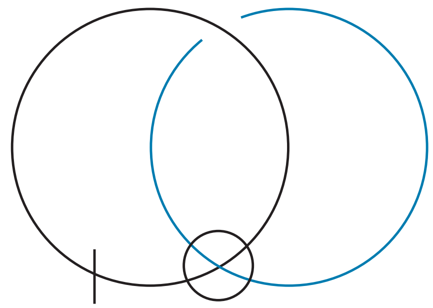


$J_1(D) = -2$, $J_{-1}(D) = -2$, $J(D) = -4$,
$Q_D(t) = -4(t-1)$

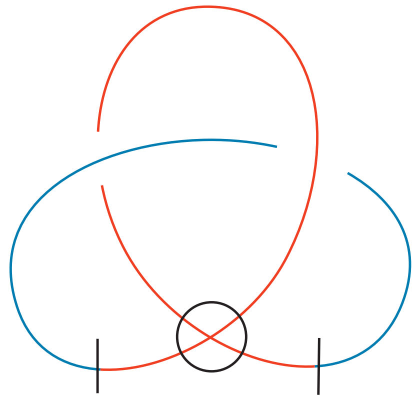
Index diagrams of twisted links

D : a twisted link diagram

A component of D is said to be even(or odd) if there are even (or odd) number of bars on it.



odd and even



even

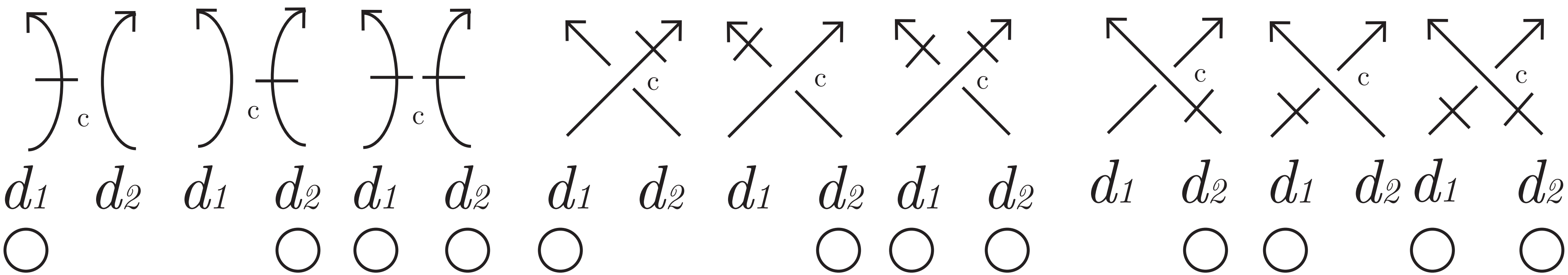
The index diagram of a real crossing c, D_c is obtained from D as follows;

- If c is the real crossing of the distinct components of D, say d_1 and d_2, $D_c = d_1 \cup d_2$
- If c is the real crossing of a component d of a twisted link diagram D, D_c is a two component link diagram $d_1 \cup d_2$ obtained from d by smoothing at c

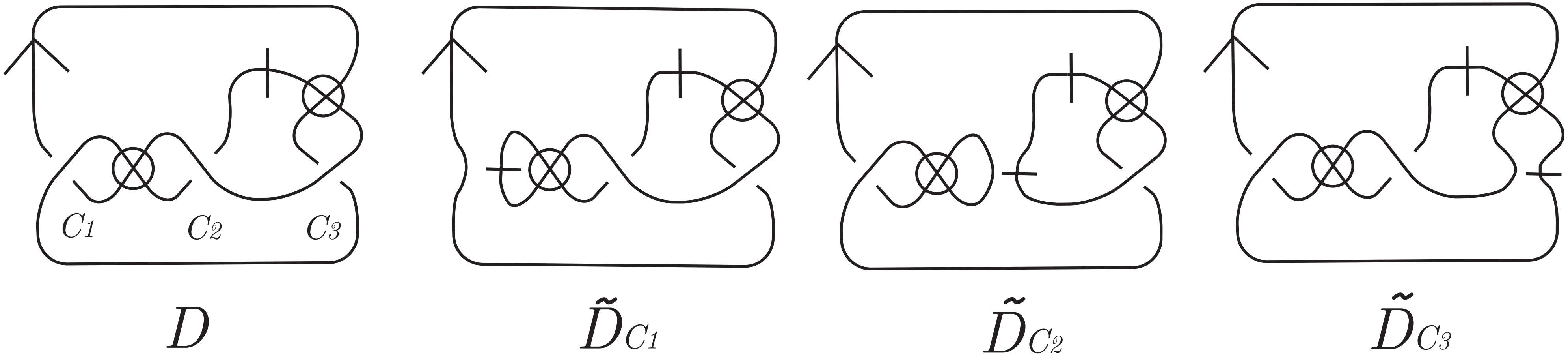
Frilled Index diagram

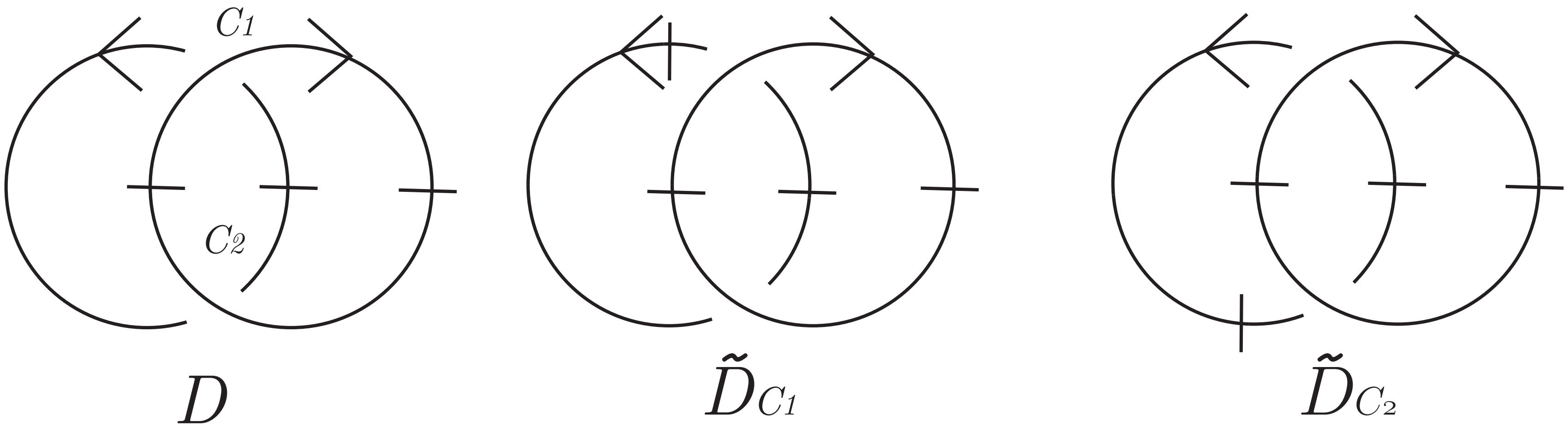
The frilled index diagram of a real crossing c, $\tilde{D}_c$ is obtained from D_c as follows;

- If d_1 and d_2 are even, $\tilde{D}_c = D_c$
- If d_i is odd, $\tilde{D}_c$ is obtained from D_c by adding a bar to d_i as in the figure below. (◯ indicates that d_i is an odd component.)



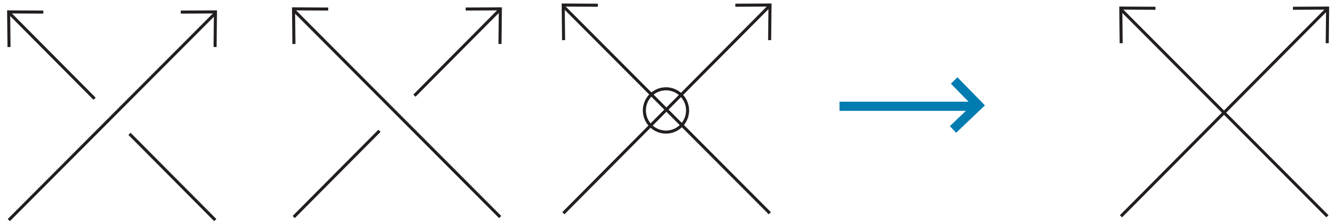
Examples


C_1
C_2
C_3
D
$\tilde{D}_{C_1}$
$\tilde{D}_{C_2}$
$\tilde{D}_{C_3}$


C_1
C_2
D
$\tilde{D}_{C_1}$
$\tilde{D}_{C_2}$

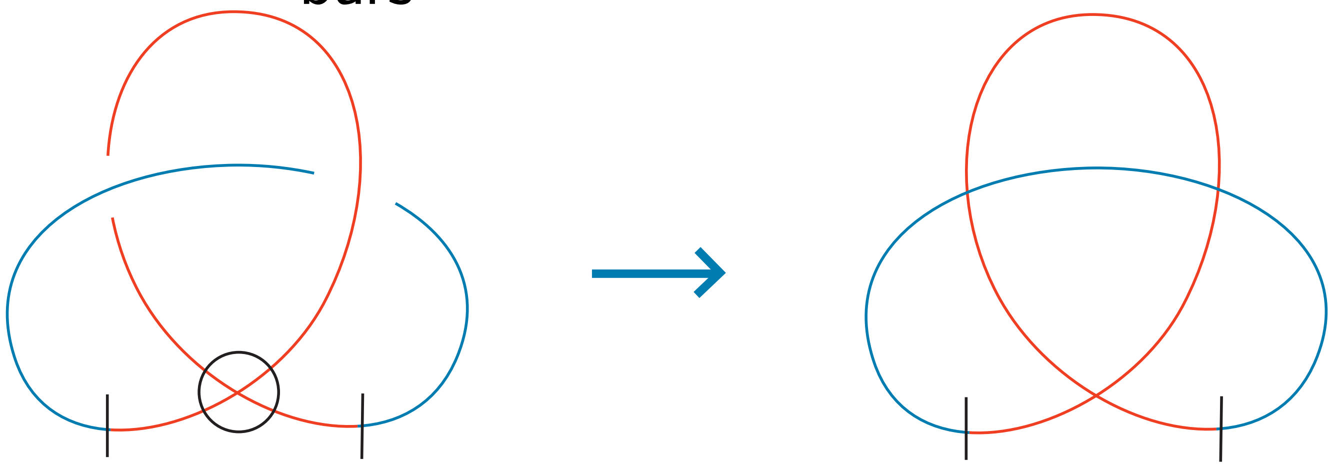
bar-edge

p: {twisted link diagrams } $\longrightarrow$ {immersed loops with some bar}



D : twisted link diagram

e: bar-edge of D $\Leftrightarrow$ Preimege of a segment of $p(D)$ between two bars

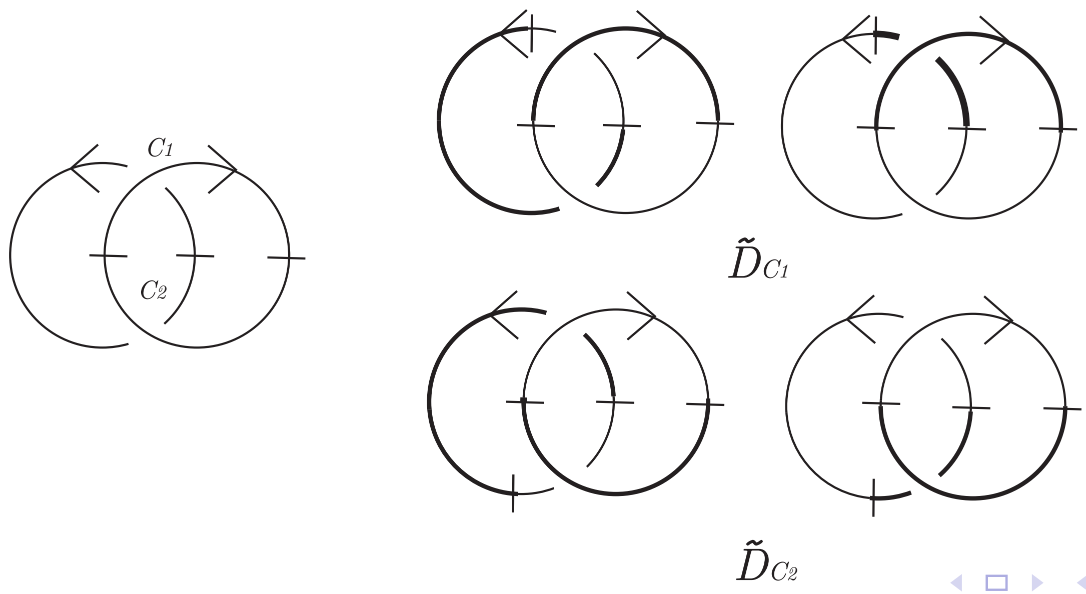


Weight map of a frilled index diagram

D : a twisted link diagram
$\tilde{D}_c$: a frilled index diagram of a real crossing c of D
$E(\tilde{D}_c)$: the set of bar-edge of $\tilde{D}_c$
σ : a weight map of $\tilde{D}_c \Leftrightarrow$

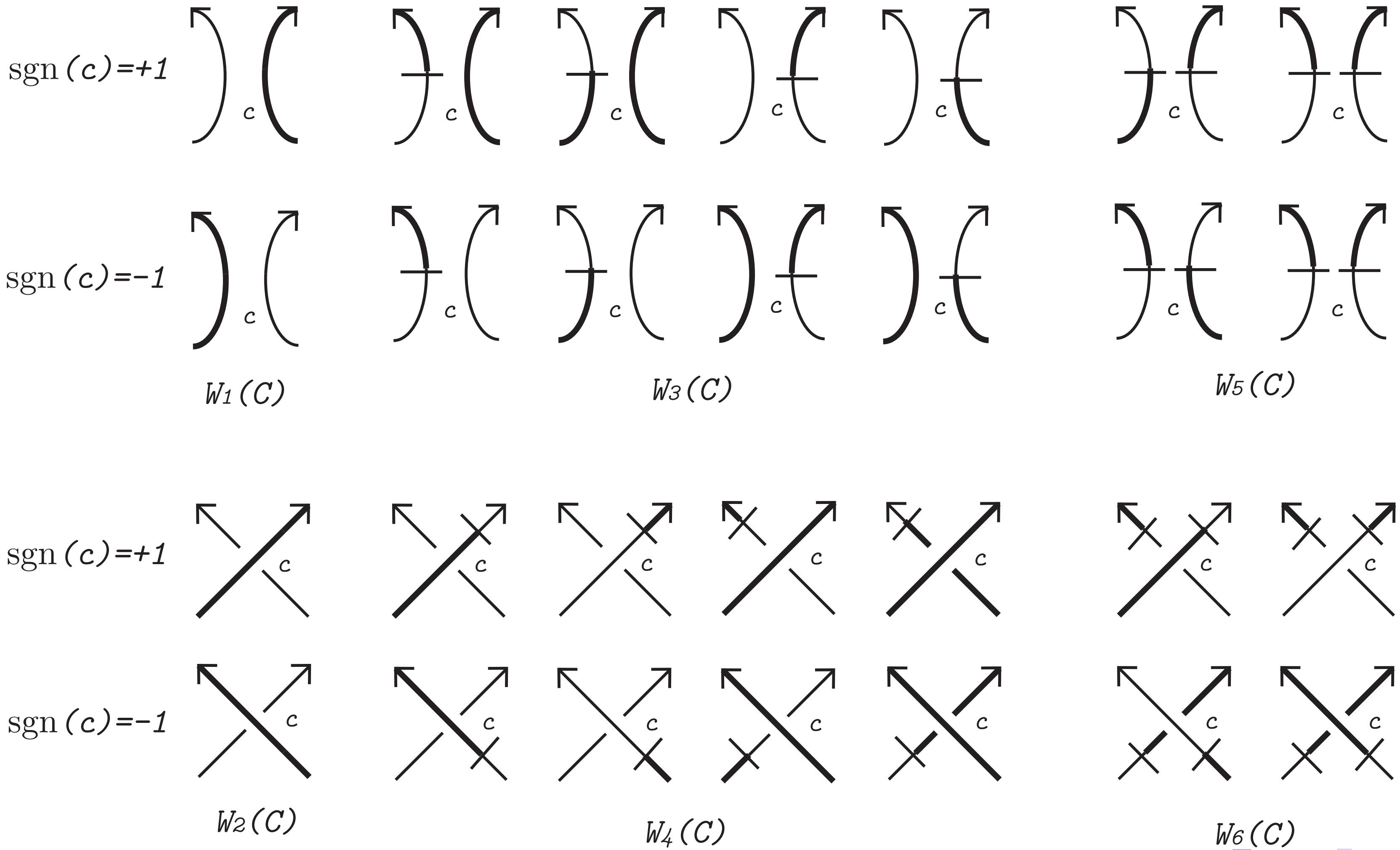
$$\sigma : E(\tilde{D}_c) \longrightarrow \{1, -1\}$$

s.t. $\sigma(e) \neq \sigma(e')$ for $e, e' \in E(\tilde{D}_c)$ if e and e' are adjacent .



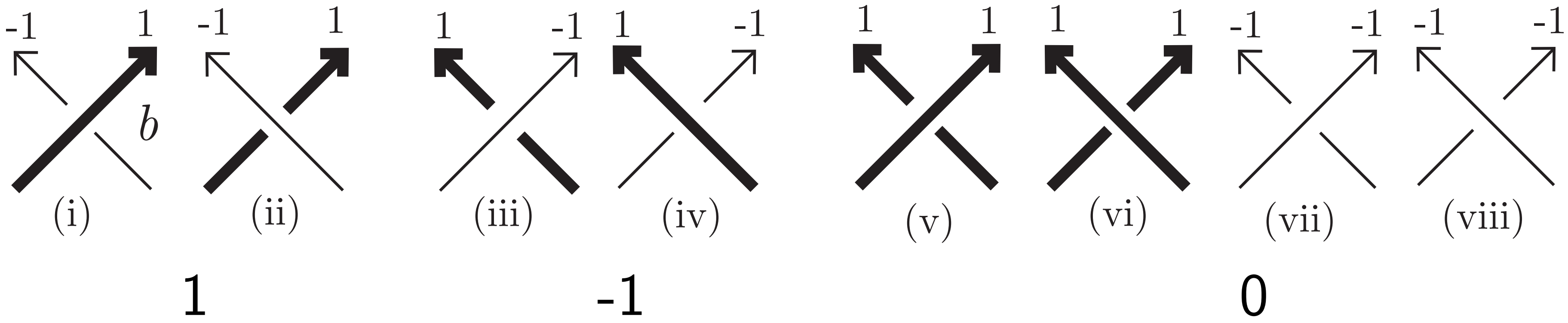
Weight map of a frilled index diagram

A weight map is admissible if the neighborhood of c is as in the following figure.



The nth partial writhe of a twisted link diagram

D : a twisted link diagram
$\tilde{D}_c = \tilde{d}_1 \cup \tilde{d}_2$: a frilled index diagram of a real crossing c of D
For a real crossing $b \in \tilde{d}_1 \cap \tilde{d}_2$, let $\tilde{\iota}(b)$ be as in follows :



The frilled index of c is define as follows

$$\widetilde{\mathrm{Ind}}(c) = \sum_{\sigma \in \mathcal{W}(c)} \sum_{b \in \tilde{d}_1 \cap \tilde{d}_2} \tilde{\iota}(b),$$

where $\mathcal{W}(c)$ is the set of admissible weight maps of $\tilde{D}_c$.

The nth partial writhe of a twisted link diagram

D: a twisted link diagram $\mathcal{R}(D)$: the set of real crossings of D
$\mathcal{R}_1(D) = \{c \in \mathcal{R}(D) |$ c: a self crossing, d_1, d_2: even$\}$,
$\mathcal{R}_2(D) = \{c \in \mathcal{R}(D) |$ c: a non self crossing, d_1, d_2: even$\}$,
$\mathcal{R}_3(D) = \{c \in \mathcal{R}(D) |$ c: a self crossing, d_i: even, d_j: odd $(i \neq j)\}$,
$\mathcal{R}_4(D) = \{c \in \mathcal{R}(D) |$ c: a non self crossing, d_i: even, d_j: odd $(i \neq j)\}$,
$\mathcal{R}_5(D) = \{c \in \mathcal{R}(D) |$ c: a self crossing, d_1, d_2: odd$\}$,
$\mathcal{R}_6(D) = \{c \in \mathcal{R}(D) |$ c: a non self crossing, d_1, d_2: odd$\}$,
where $D_c = d_1 \cup d_2$ is the index diagram for a real crossing, c of D.

For $k \in \{1, 2, 3, 4\}$ and $n \in \mathbb{Z}$, the nth partial writhe of a twisted link is defined as follows:

$$\tilde{J}_n^k(D) = \sum_{c \in \mathcal{R}_k(D), \widetilde{\mathrm{Ind}}(c) = n} \mathrm{sgn}(c)$$

Theorem

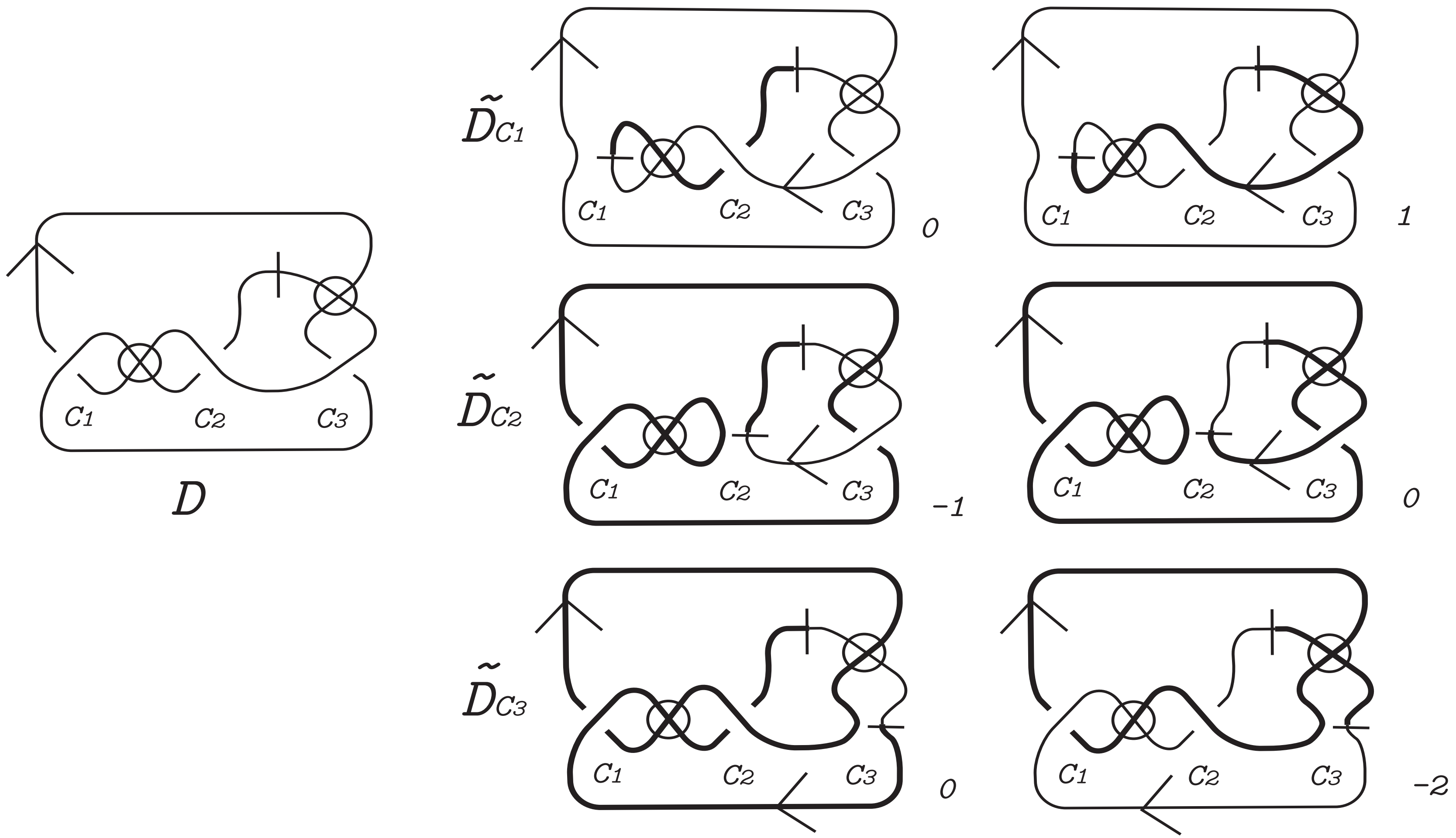
For $k \in \{5, 6\}$ and $n \in \{0, 1\}$, $\tilde{J}_n^k(D)$ is defined as follows

$$\tilde{J}_n^k(D) = \sum_{c \in \mathcal{R}_k(D), \widetilde{\mathrm{Ind}}(c) \equiv n \mod 2} \mathrm{sgn}(c)$$

Theorem

$\tilde{J}_n^k(D)$ is an invariant of twisted links for $k = 1, 2, 3, 4$ and $n \neq 0$ (or $k = 5, 6$ and $n \in \{0, 1\}$).

Example 1

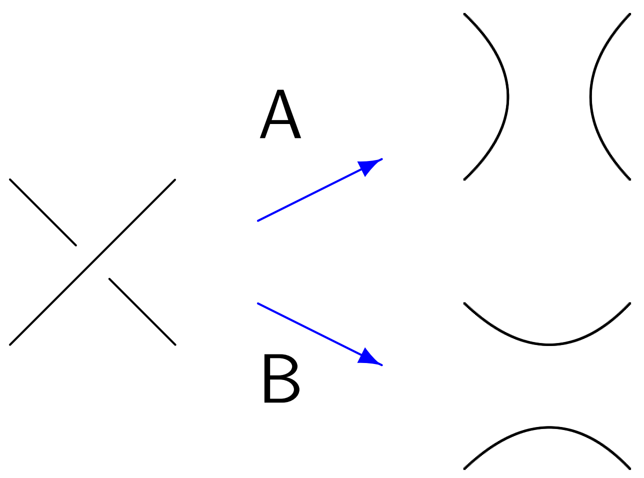


$$\tilde{J}^3_1(D) = 1,\ \tilde{J}^3_{-1}(D) = -1,\ \tilde{J}^3_{-2}(D) = 1$$

Twisted Jones polynomial

D: a twisted link diagram
S :a state S of D : a twisted link diagram which is obtained from D by applying A or B splices at all real crossings of D

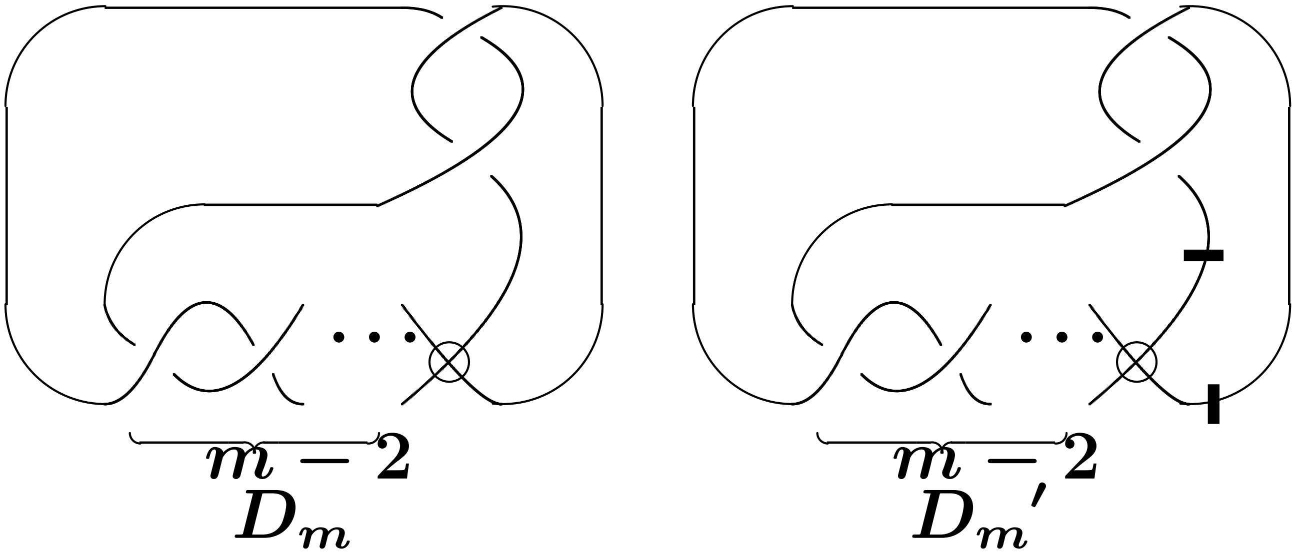
Splice



$$\langle D \rangle = \sum_S A^{\natural S}(-A^2 - A^{-2})^{\sharp S} M^{\sharp_o S}$$

where $\natural S$ is the number of A-splices minus that of B, $\natural S$ is the number of loops in S and $\sharp_o S$ is the number of loops with the odd number of bars

Example 2

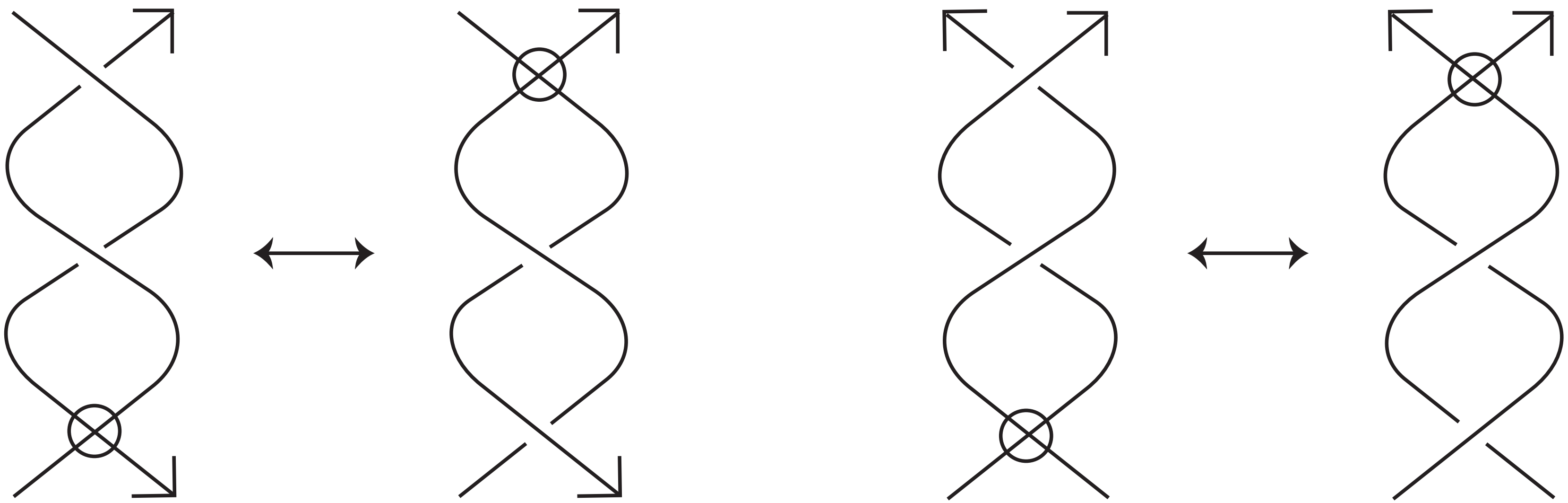


The twisted Jones polynomials of D_m and $D_m{}'$ are $-A^{-6}(A^4+A^{-4})-A^{-4m}(A^3-A^{-3})(A+A^{-1})$ (or $-A^{6}(A^4+A^{-4})+A^{-4m+12}(A^3-A^{-3})(A+A^{-1})$) if m is even (or odd). However we obtain

$\tilde{J}^1_1(D_m)=(-1)^m$, $\tilde{J}^1_{-1}(D_m)=(-1)^m$,

$\tilde{J}^5_{[1]}(D'_m)=(-1)^m\times 2$

Double Flype

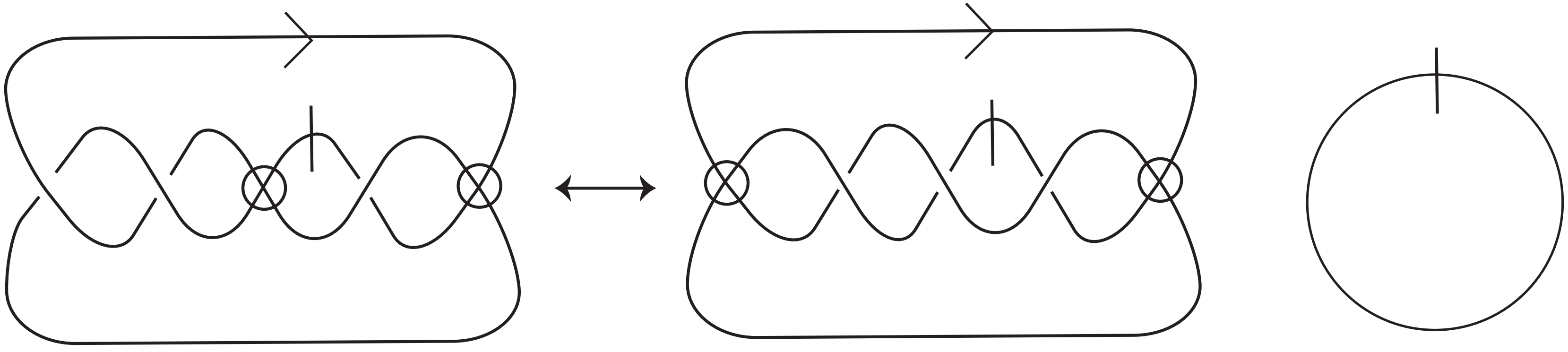


(i) parallel (ii)non parallel

Proposition

The partial writhe is invariant under parallel double flypes.

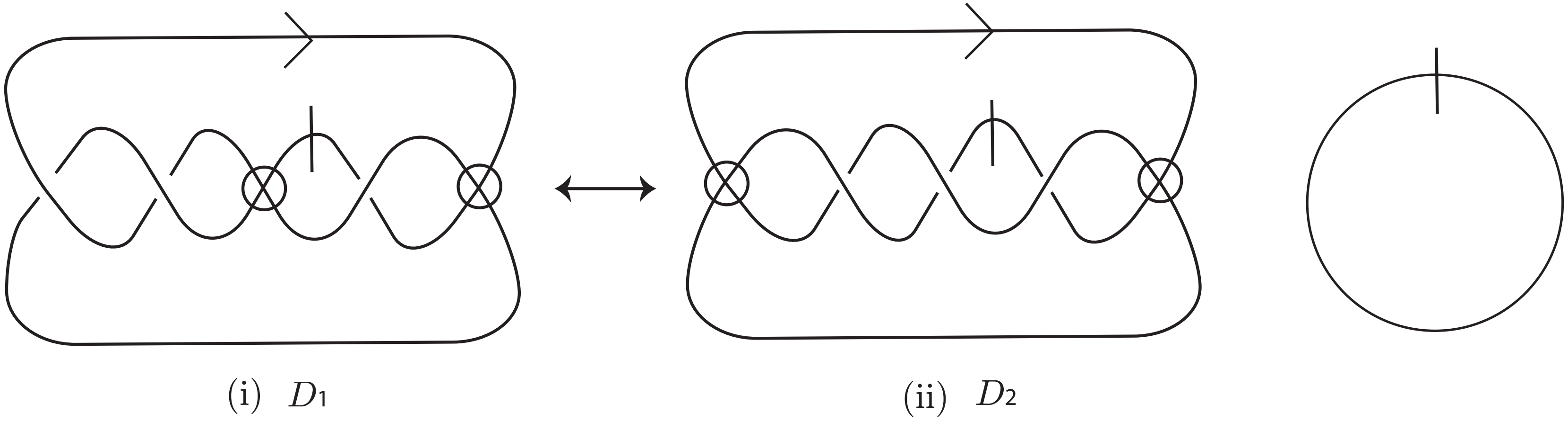
Example of diagrams related with a non parallel double flype



(i) D_1 (ii) D_2

$$\tilde{J}_2^3(D_1) = \tilde{J}^3_{-2}(D_1) = 1$$
$$\tilde{J}_i^j(D_2) = 0$$

Example of diagrams related with a non parallel double flype



$$\tilde{J}^3_2(D_1) = \tilde{J}^3_{-2}(D_1) = 1$$
$$\tilde{J}^j_i(D_2) = 0$$

THANK YOU.