Some numerical knot invariants through polynomial parametrization

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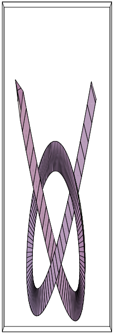
Polynomial knots

Definition

A long knot defined by an embedding of the form $t \to (f(t), g(t), h(t))$, where $f(t), g(t)$ and $h(t)$ are real polynomials, is called a polynomial knot.

It has been proved that each long knot is topologically equivalent to some polynomial knot.

Example



$$t \to (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$$

Degree of a polynomial knot

Definition

A polynomial knot defined by $t \to (f(t), g(t), h(t))$ is said to have degree d if $deg(f(t)) < deg(g(t)) < deg(h(t)) = d$.

It is easy to note that if a polynomial knot K has degree d, we can obtain polynomial knots of degree $d + k$ for each $k \geq 1$ which are topologically equivalent to K.

Minimal polynomial degree

Definition

A positive integer d is said to be the minimal degree for a knot K if there is a polynomial knot defined by $t \to (f(t), g(t), h(t))$ which is topologically equivalent to K with $deg(f(t)) < deg(g(t)) < deg(h(t))$ and $deg(h(t)) = d$ and no polynomial knot with degree less than d is equivalent to K.

Space of polynomial knots

Let us denote: $\mathcal{P}$: The set of all polynomial knots.
$\mathcal{P}_d$: The set of all polynomial knots of degree d.
The set $\mathcal{P}_d$ can be seen as an open subset of $\mathbb{R}^{3d}$ and hence has a nice subspace topology.
Also as $\mathcal{P} = \cup_d \mathcal{P}_d$ so can be given the inductive limit topology.
$\mathcal{P}_d$ and $\mathcal{P}$ are topological spaces.

Equivalence

Definition

Two polynomial knots Φ_0 and Φ_1 are said to be polynomially isotopic if there exists a one parameter family of polynomial knots $\{P_t | t \in [0, 1]\}$ such that $P_0 = \Phi_0$ and $P_1 = \Phi_1$

Being polynomially isotopic is an equivalence relation in $\mathcal{P}$ for which it is easy to note that the equivalence classes are nothing but the path components of the space $\mathcal{P}$. it was proved that:
if two polynomial knots are topologically equivalent as long knots then they are polynomially isotopic. Thus they will lie in the same path component of $\mathcal{P}$.

Equivalence

Two polynomial knots of different degree may represent equivalent long knots and the polynomial isotopy may pass through polynomial knots of various degrees. For the space P_d of polynomial knots of degree d, there is another equivalence defined as:

Definition

Two polynomial knots in P_d are said to be path equivalent if they belong to the same path component of P_d.

It can be proved easily that if two polynomial knots in P_d are path equivalent then they are topologically equivalent. Would like to explore if the converse is true or not?

Main questions

In connection with polynomial representation of knots, two main important questions are of interest namely:

- Question 1: Given a knot K what is the least degree d such that K has a polynomial representation in P_d?
- Question 2: Given a positive integer d what are the knots that can have a polynomial representation in P_d?

For both the questions only partial answers are known.

Note: the number of topologically distinct knots in P_d provides us a lower bound on the number of path components of P_d. Answer to each question helps in answering the other question.

Polynomial degree in relation with other invariants

- Result 1: If a knot K has a polynomial representation in degree d then *the minimal crossing number* $c(K)$ satisfies

$$c(K) \leq \frac{(d-2)(d-3)}{2}.$$

- Result 2: If K is a polynomial knot in degree d and *bridge number* $b(K)$ then

$$b(K) \leq \frac{(d-1)}{2}.$$

- Result 3: If K is a polynomial knot in degree d and *super-bridge number* $s(K)$ then

$$s(K) \leq \frac{(d+1)}{2}.$$

More bounds on the degree

The nature of crossing data also puts some condition on the degree of a polynomial knot. For instance, we have the following result for alternating knots.

Theorem

Suppose a polynomial knot has a regular projection $t \to (f(t), g(t))$ with n transversal double points. Suppose a polynomial $h(t)$ of degree d is such that the polynomial knot $t \to (f(t), g(t), h(t))$ is an alternating knot. Then $d \leq n + 2$.

Answer to Question 2

Question 2 has been addressed for $d \leq 5$ and the known theorems are:

Theorem

The only knot that can be represented as a polynomial knot of degree less than or equal to 4 is the trivial knot.

In fact for $d \leq 4$ there is a stronger result:

Theorem

The space P_d of all polynomial knots in degree d for $d \leq 4$ is path connected.

The spaces P_d for $d > 4$

The space P_d of all polynomial knots in degree $d > 4$ are not path connected. Estimating the number of path component in each space is an interesting question.

Information on other invariants

- Knot invariants are main tools to use knot theory anywhere.
- To use knot theory in a physical scenario we need to use the knot invariants which are dependent on the knot conformations.
- Some of such important knot invariants have been the crossing number, the bridge number and the unknotting number.
- Idea behind defining each of them is similar and is based of the following theme:

  Define a quantity as minimum for a conformation and minimize it over all conformations in a knot type.

Super knot invariants

- In place of defining a quantity as minimum if it is defined as maximum (it is exist in fact) for a conformation and minimize it over all knot conformations in a knot type results in a new knot invariant. Such knot invariants are called as **super invariants**.
- **Superbridge index** is one of the super invariant introduced by **N. Kuiper**.
- There are other super invariants such as the *super crossing number* and the *super unknotting number*, studied by Colin Adams and Others.

Super invariants

- $\phi' : S^1 \to \mathbb{R}^3$ represents a knot conformation $\mathcal{K}'$.
- $v : \mathbb{R}^3 \to \mathbb{R}$ be a linear function with $||v|| = 1$ and is such that the number of critical points of the function $v \circ \phi' : S^1 \to \mathbb{R}$ are finite.
- $\mathcal{A}$ be a set of such linear functions.
- The restriction of v to $\mathcal{K}'$ can be thought of as a projection of $\mathcal{K}'$ on the line L_v perpendicular to the plane $v(x, y, z) = 0$ and passing through the origin.
- $m_v(\mathcal{K}')$ be the number of local maxima [or minima] of $\mathcal{K}'$ along the line L_v. This is same as the number of local maxima [or minima] of the fuction $v \circ \phi'$.

Bridge Number & Super Bridge Number

1. The **bridge number** $b(\mathcal{K}')$ of a knot conformation $\mathcal{K}'$ is defined as, the minimum of $\{m_v(\mathcal{K}')\}$ where v takes all possible values from $\mathcal{A}$, i.e. Take an arbitrary line L_v. Calculate the number of local maxima [or minima] of $\mathcal{K}'$ along L_v. Then minimize it over all the possible directions L_v in $\mathbb{R}^3$. The resulting number is the bridge number of $\mathcal{K}'$.
2. The **super bridge number** $sb(\mathcal{K}')$ of a knot conformation $\mathcal{K}'$ is defined as, the maximun $\{m_v(\mathcal{K}')\}$ where v takes all possible values from $\mathcal{A}$, i.e. Take an arbitrary line L_v. Calculate the number of local maxima [or minima] of $\mathcal{K}'$ along L_v. Then maximize it over all the possible directions L_v in $\mathbb{R}^3$. The resulting number is the super bridge number of $\mathcal{K}'$.

Bridge Index & Super Bridge Index

1. The **bridge index** of a knot type $[\mathcal{K}]$ is defined as,
   $b[\mathcal{K}] := \min_{\mathcal{K}' \in [\mathcal{K}]} b(\mathcal{K}')$ i.e. Take an arbitrary conformation $\mathcal{K}'$ contained in the knot type $[\mathcal{K}]$. Calculate the bridge number of $\mathcal{K}'$. Then minimize it over all conformations contained in the knot type $[\mathcal{K}]$. The resulting number is the bridge index of $[\mathcal{K}]$.
2. A **super bridge index** of a knot type $[\mathcal{K}]$ is defined as,
   $sb[\mathcal{K}] := \min_{\mathcal{K}' \in [\mathcal{K}]} sb(\mathcal{K}')$ i.e. Take an arbitrary conformation $\mathcal{K}'$ contained in the knot type $[\mathcal{K}]$. Calculate the super bridge number of $\mathcal{K}'$. Then minimize it over all conformations contained in the knot type $[\mathcal{K}]$. The resulting number is the super bridge index of $[\mathcal{K}]$.

relation between Bridge index and Super bridge index

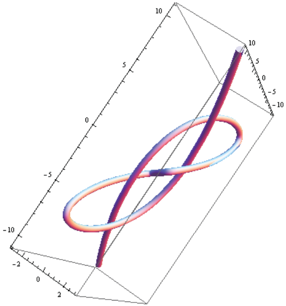
- Clearly $b(K) \leq Sb(K)$.
- Kuiper proved that $b(K) < Sb(K)$.
- Colin Adams and their group has proved that $Sb(K) \leq 2b(K)$.
- Kuiper proved in his paper that a torus knot of type (p, q) with $p < q$ has super bridge index $\min\{2p, q\}$

Bridge Index & Super Bridge Index with polynomial degree:

Looking at a polynomial knot we immediately know its bridge number and the super bridge number and hence get an upper bound for the bridge index and the super bridge index. For instance the polynomial trefoil knot shown in the begining has bridge number 2 and super bridge number 3 and these happen to be the bridge index and the super bridge index as well.

Bridge Index & Super Bridge Index

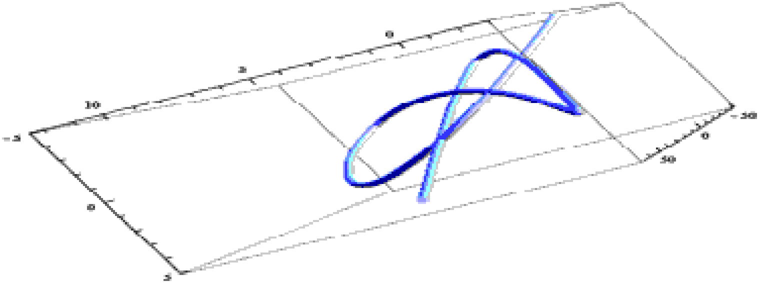
Here is a polynomial figure eight knot with bridge number 2 and super bridge number 4:



$$t \mapsto (t(t-2)(t+2), (t-2.1)(t+2.1)t^3, -12.8064t + 22.4679t^3 - 8.90928t^5 + t^7).$$

Bridge Index & Super Bridge Index

However the figure eight knot in degree 6 shown below has bridge number 2 and super bridge number 3:



$$t \to (-t^4 + 2.279283653 * t^3 + 5 * t^2 - 8.63068748 * t + .35140383, t^5 - 5 * t^3 + 4 * t, (t + 2.06) * (t + 1.916737670) * (t + .2122155248) * (t - 1.379221313) * (t - 2.05) * (t + 10))$$

3 superbridge knots

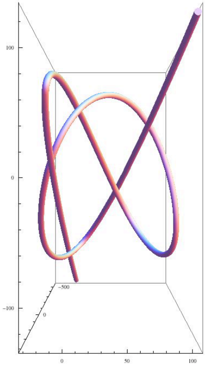
- From Kuiper's result it can be shown that for each even positive integer $n \geq 4$ there are infinitely many knots with super bridge index n.
- it has been conjectured that for every integer $n \geq 4$ there are infinitely many knots with super bridge index n.
- For $n = 3$ the story is very different.
- Jin and Geon proved that there are only finitely many knots with super bridge index 3.
- They showed that knots other than $3_1, 4_1, 5_2, 6_1, 6_2, 6_3, 7_2, 7_3, 7_4, 8_4, 8_7$ and 8_9 can not have super bridge index 3.

3 superbridge knots

- From this list 3_1 and 4_1 are confirmed to have super bridge index 3, it is evident from their polynomial parametrizations as well!!.
- Other than 5_2, 6_1, 6_2 and 6_3 all other knots are proved to have super bridge index 4.
- Jin and Geon have conjectured that 3_1 and 4_1 are the only two knots who have super bridge index 3.

The 5_2 knot

- It is a 2 bridge knot so its super bridge must be either 3 or 4.
- we have a polynomial representation of 5_2 in deg 7 shown below:



$$5_2\colon\ t \to (t^4 + 2t^3 - 21t^2 - 24t + 96, t^5 - 22t^3 + 95, .340592t^7 - .092293t^6 - 8.35577t^5 + 1.89418t^4 + 43.6898t^3 - 7.41241t^2 + 10.2056t)$$

The 5_2 knot

- We still do not know if 7 is the least polynomial degree to represent this knot.
- May be this can give us a proof that 5_2 is a 4 super bridge knot.
- Similarly one can try for all 6 crossing knots.

Unknotting number

Definition

Given a knot diagram D_K the least number of crossing changes required to convert it into a knot diagram of an unknot is called the *unknotting number of that diagram* denoted by $u(D_K)$.

Definition

The *unknotting number of a knot* K is defined as minimal number of crossing changes required among all possible diagrams of$K$ to be able to convert it into the unknot.

It is a knot invariant and is denoted by u(K).

Unknotting number using Polynomial knots

Definition

Two polynomial knots ϕ_0 and ϕ_1 are said to be *p-isoptopic* if there exists a one parameter family $\{p_s, 0 \leq s \leq 1\}$ of polynomial knots (embeddings) such that $p_0 = \phi_0$ and $p_1 = \phi_1$. This family $\{p_s, 0 \leq s \leq 1\}$ is called a *p-isotopy* between ϕ_0 and ϕ_1.

1. Given a polynomial knot $t \mapsto (f(t), g(t), h(t)$, up to p-isotopy we can always assume that the degree of $h(t)$ is odd.
2. Every polynomial knot is p-isotopic to some polynomial knot defined as $\phi(t) = (f(t), g(t), h(t))$, the projection of ϕ into xy plane is a regular projection. A polynomial knot with this property will be referred as a *good polynomial knot*.
3. Given a good polynomial knot $t \mapsto (f(t), g(t), h(t))$ there is a naturally associated knot diagram drawn on xy plane.

Unknotting number using Polynomial knots

Definition

Two polynomial knots are said to be *strongly p-regular homotopic* if there exists a one parameter family $\{p_s = (f_s, g_s, h_s), 0 \leq s \leq R\}$ of polynomial maps from $\mathbb{R}$ to $\mathbb{R}^3$ such that $p_0 = \phi_0$ and $p_R = \phi_1$ and for each s, the map $t \mapsto (f_s(t), g_s(t))$ have the same crossing data, i.e., the pairs (t_1, t_2) for which $f_s(t_1) = f_s(t_2)$ and $g_s(t_1) = g_s(t_2)$ is same for all $s \in [0, R]$.

Thus if two polynomial knots are strongly p-regular homotopic then their diagrams differ in terms of change in the nature of crossings, i.e., the diagram of second polynomial knot can be obtained by changing some over crossings in first diagram into the under crossings or vice-versa.

Realizing Crossing change in Polynomial knots

1. Let $(f(t), g(t), h(t))$ be a good polynomial knot. Let (s_i, t_i) be the parameters where there is a crossing, i.e., $f(s_i) = f(t_i)$ and $g(s_i) = g(t_i)$.
2. Let $m_i(h) = \frac{|h(s_i) - h(t_i)|}{|s_i - t_i|}$. Each $m_i(h)$ is a positive real number. Given a polynomial knot $(f(t), g(t), h(t))$ we can compare $m_i(h)$ and $m_j(h)$ for each $i \neq j$.
3. Suppose $m_{i_1}(h) < m_{i_2}(h) < \ldots < m_{i_n}(h)$. Then $\{i_1, i_2, \ldots, i_n\}$ defines an order on the set $\{1, 2, \ldots, n\}$. In the next proposition we show that it is possible to attain each order among $m_i(h)$s by choosing a suitable good polynomial representation of a knot diagram.

Realizing Ceossing change in Polynomial knots

Theorem

Let D be a knot diagram of a knot K with n crossings. Let σ be an order on $\{1, 2, \ldots, n\}$. Then there exists a good polynomial knot $t \mapsto (f(t), g(t), h_\sigma(t))$ representing the diagram D with crossings at parametric pairs of values (s_i, t_i), $i = 1, 2, \ldots, n$ for which $m_i(h_\sigma) = \frac{|h_\sigma(s_i) - h_\sigma(t_i)|}{|s_i - t_i|}$ satisfy the order σ.

Realizing Crossing change in Polynomial knots

Theorem

Every polynomial knot is strongly p-regular homotopic to a polynomial unknot.

Proof. Let $\phi(t) = (f(t), g(t), h(t))$, be a polynomial knot such that the map $t \mapsto (f(t), g(t)$ is an immersion and the $deg(h(t))$ is odd. For each $s \in \mathbb{R}$ consider a family of maps $\Phi_s : \mathbb{R} \hookrightarrow \mathbb{R}^3$ as
$\Phi_s(t) = (f(t) + s, g(t) + s, h(t) + s^2 t)$. The proposition now follows from the following two claims.

Claim 1. For each $s \in \mathbb{R}$ the map $\phi_s(t) = (f(t) + s, g(t) + s, h(t) + s^2 t)$ is an immersion and the map $t \mapsto (f(t) + s, g(t) + s)$ have the same crossing data as that of $t \mapsto (f(t), g(t))$.

Claim 2. There exists some real number R such that for $s \geq R$ the maps $\phi_s : \mathbb{R} \hookrightarrow \mathbb{R}^3$ represent the trivial knot.

Crossing Change

1. The proof of the previous theorem demonstrates that we have a continuous map $\Phi : \mathbb{R} \times [0, R] \longrightarrow \mathbb{R}^3$ such that $\Phi(t, 0) = (f(t), g(t), h(t))$, the given knot and $\Phi(t, R) = (f(t) + R, g(t) + R, h(t) + R^2 t)$ a trivial knot and for each $s \in [0, R]$ $\Phi(t, s) = (f(t) + s, g(t) + s, h(t) + s^2 t)$ is an immersion. The values of s for which $\Phi(t, s) = (f(t) + s, g(t) + s, h(t) + s^2 t)$ fails to be an embedding are called *singular values*.
2. If the given polynomial knot is non trivial then from the above proposition it follows that we can obtain a polynomial unknot with the same crossing data whose diagram is obtained by switching some of the crossings of the given knot from over crossing to under crossing or vice versa. As it is a continuous deformation, for some finite number of values of $s \in [0, R]$ the maps $\Phi(-, s) : \mathbb{R} \longrightarrow \mathbb{R}^3$ must be singular knots,i.e., must have double points.

Singularity Index

Definition

Let $\phi(t) = (f(t), g(t), h(t))$, be a polynomial knot with say n crossings. Let R_σ be the least positive real number such that the map $\Phi_s : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by $t \mapsto (f(t) + s, g(t) + s, h(t) + s^2 t)$ represents a trivial knot for $s = R + \epsilon$, for $\epsilon > 0$. Then the minimum number of singular values, i.e., the values of $s \in [0, R_\sigma]$ for which the map Φ_s is a singular knot is defined as the *singularity index* of ϕ denoted by SI_ϕ.

Definition

The minimum value of all SI_phi, minimum taken over all ϕ that represent a knot diagram D is defined as the *singularity index* of the diagram D denoted by $SI(D)$.

Singularity Index

Definition

The minimum value of all $SI(D)$, minimum taken over all knot diagrams that represent a knot K is defined as the *singularity index* of the knot K and is denoted by $SI(K)$.

1. Singularity index $SI(K)$ of a knot is a knot invariant.
2. Singularity index of the unknot is zero.
3. for any nontrivial knot K, $SI(K) \geq 1$.

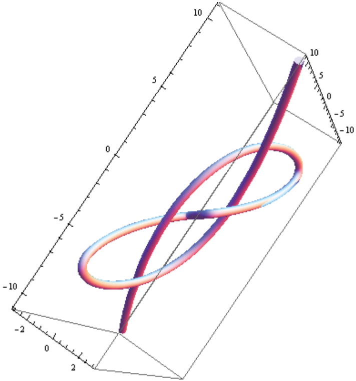
Singularity Index and Unknotting number

Theorem

Singularity index of a knot diagram is less than or equal to its unknotting number.

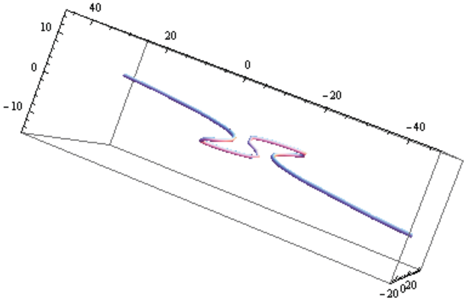
Computing Singularity Index

Consider a polynomial representation of *figure eight knot* given by $t \mapsto (t(t-2)(t+2), (t-2.1)(t+2.1)t^3, -12.8064t + 22.4679t^3 - 8.90928t^5 + t^7)$. It has a knot diagram as shown below.



Computing

In the deformation $\phi_s = t \mapsto (t(t-2)(t+2) + s, (t-2.1)(t+2.1)t^3 + s, -12.8064t + 22.4679t^3 - 8.90928t^5 + t^7 + s^2 * t)$ we see that for $s > 1.48$ each ϕ_s is an unknot (Figure 4) and there is only one singular knot corresponding to $s = 1.48$. Thus the singularity index of this diagram is 1 which is same as its unknotting number.



Thank you