On Heegaard splittings of knot exteriors with tunnel number degenerations

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1. Introduction

K : a knot in S^3

$E(K)$: the exterior

$g(E(K))$: the Heegaard genus of $E(K)$

$t(K)$: the tunnel number of K

Then, by the definition, we have :

$$t(K) = g(E(K)) - 1$$

i.e.

$E(K)$ has an unknotting tunnel system consisting of $t(K)$ arcs as in the following figure.

$E(K) = H_1 \cup H_2$: genus 4 Heegaard splitting

$N(K)$ an unknotting tunnel system

$g(E(K)) = 4$ and $t(K) = 3$

Example

The following knot is 8_{16}, and the tunnel number is 2.

The system of two red arcs is an u. t. s.

2. History

K_1, K_2 : two knots in S^3

$K_1 \# K_2$: the connected sum of K_1, K_2

There has been the following problem on the additivity of tunnel numbers.

Problem

Are the tunnel numbers of knots additive under connected sum?

i.e.

$$t(K_1 \# K_2) = t(K_1) + t(K_2) \ ?$$

The first result was :

Theorem 1 ('82 Norwood)

Every tunnel number one knot is prime.

i.e.

$$t(K_1 \# K_2) > 1 \text{ for any non-trivial knots } K_1 \text{ and } K_2$$

The second resut was :

Theorem 2 ('92 M.)

Let K_n be the knot as in the following figure ($n \neq 0$).

Then $t(K_n) = 2$ and $t(K_n \# K) = 2$ for any 2-bridge knot K.

Thus we have : $t(K_n \# K) = 2 < 3 = t(K_n) + t(K)$.

K_n $2n+1$

By the above result,

We can ask what kind of types such knots are.

On this question,

we showed the following characterization theorem :

Theorem 3 ('94 M.)

Suppose $t(K_1) = 1, t(K_2) = 2$ and $t(K_1 \# K_2) = 2$. Then,

K_1 is a 2-bridge knot and

K_2 is a knot with a 2-string essential free tangle decomposition such that at least one of the two tangles has an unknotted component.

Example

2-string essential free tangles.

with an unknotted component

with no unknotted component

Example

Knots with a 2-string essential free tangle decomposition.

Example

8_{16} has a 2-string essential free tangle decomposition as follows:

go-stop reset

Recently, Nogueira got the following :

Theorem 4 (2012, Nogueira)

There are infinitely many knots K_1 such that $t(K_1) = 3$ and $t(K_1 \# K_2) = 3$ for any 3-bridge knot K_2.

Thus we have : $t(K_1 \# K_2) = 3 < 5 = t(K_1) + t(K_2)$.

Theorem 1 : " $t = 1 \Rightarrow$ prime " $\cdots$ 1982

Theorem 2 : " 2+1 = 2 " $\cdots\cdots\cdots$ 1992

Theorem 4 : " 3+2 = 3 " $\cdots\cdots\cdots$ 2012

Theorem ? : " 4+3 = 4 " $\cdots\cdots\cdots$????

Theorem ?? : " $(n+1)+n = n+1$ " $\cdots$????

3. Results

In the present talk,

we consider genus three Heegaard splittings of the knot exterior $E(K_1 \# K_2)$ for the two knots in Theorem 3, and classify the unknotting tunnel systems up to homeomorphism.

First we get :

Theorem 5

Let K_1 and K_2 be two knots in Theorem 3.

Then any genus three Heegaard splitting of $E(K_1 \# K_2)$ is strongly irreducible.

Sketch proof of Theorem 5

Suppose a genus three Heegaard splitting (H_1, H_2) of $E(K_1\#K_2)$ is weakly reducible.

Then, by taking two meridian disks $D_1 \in H_1$ and $D_2 \in H_2$ with $\partial D_1 \cap \partial D_2 = \emptyset$, we have a two genus two Heegaard splittings (V_1, V_2) and (W_1, W_2).

Then $V_2 \cap W_1 = T$ is a separating essential torus in $E(K_1\#K_2)$, and not a swallow-follow torus in $E(K_1\#K_2)$.

However, $E(K_1\#K_2)$ contains only swallow-follow torus.

This contradiction completes the proof.

Next, concerning the homeomorphism classes of those Heegaard splittings, we get :

Theorem 6

Let K_1 and K_2 be two knots in Theorem 3.
Then $E(K_1 \# K_2)$ contains at most four genus three Heegaard splittings up to homeomorphism.

Sketch proof of Theorem 6

Let (H_1, H_2) be a genus three Heegaard spliting of $E(K_1 \# K_2)$. Then we can see that this splitting is constructed by a 2-bridge decomposition of $E(K_1)$ and a 2-string essential free tangle decomposition of $E(K_2)$.

Then, by the uniqueness of those decompositions, and by $2 \times 2 = 4$, we have at most four genus three Heegaard splittings up to homeomorphism.

2-bridge decomposition

2-string tangle decomposition

To give a complete classification of those four genus three Heegaard splittings, we assume :

K_1 is a 2-bridge knot $S(\alpha, \beta)$ (Schubert's notation).

K_2 has a 2-string essential free tangle decomposition

$$(S^3, K_2) = (C_1, t_1 \cup s_1) \cup (C_2, t_2 \cup s_2) \quad \text{such that}$$

C_1 contains an unknotted component, i.e., t_1 or s_1.

To state the classification theorem, we put the following cases :

Case 1: C_2 contains no unknotted component.

Case 2: C_2 contains an unkontted component, i.e., t_2 or s_2.

In Case2, we have the following two subcases :

Case 2a: there is a homeomorphism exchanging the two tangles $(C_1, t_1 \cup s_1)$ and $(C_2, t_2 \cup s_2)$.

Case 2b: there is no homeomorphism exchanging the two tangles $(C_1, t_1 \cup s_1)$ and $(C_2, t_2 \cup s_2)$.

Then we get :

Theorem 7

Let K_1 and K_2 be two knots in Theorem 3.

Then we have the following complete classification of genus three Heegaard splittings of $E(K_1 \# K_2)$ up to homeomorphism, where n is the number of homeomorphism classes.

$$\text{Case 1} \quad \begin{cases} n = 1 & \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\ n = 2 & \text{if } \beta \not\equiv \pm 1 \pmod{\alpha} \end{cases}$$

$$\text{Case 2a} \begin{cases} n = 1 & \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\ n = 2 & \text{if } \beta \not\equiv \pm 1 \pmod{\alpha} \end{cases}$$

$$\text{Case 2b} \begin{cases} n = 2 & \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\ n = 4 & \text{if } \beta \not\equiv \pm 1 \pmod{\alpha} \end{cases}$$

Hereafter, we show the unknotting tunnel systems of $K_1 \# K_2$ corresponding to those Heegaard splittings in the following cases of Theorem 7.

Case 1 $\beta \not\equiv \pm 1 \pmod{\alpha}$, i.e. $n = 2$

Case 2b $\beta \not\equiv \pm 1 \pmod{\alpha}$, i.e. $n = 4$

We note that $\beta \not\equiv \pm 1 \pmod{\alpha}$ if and only if
there is no homeomorphism exchanging the upper tunnel and the lower tunnel of the given 2-bridge knot.

K_1 : a 2-bridge knot $S(23,7)$ i.e., $\beta \not\equiv \pm 1 \pmod{\alpha}$

K_1

K_2 : a knot with a free tangle decomposition in Case 1.

K_2

K_3 : a knot with a free tangle decomposition in Case 2b.

K_3

Case 1

$n = 2$

$K_1 \# K_2$

Case 2b

$n = 4$

$K_1 \# K_3$

Unknotting

tunnel

sytem of

$K_1 \# K_2$

go-stop reset