Knot Theory for spatial graphs

# [Lecture 3]
# Spatial graphs with degree one vertices attaching to a surface

cf. A. Kawauchi, Spatial graphs attaching to a surface,
in preparation.

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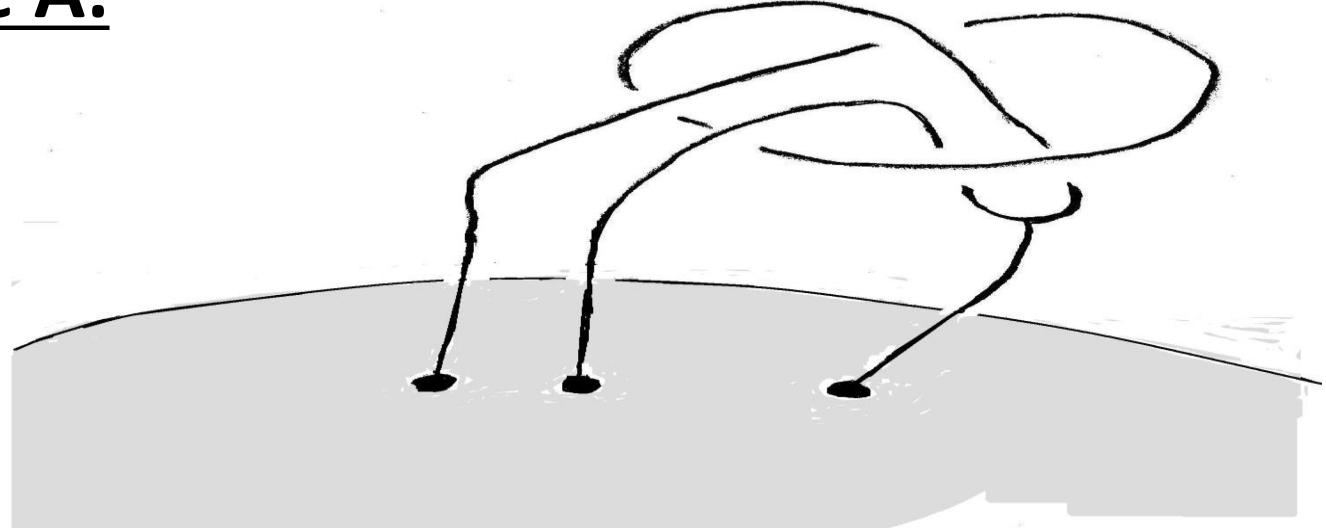
# <u>Example A:</u>



***<u>Question</u>. In what sense , this object is***

***"knotted" or "unknotted" ?***

**In this talk, the answer will be "β-unknotted" but "knotted", "γ-knotted" and "Γ-knotted" under some definitions introduced from now.**

# <u>Example B:</u> Proteins attached to a cell surface

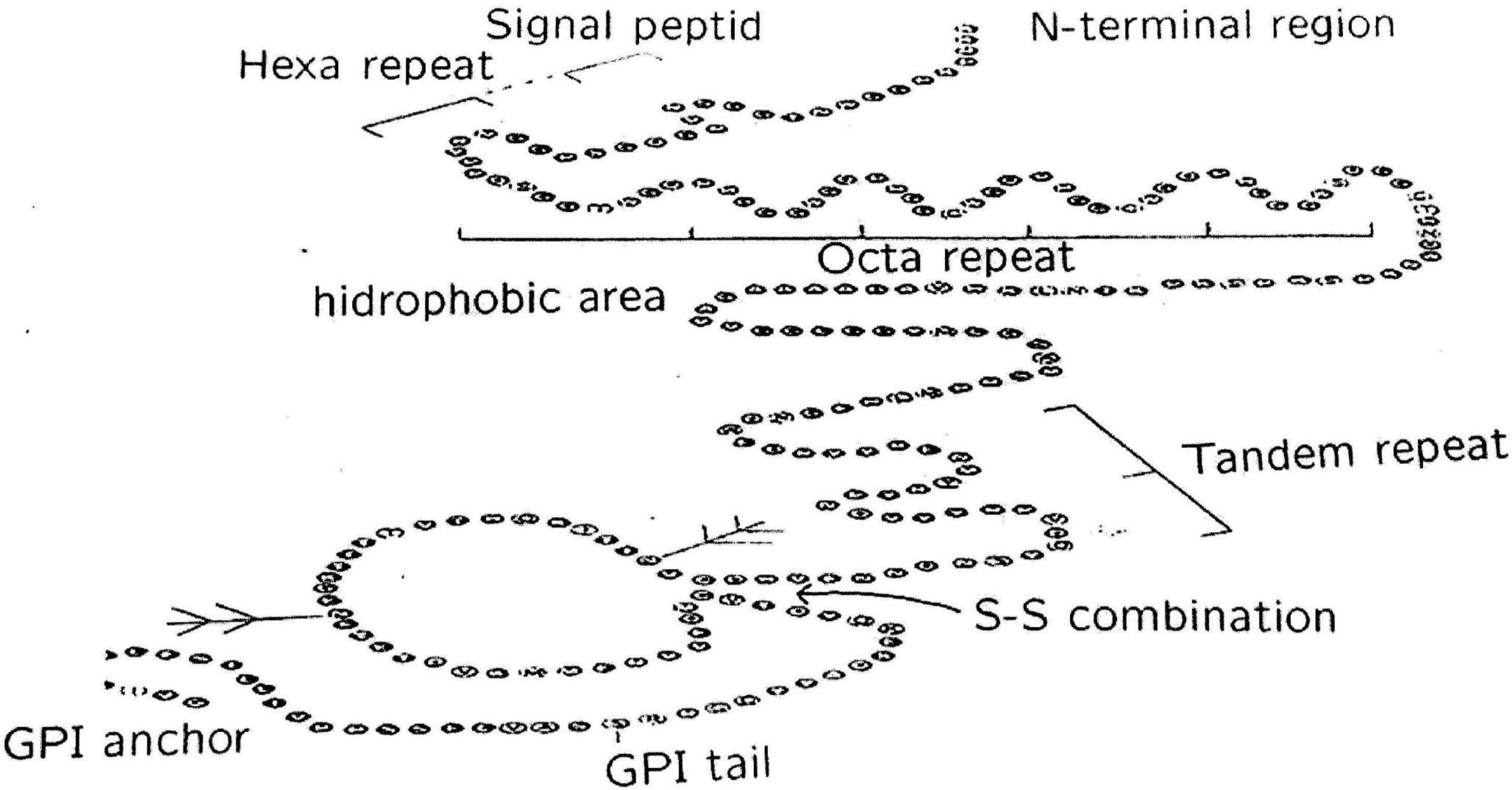
## Some points of S. B. Prusiner's theory are:

(1) By losing the N-terminal region, Prion precursor protein changes into Cellular PrP ($PrP^c$) or Scrapie PrP ($PrP^{SC}$), and α-helices change into β-sheets.
(2) The conformations of $PrP^c$ and $PrP^{SC}$ may differ although the linear structures are the same.
(3) There is one S-S combination.


● Z. Huang et al., Proposed three-dimensional Structure for the cellular prion protein, Proc. Natl. Acad. Sci. USA, 91(1994), 7139-7143.

● K. Basler et al., Scrapie and cellular PrP isoforms are encoded by the same chromosomal gene, Cell 46(1986), 417-428.

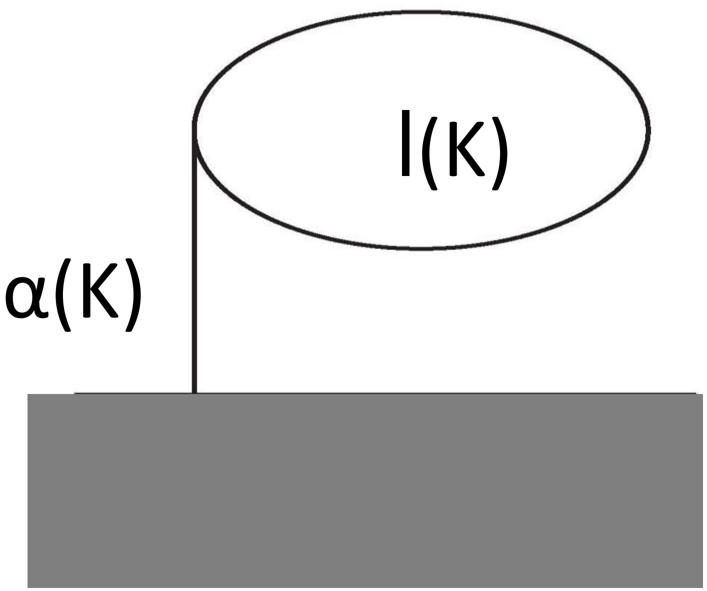
# Prion Precursor Protein

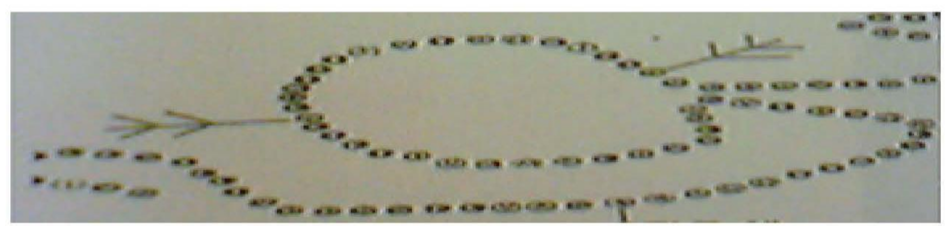


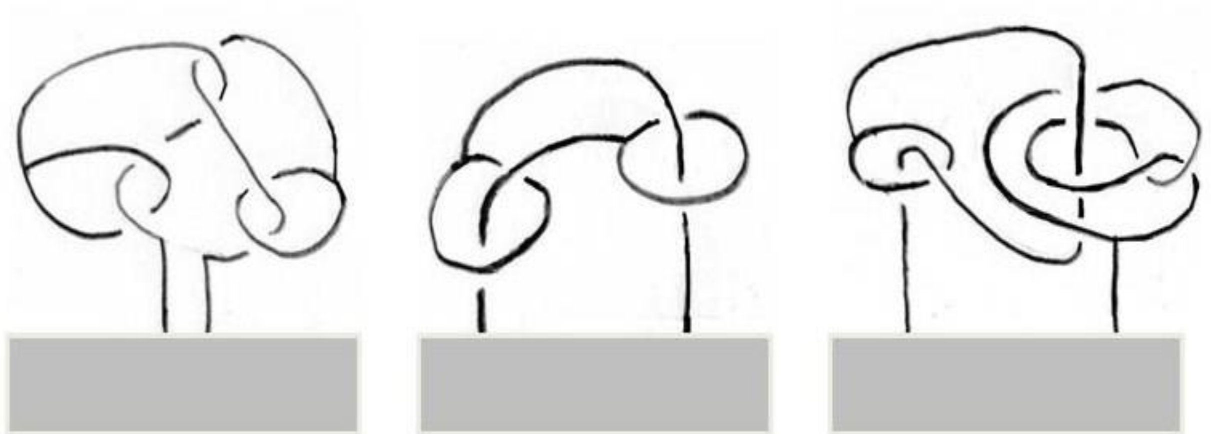

From:

K. Yamanouchi & J. Tateishi Editors, Slow Virus Infection and Prion (in Japanese), Kindaishuppan Co. Ltd. (1995)

**Definition.** A **prion-string** is a spatial graph $K = l(K) \cup \alpha(K)$ in the upper half space $\mathbb{H}^3$ consisting of **S-S loop** $l(K)$ and **GPI-tail** $\alpha(K)$ joining the S-S vertex in $l(K)$ with the **GPI-anchor** in $\partial\mathbb{H}^3$.






# Topological models of prion-proteins

（cf.　[J. Math. System Sci.  2012]）

**[J. Math. System Sci. 2012]**
**A. Kawauchi and K. Yoshida, Topology of prion proteins, Journal of Mathematics and System Science 2(2012), 237-248.**

# <u>Example C</u>: A string-shaped virus

A virus of EBOLA haemorrhagic fever

http://www.scumdoctor.com/Japanese/disease-prevention/infectious-diseases/virus/ebola/Pictures-Of-The-Effects-Of-Ebola.html

## 3.1. A spatial graph attached to a surface

**Let $\Gamma$ be a finite graph, and $v_1(\Gamma)$ the set of degree one vertices. Assume $|v_1(\Gamma)| \geqq 2$.**

**Let F be a compact surface in $R^3$.**

**Definition.**

**A spatial graph on F of $\Gamma$ is the image G of an embedding $f: \Gamma \to R^3$ such that**

**(1) G meets F with $G \cap F = f(v_1(\Gamma)) = v_1(G)$,**

**(2) $G - v_1(G)$ is contained in one component of $R^3 - F$,**

**(3) $\exists$ a homeomorphism $h: R^3 \to R^3$ such that $h(G \cup F)$ is a polyhedron.**

- F does not need $\partial F = \varphi$.
- Though Γ, G or F may be disconnected, but assume that $|F_c \cap v_1(G)| \geqq 2$ for ∀ comp. $F_c$ of F.
- Ignore the degree 2 vertices in G.

**Definition.** A spatial graph G on F is <u>equivalent</u> to a spatial graph G' on F' if ∃ an orientation-preserving homeomorphism h: $R^3 \rightarrow R^3$ such that

$$h(F \cup G) = F' \cup G'.$$

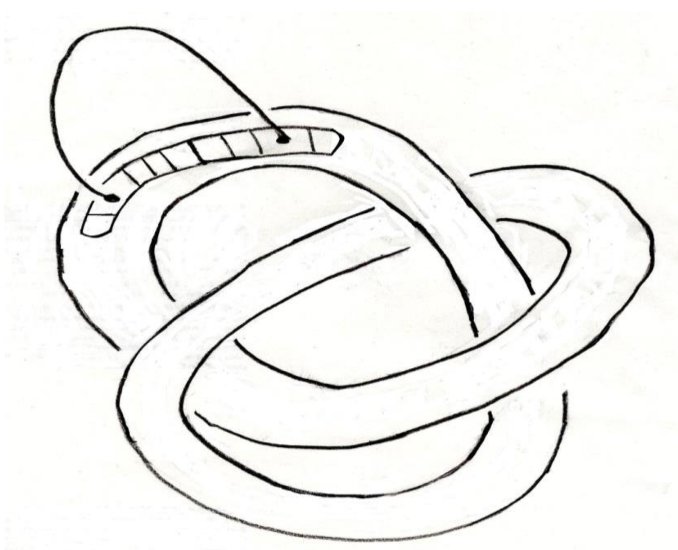
Let [G] be the class of spatial graphs G' on F' which are equivalent to G on F.

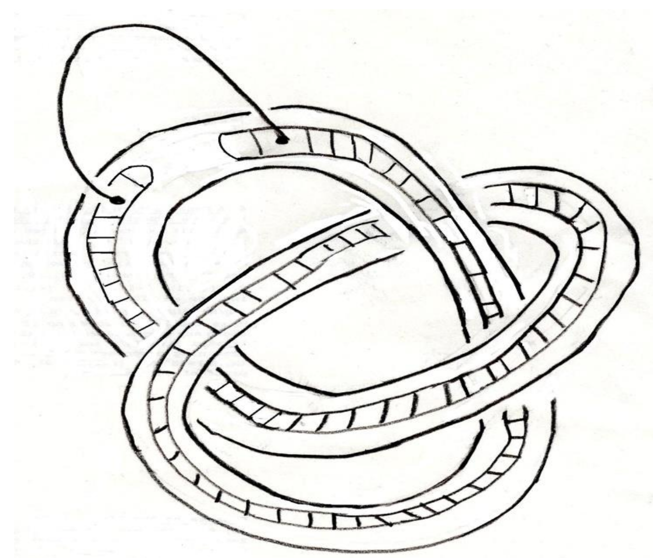
# 3.2. An unknotted graph on a surface and the induced unknotting number

**Definition.** G on F is unknotted if ∃ a 2-cell Δ' in ∀ comp. F' of F such that the union Δ of all Δ' contains $v_1(G)$ and the *shrinked spatial graph* G^ with $v_1(G\hat{})=\phi$ (i.e. a spatial graph obtained from G by shrinking ∀Δ' into a point) is unknotted in $R^3$.

__Note__. If $\forall$ F' =$S^2$ or a 2-cell, then [G^ ] does not depend on a choice of Δ.

However, in a genral F, [G^ ] depends on a choice of Δ, although the ***shrinked graph*** Γ^ with $v_1$(Γ^ )= $\phi$ associated with F is uniquely defined.





**Because ∀ G^ is a spatial graph of the same graph Γ^, we have:**

**<u>Lemma</u>. For ∀ given graph Γ and ∀ given F in $R^3$, ∃ only finitely many unknotted graphs G of Γ on F up to equivalences.**

**Let O = {unknotted graphs of Γ^}.**

**Definition.**

**The unknotting number u(G) of a spatial graph G of Γ on F is the distance from the set {G^} to O by crossing changes on edges attaching to a base:**

$$u(G) = \rho(\{G^\wedge\}, O).$$

## 3.3. A β-unknotted graph on a surface and the induced unknotting number

**Definition.** G on F is β-unknotted if ∃ a 2-cell Δ' in ∀ component F' of F such that the union Δ of all Δ' contains $v_1(G)$ and the shrinked spatial graph G^ with $v_1(G\hat{})=\phi$ is β-unknotted in $R^3$.

unknotted ⇒ β-unknotted

**Let $O_\beta$ = {β-unknotted graphs of Γ^}.**

**Definition.**

**The β-unknotting number $u_\beta(G)$ of a spatial graph G of Γ on F is the distance from the set {G^} to $O_\beta$ by crossing changes on edges attaching to a base:** $$u_\beta(G) = \rho(\{G\hat{}\}, O_\beta).$$

# 3.4. A γ-unknotted graph on a surface and the induced unknotting number

**Definition.** G on F is γ-unknotted if ∃ a 2-cell Δ' in ∀ component F' of F such that the union Δ of all Δ' contains $v_1(G)$ and the shrinked spatial graph G^ with $v_1(G^\wedge)=\phi$ is γ-unknotted in $R^3$.

γ-unknotted ⇒ unknotted ⇒ β-unknotted

**Given G, let**

$$\{D_{G^\wedge,\gamma}\} = \{(D;T) \in [D_{G^\wedge}] \mid c(D;T) = c_\gamma(G^\wedge),\ \forall G^\wedge\}.$$

**Definition.**

**The γ-unknotting number $u_\gamma(G)$ of a spatial graph G of Γ on F is the distance from $\{D_{G^\wedge,\gamma}\}$ to O by crossing changes on edges attaching to a base:**

$$u_\gamma(G) = \rho(\{D_{G^\wedge,\gamma}\}, O).$$

**Note. G on F is γ-unknotted ⇔ $u_\gamma(G) = 0$.**

# 3.5. Γ-unknotted graph on a surface and the induced unknotting numbers

**Definition.** G on F is Γ-unknotted if ∃ a 2-cell Δ' in ∀ component F' of F such that the union Δ of all Δ' contains $v_1(G)$ and the shrinked spatial graph G^ with $v_1(G\^{})=\phi$ obtained from G by shrinking ∀Δ' into a point is Γ^-unknotted in $R^3$.

Γ-unknotted⇒γ-unknotted⇒unknotted

⇒ β-unknotted

**Let $O_{\Gamma\wedge}$={Γ^-unknotted graphs}. Then $O_\beta \supset O \supset O_{\Gamma\wedge}$.**

**<u>Definition.</u>**

**The Γ-<u>unknotting number</u> $u^\Gamma(G)$ of G on F is the distance from the set {G^} to $O_{\Gamma\wedge}$ by crossing changes on edges attaching to a base:**

**$u^\Gamma(G) = \rho(\{G\wedge\}, O_{\Gamma\wedge})$**

**The (γ,Γ)-<u>unknotting number</u> $u^\Gamma_\gamma(G)$ of G on F is the distance from $\{D_{G\wedge,\gamma}\}$ to $O_\Gamma$ by crossing changes on edges attaching to a base: $u^G_\gamma(G) = \rho(\{D_{G\wedge,\gamma}\}, O_{\Gamma\wedge})$.**

## 3.6. Properties on the unknotting numbers

**Theorem 3.6.1.** The topological invariants

$$u_\beta(G),\ u(G),\ u^\Gamma(G),\ u_\gamma(G),\ u^\Gamma_\gamma(G)$$

of ∀ spatial graph G of ∀ graph Γ on ∀ surface F satisfy the following inequalities :

$$u_\beta(G) \leqq u(G) \leqq \{u_\gamma(G), u^\Gamma(G)\} \leqq u^\Gamma_\gamma(G),$$

and are distinct for some graphs G of some Γ on $F=S^2$.

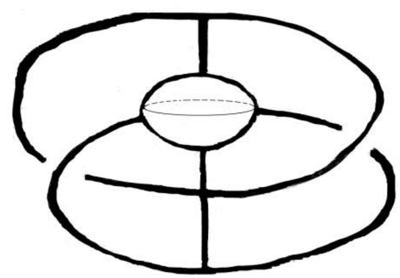
**<u>Theorem 3.6.2.</u>** **For $\forall$ given graph $\Gamma$ , $\forall$ surface F in $R^3$ and $\forall$ integer $n \geqq 1$, $\exists$ $\infty$-many spatial graphs G of $\Gamma$ on F such that**

$$u_\beta(G) = u(G) = u_\gamma(G) = u^\Gamma(G) = u^\Gamma_\gamma(G) = n.$$

**<u>Proof of Theorem 4.6.1.</u> The inequalities are direct from definitions.**

**We show that these invariants are distinct.**

**(1)**

**G=** 

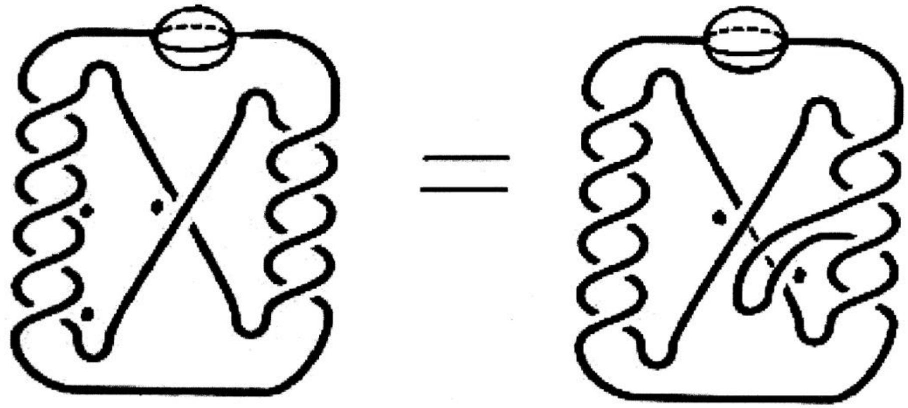
**G^ has $c_\gamma$(G^)=2 and hence $u_\beta$(G)=u(G)=$u_\gamma$(G)=0.**

**On the other hand, we have**

$$u^\Gamma(G)=u^\Gamma_\gamma(G)=1,$$

**for G^ is a spatial graph of a plane graph with a Hopf link as a constituent link and hence not Γ-unknotted.**

(2)

G= 

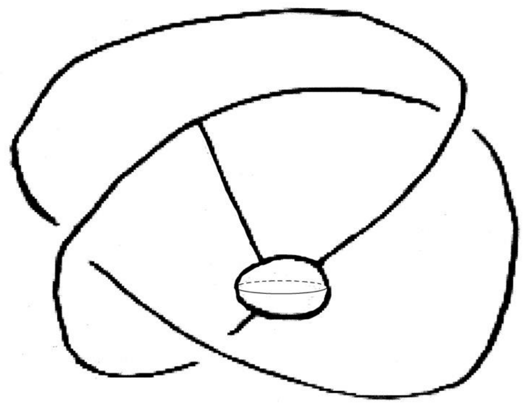
**$\hat{G}=10_8$ has $u(10_8)=2$ and $u_\gamma(10_8)=3$**

**by [Nakanishi 1983] and [Bleiler 1984].**

**Hence**

**$u_\beta(G)= u(G) =u^\Gamma(G)=2< u_\gamma (G)=u^\Gamma_\gamma (G)=3.$**

(3)

G= 

Then $u_\beta(G) = 0$. Since G^ is a Θ-curve,

$u(G^\wedge)=0 \Leftrightarrow$ G^ is isotopic to a plane graph.

Thus, $u(G) \geqq 1$ and we have

$$u(G) = u^\Gamma(G) = u_\gamma(G) = u^\Gamma_\gamma(G) = 1.$$ //

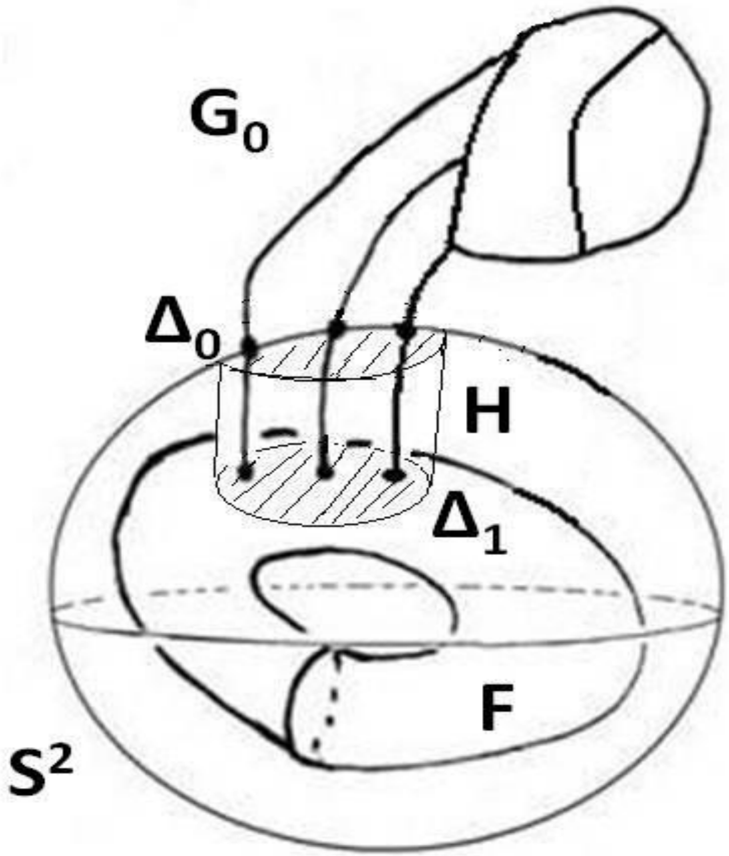
**Proof of Theorem 3.6.2.**

**Assume $v_1(\Gamma) \neq \phi$.**

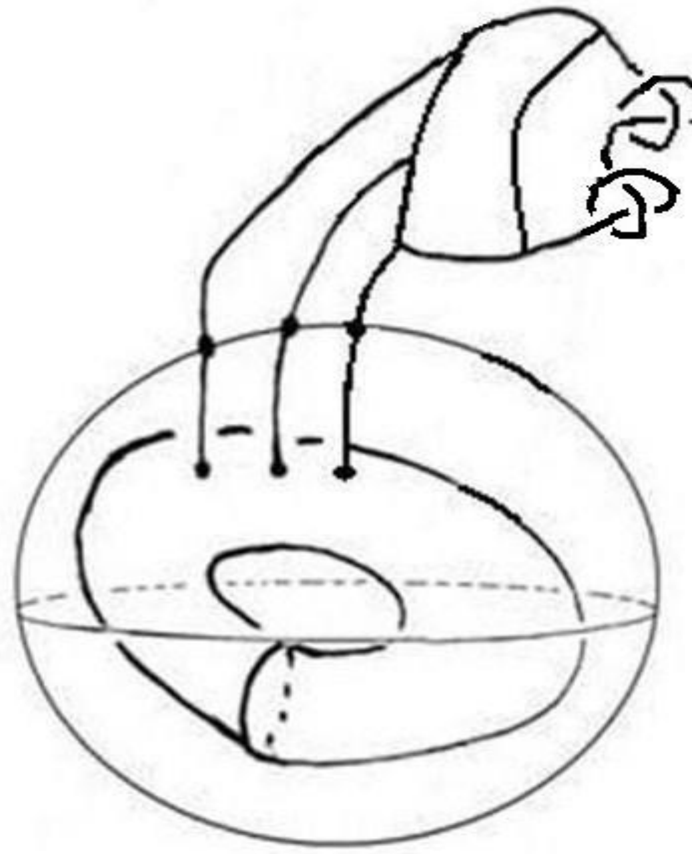
**Assume Γ and F are connected for simplicity.**

**Let F be in the interior of a 3-ball $B \subset S^3$, and $S^2 = \partial B$.**

**Let $G_0$ be a Γ-unknotted graph on $S^2$ in $B^c = cl(S^3 - B)$ and extend it to a Γ-unknotted graph $G_1$ on F by taking in B a 1-handle H joining a 2-cell $\Delta_0$ of $S^2$ and a 2-cell $\Delta_1$ of F and then taking $|v_1(\Gamma)|$ parallel arcs in H.**




**A Γ-unknotted graph $G_1$ on F**



**A Γ-spatial graph G on F**

**Note that $G_0$^= $G_0$ / $\Delta_0$ and $G_1$^= $G_1$ / $\Delta_1$ are isotopic Γ-unknotted graphs in $S^3$.**

**We take a Γ-spatial graph G on F with $v_1(G) \subset \Delta_1$ such that G^ =G / $\Delta_1$ is a connected sum $G_1$^#K(n) of an edge of $G_1$^ (in a part of $G_0$ ) and K(n) attaching to a base of $G_1$^, where K(n) is the n-fold connected sum of a trefoil knot K.**

**Then $u^{\Gamma}_{\gamma}(G) \leqq n$.**

**We show $u_\beta(G) \geqq n$.**
**Let $u_\beta(G) = u_\beta(G^\wedge{}')$ for $G^\wedge{}' = G / \Delta'$ for a 2-cell $\Delta'$ in F.**
**Assume that $u_\beta(G)=k$ and a β-unknotted graph $(G^\wedge{}')'$ is obtained from $G^\wedge{}'$ by k crossing changes on edges $\alpha_i$ (i=1,2,…,m) attaching to a base T' in $G^\wedge{}'$.**
**As it is explained in the case $v_1(\Gamma) = F = \phi$, we take orientations on the edges $\alpha_i$ (i=1,2,…,m) and take an epimorphism $\chi: H_1(E(G^\wedge{}')) \to Z$.**

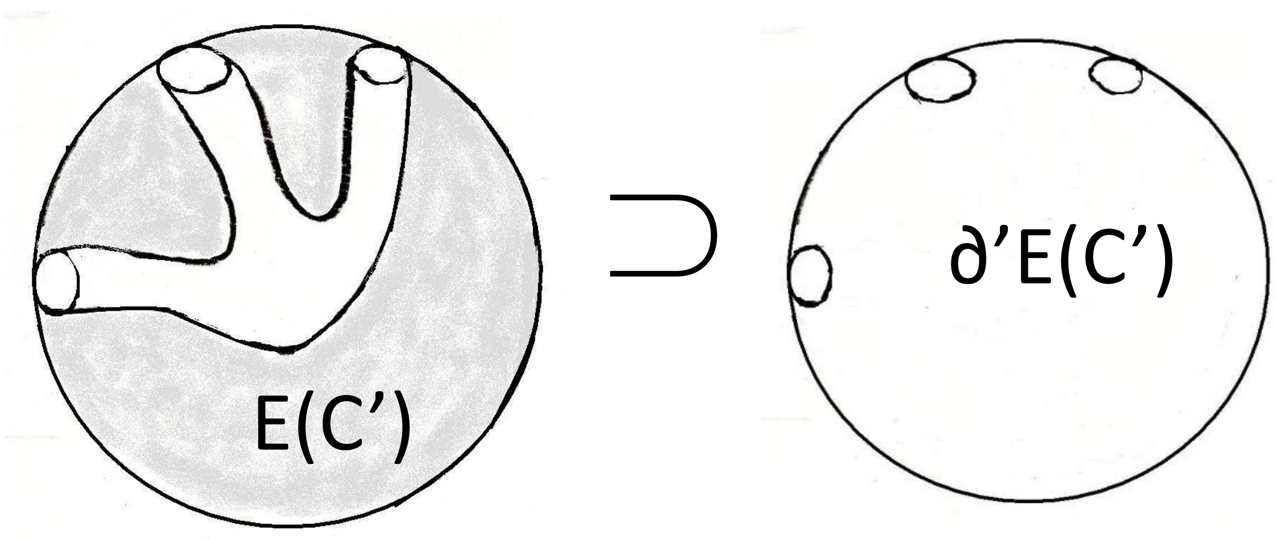
By Lemma A, $|m(G^{\wedge\prime},T')_\infty - m((G^{\wedge\prime})',T')_\infty| \leqq k$.

Note that $m((G^{\wedge\prime})',T')_\infty = m-1$.

Let $C' = G^{\wedge\prime} \cap B$ and $G' = G^{\wedge\prime} \cap B^c$. Then $G^{\wedge\prime} = G' \cup C'$.

Let $E(G') = cl(B^c - N(G'))$, $E(C') = cl(B - N(C'))$ and

$\partial' E(C') = E(C') \cap \partial B$.



**Let $E(G')_\infty$, $E(C')_\infty$ and $\partial' E(C')_\infty$ be the lifts of $E(G')$, $E(C')$ and $\partial' E(C')$ under the covering $E(G\hat{}')_\infty \to E(G\hat{}')$, respectively.**

**Let**

$$M(G')_\infty = H_1(E(G')_\infty) \text{ and}$$

$$M(C',\partial' C')_\infty = H_1(E(C')_\infty, \partial' E(C')_\infty).$$

**Lemma B.** $\exists$ a short exact sequence

$$0 \to M(G')_\infty \to M(G^{\wedge\prime},T')_\infty \to M(C',\partial' C')_\infty \to 0,$$

Further, the finite Λ-torsion part $DM(C',\partial' C')_\infty = 0$.

**Proof**. By excision,

$$H_d(E(G^{\wedge\prime})_\infty, E(G')_\infty) = H_d(E(C')_\infty, \partial' E(C')_\infty).$$

Since $H_d(E(C'),\partial' E(C'))=0$ for $d=1,2$, we see from

A. Kawauchi, Three dualities on the integral homology of infinite cyclic coverings of manifolds, Osaka J. Math. 23(1986),633-651.

that $H_2(E(C')_\infty,\partial' E(C')_\infty)=0$ and $M(C',\partial' C')_\infty$ is a torsion Λ-module with $DM(C',\partial' C')_\infty=0$.

**The homology exact sequence of the pair $(E(G\hat{}')_\infty, E(G')_\infty)$ induces an exact sequence:**

$$0 \to H_1(E(G')_\infty) \to H_1(E(G\hat{}')_\infty) \to H_1(E(G\hat{}')_\infty, E(G')_\infty) \to 0.$$

**This sequence is equivalent to an exact sequence**

$$0 \to M(G')_\infty \to M(G\hat{}', T')_\infty \to M(C', \partial' C')_\infty \to 0. \ //$$

**Note that $M(G')_\infty = M(G^\wedge, T)_\infty$ for a base T of G^ corresponding to the base T'of G^'.**

**By an argument of the case $v(\Gamma) = F = \phi$ ,**

$$m(G')_\infty = m(G^\wedge, T)_\infty = m+n-1$$

**for the minimal number $m(G')_\infty$ of Λ-generators of $M(G')_\infty$.**

## Lemma C

**A. Kawauchi, On the integral homology of infinite cyclic coverings of links, Kobe J. Math. 4(1987),31-41.**

**Let M' be a Λ-submodule of a finitely generated Λ-module M. Let m' and m be the minimal numbers of Λ-generators of M' and M, respectively. If D(M/M') =0, then m' ≦ m.**

**Proof.** For a Λ-epimorphism f: $\Lambda^m \to$ M, let B=$f^{-1}$(M')⊂$\Lambda^m$, which is mapped onto M'. Since $\Lambda^m$/B is isomorphic to M/M' which has projective dimension ≦1, B is Λ-free, i.e., B=$\Lambda^b$ with b≦m. Hence m' ≦b ≦m. //

**By Lemma C,**

$$m(G^{\wedge\prime},T')_\infty \geqq m(G')_\infty = m+n-1.$$

**Since** $m((G^{\wedge\prime})',T')_\infty = m-1$, **we have**

$$k \geqq m(G^{\wedge\prime},T')_\infty - m((G^{\wedge\prime})',T')_\infty \geqq n.$$

**Hence** $u_\beta(G) \geqq n$ **and**

$$u_\beta(G)= u(G)= u_\gamma(G)= u^\Gamma(G)= u^\Gamma_\gamma(G)=n.//$$